J. OPERATOR THEORY
48(2002), 105–119

SCHATTEN CLASS COMPOSITION OPERATORS ON PLANAR DOMAINS

VINH-THY M. TRAN

Communicated by Norberto Salinas

Abstract. We extend to finitely connected planar domains a result of Kehe Zhu which characterizes the Schatten class composition operators on the Hardy space of the disc. In the process, we characterize the positive compact and Schatten class Toeplitz operators on a weighted Bergman space.

Keywords: *Compact composition operator, planar domains, Hardy space.*

MSC (2000): Primary 47B38; Secondary 30H05.

1. INTRODUCTION

Let Ω be a domain in the plane. The Hardy space $H^2 = H^2(\Omega)$ is defined to be those analytic functions f on Ω for which the subharmonic function $|f(z)|^2$ has a harmonic majorant. Once we specify a base point $t_0 \in \Omega$, we define the norm of f to be square root of the value at t_0 of the (unique) least harmonic majorant of $|f|^2$. The norm depends on t_0 but, by an application of Harnack's inequality, the resulting topology does not. For more on the Hardy spaces, see [9].

An analytic function φ that maps Ω into itself determines a composition operator C_φ on H^2 given by

$$C_\varphi f = f \circ \varphi.$$

That C_φ is bounded follows from Harnack's inequality.

Since C_φ depends intimately on φ, it is natural to ask how the function-theoretic properties of φ relate to the operator-theoretic properties of C_φ. As an easy illustration, suppose $\varphi(t_0) = t_0$. Now if u_f is the least harmonic majorant of $|f|^2$, then $u_f \circ \varphi$ is a harmonic majorant of $|f \circ \varphi|^2$; hence

$$\|C_\varphi(f)\|^2 \leqslant u_f(\varphi(t_0)) = u_f(t_0) = \|f\|^2$$

so that C_φ is a contraction.

In this paper the operator-theoretic properties that concern us are defined as follows. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ a bounded linear operator. For $n \geqslant 0$, let $\mathcal{F}_n$ denote the set of bounded linear operators on $\mathcal{H}$ with rank less than or equal to n. Define

$$s_{n+1} = \inf\left\{\|T - F\| : F \in \mathcal{F}_n\right\}. \tag{1.1}$$

We call T compact if $\{s_n\} \in c_0$. Let $1 \leqslant p < \infty$. We call T Schatten p-class if $\{s_n\} \in l^p$. The number s_n is the nth singular value of T.

When Ω is the open unit disc, whether C_φ is compact or Schatten p-class depends on the value distribution of φ near the boundary of Ω. More precisely, from [10] and [12]:

DEFINITION 1.1. Let Δ be the open unit disc and suppose $\varphi : \Delta \to \Delta$ is analytic. Define

$$N_\varphi(w) = \sum_{\varphi(z)=w} \log \frac{1}{|z|}.$$

N_φ is the Nevanlinna counting function for φ.

THEOREM 1.2. *Suppose $\varphi : \Delta \to \Delta$ is analytic with $\varphi(0) = 0$.*
(a) *C_φ is compact on $H^2(\Delta)$ if and only if*

$$\lim_{w \to \partial\Delta} \frac{N_\varphi(w)}{\log \frac{1}{|w|}} = 0.$$

(b) *C_φ is Schatten p-class on $H^2(\Delta)$, $2 \leqslant p < \infty$, if and only if*

$$\int_\Delta \left[\frac{N_\varphi(w)}{\log \frac{1}{|w|}}\right]^{\frac{p}{2}} \frac{\mathrm{d}A(w)}{(1-|w|^2)^2} < \infty$$

where $\mathrm{d}A$ *is Lebesgue area measure.*

In [4], Definition 1.1 and Theorem 1.2 (a) were extended to finitely connected planar domains. The main result of this paper is the extension of Theorem 1.2 (b) to such domains. Our arguments closely follow those in [12]; in particular, we prove our extension by connecting the composition operator on the Hardy space to a Toeplitz operator on a weighted Bergman space. As an added bonus, our methods allow us to present a different proof of the extension of (a) from that in [4].

2. BACKGROUND

2.1. Definitions and conventions. In this paper we are concerned with a planar domain Ω whose complement consists of a finite number of disjoint non-trivial continua. Such a domain is conformally equivalent to a domain whose boundary components are circles. Since the conformal mapping gives an isometry of the corresponding Hardy spaces, we may assume, and shall do so, that the components $\Gamma_0, \dots, \Gamma_m$ of $\partial\Omega$ are circles, with Γ_0 the boundary of the unbounded component of the complement of Ω.

We let Ω_j be the region outside Γ_j, $j = 1, \dots, m$, including the point at ∞, and Ω_0 be the region inside Γ_0. A glance at Theorem 1.2 shows that behavior at the boundary is what concerns us here. Accordingly, we let A_j be a very thin annulus in Ω_j where $\Gamma_j = \partial\Omega_j$ is one component of ∂A_j and we set $A_{\partial\Omega} = \bigcup_{j=0}^{m} A_j$.

Each of the regions Ω_j is conformally equivalent to the open unit disc Δ via a linear fractional transformation φ_j; we will assume that $\varphi_j(t_0) = 0$. By explicitly writing down the linear fractional transformation ϕ_j, it is easy to see that ϕ_j' is non-vanishing in a full neighborhood of A_j. We will use this fact repeatedly in the sequel.

We let $g_\Omega(z, t_0)$ denote the Green's function for Ω with pole at t_0. The weighted Bergman space $A^2_{1,s}(\Omega)$ is defined to be those analytic functions f on Ω for which f has single-valued integral and for which

$$\|f\|_{A^2_{1,s}(\Omega)} = \left[\frac{2}{\pi}\int_\Omega |f(z)|^2 g_\Omega(z, t_0)\, \mathrm{d}A(z)\right]^{\frac{1}{2}}$$

is finite. Again, the norm depends on t_0 but, by the Littlewood-Paley identity (4.5), the resulting topology does not. Standard techniques ([3]) reveal that $A^2_{1,s}(\Omega)$ is a reproducing kernel Hilbert space; further, if K^Ω_a is the reproducing kernel for the point $a \in \Omega$ and $\{e_n\}_{n=1}^\infty$ is an orthonormal basis, then $K^\Omega_a(z) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(a)}$. We denote the normalized reproducing kernel by $k^\Omega_a = K^\Omega_a / \sqrt{K^\Omega_a(a)}$.

