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ALGEBRAS OF APPROXIMATION SEQUENCES: FREDHOLMNESS

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Abstract. In this paper, a Fredholm theory for approximation sequences is proposed. A sequence is called Fredholm if it is invertible modulo a certain ideal which plays the role of the ideal of the compact operators in the Fredholm theory of operators. With every Fredholm sequence, there are associated three integers which are the analogues of the nullity, the deficiency and the index of a Fredholm operator. The nullity of a Fredholm sequence (A_n) is interpreted as a quantity which describes the asymptotic behaviour of the small singular values of the matrices A_n as $n \to \infty$, and an identity is derived which allows the computation of this nullity in many situations. Several examples and applications are discussed.

Keywords: *Fredholm theory, approximation methods, asymptotic behaviour of singular values.*

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1. INTRODUCTION

Approximation methods. Let H be a separable complex Hilbert space, $L(H)$ the C^*-algebra of the linear and bounded operators on H, and (P_n) a sequence of orthogonal projections on H which converges strongly to the identity operator I on H:

$$\text{s-lim}\, P_n = I \quad \Leftrightarrow \quad P_n x \to x \quad \text{for all } x \in H.$$

Assume that $\dim \operatorname{Im} P_n = n$; so $\operatorname{Im} P_n$ and $L(\operatorname{Im} P_n)$ can be identified with the linear space $\mathbb{C}^n$ and with the algebra $\mathbb{C}^{n\times n} = L(\mathbb{C}^n)$, respectively. The $n \times n$ identity matrix will be denoted by I_n.

Let $A \in L(H)$. An *approximation method for* A is a sequence (A_n) of matrices $A_n \in \mathbb{C}^{n\times n}$ such that $A_n P_n \to A$ and $A_n^* P_n \to A^*$ strongly as $n \to \infty$. This method *converges*, or *is applicable to* A, if the equations

$$A_n x^{(n)} = P_n y \tag{1.1}$$

possess unique solutions $x^{(n)} \in \operatorname{Im} P_n$ for all sufficiently large n and all right hand sides $y \in H$, and if these solutions converge in the norm of H to a solution of the equation

$$Ax = y. \tag{1.2}$$

By the Banach-Steinhaus theorem, the method (A_n) for A is applicable if and only if the sequence (A_n) is *stable* in the sense that the matrices A_n are invertible for all sufficiently large n and that

$$\sup \|A_n^{-1}\| < \infty.$$

For another characterization of stability, introduce the set $\mathcal{F}$ of all bounded sequences (A_n) of matrices $A_n \in \mathbb{C}^{n\times n}$. Provided with elementwise operations and the supremum norm, this set becomes a C^*-algebra with identity element (I_n), and the set $\mathcal{G}$ of all sequences $(A_n) \in \mathcal{F}$ with $\lim \|A_n\| = 0$ forms a closed two-sided ideal in $\mathcal{F}$. A Neumann series argument shows that the sequence $(A_n) \in \mathcal{F}$ is stable if and only if its coset $(A_n) + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$.

STANDARD ALGEBRAS OF APPROXIMATION METHODS. One way to study the stability properties of a class of approximation methods is to describe the subalgebra of $\mathcal{F}/\mathcal{G}$ which is generated by the approximation sequences under consideration. For several concrete approximation methods such descriptions were given, e.g., in [3], [5], [10] and [11]. To mention at least one example, we recall the results pertaining the polynomial collocation method for singular integral operators on the unit circle obtained in [6] and [10].

Consider the operator of singular integration against the unit circle $\mathbb{T}$,

$$(Su)(t) := \frac{1}{\pi \mathrm{i}} \int_{\mathbb{T}} \frac{u(s)}{s-t}\, \mathrm{d}s, \quad t \in \mathbb{T},$$

and let a and b be piecewise continuous functions on $\mathbb{T}$ (i.e. a and b possess finite one-sided limits at each point of $\mathbb{T}$). The operator $aI + bS$ is bounded on the Lebesgue space $L^2(\mathbb{T})$. Let Π_n stand for the subspace of $L^2(\mathbb{T})$ spanned by the polynomials z^j with $-n \leqslant j \leqslant n$. For Riemann integrable and bounded right hand sides f, we seek approximate solutions $u^{(n)} \in \Pi_n$ of the singular integral equation

$$(aI + bS)u = f \tag{1.3}$$

by replacing (1.3) by the linear system

$$a(z_j)u(z_j) + b(z_j)(Su^{(n)})(z_j) = f(z_j), \quad j = -n, \dots, n, \tag{1.4}$$

where $z_j := \exp(2\pi \mathrm{i} j/(2n+1))$. Introducing the orthogonal projection P_n from $L^2(\mathbb{T})$ onto Π_n as well as the interpolation projection L_n which associates with every Riemann integrable function f the function $L_n f$ in Π_n such that $f(z_j) = (L_n f)(z_j)$ for $j = -n, \dots, n$, one can rewrite (1.4) as

$$L_n(aI + bS)P_n u^{(n)} = L_n f.$$

Thus, what is crucial is the stability of the sequence $(L_n(aI + bS)P_n)$.

Define operators $R_n : L^2(\mathbb{T}) \to \Pi_n$ by

$$\sum c_k z^k \mapsto c_{-1}z^{-n} + \dots + c_{-n}z^{-1} + c_0 z^n + \dots + c^n z^0,$$

and let $\mathcal{A}$ stand for the smallest closed subalgebra of the algebra $\mathcal{F}$ which contains all sequences $(L_n(aI+bS)P_n)$ with piecewise continuous functions a and b as well as all sequences $(P_nKP_n + R_nLR_n + G_n)$ with K and L compact and with (G_n) tending to zero in the norm. Actually, the algebra $\mathcal{F}$ consists of sequences (A_n) of $(2n+1)\times(2n+1)$ matrices A_n, but this difference to the above definition of $\mathcal{F}$ doesn't matter. The algebra $\mathcal{A}$ is a C^*-algebra, and there are two natural $*$-homomorphisms W and $\widetilde{W}$ from $\mathcal{A}$ into $L(L^2(\mathbb{T}))$:

$$W : (A_n) \mapsto \text{s-lim}\, A_nP_n \quad \text{and} \quad \widetilde{W} : (A_n) \mapsto \text{s-lim}\, R_nA_nR_n.$$

In particular, $W(L_n(aI+bS)P_n) = aI+bS$ and $\widetilde{W}(L_n(aI+bS)P_n) = \widetilde{a}I + \widetilde{b}S$ with $\widetilde{a}(t) := a(1/t)$. The following stability criterion is derived in [6] and [10].

