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ASCENT, DESCENT, AND ERGODIC PROPERTIES OF LINEAR OPERATORS

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ABSTRACT. We study the role of ascent, descent, and closedness of operator ranges in the ergodic behaviour of linear operators with respect to various operator topologies.

KEYWORDS: *Ascent, descent, spectrum, resolvent, convergence of the Cesàro means, closed range.*

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1. INTRODUCTION

Let V be a bounded linear operator on a Banach space X (in brief, $V \in B(X)$). Let $N(V)$ and $R(V)$ denote the null-space and the range of V, respectively. Recall that the *ascent* of V is the smallest nonnegative integer n such that $N(V^n) = N(V^{n+1})$; if no such n exists, we write $\operatorname{asc} V = \infty$. Similarly, the *descent* of V is the smallest nonnegative integer n such that $R(V^n) = R(V^{n+1})$; if there is no such n, we write $\operatorname{des} V = \infty$. (See e.g. [6], p. 10.)

It may be instructive to remark that one may have $\operatorname{des} V = d < \infty$ without $R(V^d)$ being closed in norm (see Example 5 from [22], or another example at the end of this paper). On the other hand, $\operatorname{asc} V < \infty$ is equivalent to the closedness of the union of all the null-spaces $N(V^n)$ (see Lemma 1 from [9]). However, if both $\operatorname{asc} V$ and $\operatorname{des} V$ are finite, then they are equal (see Lemma 1.4.2 from [6]), and the common value d gives a splitting of X into the direct sum of the closed subspaces $N(V^d)$ and $R(V^d)$ (see Proposition 1.4.3 and Lemma 3.2.4 from [6]). In other words (see Lemma 3.4.2 from [6] or p. 330 from [27]), this means that 0 is a pole of order d of the resolvent of V. This, in turn, is relevant to the behaviour of the powers of $I - V$ and their various means (see, for instance, [21] and [5]).

We shall need the following, apparently less well-known, characterization of finite ascent and descent considered separately.

LEMMA 1.1. *Given a nonnegative integer d and $V \in B(X)$, we have:*

(i) $\operatorname{asc} V \leqslant d$ *if and only if* $R(V^d) \cap N(V^m) = 0$ *for some (equivalently, all)* $m \geqslant 1$;

(ii) $\operatorname{des} V \leqslant d$ *if and only if* $R(V^m) + N(V^d) = X$ *for some (equivalently, all)* $m \geqslant 1$.

Proof. The easiest proof comes from the following simple formulas (cf. Lemma 2.1 from [12]), which we shall need repeatedly. Whenever A and B are linear operators on a vector space, we have

$$A(N(BA)) = R(A) \cap N(B), \tag{1.1a}$$

$$A^{-1}(R(AB)) = R(B) + N(A). \tag{1.1b}$$

By (1.1a), we get $V^d(N(V^{d+m})) = R(V^d) \cap N(V^m)$, and since also $V^{-d}(R(V^d) \cap N(V^m)) = N(V^{d+m})$, we see that $N(V^d) = N(V^{d+m})$ if and only if $R(V^d) \cap N(V^m) = 0$. This proves (i). Similarly, $V^{-d}(R(V^{d+m})) = R(V^m) + N(V^d)$ and $V^d(R(V^m) + N(V^d)) = R(V^{d+m})$ imply (ii). An alternate proof can be found in Theorems V.6.3 and V.6.4 from [27]. ∎

Ascent, descent, Fredholm properties, and their generalizations have been vigorously studied for many years and have been applied widely in analysis.

In this paper we are motivated partly by applications to the ergodic theory of linear operators on Banach spaces. Ergodic theory is concerned with the existence of limits, in various operator topologies, of the Cesàro means

$$A_n(T) = \frac{I + T + T^2 + \cdots + T^{n-1}}{n},$$

where T is a bounded operator on a Banach space, and $n = 1, 2, \ldots$. Frequent use is made of the following simple formulas (see Chapter 2 in [16], [21] and [23]):

$$\frac{T^n}{n} = \frac{n+1}{n} A_{n+1}(T) - A_n(T), \tag{1.2a}$$

$$(I - T)A_n(T) = \frac{I - T^n}{n}, \tag{1.2b}$$

$$I - A_n(T) = (I - T)\frac{T^{n-2} + 2T^{n-3} + \cdots + (n-2)T + (n-1)I}{n}. \tag{1.2c}$$

If the sequence $\{A_n(T)\}$ converges in some operator topology then, by the Banach-Steinhaus principle, it is bounded in the operator norm. Moreover, it follows from (1.2a) that $\{n^{-1}T^n\}$ converges to 0 in the same operator topology. Much of operator ergodic theory is concerned with determining when, conversely, the convergence to 0 of $\{n^{-1}T^n\}$ and/or the boundedness of $\{A_n(T)\}$ implies the convergence of $\{A_n(T)\}$ in the operator topology considered. The following main result, which will be proved in Section 3, clarifies the relationship between the equivalent conditions in the uniform ergodic theorem (Théorème 1 from [23]), and shows their role in the other operator topologies. See also Theorem 3.4 from [5]. Our proof relies on Theorem 2.1, which seems to be of independent interest.