When we write $H^2(\Omega_j)$ or $A^2_1(\Omega_j) = A^2_{1,s}(\Omega_j)$ for the Hardy space or weighted Bergman space for this region, we will always assume that the norm is taken with respect to the base point t_0.

Let $\Pi : \Delta \to \Omega$ be an analytic covering map. The (unique maximal) ultra-hyperbolic metric for Ω is defined at $w = \Pi(z) \in \Omega$ by

$$\lambda_\Omega(w) = \frac{1}{(1 - |z|^2)|\Pi'(z)|}.$$

It is standard ([6]) that the value of $\lambda_\Omega(w)$ is independent of the covering map Π and the particular choice of $z \in \Delta$ with $\Pi(z) = w$.

2.2. DISTANCE TO THE BOUNDARY. Let $\operatorname{dist}(z, \partial\Omega)$ denote the distance from z to the boundary of Ω. The next result is standard ([7]).

PROPOSITION 2.1. *Let U be a bounded, simply connected domain in the plane. If ϕ maps U conformally onto Δ then for all $z \in U$,*

$$\frac{1}{4}(1-|\phi(z)|^2)\frac{1}{|\phi'(z)|} \leqslant \operatorname{dist}(z, \partial U) \leqslant (1-|\phi(z)|^2)\frac{1}{|\phi'(z)|}.$$

Basically, this means that $1-|\phi(z)|^2$ and $\operatorname{dist}(z, \partial U)$ are comparable when ϕ has non-vanishing derivative in a full neighborhood of $\Gamma \subset \partial U$; in particular, when ϕ may be conformally extended across Γ.

THEOREM 2.2. *For all $z \in A_{\partial\Omega}$:*
(a) $g_\Omega(z, t_0) \approx \operatorname{dist}(z, \partial\Omega)$*;*
(b) $\lambda_\Omega(z) \approx 1/\operatorname{dist}(z, \partial\Omega)$*;*
(c)$K_z^\Omega(z) \approx 1/\big[\operatorname{dist}(z, \partial\Omega)\big]^3$.

Proof. The general idea is to bound $g_\Omega(z, t_0), \lambda_\Omega(z)$, and $K_z^\Omega(z)$ above and below by their counterparts on simply connected domains, and then to estimate using these more tractable quantities.

Fix $a \in \partial\Omega$. Put $B_a(\varepsilon) = \{z : |z-a| < \varepsilon\}$ where ε is small. Define $U = \Omega \cap B_a(\varepsilon)$ and note that $U \subset \Omega \subset \Omega_j$ where $a \in \Gamma_j = \partial\Omega_j$. Let ϕ be a conformal map of U onto Δ.

To prove (c): Standard techniques ([3]) reveal that

$$K_z^{\Omega_j}(z) \leqslant K_z^\Omega(z) \leqslant K_z^U(z). \tag{2.1}$$

So if we can show that $K_z^{\Omega_j}(z)$ and $K_z^U(z)$ are both comparable to $1/[\operatorname{dist}(z, \partial\Omega)]^3$ for $z \in U \cap B_a(\varepsilon/2)$ then we are done, for $\partial\Omega$ is compact and by a standard compactness argument, (c) will be proven.

A straightforward calculation shows that

$$K_z^U(z) = K_{\phi(z)}^\Delta(\phi(z))|\phi'(z)|^2 = \frac{(1+|\phi(z)|^2)|\phi'(z)|^2}{(1-|\phi(z)|^2)^3}. \tag{2.2}$$

Now consider the portion of ∂U defined by $\partial\Omega \cap B_a(\varepsilon)$. Since this portion is an arc of a circle, ϕ may be conformally extended across $\partial\Omega \cap B_a(\varepsilon)$; therefore ϕ' is non-vanishing in a full neighborhood of $\partial\Omega \cap B_a(\varepsilon/2)$. Applying Proposition 2.1 to (2.2) we have, for $z \in U \cap B_a(\varepsilon/2)$,

$$K_z^U(z) \approx \frac{1}{\big[1-|\phi(z)|^2\big]^3} \approx \frac{1}{\big[\operatorname{dist}(z, \partial U)\big]^3} = \frac{1}{\big[\operatorname{dist}(z, \partial\Omega)\big]^3}.$$

By a similar argument (or by explicitly writing down a linear fractional transformation ϕ_j which maps Ω_j onto Δ with $\phi_j(t_0) = 0$), it follows that $K_z^{\Omega_j}(z) \approx 1/\big[\operatorname{dist}(z, \partial\Omega)\big]^3$ for $z \in U \cap B_a(\varepsilon/2)$. Thus (c) is proven.

Examining the argument above, we clearly only need to establish that $g_\Omega(z, t_0)$ and $\lambda_\Omega(z)$ have properties akin to (2.1) and (2.2) to prove (a) and (b). For ultra-hyperbolic metrics, it is standard ([6]) that:

(d) $\lambda_{\Omega_j}(z) \leqslant \lambda_\Omega(z) \leqslant \lambda_U(z)$;

(e) $\lambda_U(z) = \lambda_\Delta(\phi(z))|\phi'(z)| = \frac{|\phi'(z)|}{1-|\phi(z)|^2}$.

For the Green's function, first modify U so that $t_0 \in U$ (for instance, let U be a thin tube in Ω containing $\Omega \cap B_a(\varepsilon)$ and t_0); next, choose ϕ so that $\phi(t_0) = 0$. With these modifications, it is standard ([8]) that:

(f) $g_U(z, t_0) \leqslant g_\Omega(z, t_0) \leqslant g_{\Omega_j}(z, t_0)$;
(g) $g_U(z, t_0) = g_\Delta(\phi(z), \phi(t_0)) = \log 1/|\phi(z)|$.

Now note that $\log 1/|\phi(z)| \approx 1 - |\phi(z)|$ near ∂U. ∎

2.3. Ultrahyperbolic discs. The ultrahyperbolic length of a smooth curve γ in Ω is defined to be $\int_\gamma \lambda_\Omega(z)\,\mathrm{d}z$. For $a, b \in \Omega$, the ultrahyperbolic distance from a to b, denoted by $\lambda_\Omega(a, b)$, may then be defined as the infimum over all ultrahyperbolic lengths of smooth curves from a to b. For $a \in \Omega$ and $r > 0$, define

(a) $U_\Omega(a, r) = \{z : \lambda_\Omega(a, z) < r\}$,
(b) $|U_\Omega(a, r)| = \frac{1}{\pi} \int_{U_\Omega(a,r)} \mathrm{d}A(z)$.

