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NORM INEQUALITIES FOR SUMS OF POSITIVE OPERATORS

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Abstract. We use certain norm inequalities for 2×2 operator matrices to establish norm inequalities for sums of positive operators. Among other inequalities, it is shown that if A and B are positive operators on a Hilbert space, then

$$\|A+B\| \leqslant \frac{1}{2}\Big(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\|A^{1/2}B^{1/2}\|^2}\Big).$$

This inequality, which is sharper than the triangle inequality, improves upon some earlier related inequalities. Applications of these inequalities are also considered.

Keywords: *Operator matrix, positive operator, unitarily invariant norm, triangle inequality.*

MSC (2000): 47A30, 47B10, 47B15.

1. INTRODUCTION

In a recent work ([11]), extending an improved version of an inequality of Davidson and Power (Lemma 3.3 from [6]), it has been shown that if A and B are positive operators in a norm ideal of operators on a Hilbert space, then

$$|||(A+B)\oplus 0||| \leqslant |||A\oplus B||| + \Big|\Big|\Big| A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2} \Big|\Big|\Big|, \tag{1.1}$$

where $|||\cdot|||$ is an associated unitarily invariant norm. In particular, for the usual operator norm $\|\cdot\|$, we have

$$\|A+B\| \leqslant \max(\|A\|,\|B\|) + \big\|A^{1/2}B^{1/2}\big\|. \tag{1.2}$$

A weaker version of (1.2), where $\big\|A^{1/2}B^{1/2}\|$ is replaced by $\|AB\|^{1/2}$, has been useful in the theory of best approximation in C^*-algebras given in [6]. In this paper we employ certain norm inequalities for 2×2 operator matrices to prove norm inequalities for sums of positive operators. In Section 2 we establish a general norm inequality involving 2×2 operator matrices, from which (1.1), among other inequalities, follows as a special case. In Section 3 we present a refinement of the inequality (1.2). In fact, our refined inequality is finer than both the inequality (1.2) and the triangle inequality. We use this inequality to investigate the equality condition of the triangle inequality for the usual operator norm and positive operators. We also use it to derive an inequality for the sum of two operators having orthogonal ranges.

Let $B(H)$ denote the space of all bounded linear operators on a complex separable Hilbert space H. In addition to the usual operator norm $\|\cdot\|$, which is defined on all of $B(H)$, we consider unitarily invariant (or symmetric) norms $|||\cdot|||$. Each of these norms is defined on an ideal in $B(H)$, and for the sake of brevity, we will make no explicit mention of this ideal. Thus, when we talk of $|||T|||$, we are assuming that the operator T belongs to the norm ideal associated with $|||\cdot|||$. For the theory of unitarily invariant norms, we refer to [1], [7], or [15].

If X and Y are operators in $B(H)$, we write the direct sum $X \oplus Y$ for the 2×2 operator matrix $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$, regarded as an operator on $H \oplus H$. Thus,

$$\|X \oplus Y\| = \max(\|X\|, \|Y\|). \tag{1.3}$$

It follows easily from the basic properties of unitarily invariant norms that

$$|||X \oplus Y||| = \left|\left|\left| \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right|\right|\right|, \tag{1.4}$$

$$|||X \oplus X^*||| = |||X \oplus X|||, \tag{1.5}$$

and it follows from the Fan dominance principle (see e.g., [10]) that the following three inequalities are equivalent:

$$|||X||| \leqslant |||Y||| \text{ for all unitarily invariant norms,} \tag{1.6}$$

$$|||X \oplus 0||| \leqslant |||Y \oplus 0||| \text{ for all unitarily invariant norms,} \tag{1.7}$$

$$|||X \oplus X||| \leqslant |||Y \oplus Y||| \text{ for all unitarily invariant norms.} \tag{1.8}$$

The equivalence of these inequalities will be frequently used in the rest of the paper.

2. A GENERAL NORM INEQUALITY

Our general norm inequality is based on the following lemma, which has played a central role in proving the arithmetic-geometric mean inequality and other related norm inequalities (see [10], [11], and [13]).

LEMMA *If X and Y are operators in $B(H)$ such that XY is self-adjoint, then*

$$|||XY||| \leqslant |||\mathrm{Re}(YX)||| \tag{2.1}$$

for every unitarily invariant norm.