THEOREM 1.1. *A sequence $(A_n) \in \mathcal{A}$ is stable if and only if both associated operators $W(A_n)$ and $\widetilde{W}(A_n)$ are invertible.*

Similar results hold for other approximation methods. The common structure of the C^*-algebras $\mathcal{A} \subseteq \mathcal{F}$ associated with the approximation methods considered in the above cited papers can be summarized as follows:

1. There is a (possibly infinite) set T, and for every $t \in T$, there is a Hilbert space H^t and a sequence (E_n^t) of partial isometries $E_n^t : H^t \to H$ such that
– the initial projections $(E_n^t)^*E_n^t$ converge strongly to the identity on H^t,
– the range projections $E_n^t(E_n^t)^*$ coincide with P_n, and
– the separation condition holds:

$$(E_n^s)^*E_n^t \to 0 \quad \text{weakly for all } s \neq t.$$

2. For every $t \in T$ and every $(A_n) \in \mathcal{A}$, there exists the strong limit

$$\text{s-lim}(E_n^t)^*A_nE_n^t =: W^t(A_n).$$

3. $\mathcal{A}$ is unital and contains all sequences $(E_n^tK(E_n^t)^*)$ with K compact on H^t as well as all sequences $(G_n) \in \mathcal{F}$ tending to zero in the norm. The closed linear span of all of these sequences forms a closed ideal $\mathcal{J}$ of $\mathcal{A}$.

4. A sequence $(A_n) \in \mathcal{A}$ is stable if and only if all associated operators $W^t(A_n)$ are invertible on H^t.

An algebra $\mathcal{A}$ which satisfies these axioms will be called a *standard algebra* in what follows. The algebra of the polynomial collocation method for singular integral operators is a standard algebra: choose $T = \{0,1\}$, and set $E_n^0 = P_n$ and $E_n^1 = R_n$.

FREDHOLM THEORY IN STANDARD ALGEBRAS. With every sequence (A_n) in a standard algebra $\mathcal{A}$, we associate the function $t \mapsto W^t(A_n)$ which is defined on T and takes a value in $L(H^t)$ at $t \in T$. A remarkable consequence of the C^*-property of $\mathcal{A}$ and of axiom 4 of a standard algebra is that the mapping

$$(A_n) + \mathcal{G} \mapsto (t \mapsto W^t(A_n))$$

is an isometry from $\mathcal{A}/\mathcal{G}$ into the algebra of all operator-valued functions on T:

$$\|(A_n) + \mathcal{G}\| = \sup_{t\in T} \|W^t(A_n)\|.$$

This isometry has been employed in [12] in order to establish a Fredholm theory for approximation sequences in standard algebras. A sequence $(A_n) \in \mathcal{A}$ is *Fredholm* in this sense if every operator $W^t(A_n)$ is a Fredholm operator. It turns out that, for every Fredholm sequence, the quantities

$$\alpha(A_n) := \sum_{t \in T} \dim \operatorname{Ker} W^t(A_n) \quad \text{and} \quad \beta(A_n) := \sum_{t \in T} \dim \operatorname{Coker} W^t(A_n) \tag{1.5}$$

are finite. This suggests to define the index of a Fredholm sequence by

$$\operatorname{ind}(A_n) := \alpha(A_n) - \beta(A_n).$$

The so-defined Fredholm sequences and the functionals α, β and ind obviously satisfy all the properties which are known from common Fredholm theory for operators: stability of the index under small perturbations as well as under perturbations belonging to the ideal $\mathcal{J}$ (which serves as a substitute for the ideal of the compact operators), and

$$\operatorname{ind}(A_n)^* = -\operatorname{ind}(A_n)^*, \quad \operatorname{ind}(A_n B_n) = \operatorname{ind}(A_n) + \operatorname{ind}(B_n)$$

for arbitrary Fredholm sequences $(A_n), (B_n)$. Moreover, if (A_n) is a Fredholm sequence and if both numbers $\alpha(A_n)$ and $\beta(A_n)$ are zero, then the sequence (A_n) is stable (which is a consequence of the 4th axiom of a standard algebra).

Furthermore, in [12] there is derived a characterization of the α-number of a Fredholm sequence in terms of the asymptotic behaviour of the singular values of A_n. Let $0 \leqslant \lambda_1^{(n)} \leqslant \cdots \leqslant \lambda_n^{(n)}$ denote the eigenvalues of $A_n^* A_n$. If $(A_n) \in \mathcal{A}$ is a Fredholm sequence, and if $\alpha(A_n) = k$, then

$$\lim \lambda_k^{(n)} = 0 \quad \text{but} \quad \liminf \lambda_{k+1}^{(n)} > 0. \tag{1.6}$$

CONTENTS OF THIS PAPER. The theory of Fredholm sequences as sketched above has interesting consequences and applications (see [12] and [13] for a first discussion of applications to the regularization of ill-posed approximation sequences and to the asymptotic behaviour of the singular values of Cauchy-Toeplitz matrices), but it is still unsatisfactory. The main point is that, so far, Fredholmness is defined only for sequences in a standard algebra. Thus, at least formally, the Fredholmness of a sequence (A_n) depends on the algebra as an element of which (A_n) is regarded. Of course, the characterization (1.6) reveals that, actually, the quantities $\alpha(A_n)$ and $\beta(A_n)$ do not depend on the embedding of (A_n) into a standard algebra. But, for example, the identity $\operatorname{ind}(A_n B_n) = \operatorname{ind}(A_n) + \operatorname{ind}(B_n)$ can only be guaranteed if both (A_n) and (B_n) are elements of one and the same standard algebra.

The goal of the present paper is to propose a general Fredholm theory which principally applies to every approximation sequence $(A_n) \in \mathcal{F}$, and which reduces to the above sketched theory in case of sequences in a standard algebra. In particular, the identities (1.5) (which are no longer definitions but consequences of the theory) will be generalized to a much larger class of algebras which includes standard algebras. Moreover, it will be pointed out how the Fredholm theory of approximation sequences is related to the theory of Fredholm elements in Banach and C^*-algebras as described, e.g., in [1]. And finally, a few new insights into the structure of algebras of approximation sequences (i.e. of subalgebras of $\mathcal{F}$) will be derived.

