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# AN INVERSE SPECTRAL PROBLEM FOR HANKEL OPERATORS

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Abstract. We prove that given any compact subset of the complex plane containing zero, there exists a Hankel operator having this set as its spectrum.

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## 1. INTRODUCTION

A classical Hankel operator is a bounded linear operator $\Gamma$ in $\ell^2$ whose matrix with respect to the standard basis in $\ell^2$ is constant on the diagonals perpendicular to the main one. In other words, the matrix entries depend only on the sum of the indices; i.e.

$$\Gamma = (\gamma_{j+k})_{j,k=0}^{\infty}.$$

Hankel operators play an important role in analysis, operator theory, probability, control theory, etc.; see, for example, [4].

A significant interest in Hankel operators can be also explained by the fact that they serve as a bridge between operator theory and function theory. For example, the classical Kronecker Theorem asserts that the Hankel operator $\Gamma$ has finite rank if and only if the function $\sum\limits_{k=0}^{\infty} \gamma_k z^{-k-1}$ is rational and, moreover, the rank of $\Gamma$ is exactly the number of poles (counting multiplicities) of this function in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

In this note we will prove that, given any compact subset of the complex plane containing zero, there exists a Hankel operator having this set as its spectrum. The main ideas can be traced back to [11], which deals with the case of the spectrum of the Hermitian square $\Gamma^*\Gamma$. The precise statement is:

Theorem 1.1. *Let $\sigma$ be any compact subset of the complex plane containing zero. Then there exists a Hankel operator $\Gamma$ such that $\sigma(\Gamma) = \sigma$.*

Let us mention that a Hankel operator $\Gamma$ is never invertible, so $0 \in \sigma(\Gamma)$. Indeed, let $\{e_k\}_{k=0}^{\infty}$ be the standard basis in $\ell^2$. Then $\Gamma e_n = \{\gamma_n, \gamma_{n+1}, \gamma_{n+2}, \ldots\}$. Since $\Gamma e_0 = \{\gamma_0, \gamma_1, \gamma_2, \ldots\} \in \ell^2$, one can conclude that

$$\|\Gamma e_n\|^2 = \sum_{k=n}^{\infty} |\gamma_k|^2 \to 0 \quad \text{as } n \to 0,$$

and so $0 \in \sigma(\Gamma)$.

Before moving further, let us discuss some related results. In some particular cases, the spectral structure of Hankel operators is very well understood. For example, a complete unitary invariant description of self-adjoint Hankel operators (all $\gamma_k$ are real) was obtained in [6].

In the general case, the situation is far from clear. Although our theorem says that there are no constraints on the spectrum of $\Gamma$, except the trivial $0 \in \sigma(\Gamma)$, there are constraints on finer spectral properties of Hankel operators. For example, (see [6]), the spectrum of a Hankel operator is *almost symmetric*: namely, $|\dim\ker(\Gamma - \lambda I) - \dim\ker(\Gamma + \lambda I)| \leqslant 1$. In particular, this implies: if $\lambda$ is a multiple eigenvalue of a Hankel operator $\Gamma$ ($\dim\ker(\Gamma - \lambda I) > 1$), then the point $-\lambda$ has to be an eigenvalue.

It is interesting to compare this to a result by E. Abakumov ([1]), which states that given a finite number of non-zero points $\lambda_1, \lambda_2, \ldots, \lambda_n$ and multiplicities $k_1, k_2, \ldots, k_n$, there exists a finite rank Hankel operator $\Gamma$ such that its non-zero eigenvalues are exactly $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the corresponding *algebraic* multiplicities are exactly $k_1, k_2, \ldots, k_n$ with 0 an eigenvalue of infinite multiplicity (The *algebraic* multiplicity of an eigenvalue $\lambda$ is the dimension of the space of all generalized eigenvectors; namely, the dimension of the space $\bigcup\limits_{n \geqslant 1} \ker(\Gamma - \lambda I)^n$.).

Let us mention two other interesting results. First of all, it was shown by S. Power ([9]) that there exist no non-trivial nilpotent ($\Gamma^n = 0$ for some $n > 0$) Hankel operators. On the other hand, answering a question of S. Power, A. Megretskiĭ ([5]) constructed an example of a nontrivial *quasinilpotent* Hankel operator; i.e., a Hankel operator $\Gamma$ such that $\|\Gamma^n\|^{1/n} \to 0$ as $n \to \infty$, or, equivalently, $\sigma(\Gamma) = \{0\}$.

### 1.1. Hankel operators on the Hardy space $\mathbf{H}^2$.

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$, and let $\mathbb{T}$ be the unit circle, $\mathbb{T} := \partial\mathbb{D}$. Let $m$ be the normalized ($m(\mathbb{T}) = 1$) Lebesgue measure on $\mathbb{T}$, and let $L^2 = L^2(\mathbb{T}, m)$ be the usual $L^2$ space.

For a function $f$ on $\mathbb{T}$ let $\widehat{f}(k)$ denote its $k$-th Fourier coefficient, $\widehat{f}(k) := \int\limits_{\mathbb{T}} f(z) z^{-k}\, \mathrm{d}m(z)$.