We call $U_\Omega(a, r)$ the ultrahyperbolic disc centered at a with radius r.

When Ω is the open unit disc Δ, $U_\Delta(a, r)$ is just the familiar hyperbolic disc with center a, radius r; in this case, it is standard ([12]) that $U_\Delta(a, r)$ is a Euclidean disc with area

$$|U_\Delta(a, r)| = \frac{(1 - |a|^2)^2 s^2}{(1 - |a|^2 s^2)^2}, \quad s = \tanh r. \tag{2.3}$$

Lemma 2.3. *If ϕ_j is the conformal map of Ω_j onto Δ, then:*
(a) *$\phi_j(U_{\Omega_j}(z, r)) = U_\Delta(\phi_j(z), r)$;*
(b) *$|U_{\Omega_j}(z, r)| \approx |U_\Delta(\phi_j(z), r)|$, $z \in A_j$, $0 < r < 1$.*

Proof. For (a): It is easy to verify that $\lambda_{\Omega_j}(a, b) = \lambda_\Delta(\phi_j(a), \phi_j(b))$; that is, ultrahyperbolic distance is conformally invariant.

For (b): By the change of variable $w = \phi_j(z)$,

$$\frac{1}{\pi} \int_{U_\Delta(\phi_j(a),r)} \mathrm{d}A(w) = \frac{1}{\pi} \int_{U_{\Omega_j}(a,r)} |\phi_j'(z)|^2 \,\mathrm{d}A(z).$$

Since ϕ_j' is non-vanishing in a full neighborhood of A_j, the integral above on the right is comparable to $|U_{\Omega_j}(a, r)|$. ∎

The next result may be proved using Theorem 2.2, Lemma 2.3, and standard estimates ([12]) for hyperbolic discs. For details, see [5] or [11].

Lemma 2.4. (a) *For any $R_2 > 0$,*

$$|U_{\Omega_j}(z, r_1)| \approx |U_{\Omega_j}(w, r_2)|$$

if $r_1, r_2 < 1$, $1/R_2 \leqslant r_1/r_2 \leqslant R_2$, $z \in A_j$, $w \in \Omega$, and $\lambda_{\Omega_j}(z, w) < 1$.

(b) *There exists a constant $S > 1$ such that*

$$U_{\Omega_j}\left(w, \frac{r}{S}\right) \subset U_\Omega(w, r) \subset U_{\Omega_j}(w, r)$$

for all $0 < r < 1$, $z \in A_j$, *and* $w \in \Omega$ *with* $\lambda_{\Omega_j}(z,w) < r$.

PROPOSITION 2.5. *Let* $0 < r < 1$.
(a) *For all* $z \in A_{\partial\Omega}$ *and* $w \in \Omega$ *with* $\lambda_\Omega(z,w) < r$,

$$|U_\Omega(z,r)| \approx |U_\Omega(w,r)|.$$

(b) *There exists a constant* $C_r > 0$ *such that for all* $z \in A_{\partial\Omega}$,

$$\frac{1}{C_r} \leqslant \frac{|U_\Omega(z,r)|^{\frac{1}{2}}}{g_\Omega(z,t_0)} \leqslant C_r.$$

(c) *There exists a constant* $C_r > 0$ *such that for all* $z \in A_j$,

$$\frac{1}{|U_\Omega(z,r)|^{\frac{3}{2}}} \leqslant C_r \inf_{w\in U_\Omega(z,r)} |k_z^{\Omega_j}(w)|^2.$$

Proof. For (a): Using the standard inequality $\lambda_{\Omega_j} \leqslant \lambda_\Omega$ and Lemma 2.4,

$$|U_\Omega(z,r)| \approx |U_{\Omega_j}(z,r)| \approx |U_{\Omega_j}(w,r)| \approx |U_\Omega(w,r)|. \tag{2.4}$$

For (b): Using (2.4), Lemma 2.3 (b), and (2.3) we obtain

$$|U_\Omega(z,r)| \approx |U_{\Omega_j}(z,r)| \approx |U_\Delta(\phi_j(z),r)| \approx (1-|\phi_j(z)|^2)^2 s^2,$$

where $s = \tanh r$. And by Proposition 2.1 and Theorem 2.2,

$$1-|\phi_j(z)|^2 \approx \operatorname{dist}(z,\partial\Omega_j) = \operatorname{dist}(z,\partial\Omega) \approx g_\Omega(z,t_0). \tag{2.5}$$

For (c): We will need the following standard estimate ([12]),

$$\inf_{w\in U_\Delta(z,r)} \left[\frac{1-|z|^2}{|1-\overline{z}w|^2}\right]^2 = \frac{(1-s|z|)^4}{(1-|z|^2)^2}, \quad s = \tanh r. \tag{2.6}$$

Using Lemma 2.3 (a), a straightforward calculation reveals that

$$\begin{aligned} \inf_{w\in U_{\Omega_j}(z,r)} |k_z^{\Omega_j}(w)|^2 &= \inf_{\phi_j(w)\in U_\Delta(\phi_j(z),r)} |k^\Delta_{\phi_j(z)}(\phi_j(w))|^2 |\phi_j'(w)|^2 \\ &\approx \inf_{\phi_j(w)\in U_\Delta(\phi_j(z),r)} \left[\frac{1-|\phi_j(z)|^2}{|1-\overline{\phi_j(z)}\phi_j(w)|^2}\right]^3, \end{aligned} \tag{2.7}$$

since ϕ_j' is non-vanishing in a neighborhood of A_j. Applying (2.6), the quantity in (2.7) is comparable to $1/\big[1-|\phi_j(z)|^2)\big]^3$. So by (2.5) and (b),

$$\frac{1}{|U_\Omega(z,r)|^{\frac{3}{2}}} \leqslant C_r \inf_{w\in U_{\Omega_j}(z,r)} |k_z^{\Omega_j}(w)|^2.$$

To complete the proof note, by Lemma 2.4 (b), $U_\Omega(z,r) \subset U_{\Omega_j}(z,r)$; so the definition of infimum shows that

$$\inf_{w\in U_{\Omega_j}(z,r)} |k_z^{\Omega_j}(w)|^2 \leqslant \inf_{w\in U_\Omega(z,r)} |k_z^{\Omega_j}(w)|^2. \quad ∎$$

3. COMPACT AND SCHATTEN CLASS TOEPLITZ OPERATORS

3.1. DEFINITIONS AND CONVENTIONS. In the sequel, $\mathrm{d}A$ will denote Lebesgue area measure normalized by a factor of $2/\pi$.