Theorem 1.2. *Let $T \in B(X)$. Suppose that either*

(i) *$\{A_n(T)\}$ is bounded, or*

(ii) *$\{n^{-1}T^n\}$ converges to* 0 *in some operator topology.*

If either

(iii) *$R((I-T)^n)$ is closed for some $n \geqslant 2$, or*

(iv) *$R((I-T)^n) + N(I-T)$ is closed for some $n \geqslant 1$,*

then X is the direct sum of the closed subspaces $R(I-T)$ and $N(I-T)$. Moreover, in this case, the sequence $\{A_n(T)\}$ converges in some operator topology if and only if (ii) *holds in the same operator topology; the limit P is the projection of X onto $N(I-T)$ along $R(I-T)$ or, in other words,*

$$P = -\frac{1}{2\pi \mathrm{i}} \int (T - \lambda I)^{-1}\, \mathrm{d}\lambda,$$

the Riesz projection corresponding to the at most simple pole 1 *of the resolvent of T.*

Remark 1.3. If the convergence in (ii) is uniform (in the operator norm), it is well-known ([20]) that also $n = 1$ may be allowed in (iii) to get the same conclusion. It is not so in the case of weak or strong operator topology. Indeed, Laura Burlando (Example 3.8 from [5]) constructs an operator T ($= A$ in her notation) satisfying condition (ii) above, in the strong operator topology, with $R(I-T)$ closed, and with the spectrum of T being $[0,1]$. Therefore, 1 is not a pole of the resolvent of T, consequently, by Theorem 4.4 below, $\{A_n(T)\}$ cannot be bounded.

Moreover, such an example with spectrum equal to $\{1\}$ (as originally asked for in [28], p. 376) is provided by the operator

$$T(x,y) = ((I-V)x, y-x)$$

defined for $(x,y) \in L_2(0,1) \times L_2(0,1)$, where V is the Volterra operator

$$(Vx)(t) = \int_0^t x(s)\, \mathrm{d}s.$$

Note that

$$T^n(x,y) = ((I-V)^n x, y - nA_n(I-V)x)$$

for $n = 1, 2, \ldots$. Since $I - V$ is power-bounded (see [28], p. 370 and [1], p. 15), it follows, from Theorem 2.1.2 of [16], that $\{A_n(I-V)x\}$ converges. The limit is actually zero, by Theorem 2.1.3 from [16], because $N(V) = 0$. Thus, T satisfies condition (ii) above, in the strong operator topology. The definition of T guarantees that $R(I-T)$ is closed, while $R((I-T)^n)$ is not for $n \geqslant 2$. Consequently, by Theorem 1.2, the sum $R(I-T) + N(I-T)$ is not closed, and the point 1 is not a pole of the resolvent of T. By Theorem 4.4, the sequence $\{\|A_n(T)\|\}$ is not even bounded. It also follows from the definition of T that $I-T$ is quasinilpotent, hence the spectrum of T is $\{1\}$. This example is a modification of Example 3.5 from [5]. In view of [14], p. 247, one can also use the space $L_1(0,1)$ in place of $L_2(0,1)$.

Further examples in this direction will be given in Section 4.

REMARK 1.4. If (i) or (ii) holds, then the spectral radius of T is no more than 1. Moreover, the arguments in [21], p. 94 or Lemme from [23] show that each of the assumptions (i) or (ii) alone implies that

$$N(I-T)\cap R(I-T)=0, \tag{1.3}$$

which in turn yields $\operatorname{asc}(I-T)\leqslant 1$ (and similarly $\operatorname{asc}(I-T^*)\leqslant 1$). Thus, each of the conditions (iii) or (iv) in Theorem 1.2 just ensures that $\operatorname{des}(I-T)<\infty$. We shall analyze this step in Section 2.