Let us recall that the Hardy space $\mathbf{H}^2$ is the *analytic* subspace of $L^2$:

$$\mathbf{H}^2 := \{f \in L^2 : \widehat{f}(k) = 0 \text{ for } k < 0\};$$

the norm in $\mathbf{H}^2$ is just the regular $L^2$-norm.

The space $\mathbf{H}^2$ can be also identified with the space of analytic function on the unit disc $\mathbb{D}$:

$$\mathbf{H}^2 := \Big\{ f(z) = \sum_{k=0}^{\infty} a_k z^k : \|f\|_{\mathbf{H}^2}^2 := \sum_{k=0}^{\infty} |a_k|^2 < \infty \Big\};$$

here $a_k = \widehat{f}(k)$.

Clearly, the Fourier Transform $\{a_k\} \mapsto \sum a_k z^k$ maps the space $\ell^2$ onto the Hardy space $\mathbf{H}^2$, and it is only natural to study Hankel operators acting on $\mathbf{H}^2$.

Let $P_+$ be the orthogonal projection of $L^2$ onto $\mathbf{H}^2$, and let $P_-$ be the orthogonal projection onto $(\mathbf{H}^2)^\perp$.

For a function $\varphi \in L^\infty$, define the operator $H_\varphi : \mathbf{H}^2 \to (\mathbf{H}^2)^\perp$ by the formula

$$H_\varphi f = P_-(\varphi f), \quad f \in \mathbf{H}^2.$$

It is easy to see, that the matrix of the operator $H_\varphi$, with respect to the standard bases $\{z^n\}_{n\geqslant 0}$ and $\{z^n\}_{n<0}$ in $\mathbf{H}^2$ and $(\mathbf{H}^2)^\perp$ respectively, has Hankel structure; i.e., its entries depend only on sum of indices (and $\gamma_k = \widehat{\varphi}(-k-1)$).

Such operators $H_\varphi$ are also called Hankel operators. To make the operator $H_\varphi$ act in $\mathbf{H}^2$, we just compose it with the standard involution $J$ on $L^2$,

$$Jf(z) = \overline{z} f(\overline{z}), \quad z \in \mathbb{T}.$$

The involution $J$ maps $(\mathbf{H}^2)^\perp$ onto $\mathbf{H}^2$ ($J(z^{-1}) = 1$, $J(z^{-2}) = z$, $J(z^{-3}) = z^2, \ldots$), and clearly the operator $\Gamma_\varphi = JH_\varphi$ is a Hankel operator on $\mathbf{H}^2$ (with respect to the standard basis). The function $\varphi$ is called a symbol of the Hankel operator $\Gamma_\varphi$. The symbol is clearly not unique: if we add an analytic function to $\varphi$, the Hankel operator does not change.

The famous Nehari Theorem ([7]; for a modern treatment see Power ([8])) says that any Hankel operator $\Gamma$ in $\mathbf{H}^2$ can be represented as $\Gamma_\varphi$, and, moreover, a symbol $\varphi$ can be chosen in such a way that $\|\varphi\|_\infty = \|\Gamma\|$ (the inequality $\|\Gamma_\varphi\| \leqslant \|\varphi\|_\infty$ is trivial).

## 2. IDEA OF THE PROOF AND MAIN CONSTRUCTION

It is very easy to construct a Hilbert space operator $T$ with prescribed spectrum $\sigma$. One can take a sequence $\{b_n\}_{n=1}^\infty$ which is dense in $\sigma$ and make $T$ a diagonal operator with $\{b_n\}_{n=1}^\infty$ on the diagonal. In other words, one picks an orthonormal basis $\{e_n\}_{n=1}^\infty$ and defines an operator $T$ by $Te_n = b_n e_n$ for all $n$. Clearly, $\sigma(T) = \operatorname{clos}(\{b_n\}_{n=1}^\infty) = \sigma$.

Note that if $0 \in \sigma$, we do not have to assume that the system $\{e_n\}_{n=1}^\infty$ is complete: any orthogonal system will work.

We also do not have to assume that the system $\{e_n\}_{n=1}^\infty$ is orthogonal: it is enough to assume that the system is a Riesz basis. Let us recall that a system of vectors $\{f_n\}$ in a Hilbert space $\mathcal{H}$ is called a *Riesz basis* if there exists a bounded invertible operator $R$ (the so-called *orthogonalizer*) which maps the system into an orthonormal basis (some basic facts about Riesz bases can be found in Young ([12])).

The orthogonalizer is unique up to a unitary factor on the left. The quantity $\|R\| \cdot \|R^{-1}\|$ is called the *measure of non-orthogonality* of the Riesz basis: $\|R\| \cdot \|R^{-1}\| = 1$ if and only if the system is a multiple of an orthonormal basis.

Again, if $0 \in \sigma$ we do not need the system to be complete: it is enough to require that it is a Riesz basis in its closed linear span.

Unfortunately, we do not have simple "building blocks" to construct a Hankel operator with prescribed spectrum. For example, if we take eigenvectors of a Hankel operator and change the corresponding eigenvalues, the resulting operators generally will not be Hankel. This makes the construction complicated.

We begin with a simple problem: given a non-zero (complex) number $b$, construct a rank one Hankel operator whose non-zero eigenvalue is exactly $b$.