2. CENTRALLY COMPACT AND FREDHOLM SEQUENCES

Compact elements in C^*-algebras. Let $\mathcal{B}$ be a C^*-algebra. An element $k \in \mathcal{B}$ is of *rank one* if, for every $b \in \mathcal{B}$, there is a complex number $\mu(b)$ such that $kbk = \mu(b)k$. An element of $\mathcal{B}$ is of *finite rank* if it is the sum of a finite number of elements of rank one, and it is *compact* if it lies in the closure of the set of all finite rank elements. We denote the set of all compact elements in $\mathcal{B}$ by $C(\mathcal{B})$. It is easy to check that both the elements of finite rank and the compact elements form two-sided ideals in $\mathcal{B}$. In case $\mathcal{B} = L(H)$, an element $b \in \mathcal{B}$ is of rank one, of finite rank, or compact if and only if the operator b has range dimension less than or equal to one, finite range dimension, or is compact, respectively.

Proposition 2.1. *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{F}$ which contains the ideal $\mathcal{G}$. Then $C(\mathcal{A}) = \mathcal{G}$.*

Proof. Let $(A_n) \neq 0$ be a rank one element of $\mathcal{F}$. Then $A_k \neq 0$ for a certain k. Let A_k^+ denote the Moore-Penrose inverse of A_k, and consider the sequence

$$B := (0, \ldots, 0, A_k^+, 0, \ldots) \in \mathcal{G}$$

with the A_k^+ standing at the kth position. By assumption, there is a $\mu(B) \in \mathbb{C}$ such that

$$(A_n)B(A_n) = \mu(B)(A_n),$$

whence $\mu(B) = 1$ and

$$A_1 = \cdots = A_{k-1} = A_{k+1} = A_{k+2} = \cdots = 0.$$

Thus, every rank one sequence in $\mathcal{F}$ is necessarily of the form

$$(0, \ldots, 0, A_k, 0, \ldots) \tag{2.1}$$

with some $A_k \in \mathbb{C}^{k \times k}$. Further, A_k must be a rank one element in $\mathbb{C}^{k \times k}$, that is, it is zero or has one-dimensional range. It is clear that, conversely, all sequences (2.1) with $\dim \operatorname{Im} A_k \leqslant 1$ are elements of rank one in $\mathcal{F}$. Now the assertion follows immediately from the definitions. ∎

Centrally compact elements. One might call a sequence $(A_n) \in \mathcal{F}$ Fredholm if it is invertible modulo the ideal $C(\mathcal{F}) = \mathcal{G}$. This indeed yields a reasonable Fredholm theory (see the following section), but it doesn't give the desired notion of Fredholmness, since Fredholmness of a sequence in this sense simply means stability of that sequence. Here is a modified notion of compactness which fits exactly to our purposes. Recall that the center of an algebra is the set of all elements which commute with every element of the algebra.

DEFINITION 2.2. Let $\mathcal{B}$ be a unital C^*-algebra. An element $k \in \mathcal{B}$ is of *central rank one* if, for every $b \in \mathcal{B}$, there is an element $\mu(b)$ belonging to the center of $\mathcal{B}$, such that $kbk = \mu(b)k$. An element of $\mathcal{B}$ is of *finite central rank* if it is the sum of a finite number of elements of central rank one, and it is *centrally compact* if it lies in the closure of the set of all elements of finite central rank.

We denote the set of all centrally compact elements in $\mathcal{B}$ by $J(\mathcal{B})$. It is again easy to check that both the elements of finite central rank and the centrally compact elements form two-sided ideals in $\mathcal{B}$.

In case $\mathcal{B} = L(H)$, the rank one, finite rank, and compact elements coincide with their central analogues, since the center of $L(H)$ consists of the scalar multiples of the identity operator only. On the other hand, the center of the algebra $\mathcal{F}$ coincides with ℓ^∞ (where the number sequence $(a_n) \in \ell^\infty$ is identified with the matrix sequence $(a_n I_n)$; see Lemma 4.1 below). Hence, the ideal $J(\mathcal{F})$ should be much larger than the ideal $C(\mathcal{F}) = \mathcal{G}$ of the zero sequences.

PROPOSITION 2.3. *A sequence $(A_n) \in \mathcal{F}$ is centrally compact if and only if, for every $\varepsilon > 0$, there is a sequence $(K_n) \in \mathcal{F}$ such that*

$$\sup_n \|A_n - K_n\| < \varepsilon \quad \textit{and} \quad \sup_n \dim \operatorname{Im} K_n < \infty.$$

Proof. If (A_n) is of central rank one in $\mathcal{F}$, then every matrix A_n is of rank one in $\mathbb{C}^{n\times n}$, hence $\dim \operatorname{Im} A_n \leqslant 1$.

Conversely, let $(A_n) \in \mathcal{F}$ be a sequence of matrices with $\dim \operatorname{Im} A_n \leqslant 1$ for every n, and let $(B_n) \in \mathcal{F}$ arbitrarily. Then there are numbers μ_n such that

$$A_n B_n A_n = \mu_n A_n. \tag{2.2}$$

The numbers μ_n are uniquely determined if $A_n \neq 0$; in case $A_n = 0$ we choose $\mu_n = 0$. The so-defined sequence (μ_n) is bounded. Indeed, (2.2) implies $|\mu_n|\,\|A_n\| \leqslant \|A_n\|^2 \|B_n\|$ for every n whence $|\mu_n| \leqslant \|A_n\|\,\|B_n\|$ for every n with $A_n \neq 0$. This observation identifies the elements of central rank one. It is clear now that the elements of finite central rank are just the sequences $(A_n) \in \mathcal{F}$ with

$$\sup \dim \operatorname{Im} A_n < \infty$$

which yields the assertion. ∎

Observe that $J(\mathcal{F})$ is a proper ideal of $\mathcal{F}$. Indeed, suppose for contrary, that there is a sequence $(K_n) \in \mathcal{F}$ such that

$$\sup \|I_n - K_n\| < \frac{1}{2} \quad \text{and} \quad \sup \dim \operatorname{Im} K_n < \infty.$$

Then every matrix K_n is invertible, hence, $\dim \operatorname{Im} K_n = n$, which contradicts the second condition of the choice of (K_n).

FREDHOLM SEQUENCES. Based on the ideal of the centrally compact sequences in $\mathcal{F}$ one can introduce an appropriate class of Fredholm sequences.

DEFINITION 2.4. A sequence $(A_n) \in \mathcal{F}$ is a *Fredholm sequence* if it is invertible modulo the ideal $J(\mathcal{F})$ of the centrally compact sequences.

The following properties of Fredholm sequences are obvious:

– every stable sequence is Fredholm;
– the adjoint of a Fredholm sequence is Fredholm;
– the product of Fredholm sequences is Fredholm;
– if (A_n) is Fredholm and $(K_n) \in J(\mathcal{F})$, then $(A_n + K_n)$ is Fredholm;
– the set of the Fredholm sequences is open in $\mathcal{F}$.