It would be interesting to know if (i) or (ii) can be replaced by (1.3), or by the still weaker condition $\operatorname{asc}(I-T)<\infty$. Property (1.3) together with (iv) implies (iii), for the same n, by Lemma 3.2.4 from [6] or Theorem IV.5.10 from [27].

REMARK 1.5. For the weak or strong convergence of $\{A_n(T)\}$, neither (iii) nor (iv) is necessary (see Theorem 11 from [14]). However, both assumptions (i) and (ii) together imply that $N(I-T)\cap(R(I-T))^-=0$ (see [16], p. 74), and the sum $N(I-T)\oplus(R(I-T))^-$ is precisely the closed set where $\{A_n(T)\}$ is weakly or strongly convergent (see Theorem 2.1.3 from [16]). Thus, the point is when $N(I-T)\oplus(R(I-T))^-=X$ (see also p. 59 in [10] for another role of this condition). This, however, does not suffice for the uniform convergence, as the standard backward shift shows. In other words, Theorem 1.2 explains when the closure can be omitted in the preceding important condition, as it has to be in the uniform case.

REMARK 1.6. Finite-dimensional examples showing that (i) does not imply (ii) can be derived from Theorem 8 of [28] (see, for instance, Example 3 from [21]). Conversely, Example 2 from [26] shows that (ii) does not imply (i).

2. OPERATORS WITH FINITE ASCENT

Let $V\in B(X)$ with $\operatorname{asc}V\leqslant d<\infty$. In this section we determine which additional hypotheses let us conclude that certain $R(V^n)$ or $R(V^j)+N(V^k)$ are closed, or that $X=R(V^d)\oplus N(V^d)$. The closed subspace result below is related to Lemmas 7 and 10 from [22], Théorème 1 from [23], and, in particular, Theorem 3.2 from [12]. See also Lemma 3.1 and Theorem 3.4 from [5].

THEOREM 2.1. *Let $V\in B(X)$ with $\operatorname{asc}V\leqslant d<\infty$. If there is an $n>d$ such that $R(V^n)$ is closed, or $R(V^j)+N(V^k)$ is closed for some positive integers with $j+k=n$, then $R(V^n)$ is closed for all $n\geqslant d$, and $R(V^j)+N(V^k)$ is closed for all $j+k\geqslant d$.*

Notice that the hypotheses of the above theorem require that $n>d$, while the conclusions hold for all $n\geqslant d$. We shall construct examples in Theorem 4.2 with $R(V^d)$ closed and $R(V^j)+N(V^k)$ closed for all $j+k=d$, but with $R(V^n)$ not closed for any $n>d$, and $R(V^j)+N(V^k)$ not closed for any positive integers j,k with $j+k>d$.

Proof of Theorem 2.1. We shall need two simple observations. Firstly, by formula (1.1b), $V^{-k}R(V^{j+k})=R(V^j)+N(V^k)$, so that, for every $V\in B(X)$, we have the following.

(i) If $R(V^n)$ is closed, so is $R(V^j)+N(V^k)$ whenever $j+k=n$.

Thus, it is enough to prove that $R(V^n)$ is closed for all $n \geqslant d$.

Secondly, if $n \geqslant d$ and $m \geqslant 1$, it follows from Lemma 1.1 (i) that $R(V^n) \cap N(V^m) \subset R(V^d) \cap N(V^m) = 0$. Thus Lemma 3.2.4 from [6] or Theorem IV.5.10 from [27] gives the following.

(ii) If ($\operatorname{asc} V \leqslant d$ and) $n \geqslant d$, then $R(V^n)$ is closed whenever $R(V^n)+N(V^m)$ is closed for some $m \geqslant 1$.

We now separate the hypotheses of the theorem into several cases.

Case 1. Suppose $R(V^n)$ is closed for some $n > d$. For this case, we need only show that $R(V^{n-1})$ and $R(V^{n+1})$ are both closed. It follows from (i) that $R(V^{n-1}) + N(V)$ is closed, and then from (ii) that $R(V^{n-1})$ is closed. Next, the restriction of V to the closed invariant subspace $R(V^{n-1})$ is one-to-one because, as noted above, $R(V^{n-1}) \cap N(V) = 0$. Thus this restriction is a Banach space isomorphism from the closed subspace $R(V^{n-1})$ onto the closed subspace $R(V^n)$. It carries the closed subspace $R(V^n)$ onto $R(V^{n+1})$, which must therefore also be closed.

Case 2. Suppose $R(V^n) + N(V^m)$ is closed for some $n > d$ and $m \geqslant 1$. Then (ii) implies that $R(V^n)$ is closed, reducing this case to Case 1.