We need the notion of a *reproducing kernel.* Let us recall that the function $K_a(z) := (1 - \overline{a}z)^{-1}$, $a \in \mathbb{D}$ is called the *reproducing kernel* of $\mathbf{H}^2$ at the point $a$. The reason for this name is the fact that for any $f \in \mathbf{H}^2$

$$(f, K_a) = f(a).$$

The above identity implies that $\|K_a\|^2 = (1 - |a|^2)^{-1}$, and therefore it is easy to find the normalized reproducing kernel

$$k_a(z) := \|K_a\|^{-1} K_a(z) = \frac{(1 - |a|^2)^{1/2}}{1 - \overline{a}z}.$$

Consider a Hankel operator $\Gamma = \Gamma_{\varphi_a}$, with symbol $\varphi_a(z) = (z - a)^{-1}$, $a \in \mathbb{D}$. Take $f \in \mathbf{H}^2$. To compute $\Gamma_{\varphi_a} f$, notice that

$$\frac{1}{z - a} f(z) = \frac{f(a)}{z - a} + \frac{f(z) - f(a)}{z - a}.$$

Clearly, the second term belongs to $\mathbf{H}^2$ and the first is orthogonal to $\mathbf{H}^2$. Recalling that $\Gamma_\varphi f = JP_-(\varphi f)$, where $Jf(z) = \overline{z} f(\overline{z})$, we get (for $z \in \mathbb{T}$)

$$\Gamma_{\varphi_a} f(z) = \frac{\overline{z} f(a)}{\overline{z} - a} = \frac{f(a)}{1 - az} = (f, K_a) K_{\overline{a}} = \frac{1}{1 - |a|^2} \cdot (f, k_a) k_{\overline{a}}. \tag{2.1}$$

Notice that $(\cdot\,, k_a) k_a$ is the orthogonal projection onto the one-dimensional subspace spanned by $k_a$. Therefore, if $a$ is real, we have that for $\varphi(z) = b \cdot (1 - |a|^2) \cdot (z - a)^{-1}$,

$$\Gamma_\varphi = b(\cdot\,, k_a) k_a\,, \quad \text{and so} \quad \Gamma_\varphi k_a = b \cdot k_a.$$

Thus we constructed a rank one Hankel operator with an eigenvalue $b$.

If we could find an orthogonal sequence of reproducing kernels $\{k_{a_n}\}_{n=1}^\infty$, we would be done. Unfortunately, no two reproducing kernels are orthogonal. Fortunately, they are *asymptotically orthogonal.* Namely, if we fix $a_1 \in \mathbb{D}$, then

$$(k_{a_1}, k_a) = \frac{(1 - |a_1|^2)^{1/2} (1 - |a|^2)^{1/2}}{1 - \overline{a}_1 a} \to 0 \quad \text{as } |a| \to 1^-. \tag{2.2}$$

This is the main property we need for our construction.

2.1. Main construction. Pick a dense disjoint sequence $\{b_n\}_{n=1}^\infty$ in $\sigma \setminus \{0\}$. If we constructed a sequence of finite-rank Hankel operators $\Gamma_{\varphi_n}$ (with $\Gamma_{\varphi_n}$ of rank $n$) such that

(i) range $\Gamma_{\varphi_n} = (\ker \Gamma_{\varphi_n})^\perp$; i.e., range $\Gamma_{\varphi_n}$ is a reducing subspace of $\Gamma_{\varphi_n}$;

(ii) range $\Gamma_{\varphi_n} \subset$ range $\Gamma_{\varphi_{n+1}}$;

(iii) the (non-zero) eigenvalues of $\Gamma_{\varphi_n}$ are exactly $b_1, b_2, \ldots, b_n$, with the corresponding *normalized* ($\|f_k^{(n)}\| = 1$) eigenvectors $f_1^{(n)}, f_2^{(n)}, \ldots, f_n^{(n)}$;

(iv) the measure of non-orthogonality $\|R\| \cdot \|R^{-1}\|$ of each system $\{f_k^{(n)}\}_{k=1}^n$, for each $n$, is strictly less than 2 (each system is a finite linearly independent system, so it is a Riesz basis in its linear span, so for each system one can construct an orthogonalizer $R$ and compute the measure of non-orthogonality $\|R\| \cdot \|R^{-1}\|$); and

(v) $\|f_k^{(n)} - f_k^{(n+1)}\| \leqslant 2^{-n}$;

then we are done!

Before explaining why it is so, let us mention that we will construct the operators $\Gamma_{\varphi_n}$ (or, equivalently, symbols $\varphi_n$) by induction, and to perform this induction we will need an additional condition, namely the condition (vi) from Subsection 2.2 below.

However, conditions (i)–(v) alone are sufficient for the conclusion of the main theorem, so we discuss the technical condition (vi) later in Subsection 2.2.

So, let us show that if we construct the operators $\Gamma_{\varphi_n}$ satisfying the conditions (i)–(v) above, then we obtain a Hankel operator with the prescribed spectrum.

Consider the trivial cases first. If $\sigma = \{0\}$, then we can just put $\Gamma = 0$. For an example of a nontrivial Hankel operator with zero spectrum see the paper by Megretskiĭ ([5]) mentioned above.

If $\sigma$ consists of zero and $n$ other points, notice that constructing an operator $\Gamma_{\varphi_n}$ satisfying (iii) solves the problem (for another solution see [1]).