For another characterization of Fredholm sequences, let $0 \leqslant \lambda_1^{(n)} \leqslant \cdots \leqslant \lambda_n^{(n)}$ denote the eigenvalues of $A_n^* A_n$ and write $\sigma_i^{(n)} := (\lambda_i^{(n)})^{1/2} \geqslant 0$ for the singular values of A_n.

THEOREM 2.5. *Each of the following conditions is equivalent to the Fredholmness of a sequence* $(A_n) \in \mathcal{F}$:

(i) *there is a sequence* $(B_n) \in \mathcal{F}$ *and a sequence* $(J_n) \in J(\mathcal{F})$ *of central finite rank such that*

$$B_n A_n^* A_n = I_n + J_n; \tag{2.3}$$

(ii) *there is a* k *such that*

$$\liminf_{n\to\infty} \sigma_{k+1}^{(n)} > 0. \tag{2.4}$$

Proof. Let (A_n) be a Fredholm sequence. Then $(A_n^* A_n)$ is a Fredholm sequence and, by definition, there are sequences $(B_n) \in \mathcal{F}$ and $(J_n) \in J(\mathcal{F})$ such that

$$(B_n)(A_n^* A_n) = (I_n) + (J_n). \tag{2.5}$$

One can assume that $\sup \dim \operatorname{Im} J_n < \infty$. Indeed, by Proposition 2.3, there exists a sequence $(K_n) \in \mathcal{F}$ with $\|(J_n) - (K_n)\| < 1/2$ and $\sup \dim \operatorname{Im} K_n < \infty$. Writing (2.5) as

$$(B_n)(A_n^* A_n) = (I_n) + (J_n - K_n) + (K_n)$$

and taking into account the invertibility of $(I_n) + (J_n - K_n)$ in $\mathcal{F}$, one gets

$$(I_n + J_n - K_n)^{-1}(B_n)(A_n^* A_n) = (I_n) + (I_n + J_n - K_n)^{-1}(K_n)$$

with $\dim \operatorname{Im} (I_n + J_n - K_n)^{-1} K_n \leqslant \dim \operatorname{Im} K_n$. Denoting $(I_n + J_n - K_n)^{-1}(B_n)$ and $(I_n + J_n - K_n)^{-1}(K_n)$ by (B_n) and (J_n) again, we arrive at (2.3).

Let now the sequences (B_n) and (J_n) be as in (2.3) and let

$$A_n^* A_n = U_n^* \Lambda_n U_n \quad \text{with } \Lambda := \operatorname{diag}\,(\lambda_1^{(n)}, \dots, \lambda_n^{(n)})$$

and with unitary matrices U_n refer to the diagonalization of $A_n^* A_n$. After multiplication by U_n and U_n^*, the identity (2.3) becomes $(U_n B_n U_n^*)(\Lambda_n) = (I_n) + (U_n J_n U_n^*)$. Abbreviating $C_n := U_n B_n U_n^*$ and $F_n := U_n J_n U_n^*$ we get

$$C_n \Lambda_n = C_n \operatorname{diag}\,(\lambda_1^{(n)}, \dots, \lambda_n^{(n)}) = I_n + F_n \quad \text{for all } n \tag{2.6}$$

where still $\sup \dim \operatorname{Im} F_n < \infty$. Set

$$k := \limsup_{n\to\infty} \dim \operatorname{Im} F_n.$$

We claim that $\liminf \lambda_{k+1}^{(n)} > 0$. Assume this is wrong. Then there is an infinite subsequence $(n_l)_{l\geqslant 1}$ of $\mathbb{N}$ such that $\lim\limits_{l\to\infty} \lambda_{k+1}^{(n_l)} = 0$. Multiplying (2.6) from both sides by $P_{k+1} = \mathrm{diag}\ (1, \dots, 1, 0, \dots, 0)$ (with the 1 occurring $k+1$ times) we get

$$P_{k+1} C_{n_l} \Lambda_{n_l} P_{k+1} = P_{k+1} + P_{k+1} F_{n_l} P_{k+1},$$

and, since

$$\|\Lambda_{n_l} P_{k+1}\| = \|\mathrm{diag}\ (\lambda_1^{(n_l)}, \dots, \lambda_{k+1}^{(n_l)}, 0, \dots, 0)\| = \lambda_{k+1}^{(n_l)} \to 0,$$

$\lim\limits_{l\to\infty} \|P_{k+1} + P_{k+1} F_{n_l} P_{k+1}\| = 0$. Thus, the matrices $P_{k+1} F_{n_l} P_{k+1} \in \mathbb{C}^{(k+1)\times(k+1)}$ are invertible for all sufficiently large n_l. This is impossible because of

$$\dim \mathrm{Im}\, P_{k+1} F_{n_l} P_{k+1} \leqslant \dim \mathrm{Im}\, F_{n_l} \leqslant k < k+1$$

which proves the claim (ii).

Finally, for the proof that (ii) implies the Fredholmness of (A_n), let k be a number such that

$$\liminf \sigma_{k+1}^{(n)} > 0,$$

and let $A_n = U_n \Sigma_n V_n^*$ with $\Sigma_n = \mathrm{diag}\ (\sigma_1^{(n)}, \dots, \sigma_n^{(n)})$ and with unitary matrices U_n and V_n refer to the singular value decomposition of A_n. The choice of k guarantees that the sequence $(\Sigma_n + P_k)_{n\geqslant 1}$ is stable. Then $(A_n + U_n P_k V_n^*)_{n\geqslant 1}$ is a stable sequence, too. Thus, there are sequences $(C_n) \in \mathcal{F}$ and $(G_n) \in \mathcal{G}$ such that

$$(C_n)(A_n + U_n P_k V_n^*) = (I_n) + (G_n)$$

or, equivalently,

$$(C_n)(A_n) = (I_n) + (G_n) - (C_n U_n P_k V_n^*).$$

The sequence $(C_n U_n P_k V_n^*)$ is finite central rank, hence, $(G_n) - (C_n U_n P_k V_n^*)$ is a centrally compact sequence. Thus, (A_n) is invertible modulo $J(\mathcal{F})$ from the left hand side, and its invertibility from the right hand side follows analogously. ∎

The preceding theorem suggests to introduce the *α-number* of a Fredholm sequence (A_n) (corresponding to the kernel dimension of a Fredholm operator) as the smallest number k for which (2.4) is true. Equivalently, $\alpha(A_n)$ is the smallest number for which there exists a sequence $(B_n) \in \mathcal{F}$ as well as a sequence (J_n) of finite central rank such that $B_n A_n^* A_n = I_n + J_n$ and $\limsup\limits_{n\to\infty} \dim \mathrm{Im}\, J_n = \alpha(A_n)$. The *index* of a Fredholm sequence is the quantity

$$\mathrm{ind}(A_n) := \alpha(A_n) - \alpha(A_n^*).$$

Observe that, in the case at hand, this index is always zero. This is a consequence of the fact that the entries of the sequences under consideration are finite-dimensional operators and, hence, the matrices $A_n^* A_n$ and $A_n A_n^*$ have the same eigenvalues even with respect to their multiplicity. So the most interesting quantity associated with a Fredholm sequence of matrices seems to be its α-number. On the other hand, the vanishing of the index of (A_n) also has remarkable consequences, mainly due to the identities (1.5), as it will be pointed out in Section 5.