Case 3. Suppose $R(V^j) + N(V^m)$ is closed for some $j \geqslant 1$ and $m \geqslant d$. By Theorem 3.2 from [12], which is applicable by Theorem 3.1 from [12] since $\operatorname{asc} V \leqslant d$, we conclude that

(iii) $R(V^j) + N(V^m)$ is closed for all $j \geqslant 1$.

Now we can assume that $j > d$, reducing this case to Case 2.

Case 4. Suppose, finally, that $R(V^j) + N(V^k)$ is closed for some arbitrary positive integers with $j + k > d$. Since $k \geqslant d$ was considered in Case 3, we can assume that $k < d < j + k$. Using formula (1.1b) we find that

$$\begin{aligned} R(V^{j+k-d}) + N(V^d) &= V^{-d}R(V^{j+k}) = V^{-(d-k)}V^{-k}R(V^{j+k}) \\ &= V^{-(d-k)}(R(V^j) + N(V^k)). \end{aligned}$$

Thus $R(V^{j+k-d})+N(V^d)$ is closed. This reduces Case 4 to Case 3, and completes the proof of the theorem. ∎

We can also give a direct elementary proof, avoiding Theorem 3.2 from [12], of Theorem 2.1 in the case that $R(V^d) + N(V^d)$ is assumed closed. When $d = 1$, which is the situation in Theorem 1.2, this is the only remaining case after Cases 1 and 2 are done. When $R(V^d) + N(V^d)$ is closed, it follows from (ii) that $R(V^d)$ is closed. Since $N(V^{2d}) = N(V^d)$, we see that the Kato minimum modulus of V^{2d} is also positive, hence $R(V^{2d})$ is closed. Thus we are again in Case 1.

Corollary 2.2. *If, in addition to the hypotheses of Theorem* 2.1, *also* $\operatorname{asc} V^* < \infty$, *then* $X = R(V^d) \oplus N(V^d)$.

Proof. By Theorem 2.1, the ranges $R(V^n)$ are closed for all $n \geqslant d$. Hence, by Lemma 6.4.8 from [2] or Theorem 1.2.4 from [6], we have $R(V^n) = {}^{\perp}N(V^{*n})$ for these n. Thus, $\operatorname{asc} V^* < \infty$ implies $\operatorname{des} V < \infty$, so that $X = R(V^d) \oplus N(V^d)$ as mentioned at the beginning of Section 1. ∎

Let us also mention the following related result, which is just Theorem 5.4 from [11], or a special case of Corollary 4.9 from [12]; when $\operatorname{asc} V \leqslant d$, the result follows from Theorem 2.7 in [19] and Theorem 2.1. See also Lemma 5.1 and Theorem 5.4 below.

Theorem 2.3. *Let $T \in B(X)$ and let $V = T - \lambda I$. Suppose that $N(V^d)$ has finite codimension in $N(V^{d+1})$ and that there is an $n > d$ such that either $R(V^n)$ is closed, or $R(V^j) + N(V^k)$ is closed for some positive integers with $j + k = n$.*

If λ belongs to the boundary of the spectrum of T, then it is a pole of the resolvent of T. ∎

3. ERGODIC RESULTS

In this section we prove Theorem 1.2 and some related results in operator ergodic theory. We start with a lemma that characterizes exactly when $I - T$ has ascent no more than 1 in terms of properties of the sequences $\{A_n(T)\}$ and $\{n^{-1}T^n\}$.

Lemma 3.1. *If $T \in B(X)$, then the following are equivalent:*
(i) $\operatorname{asc}(I - T) \leqslant 1$*;*
(ii) *$\{n^{-1}T^n z\}$ converges to 0 in norm (or, equivalently, is any designated weaker linear space topology) for each z in $N((I - T)^2)$;*
(iii) *$\{A_n(T)z\}$ is bounded for each z in $N((I - T)^2)$.*

Proof. Let $x = (I - T)z$ with $z \in N((I - T)^2)$. Since

$$x = A_n(T)x = A_n(T)(I - T)z = n^{-1}z - n^{-1}T^n z,$$

we see that $\{n^{-1}T^n z\}$ converges to 0 if and only if $x = 0$. In view of Lemma 1.1 (i), this proves the equivalence of (i) and (ii).