Thus we need only consider the case when $\sigma$ consists of infinitely many points. In this case, the construction gives us a Hankel operator with the desired spectral property, as we now describe.

Indeed, let us first notice that condition (v) implies that $\{f_k^{(n)}\} \to f_k$ as $n \to \infty$ for some $f_k \in \mathbf{H}^2$. Since for every fixed $N \leqslant n$ the finite set $\{f_k^{(n)}\}_{k=1}^N$ is clearly a Riesz basis, it is easy to prove that the set $\{f_k\}_{k=1}^N$ is also a Riesz basis (since $\{f_k^{(n)}\} \to f_k$ as $n \to \infty$) and its measure of non-orthogonality is also bounded by 2 (since, by condition (iv), the measure of non-orthogonality of every set $\{f_k^{(n)}\}_{k=1}^N$ is bounded by 2). Secondly, since every finite set $\{f_k\}_{k=1}^N$ is a Riesz basis, it follows that the system $\{f_k\}_{k=1}^\infty$ is also a Riesz basis (for its closed linear span), and its measure of non-orthogonality $\|R\| \cdot \|R^{-1}\|$ is at most 2 (because it is sufficient to compute the norms of $R$ and $R^{-1}$ on dense sets of linear combinations $\sum c_k f_k$ and $\sum c_k R f_k$ respectively).

Define an operator $\Gamma$ on $\mathbf{H}^2$ by

$$\Gamma f_k = b_k f_k, \quad \text{for each } k \geqslant 1,$$

and

$$\Gamma f = 0 \quad \text{when } f \text{ is orthogonal to all the } \{f_k\}.$$

Then

$$\begin{aligned}\|\Gamma f_k - \Gamma_{\varphi_n} f_k\| &\leqslant \|\Gamma f_k - \Gamma_{\varphi_n} f_k^{(n)}\| + \|\Gamma_{\varphi_n} f_k^{(n)} - \Gamma_{\varphi_n} f_k\| \\ &\leqslant |b_k|\,\|f_k - f_k^{(n)}\| + \|\Gamma_{\varphi_n}\|\,\|f_k - f_k^{(n)}\| \\ &\leqslant (\operatorname{diam}\sigma + 2\operatorname{diam}\sigma)\,\|f_k - f_k^{(n)}\|,\end{aligned}$$

where the last inequality follows because the norms of the operators $\Gamma_{\varphi_n}$ are all bounded by $2\operatorname{diam}\sigma$ (this is obtained by observing that when restricted to $\operatorname{range}\Gamma_{\varphi_n}$, $\Gamma_{\varphi_n} = R_n^{-1}\operatorname{diag}\{b_1, b_2, \dots, b_n\}R_n$ where $R_n$ is the orthogonalizer corresponding to the Riesz basis $\{f_k^{(n)}\}_{k=1}^n$). But this shows that $\Gamma_{\varphi_n} \to \Gamma$ in the strong operator topology. Therefore $\Gamma$ is a Hankel operator.

Clearly the spectrum of $\Gamma$ is $\operatorname{clos}\{b_k : k \geqslant 1\}$ if the system $\{f_k\}_{k=1}^\infty$ is complete, and $\operatorname{clos}\{b_k : k \geqslant 1\} \cup \{0\}$ if it is not complete. Since for a Hankel operator $\Gamma$ one always has $0 \in \sigma(\Gamma)$, our operator has the prescribed spectrum in either case, but notice that the system $\{f_k\}_{k=1}^\infty$ may be complete only if $0 \in \operatorname{clos}\{b_k : k \geqslant 1\}$.

Let us also mention that, since the subspace $E_n := \operatorname{range}\Gamma_{\varphi_n}$ is a reducing subspace for $\Gamma_{\varphi_n}$, one can forget about $\ker\Gamma_{\varphi_n}$, and treat $\Gamma_{\varphi_n}$ as an operator acting on the finite dimensional space $E_n$.

We will construct the operators $\Gamma_{\varphi_n}$ by induction. The symbols $\varphi_n$ will be of the form

$$\varphi_n(z) = \sum_{k=1}^n b_k \cdot \frac{1-|a_k|^2}{z-a_k} \cdot (1 + t_k^{(n)}), \quad a_k \in (0,1),\ t_k^{(n)} \in \mathbb{R} \setminus \{-1\} \tag{2.3}$$

(we will have to chose $a_k$ and $t_k^{(n)}$). It follows from (2.1) that

$$\Gamma_{\varphi_n} f = \sum_{k=1}^n b_k \cdot (1 + t_k^{(n)}) \cdot (f, k_{a_k}) k_{a_k}.$$

Therefore $\operatorname{range}\Gamma_{\varphi_n} = \operatorname{span}\{k_{a_k} : 1 \leqslant k \leqslant n\}$ and $\ker\Gamma_{\varphi_n} = (\operatorname{span}\{k_{a_k} : 1 \leqslant k \leqslant n\})^\perp$, so conditions (i) and (ii) hold automatically.