Let (A_n) be a Fredholm sequence and $k := \alpha(A_n)$. Is there an analogue of the splitting property (1.6) which holds for Fredholm sequences in standard algebras? The following simple example says that the answer is NO.

EXAMPLE 2.6. Let (a_n) be an enumeration of the rational numbers in $[0,1]$, and set

$$A_n := P_n(a_n P_1 + (I - P_1))P_n = \text{diag}\,(a_n, 1, \dots, 1).$$

Since

$$(P_n)(A_n) = (a_n)(P_1) + (P_n(I - P_1)P_n) = (P_n) - (1 - a_n)(P_1),$$

and since $(1 - a_n)(P_1)$ is a sequence of central rank one, the sequence (A_n) is Fredholm, but the smallest singular values $\sigma_1^{(n)}$ of the matrices A_n lie dense in $[0,1]$.

Thus, one cannot expect that $\lim\limits_{n\to\infty} \sigma_k^{(n)} = 0$ if (A_n) is a Fredholm sequence with $k = \alpha(A_n)$, but one obviously has $\liminf \sigma_k^{(n)} = 0$. Hence, every Fredholm sequence in $\mathcal{F}$ possesses an infinite subsequence which owns the splitting property (1.6).

Finally, let us agree upon the following. The phrase *Fredholm sequence* is reserved for sequences in $\mathcal{F}$ which are invertible modulo the ideal $J(\mathcal{F})$. Occassionally, we also will have to deal with sequences or elements which are invertible modulo other ideals J of compact or centrally compact sequences or elements. To these kind of Fredholmness we will refer as J-Fredholmness.

3. FREDHOLMNESS MODULO COMPACT ELEMENTS

In this section, a brief sketch of the Fredholm theory in a C^*-algebra $\mathcal{A}$ modulo the ideal $C(\mathcal{A})$ is given. Some of these results are well known (see [1]); they are recalled here for the reader's convenience with their (as a rule, short) proofs. As mentioned before, a direct application of this Fredholm theory to the algebra $\mathcal{F}$ does not yield anything of interest. But, as will be pointed out in the forthcoming section, applying this Fredholm theory in case of a standard algebra $\mathcal{A} \subseteq \mathcal{F}$ *twice* (namely in the algebra $\mathcal{A}/C(\mathcal{A})$ modulo the ideal $C(\mathcal{A}/C(\mathcal{A}))$), one will exactly obtain the Fredholm theory described in the introduction.

IDEALS GENERATED BY ELEMENTS OF RANK ONE. In what follows, H is a complex separable Hilbert space, $L(H)$ the C^*-algebra of the bounded linear operators on H, and $K(H)$ the ideal of the compact linear operators on H. We start with a result on irreducible representations of the ideal $J(\mathcal{A})$.

THEOREM 3.1. *Let $\mathcal{A}$ be a unital C^*-algebra and $\pi : \mathcal{A} \to L(H)$ an irreducible representation of $\mathcal{A}$. Then $\pi(J(\mathcal{A})) \subseteq K(H)$.*

Proof. We will prove that, if $k \in J(\mathcal{A})$ is of central rank one, then $\pi(k)$ is an operator with range dimension at most one. This clearly implies the assertion of the theorem.

If $\pi(k) = 0$, then nothing is to prove. So let $\pi(k) \neq 0$. For every $a \in \mathcal{A}$, there exists an element μ in the center of $\mathcal{A}$ such that $kak = \mu k$. Then $\pi(\mu)$ is in the center of $\pi(\mathcal{A})$, and the identity

$$\pi(k)\pi(a)\pi(k) = \pi(\mu)\pi(k) \tag{3.1}$$

shows that $\pi(k)$ is a central rank one element of $\pi(\mathcal{A})$. Since $\pi(\mu)$ is in the center of $\pi(\mathcal{A})$, the operator $\pi(\mu)$ is a scalar multiple of the identity operator due to the irreducibility of π (Schur's lemma; see [7], Section 5.4). Hence, the $\pi(\mu)$ in (3.1) can be chosen as a complex number, and $\pi(k)$ is a (common) rank one element of $\pi(\mathcal{A})$.

Let now $x, \widetilde{x}$ be vectors in $\operatorname{Im} \pi(k)$ with $x \neq 0$, and choose vectors $y, \widetilde{y} \in H$ such that $x = \pi(k)y$ and $\widetilde{x} = \pi(k)\widetilde{y}$. Again due to the irreducibility, $\pi(\mathcal{A})x = H$. In particular, there is an $a \in \mathcal{A}$ such that $\pi(a)\pi(k)y = \pi(a)x = \widetilde{y}$. Multiplying this identity by $\pi(k)$ we get $\pi(k)\pi(a)\pi(k)y = \pi(k)\widetilde{y}$ which, together with (3.1), yields $\pi(\mu)\pi(k)y = \pi(k)\widetilde{y}$ or $\pi(\mu)x = \widetilde{x}$. Since $\pi(\mu)$ is a number, this shows that $\operatorname{Im} \pi(k) = \operatorname{span}\{x\}$. In particular, $\pi(k)$ has range dimension one. ∎

Since $K(H)$ has no proper closed ideals besides the zero ideal, this result implies that $\pi(J(\mathcal{A}))$ is either $\{0\}$ or $K(H)$.

Now we turn over to the ideal $C(\mathcal{A})$ of the compact elements. For every non-zero rank one element k of $\mathcal{A}$, we denote by $I(k)$ the smallest closed ideal of $\mathcal{A}$ which contains this element. From Theorem 3.1 one immediately gets

COROLLARY 3.2. *Let $\mathcal{A}$ be a unital C^*-algebra. Then, for every irreducible representation $\pi : \mathcal{A} \to L(H)$ and every rank one element k of $\mathcal{A}$,*

$$\pi(I(k)) \subseteq K(H).$$

Actually, much more can be shown.