Next, denote

$$B_n(T) = \frac{T^{n-2} + 2T^{n-3} + \cdots + (n-2)T + (n-1)I}{n}, \tag{3.1}$$

an expression occurring in formula (1.2c). By (1.2c), with z and x as above, we have

$$z - A_n(T)z = B_n(T)(I - T)z = B_n(T)x = \frac{(n-1)x}{2}$$

(cf. [21], p. 94). Thus, we see that $\{A_n(T)z\}$ is bounded if and only if $x = 0$, which proves the equivalence of (i) and (iii), and completes the proof of the lemma. ∎

Proof of Theorem 1.2. Combining Remark 1.4 with Corollary 2.2, we obtain $X = R(I - T) \oplus N(I - T)$. If $x \in N(I - T)$, then $A_n(T)x = x$, which certainly converges. On the other hand, if $x = (I - T)y$ belongs to $R(I - T)$, it follows from formula (1.2b) that

$$A_n(T)x = n^{-1}y - n^{-1}T^n y,$$

so that $\{A_n(T)x\}$ and $\{n^{-1}T^n y\}$ converge simultaneously, to the same limit belonging to $N(I - T)$. This completes the proof of Theorem 1.2. ∎

The authors of [18] consider operators for which there is a positive integer d with $\lim_{n\to\infty}(I-T)^d A_n(T)=0$ in the uniform operator topology. They show that $\mathrm{asc}(I-T)\leqslant d$, and they give conditions under which $X=R((I-T)^d)\oplus N((I-T)^d)$. This is further developed in [5]. Our Theorem 1.2 motivates the following counterpart for the weak operator topology.

Theorem 3.2. *Let $T\in B(X)$ and let d be a nonnegative integer. If $\{(I-T)^d A_n(T)\}$ converges to 0 in the weak operator topology, then both $I-T$ and $I-T^*$ have ascent no more than d.*

Moreover, if there is an $n>d$ such that either $R((I-T)^n)$ is closed, or $R((I-T)^j)+N((I-T)^k)$ is closed for some positive integers with $j+k=n$, then X is the direct sum of the closed subspaces $R((I-T)^d)$ and $N((I-T)^d)$.

Proof. Suppose that x belongs to $R((I-T)^d)\cap N(I-T)$. Since $x\in R((I-T)^d)$, our hypothesis implies that $A_n(T)x\to 0$ weakly. Then $x\in N(I-T)$ forces $x=A_n(T)x=0$. Thus $R((I-T)^d)\cap N(I-T)=0$ and so, by Lemma 1.1 (i), $\mathrm{asc}(I-T)\leqslant d$. Since $(I-T^*)^d A_n(T^*)\to 0$ in the weak*-operator topology, also $\mathrm{asc}(I-T^*)\leqslant d$. The rest of the theorem follows immediately from Corollary 2.2. ∎

4. QUASICOMPLEMENTED OPERATOR RANGES

Recall that two closed subspaces E and F of the Banach space X are *quasicomplements* of each other provided that $E\cap F=0$ and $E+F$ is dense in X. Many constructions of quasicomplements which are not complements are known (see, for instance, [15] or [25]). The recent papers [3] and [4] might be useful for understanding this phenomenon. Some general conditions equivalent to the closedness of the sum of two closed subspaces can be found in Theorem III.3.9 from [7] and Theorem 8 from [24].

The preceding results on ascent and their applications to ergodic theory show that if V is an operator with finite ascent d which satisfies some auxiliary conditions, such as V^* also having finite ascent, then $R(V^d)$ and $N(V^d)$ are complementary closed subspaces, provided that $R(V^n)$ is closed for some n *strictly greater* than d.

In this section we show that if we just assume that $R(V^d)$ is closed, then all we can conclude is that $R(V^d)$ and $N(V^d)$ are quasicomplements (Theorem 4.1), and we give an example in which they are not complements (Theorem 4.2). We then consider when additional ergodic hypotheses, which means in particular that $d=1$, allow us to get complements.

Theorem 4.1. *Supppose that V is an operator on a Banach space such that $R(V^d)$ is closed. Then $R(V^d)$ and $N(V^d)$ are quasicomplements if and only if both V and V^* have ascent no more than d.*

Proof. We know from Lemma 1.1 (i) that $R(V^d)\cap N(V^d)=0$ if and only if $\mathrm{asc}\,V\leqslant d$. Similarly, $\mathrm{asc}\,V^*\leqslant d$ is equivalent to $R(V^{*d})\cap N(V^{*d})=0$. Note that $R(V^d)^\perp = N(V^{*d})$, by definition of the adjoint. Also $N(V^d)^\perp = R(V^{*d})$, by Theorem 1.2.3 from [6]. Hence

$$(R(V^d) + N(V^d))^\perp = R(V^d)^\perp \cap N(V^d)^\perp = N(V^{*d}) \cap R(V^{*d}).$$