The choice of $\varphi_1$ is trivial: in equation (2.3) above pick an arbitrary $a_1$ and put $t_1^{(1)} = 0$ (that it satisfies all our conditions follows from equation (2.1) above). Suppose we have constructed symbols $\varphi_1, \dots, \varphi_n$. If we pick a point $a_{n+1}$ close enough to the boundary of the disk, the reproducing kernel $k_{a_{n+1}}$ will be almost orthogonal to $k_{a_1}, \dots, k_{a_n}$. Therefore the Hankel operator with symbol $\varphi_n + b_{n+1} \cdot (1 - |a_{n+1}|^2)/(z - a_{n+1})$ will be almost the operator $\Gamma_{\varphi_{n+1}}$ we need (the eigenvalues are close to the desired eigenvalues, and the the other conditions are also satisfied). To get the desired operator $\Gamma_{\varphi_{n+1}}$ we just perturb the parameters $t_k^{(n)}$ a little.

To show that such perturbation is possible, we will use the implicit function theorem. For this we will need one more assumption.

2.2. An additional assumption. Consider families of symbols $\varphi_{n,\tau}$, parameterized by $\tau = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$, where

$$\varphi_{n,\tau}(z) = \sum_{k=1}^n b_k \cdot \frac{1-|a_k|^2}{z-a_k} \cdot (1 + t_k).$$

If we put $\tau^{(n)} := (t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)})$, then, in this notation, $\varphi_n = \varphi_{n,\tau^{(n)}}$.

Let $\Lambda^{(n)}(\tau) = (\lambda_1^{(n)}(\tau), \lambda_2^{(n)}(\tau), \dots, \lambda_n^{(n)}(\tau))$ be the nonzero eigenvalues of the Hankel operator $\Gamma = \Gamma_{\varphi_{n,\tau}}$ with symbol $\varphi_{n,\tau}$. Then the last condition required is

(vi) the Jacobian $\frac{\mathrm{d}\Lambda^{(n)}}{\mathrm{d}\tau} = \left\{ \frac{\partial \lambda_j^{(n)}}{\partial t_k} \right\}_{j,k=1}^{n}$ is non-singular at $\tau = \tau^{(n)}$.

Clearly, the ordering of eigenvalues here is not essential. It is natural for our purposes to order the eigenvalues such that $\lambda_k^{(n)}(\tau^{(n)}) = b_k$.

Since $b_k \neq b_j$ whenever $k \neq j$, it follows that $\lambda_k^{(n)}(\tau) \neq \lambda_j^{(n)}(\tau)$ for $j \neq k$ in a small neighborhood of $\tau^{(n)}$; so by Lemma 3.1 below, the functions $\tau \mapsto \lambda_k(\tau)$ are continuously differentiable. Therefore the Jacobian $\mathrm{d}\Lambda/\mathrm{d}\tau$ is well-defined.

## 3. TECHNICAL LEMMAS

To prove the theorem we need few technical lemmas. Although they hold for general operators, we will only need them for operators on a finite dimensional space (matrices).

Let us recall that a point $\lambda$ is called a *simple isolated eigenvalue* of an operator $A$ if it is an isolated point of $\sigma(A)$ and the dimension of the corresponding spectral subspace is 1. For matrices this means that $\lambda$ is a simple root of the characteristic polynomial $p(z) := \det(A - zI)$.

Lemma 3.1. *Let $t \mapsto A(t)$ be a continuously differentiable operator-valued function defined on an open subset $\Omega \subset \mathbb{R}^n$. Let the point $\lambda \in \mathbb{C}$ be a simple isolated eigenvalue of $A(t_0)$ for $t_0 \in \Omega$. Then there exists a continuously differentiable function $t \mapsto \lambda(t)$, defined in a small neighborhood of $t_0$, such that $\lambda(t_0) = \lambda$ and $\lambda(t)$ is a simple isolated eigenvalue of $A(t)$ for all $t$ in the neighborhood.*

*Proof.* The proof is not too difficult even in general case. In the matrix case (which is what we need) the lemma is a simple corollary of the implicit function theorem. The critical step here is that if $\lambda$ is a simple root of a polynomial $p(z)$, then $\frac{\mathrm{d}p}{\mathrm{d}z}(\lambda) \neq 0$. ∎

The next lemma (in the matrix case) is also a simple corollary of the implicit function theorem. It also can be obtained (in the general case) by application of standard techniques of perturbation theory; see, for example, [3], p. 213.

Lemma 3.2. *Let $A(t)$ and $A_n(t)$ be continuously differentiable operator-valued functions on some open subset $\Omega \subset \mathbb{R}^m$, and let*

$$\lim_{n\to\infty} \|A_n(\cdot) - A(\cdot)\|_{C^1(\Omega)} = 0.$$

*For $t_0 \in \Omega$ let $\lambda$ be a simple isolated eigenvalue of $A(t_0)$.*

*Then there exist a small neighborhood $\mathcal{U}$ of $t_0$ and continuously differentiable functions $\lambda(t)$, $\lambda^{(n)}(t)$, $n \geqslant N$ for some large $N$, on $\mathcal{U}$ such that:*

(i) *$\lambda(t)$ and $\lambda^{(n)}(t)$ are simple isolated eigenvalues of $A(t)$ and $A_n(t)$ respectively for all $t \in \mathcal{U}$;*

(ii) *$\lambda(t_0) = \lambda$; and*

(iii) $\lim\limits_{n\to\infty} \|\lambda^{(n)}(\cdot) - \lambda(\cdot)\|_{C^1(U)} = 0.$