Let $P : L^2(\Omega, g_\Omega(z,t_0)\mathrm{d}A(z)) \to A^2_{1,s}(\Omega)$ denote the orthogonal projection and let $\psi \in L^\infty(\Omega, \mathrm{d}A)$. The Toeplitz operator T_ψ on $A^2_{1,s}(\Omega)$ is defined as

$$T_\psi f = P(\psi f).$$

T_ψ is clearly bounded. Further, $\langle P(\psi f), K_z^\Omega\rangle = \langle \psi f, PK_z^\Omega\rangle = \langle \psi f, K_z^\Omega\rangle$ so that T_ψ has the integral representation

$$T_\psi f(z) = \int_\Omega \psi(w) f(w) K_w^\Omega(z) g_\Omega(w,t_0)\,\mathrm{d}A(w).$$

The goal of this section is to characterize those compact and Schatten class Toeplitz operators T_ψ whose non-negative symbol ψ possesses a certain averaging property.

DEFINITION 3.1. Let ψ be a non-negative function in $L^\infty(\Omega, \mathrm{d}A)$. Fix $0 < r < 1$ and define, for $z \in \Omega$,

$$\widehat{\psi}_r(z) = \frac{1}{|U_\Omega(z,r)|^{\frac{3}{2}}} \int_{U_\Omega(z,r)} \psi(w) g_\Omega(w,t_0)\,\mathrm{d}A(w).$$

We say ψ has the generalized sub-mean-value property near $\partial\Omega$ if there exists a constant $C_r > 0$ such that $\psi(z) \leqslant C_r \widehat{\psi}_r(z)$ for all $z \in A_{\partial\Omega}$.

The extra factor $|U_\Omega(z,r)|^{1/2}$ may seem odd, but compensates for the extra weight $g_\Omega(w,t_0)$: by Proposition 2.5 we know $|U_\Omega(z,r)| \approx |U_\Omega(w,r)|$ when $w \in U_\Omega(z,r)$ and $|U_\Omega(w,r)|^{1/2} \approx g_\Omega(w,t_0)$. Thus

$$\begin{aligned}\widehat{\psi}_r(z) &= \frac{1}{|U_\Omega(z,r)|} \int_{U_\Omega(z,r)} \psi(w) \frac{g_\Omega(w,t_0)}{|U_\Omega(z,r)|^{1/2}}\,\mathrm{d}A(w)\\ &\approx \frac{1}{|U_\Omega(z,r)|} \int_{U_\Omega(z,r)} \psi(w)\,\mathrm{d}A(w).\end{aligned} \tag{3.1}$$

So, at least near $\partial\Omega$, $\widehat{\psi}_r(z)$ is just the average value of $\psi(z)$ calculated with respect to an ultrahyperbolic, as opposed to Euclidean, disc.

With this definition, we may now state the main result of this section.

THEOREM 3.2. *Let ψ be a non-negative function in $L^\infty(\Omega, \mathrm{d}A)$ with the generalized sub-mean-value property near $\partial\Omega$.*

(a) *T_ψ is compact on $A^2_{1,s}(\Omega)$ if and only if*

$$\lim_{z\to\partial\Omega} \psi(z) = 0.$$

(b) *T_ψ is Schatten p-class on $A^2_{1,s}(\Omega)$, $1 \leqslant p < \infty$, if and only if*

$$\int_\Omega \big[\psi(z)\big]^p \big[\lambda_\Omega(z)\big]^2\,\mathrm{d}A(z) < \infty.$$

In the next two subsections, we derive necessary and sufficient conditions for a Toeplitz operator to be compact or Schatten class. Then we use these conditions, coupled with the restriction on the symbol ψ, to prove Theorem 3.2.

3.2. NECESSARY CONDITIONS. The standard argument for the open unit disc involves estimating the Berezin transform (The Berezin transform of T_ψ is defined for $z \in \Omega$ by $\widetilde{T}_\psi(z) = \langle T_\psi k_z^\Omega, k_z^\Omega \rangle$.) of T_ψ near the boundary. Our argument for finitely connected domains is essentially the same, except we use "pseudo-Berezin" transforms to estimate one boundary component at a time.

DEFINITION 3.3. Let T_ψ be a positive Toeplitz operator on $A^2_{1,s}(\Omega)$ and let $I : A^2_1(\Omega_j) \to A^2_{1,s}(\Omega)$ be the inclusion map and I^* the adjoint of I. Define $T_j : A^2_1(\Omega_j) \to A^2_1(\Omega_j)$ by

$$T_j = I^* T_\psi I.$$

Note that for $f \in A^2_1(\Omega_j)$,

$$\langle T_j f, f\rangle = \langle I^* T_\psi I f, f\rangle = \langle T_\psi I f, I f\rangle = \langle T_\psi f, f\rangle. \tag{3.2}$$

Hence T_j is also positive. Specializing to $f = k_z^{\Omega_j}$ in (3.2) we obtain

$$\langle T_j k_z^{\Omega_j}, k_z^{\Omega_j}\rangle = \langle T_\psi k_z^{\Omega_j}, k_z^{\Omega_j}\rangle. \tag{3.3}$$

PROPOSITION 3.4. *Let $1 \leqslant p < \infty$. If T_ψ is a positive Toeplitz operator that is Schatten p-class on $A^2_{1,s}(\Omega)$, then:*

(a) *T_j is Schatten p-class on $A^2_1(\Omega_j)$;*

(b) $\displaystyle\int_{\Omega_j} \langle T_\psi k_z^{\Omega_j}, k_z^{\Omega_j}\rangle^p \big[\lambda_{\Omega_j}(z)\big]^2 \,\mathrm{d}A(z) < \infty.$

Proof. For (a): If $\{s_n\}$ denotes the singular values of T_j and $\{t_n\}$ denotes the singular values of T_ψ, a straightforward argument using the definition of singular values gives $s_n \leqslant t_n$.