Now we are in a position to prove our general norm inequality, which includes several norm inequalities as special cases.

THEOREM *If A_1, A_2, B_1, B_2, X, and Y are operators in $B(H)$, then*

$$2|||(A_1XA_2^*+B_1YB_2^*)\oplus 0||| \leqslant \left|\left|\left| \begin{bmatrix} A_1^*A_1X+XA_2^*A_2 & A_1^*B_1Y+XA_2^*B_2 \\ B_1^*A_1X+YB_2^*A_2 & B_1^*B_1Y+YB_2^*B_2 \end{bmatrix} \right|\right|\right| \tag{2.2}$$

for every unitarily invariant norm.

Proof. On $H^4 = H \oplus H \oplus H \oplus H$, let

$$T = \begin{bmatrix} A_1 & B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & B_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & 0 & X & 0 \\ 0 & 0 & 0 & Y \\ X^* & 0 & 0 & 0 \\ 0 & Y^* & 0 & 0 \end{bmatrix}.$$

Then

$$TRT^* = \begin{bmatrix} 0 & 0 & A_1XA_2^*+B_1YB_2^* & 0 \\ 0 & 0 & 0 & 0 \\ A_2X^*A_1^*+B_2Y^*B_1^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T^*TR = \begin{bmatrix} 0 & 0 & A_1^*A_1X & A_1^*B_1Y \\ 0 & 0 & B_1^*A_1X & B_1^*B_1Y \\ A_2^*A_2X & A_2^*B_2Y^* & 0 & 0 \\ B_2^*A_2X^* & B_2^*B_2Y^* & 0 & 0 \end{bmatrix},$$

and

$\mathrm{Re}(T^*TR)$

$$= \frac{1}{2} \begin{bmatrix} 0 & 0 & A_1^*A_1X+XA_2^*A_2 & A_1^*B_1Y+XA_2^*B_2 \\ 0 & 0 & B_1^*A_1X+YB_2^*A_2 & B_1^*B_!Y+YB_2^*B_2 \\ A_2^*A_2X^*+X^*A_1^*A_1 & A_2^*B_2Y^*+X^*A_1^*B_1 & 0 & 0 \\ B_2^*A_2X^*+Y^*B_1^*A_1 & B^*B_2Y^*+Y^*B_1^*B_1 & 0 & 0 \end{bmatrix}.$$

Since TRT^* is self-adjoint, it follows by Lemma 2.1 that $|||TRT^*||| \leqslant |||\mathrm{Re}(T^*TR)|||$. Now the desired inequality follows by (1.4), (1.5), and the equivalence of the inequalities (1.6) and (1.8).

It should be mentioned here that the inequality (2.2) yields several inequalities as special cases. This is demonstrated in the following remarks.

REMARK If $X = Y = I$ (the identity operator), if A and B are positive operators, and if $A_1 = A_2 = A^{1/2}$ and $B_1 = B_2 = B^{1/2}$, then employing (2.2) and the triangle inequality, we obtain the inequality (1.1).

REMARK If A_i and B_i have orthogonal ranges for $i = 1, 2$, i.e., if $A_i^*B_i = 0$ for $i = 1, 2$, then we have the inequality

$$2|||A_1XA_2^*+B_1YB_2^*)\oplus 0||| \leqslant |||(A_1^*A_1X+XA_2^*A_2)\oplus(B_1^*B_1Y+YB_2^*B_2)|||. \tag{2.3}$$

In particular, if $A_i = 0$ or $B_i = 0$ for $i = 1, 2$, then using the equivalence of the inequalities (1.6) and (1.7), we get the arithmetic-geometric mean inequality. Recall that this inequality asserts that if A, B and X are operators in $B(H)$, then

$$2|||AXB^*||| \leqslant |||A^*AX+XB^*B||| \tag{2.4}$$

for every unitarily invariant norm. See [1] for a comprehensive account on this inequality.