THEOREM 3.3. *Let $\mathcal{A}$ be a unital C^*-algebra and k a non-zero rank one element of $\mathcal{A}$. Then there exists an irreducible representation $\pi : \mathcal{A} \to L(H)$ such that*

$$\pi(I(k)) = K(H) \quad \textit{and} \quad \operatorname{Ker}(\pi|I(k)) = \{0\}.$$

In particular, every ideal $I(k)$ is $*$-isomorphic to the ideal of the compact operators on a Hilbert space. We split the proof into several steps. The first partial result says that every ideal $I(k)$ is generated by a rank one *projection.*

PROPOSITION 3.4. *Let $\mathcal{A}$ be a unital C^*-algebra and let $k \in \mathcal{A} \setminus \{0\}$ be a non-zero rank one element. Then there exists a rank one projection $p \in \mathcal{A}$ such that $I(k) = I(p)$.*

Proof. Let k be a non-zero rank one element of $\mathcal{A}$, i.e. given $A \in \mathcal{A}$ there is a complex number $\mu(a)$ such that $kak = \mu(a)k$. Then the elements k^*, kk^* and k^*k are rank one and non-zero, too. Indeed, for every $a \in \mathcal{A}$,

$$k^*ak^* = \overline{\mu(a^*)}k^*, \quad k^*kak^*k = \mu(ak^*)k^*k \quad \text{and} \quad kk^*akk^* = \mu(k^*a)kk^*.$$

So, these elements are rank one, and moreover

$$0 \neq \|k\|^2 = \|k^*\|^2 = \|kk^*\| = \|k^*k\|.$$

In the next step we verify that $I(k) = I(k^*k)$. The inclusion $I(k^*k) \subseteq I(k)$ is obvious. For the reverse inclusion, consider

$$kk^*k = \mu(k^*)k. \tag{3.2}$$

Since $k \neq 0$, the number $\mu(k^*)$ is uniquely determined. Assume that $\mu(k^*) = 0$. Then $kk^*k = 0$ and, consequently, $\|k\|^4 = \|k^*k\|^2 = \|k^*kk^*k\| = 0$ which contradicts $k \neq 0$. Thus, $\mu(k^*) \neq 0$, which implies

$$k = \mu(k^*)^{-1}kk^*k \in I(k^*k)$$

and hence, $I(k) \subseteq I(k^*k)$. From (3.2) we further conclude

$$k^*kk^*k = \mu(k^*)k^*k. \tag{3.3}$$

Both sides of (3.3) are non-negative elements of $\mathcal{A}$ and $\mu(k^*) \neq 0$. Thus $\mu(k^*) > 0$, and taking norms in (3.3) gives

$$\|k^*kk^*k\| = \|k^*k\|^2 = \mu(k^*)\|k^*k\|.$$

Since $\|k^*k\| = \|k\|^2 \neq 0$, this implies $\mu(k^*) = \|k^*k\|$. Now it is evident from (3.3) that $p := \|k^*k\|^{-1}k^*k$ is a projection in $\mathcal{A}$ which is rank one and that $I(p) = I(k^*k) = I(k)$. ∎

Proposition 3.5. *Let $\mathcal{A}$ be a C^*-algebra with unit element e and let $p \in \mathcal{A} \setminus \{0\}$ be a non-zero rank one projection. Then the identity*

$$pap = \tau(a)p$$

defines uniquely a pure state τ of $\mathcal{A}$.

Proof. The uniqueness follows from $p \neq 0$. For $a = e$ one gets $\tau(e)p = pep = p^2 = p$, hence $\tau(e) = 1$. Since $\|p\| = 1$, one moreover has $|\tau(a)| = \|\tau(a)p\| = \|pap\| \leqslant \|a\|$ for every $a \in \mathcal{A}$, whence $\|\tau\| = 1$. It is also clear that the functional τ is linear, hence τ is a state of $\mathcal{A}$. It remains to show that this state is pure. Let

$$L_\tau := \{a \in \mathcal{A} : \tau(a^*a) = 0\}$$

denote the left kernel of τ. The state τ is a pure if and only if

$$\operatorname{Ker}\tau = L_\tau + L_\tau^* \tag{3.4}$$

([7], Theorem 10.2.8). Since the inclusion $L_\tau + L_\tau^* \subseteq \operatorname{Ker}\tau$ holds for every state, it remains to check the reverse inclusion. Let $a \in \operatorname{Ker}\tau$, i.e. $pap = \tau(a)p = 0$. Since $pap = 0$, $a = pa + qa = paq + qa$ with $q = e - p$. For $b := paq$ one gets $\tau(b^*b)p = pb^*bp = pqa^*paqp = 0$, whence $\tau(b^*b) = 0$ and $b \in L_\tau$. Analogously, for $c := qa$ one finds

$$\tau(cc^*)p = pcc^*p = pqaa^*qp = 0,$$

hence, $\tau(cc^*) = 0$ and $c \in L_\tau^*$. Consequently, $a = b + c \in L_\tau + L_\tau^*$, and τ is a pure state. ∎

Since $\tau(a^*a)p = pa^*ap = (ap)^*(ap)$, it is $\tau(a^*a) = 0$ if and only if $ap = 0$. Thus,

$$L_\tau = \mathcal{A}q = \{aq : a \in A\},$$

and the Hilbert space associated via the GNS construction with the pure state τ is $H := \mathcal{A}/L_\tau = \mathcal{A}/\mathcal{A}q$ with inner product

$$\langle a + \mathcal{A}q, b + \mathcal{A}q\rangle := \tau(b^*a)$$

(it is not necessary to take the completion since τ is pure, cf. [7], Theorem 10.2.3). The pureness of τ also guarantees that the representation

$$\pi : \mathcal{A} \to L(H), \quad a \mapsto (b + \mathcal{A}q \mapsto ab + \mathcal{A}q) \tag{3.5}$$

is irreducible. The following proposition finishes the proof of Theorem 3.4.

PROPOSITION 3.6. *Let* π *as in* (3.5). *Then* $\operatorname{Ker}(\pi|I(k)) = \{0\}$.