For (b): From [1] and [2], the formulas

(c) $\langle T k_z^\Omega, k_z^\Omega\rangle^p \leqslant \langle T^p k_z^\Omega, k_z^\Omega\rangle$;

(d) $\operatorname{trace}(T) = \int_\Omega \langle T k_z^\Omega, k_z^\Omega\rangle K_z^\Omega(z) g_\Omega(z, t_0)\,\mathrm{d}A(z)$,

are valid for any positive operator T on $A^2_{1,s}(\Omega)$. So using (c), Theorem 2.2, and then (d), we obtain

$$\begin{aligned}\int_\Omega \langle T k_z^\Omega, k_z^\Omega\rangle^p \big[\lambda_\Omega(z)\big]^2 \,\mathrm{d}A(z) &\leqslant \int_\Omega \langle T^p k_z^\Omega, k_z^\Omega\rangle \big[\lambda_\Omega(z)\big]^2 \,\mathrm{d}A(z)\\ &\leqslant C \int_\Omega \langle T^p k_z^\Omega, k_z^\Omega\rangle K_z^\Omega(z) g_\Omega(z, t_0)\,\mathrm{d}A(z)\\ &= C\operatorname{trace}(T^p).\end{aligned}$$

By a standard result ([12]) of operator theory, if T is Schatten p-class, then T^p is Schatten 1-class; that is, $\operatorname{trace}(T^p)$ is finite.

To complete the proof of (b), apply the above chain of inequalities with $\Omega = \Omega_j$, $T = T_j$, and use (3.3). ∎

3.3. SUFFICIENT CONDITIONS. Since the arguments for the open unit disc adapt readily to finitely connected domains, we will only sketch the proofs of the next two results.

PROPOSITION 3.5. *Let ψ be a non-negative function in $L^\infty(\Omega, \mathrm{d}A)$. If $\psi \in C_0(\Omega)$, then T_ψ is compact on $A^2_{1,s}(\Omega)$.*

Proof. Let $\{f_n\}$ be a sequence of functions in $A^2_{1,s}(\Omega)$ with $\|f_n\| \leqslant 1$ and $f_n \to 0$ uniformly on compact subsets of Ω. We wish to show that $\langle T_\psi f_n, f_n\rangle \to 0$.

Using hypothesis, we may choose a compact set $K \subset \Omega$ such that

$$\begin{aligned}\langle T_\psi f_n, f_n\rangle &= \int_\Omega \psi(z)|f_n(z)|^2 g_\Omega(z,t_0)\,\mathrm{d}A(z)\\ &\leqslant \int_{\Omega\setminus K} \psi(z)\varepsilon g_\Omega(z,t_0)\,\mathrm{d}A(z) + \int_K \varepsilon|f_n(z)|^2 g_\Omega(z,t_0)\,\mathrm{d}A(z)\\ &\leqslant \varepsilon\|\psi\|_\infty \int_\Omega g_\Omega(z,t_0)\,\mathrm{d}A(z) + \varepsilon \int_\Omega |f_n(z)|^2 g_\Omega(z,t_0)\,\mathrm{d}A(z).\end{aligned}$$

Since the singularity of $g_\Omega(z,t_0)$ at t_0 is integrable, and since $\|f_n\| \leqslant 1$,

$$\langle T_\psi f_n, f_n\rangle \leqslant \varepsilon\|\psi\|_\infty \cdot C_\Omega + \varepsilon.$$

By a standard result ([12]) of operator theory, T_ψ is compact. ∎

PROPOSITION 3.6. *Let ψ be a non-negative function in $L^\infty(\Omega, \mathrm{d}A)$ and suppose $1 \leqslant p < \infty$. If $\psi \in L^p(\Omega, [\lambda_\Omega]^2\mathrm{d}A)$, then T_ψ is Schatten p-class on $A^2_{1,s}(\Omega)$.*

Proof. Let $\{e_n\}$ be any orthonormal set in $A^2_{1,s}(\Omega)$. We wish to show that $\sum_n \langle T_\psi e_n, e_n\rangle^p$ is finite.

A standard calculation reveals that

$$\begin{aligned}\sum_n \langle T_\psi e_n, e_n\rangle^p &\leqslant \int_\Omega \sum_n |e_n(z)|^2[\psi(z)]^p g_\Omega(z,t_0)\,\mathrm{d}A(z)\\ &= \int_\Omega K^\Omega_z(z)[\psi(z)]^p g_\Omega(z,t_0)\,\mathrm{d}A(z).\end{aligned}$$

But $K^\Omega_z(z) g_\Omega(z,t_0) \leqslant C\,[\lambda_\Omega(z)]^2$ near $\partial\Omega$ by Theorem 2.2. Therefore,

$$\sum_n \langle T_\psi e_n, e_n\rangle^p \leqslant C \int_\Omega [\psi(z)]^p[\lambda_\Omega(z)]^2\,\mathrm{d}A(z) < \infty.$$

By a standard result ([12]) of operator theory, T_ψ is Schatten p-class. ∎

3.4. PROOF OF THEOREM 3.2. We begin by connecting the averaging function for ψ with our "pseudo-Berezin" transform for T_ψ.

LEMMA 3.7. *Let $0 < r < 1$. There exists a constant $C_r > 0$ such that*

$$\widehat{\psi}_r(z) \leqslant C_r \langle T_\psi k_z^{\Omega_j}, k_z^{\Omega_j} \rangle, \quad z \in A_j.$$

Proof. By Proposition 2.5 (c),

$$\begin{aligned} \widehat{\psi}_r(z) &= \frac{1}{|U_\Omega(z,r)|^{\frac{3}{2}}} \int\limits_{U_\Omega(z,r)} \psi(w) g_\Omega(w,t_0)\, \mathrm{d}A(w) \\ &\leqslant C_r \inf_{w \in U_\Omega(z,r)} |k_z^{\Omega_j}(w)|^2 \int\limits_{U_\Omega(z,r)} \psi(w) g_\Omega(w,t_0)\, \mathrm{d}A(w) \\ &= C_r \int\limits_{U_\Omega(z,r)} \psi(w) \Big[\inf_{w \in U_\Omega(z,r)} |k_z^{\Omega_j}(w)|^2 \Big] g_\Omega(w,t_0)\, \mathrm{d}A(w). \end{aligned}$$

Therefore,

$$\widehat{\psi}_r(z) \leqslant C_r \int\limits_\Omega \psi(w) |k_z^{\Omega_j}(w)|^2 g_\Omega(w,t_0)\, \mathrm{d}A(w) = C_r \langle T_\psi k_z^{\Omega_j}, k_z^{\Omega_j} \rangle. \quad ∎$$

We are now ready to prove the main theorem of this section.