Consequently, $R(V^d) + N(V^d)$ is dense if and only if $\operatorname{asc} V^* \leqslant d$. This completes the proof. ∎

THEOREM 4.2. *For each positive integer d, there is a bounded operator V on a Banach space, with the following properties:*

(i) *both V and V^* have ascent no more than d;*
(ii) *$R(V^n)$ is closed as are all $R(V^j) + N(V^k)$ with $j + k = n$ when $n \leqslant d$;*
(iii) *$R(V^n)$ and $R(V^j) + N(V^k)$ are not closed if $n > d$ and $j + k = n$;*
(iv) *$R(V^d)$ and $N(V^d)$ are quasicomplements but not complements.*

The following lemma essentially gives the construction of V when $d = 1$.

LEMMA 4.3. *Suppose that E and F are closed subspaces of the Banach space X. There is a $V \in B(X)$ with $N(V) = E$ and $R(V) = F$ if and only if X/E and F are isomorphic Banach spaces.*

Proof. If $N(V) = E$ and $R(V) = F$, then V induces an isomorphism from X/E onto F. Conversely, if S is a linear isomorphism from X/E onto F, and if π is the canonical projection from X onto X/E, then $V = S\pi$ has $N(V) = E$ and $R(V) = F$. ∎

Proof of Theorem 4.2. First notice that it is enough to prove (iv) and to prove that $R(V^n)$ is closed for all $n \leqslant d$. Then part (i) follows from Theorem 4.1, part (ii) follows from (i) in the proof of Theorem 2.1, and part (iii) follows from Corollary 2.2.

We start by considering the basic case where $d = 1$. By Lemma 4.3, we just have to find closed subspaces E and F of a Banach space X, which are quasicomplementary but not complementary, and such that X/E is isomorphic to F.

Since all infinite-dimensional quotient spaces and closed subspaces of a separable Hilbert space are isomorphic, in this case one need only use the simple well-known construction (see [13], p. 28–29, for instance) of two closed subspaces whose sum is not a closed subspace. See also [15] or Proposition 4.8 from [25].

Now suppose that $d > 1$, and let S be a bounded operator on a Banach space Y with $R(S)$ and $N(S)$ closed quasicomplementary but not complementary subspaces. On $X = Y^d$ we define V by $V(x_1, x_2, \ldots, x_d) = (Sx_1 + x_2, x_3, \ldots, x_d, 0)$. One can easily verify that $R(V^d)$ and $N(V^d)$ are closed subspaces of X which are quasicomplementary but not complementary, and that $R(V^n)$ is closed for all $n \leqslant d$. This completes the proof. ∎

When T satisfies appropriate ergodic hypotheses, then both $I-T$ and $I-T^*$ have ascent no more than 1 (cf. Remark 1.4). It is therefore natural to ask when these added hypotheses let us conclude that $\operatorname{des}(I-T) < \infty$, if we assume $R(I-T)$ closed rather than the stronger and always sufficient hypothesis that $R((I-T)^n)$ is closed for some $n > 1$. When $n^{-1}\|T^n\| \to 0$, it is known (see [20] and [16], p. 87) that the closedness of $R(I-T)$ implies that the restriction of $I-T$ to $R(I-T)$ is invertible so that $R((I-T)^2) = R(I-T)$. As mentioned in Remark 1.3, the latter conclusion cannot be derived under the weaker assumption that $n^{-1}T^n \to 0$ in the weak- or strong-operator topology.

However, if we assume *both* of the alternate ergodic hypotheses in Theorem 1.2, a variation of a standard argument yields the following result.

Theorem 4.4. *Let $T \in B(X)$. Suppose that $\{\|A_n(T)\|\}$ is bounded, and that $n^{-1}T^n \to 0$ in the weak-operator topology. If $R(I-T)$ is closed, then $X = R(I-T) \oplus N(I-T)$.*

Proof. The boundedness of $\{\|A_n(T)\|\}$ implies that $\{x \in X : \operatorname{weak-lim}_{n\to\infty} A_n(T)x$ exists$\}$ is a closed subspace. Moreover, this subspace is equal to $N(I-T) \oplus (R(I-T))^-$, by Theorem 2.1.3 from [16], because the corresponding argument also holds for the weak-convergence $n^{-1}T^n \to 0$. Thus, in our case, $R(I-T) \oplus N(I-T)$ is closed. By Theorem 4.1 and Remark 1.4 we know that $R(I-T)$ and $N(I-T)$ are quasicomplementary. This completes the proof. ∎

If the spectral radius of T is less than 1, then $(I-T)^{-1} = I + T + T^2 + \cdots$. We conclude this section by noting that (1.2c) and (3.1) suggest a formula for $(I-T)^{-1}$ under the weaker assumption that $A_n(T)x \to 0$ for all $x \in X$.