REMARK If $X = Y = I$ and if $A_1 = B_2 = A^*$ and $A_2 = B_1 = B^*$, then we get the inequality

$$2|||(A^*B + B^*A) \oplus 0||| \leqslant \left|\left|\left| \begin{bmatrix} AA^* + BB^* & AB^* + BA^* \\ AB^* + BA^* & AA^* + BB^* \end{bmatrix} \right|\right|\right|, \tag{2.5}$$

which has been observed in Theorem 1 from [12].

REMARK If $X = I$ and $Y = -I$, if A and B are positive operators, and if $A_1 = A_2 = A^{1/2}$ and $B_1 = B_2 = B^{1/2}$, then we have the inequality

$$|||(A - B) \oplus 0||| \leqslant |||A \oplus B|||, \tag{2.6}$$

which has been pointed out in [3].

REMARK If $X = I$ and $Y = \mathrm{i}I$, if A and B are positive operators, and if $A_1 = A_2 = A^{1/2}$ and $B_1 = B_2 = B^{1/2}$, then we have the inequality

$$|||(A + \mathrm{i}B) \oplus 0||| \leqslant \left|\left|\left| \begin{bmatrix} A & \left(\frac{1+\mathrm{i}}{2}\right) A^{1/2}B^{1/2} \\ \left(\frac{1+\mathrm{i}}{2}\right) B^{1/2}A^{1/2} & \mathrm{i}B \end{bmatrix} \right|\right|\right|. \tag{2.7}$$

This inequality can be used to establish norm inequalities for operators whose real and imaginary parts are positive. Thus, employing (1.5), (2.7), and the triangle inequality, we obtain the inequality

$$|||(A + \mathrm{i}B) \oplus 0||| \leqslant |||A \oplus B||| + \frac{1}{\sqrt{2}} |||A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2}|||, \tag{2.8}$$

which is akin to the inequality (1.1). When comparing between (1.1) and (2.8), one should bear in mind that for positive operators A and B in $B(H)$, we have $|||A + \mathrm{i}B||| \leqslant |||A + B|||$ (see Theorem 1 from [4]).

3. A REFINEMENT OF THE INEQUALITY (1.2)

To refine the inequality (1.2), we utilize the following lemma, which is essentially due to Tomiyama (see p. 41, [14]). This lemma, which concerns the usual operator norm, need not be true for the general class of unitarily invariant norms. Though the lemma holds for general $n \times n$ operator matrices (see e.g., [8]), we state it for 2×2 operator matrices, the case that we actually need. For related results involving norms of operator matrices, we refer to [2], [8], [9], and references therein.

LEMMA *If A, B, C, and D are operators in $B(H)$, then*

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leqslant \left\| \begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right\|. \tag{3.1}$$

Our refinement of the inequality (1.2) is presented in the following theorem.

THEOREM *Let A, B, X, and Y be operators in $B(H)$ such that A and B are positive and X and Y are self-adjoint. Then*

$$\begin{aligned}\|A^{1/2}XA^{1/2}+B^{1/2}YB^{1/2}\| \leqslant \frac{1}{4}\Big(&\|AX+XA\|+\|BY+YB\| \\ &+\sqrt{(\|AX+XA\|-\|BY+YB\|)^2+4\|A^{1/2}B^{1/2}Y+XA^{1/2}B^{1/2}\|^2}\Big).\end{aligned} \tag{3.2}$$

In particular, letting $X = Y = I$, we have

$$\|A+B\| \leqslant \frac{1}{2}\Big(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\|A^{1/2}B^{1/2}\|^2}\Big). \tag{3.3}$$

Proof. Letting $A_1 = A_2 = A^{1/2}$ and $B_1 = B_2 = B^{1/2}$ in Theorem 2.2, we have

$$\begin{aligned}&2\|A^{1/2}XA^{1/2}+B^{1/2}YB^{1/2}\| \\ &\quad\leqslant \left\| \begin{bmatrix} AX+XA & A^{1/2}B^{1/2}Y+XA^{1/2}B^{1/2} \\ B^{1/2}A^{1/2}X+YB^{1/2}A^{1/2} & BY+YB \end{bmatrix} \right\|.\end{aligned}$$