Proof. (Proof of Theorem 3.2 (a)) Suppose T_ψ is compact. Then $\langle T_\psi k_z^{\Omega_j}, k_z^{\Omega_j} \rangle \to 0$ as $z \to \partial\Omega_j$ since $k_z^{\Omega_j} \to 0$ weakly as $z \to \partial\Omega_j$. Now by Lemma 3.7,

$$\widehat{\psi}_r(z) \leqslant C_r \langle T_\psi k_z^{\Omega_j}, k_z^{\Omega_j} \rangle, \quad z \in A_j.$$

Hence $\widehat{\psi}_r(z) \to 0$ as $z \to \partial\Omega_j$. But by hypothesis $\psi(z) \leqslant C_r \widehat{\psi}_r(z)$, forcing $\psi(z) \to 0$ as $z \to \partial\Omega_j$.

For the converse, write $T_\psi = T_{\psi|\Omega \setminus A_{\partial\Omega}} + T_{\psi|A_{\partial\Omega}}$. Then T_ψ will be compact if we can show that $T_{\psi|\Omega \setminus A_{\partial\Omega}}$ and $T_{\psi|A_{\partial\Omega}}$ are compact.

Now by assumption $\psi(z) \to 0$ as $z \to \partial\Omega$, so (3.1) implies $\widehat{\psi}_r(z) \to 0$ as $z \to \partial\Omega$. Thus, by Proposition 3.5, $T_{\widehat{\psi}_r}$ is compact. But by hypothesis, $\psi(z) \leqslant C\widehat{\psi}_r(z)$ for z near $\partial\Omega$ so that $T_{\psi|A_{\partial\Omega}} \leqslant T_{C\widehat{\psi}_r} = CT_{\widehat{\psi}_r}$. Therefore $T_{\psi|A_{\partial\Omega}}$ is compact.

It remains to show $T_{\psi|\Omega \setminus A_{\partial\Omega}}$ is compact; but this follows from the standard argument outlined in Proposition 3.5. ∎

Proof. (Proof of Theorem 3.2 (b)) Suppose T_ψ is Schatten p-class. Then by Lemma 3.7 and Proposition 3.4,

$$\int\limits_\Omega \big[\widehat{\psi}_r(z)\big]^p \big[\lambda_{\Omega_j}\big]^2 \,\mathrm{d}A \leqslant C_r \int\limits_{\Omega_j} \langle T_\psi k_z^{\Omega_j}, k_z^{\Omega_j} \rangle^p \big[\lambda_{\Omega_j}\big]^2 \,\mathrm{d}A < \infty.$$

Now by Theorem 2.2 λ_{Ω_j} and λ_Ω are comparable near $\partial\Omega_j \subset \partial\Omega$ since each is comparable to the reciprocal of the distance to the boundary. Thus $\widehat{\psi}_r$ is in $L^p(\Omega, [\lambda_\Omega]^2 \mathrm{d}A)$. But by hypothesis, $\psi(z) \leqslant C_r \widehat{\psi}_r(z)$ for z near $\partial\Omega$ so that ψ is in $L^p(\Omega, [\lambda_\Omega]^2 \mathrm{d}A)$.

The converse follows immediately from Proposition 3.6. ∎

REMARK 3.8. Similar to the situation for the open unit disc, a little more work shows that, with no extra sub-mean-value assumption on ψ, the conditions $\widehat{\psi}_r(z) \to 0$ as $z \to \partial\Omega$ and $\widehat{\psi}_r \in L^p(\Omega, [\lambda_\Omega]^2 \mathrm{d}A)$ actually characterize the compact and Schatten class Toeplitz operators T_ψ on $A^2_{1,s}(\Omega)$ with non-negative symbol ψ. For details, see [11].

4. COMPACT AND SCHATTEN CLASS COMPOSITION OPERATORS

With the results of Section 3 in place, we need just a few more results before we can state and prove the main theorem of this paper.

4.1. THE SUB-MEAN-VALUE PROPERTY. The following definition and theorem are from [4].

DEFINITION 4.1. Let $\varphi : \Omega \to \Omega$ be an analytic function. Define, for $w \in \Omega \backslash \{\varphi(t_0)\}$,

$$N_\varphi(w) = \sum_{\varphi(z)=w} g_\Omega(z, t_0).$$

N_φ is the Nevalinna counting function for φ.

THEOREM 4.2. *Suppose* $\varphi : \Omega \to \Omega$ *is analytic with* $\varphi(t_0) = t_0$.

(a) *For all* $w \in \Omega \backslash \{t_0\}$, $N_\varphi(w) \leqslant g_\Omega(w, t_0)$.

(b) *Suppose* f *is an analytic function on an open disc* D *with center at* w. *If* $f(D) \subset \Omega$ *and* $t_0 \notin f(D)$, *then*

$$N_\varphi(f(w)) \leqslant \frac{1}{|D|} \int\limits_D N_\varphi(f(z)) \, \mathrm{d}A(z).$$

If $f(w) = w$ is the identity map, then (b) asserts that N_φ has a subharmonic mean value property on Euclidean discs in Ω which do not contain t_0. The following corollary shows that, near $\partial\Omega$, N_φ retains a similar property for ultrahyperbolic discs as well.