Proposition 4.5. *Suppose that $A_n(T)x \to 0$ for all $x \in X$. Then $(R(I-T))^- = X$, and the following are equivalent:*

(i) *$R(I-T) = X$;*

(ii) *$I-T$ is invertible;*

(iii) *$\{\|B_n(T)\|\}$ is bounded;*

(iv) *$\{B_n(T)x\}$ converges for all $x \in X$.*

In this case, $B_n(T)x \to (I-T)^{-1}x$ for all $x \in X$.

Proof. Our assumption implies that $N(I-T) = 0$. Hence, by Theorem 2.1.3 from [16], we have $(R(I-T))^- = X$, and the rest follows from formula (1.2c). ∎

Remark 4.6. It follows that 1 is an at most simple pole of (the resolvent of) a strongly or weakly ergodic operator T, if and only if the corresponding convergence in Proposition 4.5 (iv) holds for all x in $(R(I-T))^-$, or if and only if Proposition 4.5 (iii) holds for the norms of the restrictions to $(R(I-T))^-$.

5. FINITE ESSENTIAL ASCENT

In this section we shall extend some of the preceding results about operators with finite ascent to operators for which some $N(V^n)$ has finite codimension in $N(V^{n+1})$. Following [22] we shall say that such operators have *finite essential ascent*. Similarly, V has *finite essential descent* if some $R(V^{n+1})$ has finite codimension in $R(V^n)$. Operators with finite essential ascent or descent seem to have been first studied in [11] where a part of Theorem 2.3 above and its analogue for finite essential descent (Theorem 5.2 from [11]) are proven. These operators play an important role in more general studies in [12] and [22]. Also some recent papers have considered operators with $R(V) \cap N(V)$ finite-dimensional which, by Lemma 5.1 below, is equivalent to $N(V)$ having finite codimension in $N(V^2)$. Notice that the sequences $\{\dim(N(V^{n+1})/N(V^n))\}$ and $\{\dim(R(V^n)/R(V^{n+1}))\}$ are nonincreasing, so that $N(V^n)$ has finite codimension in $N(V^{n+1})$ if and only if it has finite codimension in $N(V^{n+m})$ for some (equivalently, all) $m \geqslant 1$; and the analogous observation holds for the ranges. This lets us easily prove characterizations of finite essential ascent and descent which are analogous to the characterizations of finite ascent and descent given in Lemma 1.1. We include proofs for the convenience of the reader.

LEMMA 5.1. *If $V \in B(X)$ and d is a nonnegative integer, then:*

(i) *$N(V^d)$ has finite codimension in $N(V^{d+1})$ if and only if for some (equivalently, all) $m \geqslant 1$ the space $R(V^d) \cap N(V^m)$ is finite-dimensional.*

(ii) *$R(V^{d+1})$ has finite codimension in $R(V^d)$ if and only if for some (equivalently, all) $m \geqslant 1$ the space $R(V^m) + N(V^d)$ has finite codimension in X.*

Proof. From formula (1.1a), we have $V^d N(V^{d+m}) = R(V^d) \cap N(V^m)$ so that V^d induces an isomorphism from $N(V^{d+m})/N(V^d)$ onto $R(V^d) \cap N(V^m)$. This proves (i).

Part (ii) follows analogously from the fact that V^d induces an isomorphism from $X/(R(V^m) + N(V^d))$ onto $R(V^d)/R(V^{d+m})$. ∎

We say that the subspace E of the Banach space X is an *operator range* if it is the range of a bounded operator from some Banach space to X. For characterizations and elementary properties of operator ranges see, for instance, Section 3 of [11] and [8].

We have been already using several times the simple consequence of the closed graph theorem, saying that if E and F are operator ranges in X with $E+F$ closed, then E and F are both closed when $E \cap F = 0$ (cf. Lemma 3.2.4 from [6] or Theorem IV.5.10 from [27]). For our results on closed ranges for operators with finite essential ascent we shall need the following, presumably known, extension of this result (see also Proposition 2.1.1 from [17]).