Now using Lemma 3.1, we conclude that

$$\begin{aligned}&2\|A^{1/2}XA^{1/2}+B^{1/2}YB^{1/2}\| \\ &\quad\leqslant \left\| \begin{bmatrix} \|AX+XA\| & \|A^{1/2}B^{1/2}Y+XA^{1/2}B^{1/2}\| \\ \|B^{1/2}A^{1/2}X+YB^{1/2}A^{1/2}\| & \|BY+YB\| \end{bmatrix} \right\|.\end{aligned}$$

Since X and Y are self-adjoint, it follows that $\|B^{1/2}A^{1/2}X+YB^{1/2}A^{1/2}\| = \|A^{1/2}B^{1/2}Y+XA^{1/2}B^{1/2}\|$. Thus, the matrix

$$\begin{bmatrix} \|AX+XA\| & \|A^{1/2}B^{1/2}Y+XA^{1/2}B^{1/2}\| \\ \|B^{1/2}A^{1/2}X+YB^{1/2}A^{1/2}\| & \|BY+YB\| \end{bmatrix}$$

is hermitian. Considered as an operator on a two-dimensional Hilbert space, the usual operator norm of this matrix is the same as its spectral radius, which is

$$\begin{aligned}&\frac{1}{2}\Big(\|AX+XA\|+\|BY+YB\| \\ &\qquad+\sqrt{(\|AX+XA\|-\|BY+YB\|)^2+4\|A^{1/2}B^{1/2}Y+XA^{1/2}B^{1/2}\|^2}\Big).\end{aligned}$$

This completes the proof of the theorem. ∎

It should be remarked here that the inequality (3.3) is sharper than both the inequality (1.2) and the triangle inequality. To see this, note that

$$\begin{aligned}&\frac{1}{2}\Big(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\|A^{1/2}B^{1/2}\|^2}\Big) \\ &\quad\leqslant \frac{1}{2}\Big(\|A\|+\|B\|+\sqrt{(\|A\|+\|B\|)^2}+\sqrt{4\|A^{1/2}B^{1/2}\|^2}\Big) \\ &\quad= \frac{1}{2}(\|A\|+\|B\|+|\,\|A\|-\|B\|\,|+2\|A^{1/2}B^{1/2}\|) \\ &\quad= \max(\|A\|,\|B\|)+\|A^{1/2}B^{1/2}\|\end{aligned} \tag{3.4}$$

and that

$$\begin{aligned}&\frac{1}{2}\Big(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\|A^{1/2}B^{1/2}\|^2}\,\Big)\\ &\qquad\leqslant \frac{1}{2}\Big(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\|A\|\,\|B\|}\Big)\\ &\qquad= \frac{1}{2}\Big(\|A\|+\|B\|+\sqrt{(\|A\|+\|B\|)^2}\,\Big)=\|A\|+\|B\|.\end{aligned} \tag{3.5}$$

The inequality (3.3) can be used to analyze the equality case in the triangle inequality for the usual operator norm and positive operators.

PROPOSITION *Let A and B be nonzero positive operators in $B(H)$. Then $\|A+B\|=\|A\|+\|B\|$ if and only if $\|A^{1/2}B^{1/2}\|=\|A\|^{1/2}\|B\|^{1/2}$.*

Proof. If $\|A+B\|=\|A\|+\|B\|$, then it follows from the inequalities (3.3) and (3.5) that

$$\|A\|+\|B\|=\frac{1}{2}\Big(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\|A^{1/2}B^{1/2}\|^2}\,\Big).$$

Direct computations now show that $\|A^{1/2}B^{1/2}\|=\|A\|^{1/2}\|B\|^{1/2}$.

On the other hand, if $\|A^{1/2}B^{1/2\|}=\|A\|^{1/2}\|B\|^{1/2}$, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ of unit vectors in H such that $\lim\limits_{n\to\infty}(A^{1/2}B^{1/2}y_n,x_n)=\|A\|^{1/2}\|B\|^{1/2}$. Now using the Cauchy-Schwarz inequality, and passing to subsequences if necessary, we have

$$\|A\|^{1/2}\|B\|^{1/2}=\lim_{n\to\infty}(B^{1/2}y_n,A^{1/2}x_n)\leqslant\|A\|^{1/2}\lim_{n\to\infty}\|B^{1/2}y_n\|\leqslant\|A\|^{1/2}\|B\|^{1/2}.$$