COROLLARY 4.3. *Let* $0 < r < 1$. *There exists a constant* $C > 0$ *such that for all* $w \in A_{\partial\Omega}$,

$$N_\varphi(w) \leqslant \frac{C}{|U_\Omega(w, r)|} \int\limits_{U_\Omega(w,r)} N_\varphi(z) \, \mathrm{d}A(z). \tag{4.1}$$

Proof. Let $a \in A_j$ and $U_{\Omega_j}(a, r)$ a hyperbolic disc contained in A_j. For simplicity, put $b = \phi_j(a)$. If $\tau_b(z) = \frac{b-z}{1-\bar{b}z}$, then it is easy to check that

$$\tau_b(U_\Delta(0, r)) = U_{\Delta(b,r)} = \phi_j(U_{\Omega_j}(a, r)). \tag{4.2}$$

Applying Theorem 4.2 (b) with $f = \phi_j^{-1} \circ \tau_b$ and $D = U_\Delta(0, r)$,

$$N_\varphi(a) = N_\varphi(\phi_j^{-1} \circ \tau_b(0)) \leqslant \frac{1}{|U_\Delta(0, r)|} \int\limits_{U_\Delta(0,r)} N_\varphi(\phi_j^{-1} \circ \tau_b(z)) \, \mathrm{d}A(z).$$

Since $\tau_b(\tau_b(w)) = w$, by (4.2) the change of variable $z = \tau_b(w)$ gives

$$\int_{U_\Delta(0,r)} N_\varphi(\phi_j^{-1} \circ \tau_b(z))\,\mathrm{d}A(z) = \int_{U_\Delta(b,r)} N_\varphi(\phi_j^{-1}(w)|\tau_b'(w)|^2\,\mathrm{d}A(w).$$

Letting $s = \tanh r$ and using (2.3) along with standard estimates ([12]),

$$\begin{aligned} N_\varphi(a) &\leqslant \frac{1}{s^2} \int_{U_\Delta(b,r)} N_\varphi(\phi_j^{-1}(w)|\tau_b'(w)|^2\,\mathrm{d}A(w) \\ &\leqslant \frac{1}{s^2} \int_{U_\Delta(b,r)} N_\varphi(\phi_j^{-1}(w))\Big[\sup_{w\in U_\Delta(b,r)} \Big|\frac{1-|b|^2}{(1-\bar{b}z)^2}\Big|^2\Big]\,\mathrm{d}A(w) \\ &= \frac{(1+s|b|)^4}{s^2(1-|b|^2)^2} \int_{U_\Delta(b,r)} N_\varphi(\phi_j^{-1}(w))\,\mathrm{d}A(w) \\ &= \frac{(1+s|b|)^4}{(1-|b|^2s^2)^2}\frac{1}{|U_\Delta(b,r)|} \int_{U_\Delta(b,r)} N_\varphi(\phi_j^{-1}(w))\,\mathrm{d}A(w). \end{aligned}$$

The factor $(1+s|b|)^4/(1-|b|^2s^2)^2$ is bounded by a constant independent of $b = \phi_j(a)$ and $s = \tanh r$ since $|\phi_j(a)| < 1$ and $r < 1$. Thus

$$\begin{aligned} N_\varphi(a) &\leqslant \frac{C}{|U_\Delta(b,r)|} \int_{U_\Delta(b,r)} N_\varphi(\phi_j^{-1}(w))\,\mathrm{d}A(w) \\ &= \frac{C}{|U_\Delta(\phi_j(a),r)|} \int_{U_{\Omega_j}(a,r)} N_\varphi(z)|\phi_j'(z)|^2\,\mathrm{d}A(z) \end{aligned}$$

where, by (4.2), we have made the change of variable $w = \phi_j(z)$. Since $U_{\Omega_j}(a,r) \subset A_j$ and ϕ_j' is non-vanishing in a neighborhood of A_j,

$$N_\varphi(a) \leqslant \frac{C}{|U_\Delta(\phi_j(a),r)|} \int_{U_{\Omega_j}(a,r)} N_\varphi(z)\,\mathrm{d}A(z). \tag{4.3}$$

Now fix an $S > 1$. Since (4.3) holds for $0 < r < 1$, it holds for r/S. So replacing r with r/S in (4.3) and using Lemma 2.3 (b) and Lemma 2.4 (a),

$$N_\varphi(a) \leqslant \frac{C}{|U_{\Omega_j}(a,r/S)|} \int_{U_{\Omega_j}(a,r/S)} N_\varphi(z)\,\mathrm{d}A(z). \tag{4.4}$$

If S is the constant from Lemma 2.4, then there exists a $C > 0$ such that $|U_\Omega(a,r)| \leqslant C|U_{\Omega_j}(a,r/S)|$ and $U_{\Omega_j}(a,r/S) \subset U_\Omega(a,r)$; so by (4.4),

$$N_\varphi(a) \leqslant \frac{C}{|U_\Omega(a,r)|} \int_{U_\Omega(a,r)} N_\varphi(z)\,\mathrm{d}A(z). \quad \blacksquare$$

4.2. THE MAIN THEOREM. The next lemma provides the crucial link between composition operators on the Hardy space $H^2(\Omega)$ and Toeplitz operators on the weighted Bergman space $A^2_{1,s}(\Omega)$.

LEMMA 4.4. *Suppose* $\varphi : \Omega \to \Omega$ *is analytic with* $\varphi(t_0) = t_0$. *Let* $H^2_{t_0}(\Omega)$ *denote the subspace of* $H^2(\Omega)$ *functions vanishing at the base point* t_0 *and define*

$$U : H^2_{t_0}(\Omega) \to A^2_{1,s}(\Omega) \quad \text{and} \quad D_\varphi : A^2_{1,s}(\Omega) \to A^2_{1,s}(\Omega)$$

by $Uf(z) = f'(z)$ *and* $D_\varphi f(z) = f(\varphi(z))\varphi'(z)$. *Then:*
(a) $UC_\varphi U^* = D_\varphi$; *that is,* C_φ *is unitarily equivalent to* D_φ.
(b) *Let* $\psi(w) = N_\varphi(w)/g_\Omega(w, t_0)$. *Then* $D^*_\varphi D_\varphi = T_\psi$.

Proof. For (a): Let ω_{t_0} denote the harmonic measure on $\partial\Omega$ for the base point t_0. It is standard ([9]) that each H^2 function f on Ω has boundary values almost everywhere on $\partial\Omega$, and that the correspondence of f to its boundary values is an isometry of H^2 onto a closed subspace of $L^2(\partial\Omega, \omega_{t_0})$. From the Littlewood-Paley identity ([4]),

$$\|f\|^2_{H^2(\Omega)} = \int_{\partial\Omega} |f|^2 \,\mathrm{d}\omega_{t_0} = |f(t_0)|^2 + \int_{\Omega} |f'(z)|^2 g_\Omega(z, t_0) \,\mathrm{d}A(z). \tag{4.5}$$

Hence $H^2_{t_0}(\Omega)$ is unitarily isomorphic to $A^2_{1,s}(\Omega)$ via the map $Uf = f'$ and (a) now follows from the chain rule.

For (b): From [4], the change of variable formula

$$\int_{\Omega} F(\varphi(z)) \,|\varphi'(z)|^2 \, g(z, t_0) \,\mathrm{d}A(z) = \int_{\Omega} F(w) N_\varphi(w) \,\mathrm{d}A(w)$$

is valid for any non-negative measurable function F on Ω.