Proof. Let $r \in I(k) = I(p)$ and $\pi(r) = 0$. Then, by (3.5), $rb + \mathcal{A}q = 0$, respectively $rb \in \mathcal{A}q$, respectively $rbp = 0$ for all $b \in \mathcal{A}$. This implies $rbpc = 0$ for all $b, c \in \mathcal{A}$ and, consequently, $r \sum_i b_i p c_i = 0$ for all $b_i, c_i \in \mathcal{A}$. The elements $\sum b_i p c_i$ lie densely in $I(p)$. Hence, $rj = 0$ for every $j \in I(p)$. In particular, $rr^* = 0$, i.e. $r = 0$. ∎

Since $K(H)$ has no proper closed subideals besides $\{0\}$, one has the following consequence of Theorem 3.4.

COROLLARY 3.7. *Let* k_1, k_2 *be non-zero rank one elements of the unital* C^**-algebra* $\mathcal{A}$. *Then either* $I(k_1) = I(k_2)$ *or* $I(k_1) \cap I(k_2) = \{0\}$.

LIFTING THEOREMS. The preceding results suggest to introduce an equivalence relation in the set of all non-zero rank one elements of a unital C^*-algebra $\mathcal{A}$ by calling k_1 and k_2 equivalent if $I(k_1) = I(k_2)$. Let T abbreviate the set of all equivalence classes and, given $t \in T$, choose a representative p_t of the coset t, abbreviate the ideal $I(p_t)$ by I_t, and let $\pi_t : \mathcal{A} \to L(H_t)$ stand for the associated irreducible representation (3.5). Thus, $C(\mathcal{A})$ is generated by its minimal subideals I_t where $t \in T$. With these notations, the following so-called lifting theorem holds. The results of this subsection are taken from [11].

THEOREM 3.8. (Lifting theorem, part 1.) *Let* $\mathcal{A}$ *be a unital* C^**-algebra and* T *the set of the equivalence classes of the non-zero rank one elements of* $\mathcal{A}$. *Then an element* $a \in \mathcal{A}$ *is invertible in* $\mathcal{A}$ *if and only if the operators* $\pi_t(a)$ *are invertible in* $L(H_t)$ *for every* $t \in T$ *and if the coset* $a + C(\mathcal{A})$ *is invertible in the quotient algebra* $\mathcal{A}/C(\mathcal{A})$.

In other words: If $a \in \mathcal{A}$ is a $C(\mathcal{A})$-Fredholm element, then all operators $\pi_t(a)$ are Fredholm, and the Fredholm element a is invertible if and only if all Fredholm operators $\pi_t(a)$ are invertible.

Proof. If a is invertible, then coset $a + C(\mathcal{A})$ and all operators $\pi_t(a)$ are invertible. Conversely: If $a + C(\mathcal{A})$ is invertible, then there are elements $b \in \mathcal{A}$ and $k \in C(\mathcal{A})$ such that $ba = e + k$. Since $C(\mathcal{A})$ is the smallest closed ideal which contains all ideals I_t, one finds an element $j \in C(\mathcal{A})$ as well as finitely many elements $j_{t_i} \in I_{t_i}$ such that $j = j_{t_1} + \cdots + j_{t_m}$ and $\|k - j\| < 1/2$. Multiplying the equation $ba = e + k$ from the left hand side by $(e + k - j)^{-1}$ and setting $c := (e + k - j)^{-1} b$ and $k_{t_i} := (e + k - j)^{-1} j_{t_i}$, one arrives at

$$ca = e + k_{t_1} + \cdots + k_{t_m} \quad \text{with } k_{t_i} \in I_{t_i}. \tag{3.6}$$

Since I_t is isomorphic to $\pi_t(I_t) = K(H_t)$, there are elements $r_{t_i} \in I_{t_i}$ such that $\pi_{t_i}(r_{t_i}) = \pi_{t_i}(k_{t_i}) \pi_{t_i}(a)^{-1}$. Set $\widehat{c} := c - r_{t_1} - \cdots - r_{t_m}$. Then

$$\widehat{c}a = e + (k_{t_1} - r_{t_1} a) + \cdots + (k_{t_m} - r_{t_m} a).$$

The elements $k_{t_i} - r_{t_i} a$ belong to I_{t_i} and $\pi_{t_i}(k_{t_i} - r_{t_i} a) = 0$, hence $k_{t_i} - r_{t_i} a = 0$ and $\widehat{c}a = e$. The invertibility of a from the right hand side can be checked analogously. ∎

Together with the following separation property, the lifting theorem can be essentially completed.

PROPOSITION 3.9. *Let the notation be as before. Then*
(i) *Let* $t_1, \ldots, t_m \in T$ *and* $t_i \neq t_j$ *for* $i \neq j$. *Then* $(I_{t_1} + \cdots + I_{t_{m-1}}) \cap I_{t_m} = \{0\}$.
(ii) *Let* $s, t \in T$ *with* $s \neq t$. *Then* $\pi_s(I_t) = \{0\}$.

Proof. (i) Clearly, $(I_{t_1} + \cdots + I_{t_{m-1}}) \cap I_{t_m}$ is an ideal in I_{t_m}. Since I_{t_m} is isomorphic to $K(H_{t_m})$, its only closed subideals are $\{0\}$ and I_{t_m} itself. Thus, if the assertion would be wrong then, necessarily, $I_{t_m} \subseteq I_{t_1} + \cdots + I_{t_{m-1}}$. In this case, let $I_{t_m} = I(p)$ with a rank one projection p, and choose elements $k_{t_i} \in I_{t_i}$ such that $p = k_{t_1} + \cdots + k_{t_{m-1}}$. Multiplying this identity by p from both sides yields $p = pk_{t_1}p + \cdots + pk_{t_{m-1}}p$. If $pk_{t_i}p = 0$ for every i, then $p = 0$ which is impossible. So $pk_{t_i}p \neq 0$ for some i. Since p is rank one, there is a complex number μ such that $\mu p = pk_{t_i}p \in I_{t_i}$. Hence, $p \in I_{t_i}$ which contradicts $I_{t_i} \cap I_{t_m} = \{0\}$.

(ii) Let $r \in \pi_s(I_t)$, i.e. $r = \pi_s(k_t)$ for a $k_t \in I_t$. By Theorem 3.1, r belongs to $K(H_s)$, and by Theorem 3.3, there exists a $k_s \in I_s$ such that $\pi(k_s) = r^*$. Then $\pi_s(k_s k_t) = r^* r$. On the other hand, since $I_s \cap I_t = \{0\}$, one has $k_s k_t = 0$ which implies $\pi_s(k_s k_t) = 0$. Thus, $r = 0$. ∎

THEOREM 3.10. (Lifting theorem, part 2) *Let the situation be as in Theorem* 3.8, *and let* $a \in \mathcal{A}$ *be a* $C(\mathcal{A})$*-Fredholm element. Then all operators* $\pi_t(a)$ *are Fredholm, and there are only finitely many* $t \in T$ *for which* $\pi_t(a)$ *is not invertible.*

Proof. Apply the representation π_t to both sides of identity (3.6) to obtain the invertibility of $\pi_t(a)$ from the left hand side for all $t \notin \{t_1, \ldots, t_m\}$. ∎

THE RANK OF AN ELEMENT. Let $k \in C(\mathcal{A})$ be a non-zero element of finite rank. We say that k has rank r, if k is the sum of r elements of rank one, but not a sum of $r-1$ rank one elements. The rank of k will be denoted by $\operatorname{rank} k$. Further define $\operatorname{rank} 0 = 0$.