Hence, $\lim\limits_{n\to\infty}\|B^{1/2}y_n\|=\|B\|^{1/2}$, and so $\lim\limits_{n\to\infty}(By_n,y_n)=\|B\|$. Similarly, we have $\lim\limits_{n\to\infty}(Ax_n,x_n)=\|A\|$. Let $z_n=\begin{bmatrix}ax_n\\ by_n\end{bmatrix}$ for $n=1,2,\dots$, where $a=\sqrt{\frac{\|A\|}{\|A\|+\|B\|}}$ and $b=\sqrt{\frac{\|B\|}{\|A\|+\|B\|}}$. Then $\{z_n\}$ is a sequence of unit vectors in $H\oplus H$. Using the fact that $\|T^*T\|=\|TT^*\|$ for every operator T, we have

$$\begin{aligned}\|A+B\|&=\left\|\begin{bmatrix}A+B&0\\0&0\end{bmatrix}\right\|=\left\|\begin{bmatrix}A^{1/2}&B^{1/2}\\0&0\end{bmatrix}\begin{bmatrix}A^{1/2}&0\\B^{1/2}&0\end{bmatrix}\right\|\\ &=\left\|\begin{bmatrix}A^{1/2}&0\\B^{1/2}&0\end{bmatrix}\begin{bmatrix}A^{1/2}&B^{1/2}\\0&0\end{bmatrix}\right\|=\left\|\begin{bmatrix}A&A^{1/2}B^{1/2}\\B^{1/2}A^{1/2}&B\end{bmatrix}\right\|.\end{aligned}$$

Consequently, for $n=1,2,\dots$, we have

$$\|A+B\|\geqslant\left(\begin{bmatrix}A&A^{1/2}B^{1/2}\\B^{1/2}A^{1/2}&B\end{bmatrix}z_n,z_n\right).$$

Expanding this inner product and letting $n\to\infty$, we conclude that $\|A+B\|\geqslant\|A\|+\|B\|$. But by the triangle inequality we have $\|A+B\|\leqslant\|A\|+\|B\|$. Therefore, $\|A+B\|=\|A\|+\|B\|$, which completes the proof of the proposition. ∎

Proposition 3.3 gives an equality condition of the triangle inequality for the usual operator norm and positive operators. It turns out that this is the same equality condition of the Cauchy-Schwarz type inequality $\|AB\|\leqslant\|A\|\,\|B\|$.

PROPOSITION *Let A and B be nonzero positive operators in $B(H)$. Then $\|AB\| = \|A\|\,\|B\|$ if and only if $\|A^{1/2}B^{1/2}\| = \|A\|^{1/2}\|B\|^{1/2}$.*

Proof. If $\|A^{1/2}B^{1/2}\| = \|A\|^{1/2}\|B\|^{1/2}$, then using Lemma 2.1 and the fact that $\|T\|^2 = \|T^*T\|$ for every operator T, we have

$$\|A\|\,\|B\| = \|A^{1/2}B^{1/2}\|^2 = \|B^{1/2}AB^{1/2}\| \leqslant \|\mathrm{Re}(AB)\| \leqslant \|AB\| \leqslant \|A\|\,\|B\|.$$

This implies that $\|AB\| = \|A\|\,\|B\|$.

To prove the "only if" part, assume that $\|AB\| = \|A\|\,\|B\|$. Then

$$\begin{aligned}\|A\|\,\|B\| &= \|A^{1/2}(A^{1/2}B^{1/2})B^{1/2}\|\\ &\leqslant \|A^{1/2}\|\,\|A^{1/2}B^{1/2}\|\,\|B^{1/2}\|\\ &= \|A\|^{1/2}\|A^{1/2}B^{1/2}\|\,\|B\|^{1/2},\end{aligned}$$

and so

$$\|A\|^{1/2}\|B\|^{1/2} \leqslant \|A^{1/2}B^{1/2}\| \leqslant \|A^{1/2}\|\,\|B^{1/2}\| = \|A\|^{1/2}\|B\|^{1/2}.$$

Hence, $\|A^{1/2}B^{1/2}\| = \|A\|^{1/2}\|B\|^{1/2}$, as required. ∎

Combining Propositions 3.3 and 3.4, we conclude that for nonzero positive operators A and B in $B(H)$, $\|A+B\| = \|A\|+\|B\|$ if and only if $\|AB\| = \|A\|\,\|B\|$.