Lemma 5.2. *Suppose that E and F are operator ranges in the Banach space X, and that $E+F$ is a closed subspace. If $E\cap F$ is closed (in particular, if $E\cap F$ is finite-dimensional), then E and F are both closed.*

Proof. Let M be the closed subspace $E\cap F$. Since $E+F$ is closed in X, we must have $(E+F)/M$ closed in X/M. But $(E+F)/M$ is the algebraic direct sum of the operator ranges E/M and F/M.

Hence E/M and F/M are closed in X/M, so that E and F are closed in X. This completes the proof. ∎

Every Fredholm operator V has finite essential ascent and descent but, in contrast to operators with finite ascent and descent, we need not have some $R(V^d)$ and $N(V^d)$ as complementary closed subspaces. The best we can expect seems to be the following.

Theorem 5.3. *If $V\in B(X)$ has both finite essential ascent and finite essential descent, then we have:*

(i) *for every $n\geqslant 0$, $N(V^n)$ has finite codimension in $N(V^{n+1})$ if and only if $R(V^{n+1})$ has finite codimension in $R(V^n)$;*

(ii) *whenever $R(V^{n+1})$ has finite codimension in $R(V^n)$, then $R(V^n)$ is closed.*

Proof. By hypothesis, both $N(V^{n+1})/N(V^n)$ and $R(V^n)/R(V^{n+1})$ are finite-dimensional for all sufficiently large n. For each of these n, it follows from Lemma 2.3 of [12] that

$$\begin{aligned}\dim(N(V^n)/N(V^{n-1})) &- \dim(N(V^{n+1})/N(V^n))\\ &= \dim(R(V^{n-1})/R(V^n)) - \dim(R(V^n)/R(V^{n+1})).\end{aligned}$$

This proves (i).

When $R(V^{n+1})$ has finite codimension in $R(V^n)$, it follows from part (i) that $N(V^n)$ also has finite codimension in $N(V^{n+1})$. Hence it follows from Lemma 5.1 that the operator range $R(V^n)+N(V^n)$ has finite codimension, and must therefore be closed, and that $R(V^n)\cap N(V^n)$ is finite-dimensional, so also closed. Thus Lemma 5.2 guarantees that $R(V^n)$ is closed, completing the proof. ∎

Next, we show that Theorem 2.1 above on closed ranges for operators with finite ascent carries over, with essentially the same proof, for operators with finite essential ascent (see also Lemmas 7 and 10 from [22]).

Theorem 5.4. *Let $V\in B(X)$. Suppose that $N(V^d)$ has finite codimension in $N(V^{d+1})$. If there is an $n>d$ such that $R(V^n)$ is closed, or $R(V^j)+N(V^k)$ is closed for some positive integers with $j+k=n$, then $R(V^n)$ is closed for all $n\geqslant d$, and $R(V^j)+N(V^k)$ is closed for all $j+k\geqslant d$.*

Proof. It is enough to show that observations (i), (ii) and (iii) in the proof of Theorem 2.1 hold when we assume only that $N(V^d)$ has finite codimension in $N(V^{d+1})$, instead of assuming these spaces are equal. As we pointed out in the proof of Theorem 2.1, (i) holds for all bounded operators. Observation (ii) follows from Lemma 5.1 (i) together with Lemma 5.2. Observation (iii) follows from Lemma 2.4 of [12] in the same way as Theorem 3.2 in [12]. This completes the proof. ∎

If $R(V^{d+1})$ has finite codimension in $R(V^d)$, then $R(V^j) + N(V^k)$ is closed for all $j \geqslant 1$ and $k \geqslant d$ or, which is the same, $R(V^{j+k})$ is closed in the operator range topology of $R(V^k)$. One proves this by using Lemma 2.4 from [12] exactly in the way it is used in proving Theorem 3.2 from [12].

It is also clear that $R(V^d)$ is closed, in this case, if and only if $R(V^m)$ is closed for some (equivalently, all) $m \geqslant d$.

On the other hand, it is easy to construct examples with $R(V^d)$ not closed. For instance, if S is an operator on a Banach space Y with $R(S)$ not closed, we can let $X = l^1(Y)$ and define V as the backward shift given by $V(x_1, x_2, x_3, \ldots) = (Sx_2, x_3, x_4, \ldots)$. Then $R(V) = R(V^2)$ is not closed.

Fortunately if $V = T - \lambda I$ has $R(V^d)/R(V^{d+1})$ finite-dimensional, and λ is in the boundary of the spectrum of T, then we conclude that λ is a pole of the resolvent of T, by Theorem 5.2 from [11] or Corollary 4.9 from [12], without knowing ahead of time that $R(V^d)$ is closed. This complements Theorem 2.3.

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