LEMMA 3.3. *Let $A, A_n$ be bounded operators of finite rank on a Hilbert space, and assume that $\lim\limits_{n\to\infty} \|A_n - A\| = 0$. Suppose that $\mu$ is a simple isolated eigenvalue for $A$ and $A_n$ for all $n$ larger than some fixed $N$. If $Af = \mu f$, $\|f\| = 1$, then there exists a sequence $\{f_n\}$ of unit vectors ($\|f_n\| = 1$) such that $A_n f_n = \mu f_n$ for $n > N$ and*

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

*Proof.* If $P(A_n)$ and $P(A)$ are the Riesz spectral projections corresponding to the eigenvalue $\mu$, then

$$\|P(A_n) - P(A)\| \to 0 \quad \text{whenever} \quad \|A_n - A\| \to 0,$$

(see, for example, [2], p. 80).

Thus, $\|f - P(A_n)f\| = \|P(A)f - P(A_n)f\| \to 0$ since $f$ is an eigenvector of $A$. Also, $|1 - \|P(A_n)f\|\,| \to 0$ as $n \to \infty$. Let

$$f_n = \frac{P(A_n)f}{\|P(A_n)f\|}.$$

Clearly $f_n$ is a normalized eigenvector of $A_n$ corresponding to $\mu$, and

$$\|f - f_n\| \leqslant \|f - P(A_n)f\| + |\,\|P(A_n)f\| - 1| \to 0 \quad \text{as } n \to \infty. \qquad ∎$$

## 4. PROOF OF THE MAIN THEOREM

As it was shown above in Section 2, to prove the main theorem, it is enough to construct a sequence of functions (symbols) $\varphi_n$, such that Hankel operators $\Gamma_{\varphi_n}$ satisfy the conditions (i)–(v) from Subsection 2.1.

As it was mentioned above, we will construct symbols $\varphi_n = \varphi_{n,\tau^{(n)}}$ by induction, and to perform such induction we need to impose one more condition on the symbols $\varphi_n$, namely condition (vi) from Subsection 2.2.

So, our goal is to construct by induction the sequence of symbols $\varphi_n$, such that the corresponding Hankel operators $\Gamma_{\varphi_n}$ satisfy the conditions (i)–(vi).

The case $n = 1$ is trivial: we just pick an arbitrary $a_1 \in (0, 1)$ and put $t_1^{(1)} = 0$.

Let us suppose that we have constructed vectors $\tau^{(k)} \in \mathbb{R}^k$ and real numbers $a_k$, with $1 \leqslant k \leqslant n$, satisfying conditions (i)–(vi). We must show that there is a vector $\tau^{(n+1)}$ and a real number $a_{n+1}$ such that conditions (i)–(vi) are satisfied.

First, let $\tau = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$, and let $\widetilde{\tau} = (\tau, t_{n+1}) = (t_1, t_2, \ldots, t_n, t_{n+1}) \in \mathbb{R}^{n+1}$. Define

$$\varphi^{(a)}_{n,\widetilde{\tau}}(z) = \varphi_{n,\tau}(z) + b_{n+1}\frac{1 - a^2}{z - a}(1 + t_{n+1}), \quad a \in (0, 1).$$

Define the operator-valued function $(a, \widetilde{\tau}) \mapsto \Gamma^a_{n+1}(\widetilde{\tau})$ by

$$\Gamma^a_{n+1}(\widetilde{\tau})f = \Gamma_{\varphi_{n,\tau}}f + b_{n+1}(1 + t_{n+1})(f, h_a)h_a,$$

where $h_a$ is the normalized ($\|h_a\| = 1$) projection of the reproducing kernel $k_a(z) = (1-|a|^2)^{1/2}/(1-az)$ onto the orthogonal complement of $E_n := \operatorname{range}\Gamma_{\varphi_n} = \operatorname{span}\{k_{a_j} : 1 \leqslant j \leqslant n\}$.

Notice that, for fixed $\widetilde{\tau}$, the operators $\Gamma^a_{n+1}(\widetilde{\tau})$, $a \in (0,1) \setminus \{a_1, a_2, \dots, a_n\}$, are unitarily equivalent to each other. Thus for each $a$ there exist an operator $\Gamma_{n+1}(\widetilde{\tau})$ (acting, say, on $\mathbb{C}^{n+1}$) and a unitary operator $U_a : \mathbb{C}^{n+1} \to \operatorname{span}\{E_n, h_a\} = \operatorname{span}\{h_a, k_{a_j} : 1 \leqslant j \leqslant n\}$ satisfying

$$U_a^* \Gamma^a_{n+1}(\widetilde{\tau}) U_a = \Gamma_{n+1}(\widetilde{\tau}),$$

and such that the restriction $U_a^*|E_n$ does not depend on $a$ (operators $\Gamma^a_{n+1}(\widetilde{\tau})$ for fixed $\widetilde{\tau}$ and different $a$ coincide on $E_n$).