In view of the importance of the inequality (3.3), it is desirable to extend it to general (i.e., not necessarily positive) operators. Fortunately, this is possible and the sought version asserts that if A and B are operators in $B(H)$, then

$$\begin{aligned}\|A+B\| \leqslant \frac{1}{2}\Big(&\|A\| + \|B\|\\ &+ \sqrt{(\|A\|-\|B\|)^2+4\max\left(\|\,|A|^{1/2}|B|^{1/2}\|^2, \|\,|A^*|^{1/2}|B^*|^{1/2}\|^2\right)}\,\Big),\end{aligned} \tag{3.6}$$

where $|X| = (X^*X)^{1/2}$ is the absolute value of X. Moreover, as it can be easily verified, this inequality is sharper than the triangle inequality.

To prove the inequality (3.6), we remark that one can infer from Theorem I.X.5.11 in [1] that if A and B are self-adjoint (more generally, hyponormal) operators, then

$$|||A + B||| \leqslant |||\,|A| + |B|\,||| \tag{3.7}$$

for every unitarily invariant norm. Thus, it follows from the inequality (3.7) and the inequality (3.3) applied to the positive operators $|A|$ and $|B|$ that

$$\|A + B\| \leqslant \frac{1}{2}\Big(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\|\,|A|^{1/2}|B|^{1/2}\|^2}\Big) \tag{3.8}$$

for all self-adjoint operators A and B. Now for general operators A and B, the inequality (3.6) follows by applying the inequality (3.8) to the self-adjoint operators $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$.

Finally, we conclude this section with the following remarks.

REMARK Using the inequality (2.7), Lemma 3.1, and an argument similar to that used in the proof of Theorem 3.2, it can be easily shown that if A and B are positive operators in $B(H)$, then

$$\|A+\mathrm{i}B\| \leqslant \frac{1}{2}\Big(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+2\|A^{1/2}B^{1/2}\|^2}\Big). \tag{3.9}$$

For other recent norm inequalities involving operators whose real and imaginary parts are positive, we refer to [5].

REMARK In [11] we have seen how to apply inequalities for sums of positive operators to derive inequalities for sums of operators having orthogonal ranges. In the same spirit, we can invoke the inequality (3.3) to establish a refinement of the second inequality of Corollary 7 in [11]. In fact, it can be shown that if A and B are operators in $B(H)$ with orthogonal ranges, then

$$\|A+B\|^2 \leqslant \frac{1}{2}\Big(\|A\|^2+\|B\|^2+\sqrt{(\|A\|^2-\|B\|^2)^2+4\|AB^*\|^2}\Big). \tag{3.10}$$

To see this, first observe that, since $A^*B=0$, we have $(A+B)^*(A+B)=A^*A+B^*B$. Thus, the inequality (3.10) follows from the inequality (3.3) applied to the positive operators A^*A and B^*B, together with the fact that $\|\,|A|\,|B|\,\|=\|AB^*\|$.

REMARK Our analysis in this section depends heavily on Lemma 3.1, which is concerned with the particularly important usual operator norm. It should be mentioned here that this lemma remains true for the Schatten p-norms with $p=2k$, where k is a natural number. Let $T=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $S=\begin{bmatrix} \|A\|_{2k} & \|B\|_{2k} \\ \|C\|_{2k} & \|D\|_{2k} \end{bmatrix}$. To prove that $\|T\|_{2k} \leqslant \|S\|_{2k},$, it is sufficient to show that $\mathrm{tr}(T^*T)^k \leqslant \mathrm{tr}(S^*S)^k$, where *tr* denotes the usual trace functional. However, this can be derived by invoking Hölder's inequality and the triangle inequality for the Schatten p-norms (see [1], [7], or [15]).

Consequently, it is possible to establish inequalities for the norm $\|\cdot\|_{2k}$ related to the inequalities given in this section. We leave the details to the interested reader.

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