Let $f \in A^2_{1,s}(\Omega)$. Then $\langle D^*_\varphi D_\varphi f, f\rangle = \langle D_\varphi f, D_\varphi f\rangle$ so that

$$\langle D^*_\varphi D_\varphi f, f\rangle = \int_{\Omega} |f(\varphi(z))|^2 |\varphi'(z)|^2 g_\Omega(z, t_0) \,\mathrm{d}A(z).$$

Applying the change of variable formula with $F = |f|^2$, we obtain

$$\langle D^*_\varphi D_\varphi f, f\rangle = \int_{\Omega} |f(w)|^2 N_\varphi(w) \,\mathrm{d}A(w).$$

On the other hand, $\psi(w) = N_\varphi(w)/g_\Omega(w, t_0)$ so that

$$\langle T_\psi f, f\rangle = \int_{\Omega} \psi(w) |f(w)|^2 g_\Omega(w, t_0) \,\mathrm{d}A(w) = \int_{\Omega} |f(w)|^2 N_\varphi(w) \,\mathrm{d}A(w).$$

Hence $\langle T_\psi f, f\rangle = \langle D^*_\varphi D_\varphi f, f\rangle$ for all $f \in A^2_{1,s}(\Omega)$. ∎

We are now ready to state and prove the main result of this paper.

THEOREM 4.5. *Suppose* $\varphi : \Omega \to \Omega$ *is analytic with* $\varphi(t_0) = t_0$.
(a) C_φ *is compact on* $H^2(\Omega)$ *if and only if*

$$\lim_{w \to \partial\Omega} \frac{N_\varphi(w)}{g_\Omega(w, t_0)} = 0.$$

(b) C_φ *is Schatten p-class on* $H^2(\Omega)$, $2 \leqslant p < \infty$, *if and only if*

$$\int_\Omega \left[\frac{N_\varphi(w)}{g_\Omega(w,t_0)}\right]^{\frac{p}{2}} \left[\lambda_\Omega(w)\right]^2 \mathrm{d}A(w) \ < \infty$$

where dA *is Lebesgue area measure.*

Proof. Since $H^2_{t_0}(\Omega)$ is a subspace of codimension 1 in $H^2(\Omega)$, the question of whether C_φ is compact or Schatten class on $H^2(\Omega)$ is the same as whether C_φ is compact or Schatten class on $H^2_{t_0}(\Omega)$. Hence, it suffices to consider C_φ acting on $H^2_{t_0}(\Omega)$.

Let

$$\psi(w) = \frac{N_\varphi(w)}{g_\Omega(w,t_0)}.$$

From Lemma 4.4, C_φ is unitarily equivalent to D_φ, and $D^*_\varphi D_\varphi = T_\psi$. Therefore, C_φ is compact if and only if T_ψ is compact and, for $2 \leqslant p < \infty$, C_φ is Schatten p-class if and only if T_ψ is Schatten $\frac{p}{2}$-class. So by Theorem 3.2, it simply remains to check that ψ has the generalized sub-mean-value property near $\partial\Omega$.

Multiplying both sides of (4.1) by $1/g_\Omega(w,t_0)$,

$$\frac{N_\varphi(w)}{g_\Omega(w,t_0)} \leqslant \frac{C}{g_\Omega(w,t_0)|U_\Omega(w,r)|} \int_{U_\Omega(w,r)} N_\varphi(z)\,\mathrm{d}A(z). \tag{4.6}$$

Now from Proposition 2.5 (b), $|U_\Omega(w,r)|^{1/2} \leqslant C_r g_\Omega(w,t_0)$. So by (4.6),

$$\begin{aligned}\frac{N_\varphi(w)}{g_\Omega(w,t_0)} &\leqslant \frac{C_r}{|U_\Omega(w,r)|^{\frac{3}{2}}} \int_{U_\Omega(w,r)} N_\varphi(z)\,\mathrm{d}A(z)\\ &= \frac{C_r}{|U_\Omega(w,r)|^{\frac{3}{2}}} \int_{U_\Omega(w,r)} \frac{N_\varphi(z)}{g_\Omega(z,t_0)} g_\Omega(z,t_0)\,\mathrm{d}A(z),\end{aligned}$$

for all $w \in A_{\partial\Omega}$. Therefore, ψ has the generalized sub-mean-value property near $\partial\Omega$ and the proof is complete. ∎

The material in this paper formed part of the author's dissertation prepared at Northwestern University under the direction of Stephen D. Fisher.

REFERENCES

1. J. Arazy, S.D. Fisher, J. Peetre, Hankel operators on weighted Bergman spaces, *Amer. J. Math.* **110**(1988), 989–1054.
2. J. Arazy, S.D. Fisher, J. Peetre, Hankel operators on planar domains, *Constr. Approx.* **6**(1990), 113–138.
3. S. Bergman, *The Kernel Function and Conformal Mapping*, Math. Surveys, vol. 5, Amer. Math. Soc., New York 1950.
4. S.D. Fisher, J.E. Shapiro, The essential norm of a composition operator on a planar domain, *Illinois J. Math.* **43**(1999), 113–130.

5. H.P. Li, Hankel operators on the Bergman space of multiply-connected domains, *J. Operator Theory* **28**(1992), 321–335.
6. I. Kra, *Automorphic Forms and Kleinian Groups*, W.A. Benjamin, Reading 1972.
7. Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin 1991.
8. T. Ransford, *Potential Theory in the Complex Plane*, Press Syndicate of the University of Cambridge, Cambridge 1995.
9. W. Rudin, Analytic functions of class H^p, *Trans. Amer. Math. Soc.* **78**(1955), 46–66.
10. J.H. Shapiro, The essential norm of a composition operator, *Ann. of Math.* **125** (1987), 375–404.
11. V.M. Tran, *Function-theoretic operator theory on finitely connected planar domains*, Ph.D. Dissertation, Northwestern University, 1998.
12. K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, New York 1990.

VINH-THY M. TRAN
Department of Mathematics
Vassar College
124 Raymond Ave.
Poughkeepsie, NY 12604
USA

Current address:

1785 Jonathan's Way, Apt. D
Reston, VA 20190
E-mail: vinhthy-tran@yahoo.com

Received December 6, 1999.