The asymptotic orthogonality (2.2) of reproducing kernels implies

$$\lim_{a\to 1} \|h_a - k_a\| = 0, \tag{4.1}$$

and therefore the operator-valued functions $\widetilde{\tau} \mapsto U_a^* \Gamma_{\varphi^{(a)}_{n+1,\widetilde{\tau}}} U_a$ converge in $C^1(G)$ to $\widetilde{\tau} \mapsto \Gamma_{n+1}(\widetilde{\tau})$, for every bounded domain $G$ in $\mathbb{R}^n$ as $a \to 1$.

Notice that the eigenvalues of $\Gamma_{n+1}(\widetilde{\tau})$ for $\widetilde{\tau} = (\tau^{(n)}, 0) \in \mathbb{R}^{n+1}$ are exactly the numbers $b_1, b_2, \dots, b_n, b_{n+1}$.

Since the $\{b_j\}$ are distinct, we can apply Lemma 3.2. We get that there exists a neighborhood $\mathcal{U}$ of the point $(\tau^{(n)}, 0)$ such that $\Lambda^a(\cdot) \to \Lambda(\cdot)$ in $C^1(\mathcal{U})$ as $a \to 1$, where $\Lambda^a(\widetilde{\tau}) = (\lambda^a_1(\widetilde{\tau}), \lambda^a_2(\widetilde{\tau}), \dots, \lambda^a_{n+1}(\widetilde{\tau}))$ and $\Lambda(\widetilde{\tau}) = (\lambda_1(\widetilde{\tau}), \lambda_2(\widetilde{\tau}), \dots, \lambda_{n+1}(\widetilde{\tau}))$ are eigenvalues of $\Gamma_{\varphi^{(a)}_{n+1,\widetilde{\tau}}}|\operatorname{span}\{E_n, k_a\}$, and $\Gamma_{n+1}(\widetilde{\tau})$ respectively.

The order of eigenvalues is not essential here, but it is convenient for us to order $\Lambda(\widetilde{\tau})$ such that $\lambda_k((\tau^{(n)}, 0)) = b_k$, $k = 1, 2, \dots, n+1$.

The Jacobian $\frac{\mathrm{d}\Lambda}{\mathrm{d}\widetilde{\tau}}$ is non-singular at the point $(\tau^{(n)}, 0)$ since it can be easily seen to have the following form:

$$\begin{pmatrix} \left\{ \frac{\partial \lambda^{(n)}_k}{\partial t_j} \right\}^n_{k,j=1} & \mathbb{O} \\ \mathbb{O} & b_{n+1} \end{pmatrix}.$$

The upper-left corner is non-singular by the induction hypothesis (vi), and $b_{n+1} \neq 0$ by the initial assumption.

Let $\mathbb{I} = (0,1]$ and $\Omega \subset \mathbb{I} \times \mathbb{R}^{n+1}$ be a neighborhood of the point $(1, (\tau^{(n)}, 0)) \in \mathbb{I} \times \mathbb{R}^{n+1}$. Define the function $f : \Omega \to \mathbb{C}^{n+1}$ by

$$f(a, \widetilde{\tau}) = \begin{cases} \Lambda^a(\widetilde{\tau}), & \text{if } a \in (0,1), \\ \Lambda(\widetilde{\tau}), & \text{if } a = 1. \end{cases}$$

If the neighborhood $\Omega$ is small enough, the function $f$ is well-defined and continuous. Moreover, the partial derivative $\partial f/\partial \widetilde{\tau}$ (matrix-valued function) exists and is continuous in $\Omega$. Notice that at the point $(1, (\tau^{(n)}, 0)) \in \mathbb{I} \times \mathbb{R}^{n+1}$ the partial derivative $\partial f/\partial \widetilde{\tau}$ is exactly the Jacobian $\partial \Lambda/\partial \widetilde{\tau}$, which is non-singular at this point.

Therefore, we are in position to apply the following implicit function theorem

Theorem 4.1. (Implicit function theorem) *Let $\mathcal{E}$ be a topological space, let $F$ and $G$ be Banach spaces, let $L(F,G)$ be the space of bounded linear operators from $F$ to $G$, let $\Omega$ be an open subset of $\mathcal{E} \times F$, and let $(\mathfrak{a}, b)$ be a point in $\Omega$. Let $f$ be a continuous mapping from $\Omega$ into $G$ such that*

*(1) for any fixed $x$, the function $f$ has a partial derivative $\frac{\partial f}{\partial y}(x,y)$, and the mapping $(x,y) \mapsto \frac{\partial f}{\partial y}(x,y)$ is a continuous mapping of $\Omega$ into $L(F,G)$;*

*(2) $\frac{\partial f}{\partial y}(\mathfrak{a}, b)$ is an invertible mapping from $F$ to $G$.*

*Assume, in addition, $c = f(\mathfrak{a}, b)$.*

*Then there exist neighborhoods $A$ and $B$ of the points $\mathfrak{a}$ and $b$ such that for any $x \in A$, the equation $f(x,y) = c$ has a unique solution $y = g(x)$ belonging to $B$, and the function $g$ defined in this manner is a continuous mapping from $A$ to $B$.*

This formulation of the implicit function theorem can be found, for example, in Schwartz' book ([10]).

If we apply this theorem to the function $f$ defined above (with $x = a$, $y = \widetilde{\tau}$, $\mathfrak{a} = 1$, $b = (\tau^{(n)}, 0)$, $c = (b_1, b_2, \ldots, b_{n+1})$), we get that for $a$ sufficiently close to 1 there exists a vector $\widetilde{\tau}(a)$ such that

$$\Lambda^a(\widetilde{\tau}(a)) = \Lambda(\tau^{(n)}, 0) = (b_1, b_2, \ldots, b_{n+1}) \in \mathbb{R}^{n+1}.$$

This satisfies condition (iii). Of course,

$$\lim_{a \to 1} \widetilde{\tau}(a) = (\tau^{(n)}, 0)$$

by continuity. Thus, if we choose $a$ close enough to 1, the operator $U_a^* \Gamma_{\varphi^{(a)}_{n+1,\tilde{\tau}(a)}} U_a$ can be as close as we want to $\Gamma_{n+1}(\tau^n, 0)$. Therefore, by Lemma 3.3, the normalized eigenvectors of $\Gamma_{\varphi^{(a)}_{n+1,\tilde{\tau}(a)}}$ are as close as we want to the corresponding normalized eigenvectors of $\Gamma^a_{n+1}(\tau^{(n)}, 0)$, which are exactly the normalized eigenvectors of $\Gamma_{\varphi_n}$ and the normal eigenvector $h_a$ (which is orthogonal to all the other eigenvectors) corresponding to the eigenvalue $b_{n+1}$. So condition (v) is satisfied.

If we add to a Riesz basis of unit vectors (in a subspace) a unit vector orthogonal to it, we get a Riesz basis for the higher-dimensional subspace with the same measure of non-orthogonality $\|R\| \cdot \|R^{-1}\|$ (because, as can easily be seen, neither $\|R\|$ nor $\|R^{-1}\|$ change when adding an orthonormal vector). So, by the induction hypothesis (iv), the measure of non-orthogonality of the system of normalized eigenvectors of $\Gamma^a_{n+1}(\tau^{(n)}, 0)$ is strictly less than 2.

Therefore, for $a$ close to 1, the measure of non-orthogonality of the system of normalized eigenvectors of $\Gamma_{\varphi^{(a)}_{n+1,\tilde{\tau}(a)}}$ is strictly less than 2 as well, since we can make the eigenvectors as close as we want to the system of normalized eigenvectors of $\Gamma^a_{n+1}(\tau^{(n)}, 0)$, and for finite systems this implies that the orthogonalizers can be made as close as we desire. Hence condition (iv) is satisfied.

Finally, since $\Lambda^a(\,\cdot\,) \to \Lambda(\,\cdot\,)$ in $C^1(\Omega)$ as $a \to 1$, for $a$ close enough to 1 the Jacobian $\partial \Lambda^a(\widetilde{\tau})/\partial \widetilde{\tau}$ is non-singular at $\widetilde{\tau} = \widetilde{\tau}(a)$, which is just condition (vi).

Therefore if we put $\tau^{(n+1)} = \widetilde{\tau}(a)$, where $a$ is close enough to 1 such that all of the above hold, we get that the symbol $\varphi_{n+1} = \varphi^{(a)}_{n+1,\tau^{(n+1)}} = \varphi^{(a)}_{n+1,\tilde{\tau}(a)}$ satisfies all the conditions (i)–(vi). This finishes the proof. ∎

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## REFERENCES

1. E.V. Abakumov, *The inverse spectral problem for Hankel operators of finite rank*, [Russian], *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **217**(1994), *Issled. po Linein. Oper. i Teor. Funktsii.* **22**, 5–15; English transl., *J. Math. Sci. (New York)* **85**(1997), 1759–1766
2. H. Baumgärtel, *Analytic Perturbation Theory for Matrices and Operators*, Birkhäuser, Basel 1985.
3. T. Kato, *Perturbation Theory for Linear Operators*, Springer–Verlag, New York 1966.
4. S.V. Khrushchev, V.V. Peller, Hankel operators, best approximation and stationary Gaussian processes, *Uspekhi Mat. Nauk* **37**(1982), 53–124; English transl., *Russian Math. Surveys* **37**(1982), 61–144.
5. A.V. Megretskiĭ, A quasinilpotent Hankel operator, *Leningrad Math. J.* **2**(1991), 879–889.
6. A.V. Megretskiĭ, V.V. Peller, S.R. Treil, The inverse spectral problem for self-adjoint Hankel operators, *Acta Math.* **174**(1995), 241–309.
7. Z. Nehari, On bounded bilinear forms, *Ann. of Math.* **65**(1957), 153–162.
8. S.C. Power, *Hankel Operators on Hilbert Space*, Pitman, Boston 1982.
9. S.C. Power, Quasinilpotent Hankel operators, in *Linear and Complex Analysis Problem Book*, Lect. Notes Math., vol. 1043, Springer-Verlag, 1984, pp. 259–261.
10. L. Schwartz, *Analyse. Deuxième partie: Topologie générale et analyse fonctionnelle*, Collect. Enseign. Sci., vol. 11, Hermann, Paris 1970.
11. S.R. Treil, Moduli of Hankel operators and a problem of V.V. Peller and S.V. Khruschev, [Russian], Investigations on linear operators and the theory of functions, vol. 14, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **141**(1985), 39–55; English transl., *J. Soviet. Math.* **37**(1987), 1287–1296.
12. R. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York 1980.

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