PROPOSITION 3.11. *Let* $k \in C(\mathcal{A})$ *be of finite rank. Then, for every* $t \in T$, *there exist finite rank elements* $k_t \in I_t$ *with* $k_t = 0$ *for all but a finite number of* t *such that* $k = \sum_{t \in T} k_t$. *The* k_t *are uniquely determined, and*

$$\operatorname{rank} k = \sum_{t \in T} \operatorname{rank} k_t.$$

Proof. Let $k \in C(\mathcal{A})$ be the sum of the rank one elements $k_1, \ldots, k_r$. Every k_j belongs to exactly one of the ideals I_t (namely to $I(k_j)$). So one gets a decomposition of k as a sum $\sum_{t \in T} k_t$ with only finitely many non-vanishing elements $k_t \in I_t$ of finite rank.

The uniqueness of this decomposition can be checked as follows: let $h_1 + \cdots + h_m = 0$ for certain elements $h_i \in I_{t_i}$ with $t_i \neq t_j$ for $i \neq j$. Then $h_{t_1} = -h_{t_2} - \cdots - h_{t_m}$, i.e. $h_{t_1} \in I_{t_1} \cap (I_{t_2} + \cdots + I_{t_m})$. By Proposition 3.9 (i), $h_{t_1} = 0$.

It remains to show the rank identity. Let $k = \sum k_t$ with $k_t \in I_t$ and let $\operatorname{rank} k_t = r_t$. Then every k_t is the sum of r_t rank one elements, hence, k is the sum of $\sum r_t$ rank one elements. Consequently,

$$\operatorname{rank} k \leqslant \sum r_t = \sum \operatorname{rank} k_t.$$

Conversely, let k be the sum of $r = \operatorname{rank} k$ elements $k_1, \ldots, k_r$ of rank one. For every $t \in T$ and every rank one element h, define $n_t(h)$ to be 1 if $h \in I_t$ and set $n_t(h) = 0$ if $h \notin I_t$. Further let $k_t := \sum\limits_{i=1}^{r} n_t(k_i)k_i$. Then every k_t is the sum of r_t rank one elements where r_t is the number of the k_i which lie in I_t. Since every k_i belongs to exactly one of the ideals I_t, one has $\sum r_t = r$. Thus,

$$\sum \operatorname{rank} k_t \leqslant \sum r_t = r = \operatorname{rank} k$$

which verifies the rank identity. ∎

We proceed with relations between the rank of an element and the range dimension of its image under irreducible representations.

PROPOSITION 3.12. *If $k \in I_t$ is of finite rank then* $\operatorname{rank} k = \dim \operatorname{Im} \pi_t(k)$.

Proof. Recall from the proof of Theorem 3.1 that the irreducible representation π_t of $\mathcal{A}$ maps elements of rank one onto operators with range dimension at most one. Hence, if k is the sum of r rank one elements, then $\pi_t(k)$ is the sum of r operators of rank one, whence

$$\dim \operatorname{Im} \pi_t(k) \leqslant r = \operatorname{rank} k.$$

Conversely, suppose $\pi_t(k)$ is a compact operator with range dimension r. Choose an orthonormal basis $e_1, \ldots, e_r$ in $\operatorname{Im} \pi_t(k)$, let P_i stand for the orthogonal projection from H_t onto $\mathbb{C}e_i$, and let p_i denote the (uniquely determined) element in I_t such that $\pi(p_i) = P_i$. Since $\sum P_i$ is the orthogonal projection from H_t onto the range of $\pi_t(k)$, one has

$$\pi_t(k) = \sum_{i=1}^{r} P_i \pi_t(k) = \sum_{i=1}^{r} \pi_t(p_i k) = \pi_t\Big(\sum_{i=1}^{r} p_i k\Big).$$

Due to Proposition 3.6, this implies $k = \sum p_i k$. One easily checks that every $p_i k$ is a rank one element. Thus, k is the sum of r rank one elements whence the estimate $\operatorname{rank} k \leqslant \dim \operatorname{Im} \pi_t(k)$. ∎

FREDHOLMNESS MODULO $C(\boldsymbol{\mathcal{A}})$. Now we will have a closer look at the Fredholm theory associated with the ideal $C(\mathcal{A})$ of the compact elements. The remainder of this section is not needed in what follows. Recall that $a \in \mathcal{A}$ is $C(\mathcal{A})$-*Fredholm* if the coset $a + C(\mathcal{A})$ is invertible in the quotient algebra $\mathcal{A}/C(\mathcal{A})$. If a is $C(\mathcal{A})$-Fredholm then a^*a and aa^* are $C(\mathcal{A})$-Fredholm, too, and there exist elements $b, c \in \mathcal{A}$ as well as elements k_1, k_2 of finite rank such that

$$ba^*a = e + k_1 \quad \text{and} \quad aa^*c = e + k_2.$$

Let $\alpha(a)$ stand for the smallest non-negative integer which owns the following property: there are a finite rank element k_1 with $\operatorname{rank} k_1 = \alpha(a)$ and an element $b \in \mathcal{A}$ such that $ba^*a = e + k_1$. Analogously, $\beta(a)$ is defined as the smallest possible rank of k_2. Finally, define the *index* of a by $\operatorname{ind} a := \alpha(a) - \beta(a)$. In case $\mathcal{A} = L(H)$, one has $C(L(H)) = K(H)$ and

$$\alpha(A) = \operatorname{Ker} A \quad \text{and} \quad \beta(A) = \dim \operatorname{Coker} A$$

for every Fredholm operator. Here is a generalization of these results to arbitrary $C(\mathcal{A})$-Fredholm elements.

THEOREM 3.13. *Let $a \in \mathcal{A}$ be a Fredholm element modulo $C(\mathcal{A})$. Then*

$$\alpha(a) = \sum_{t \in T} \dim \operatorname{Ker} \pi_t(a) \quad \textit{and} \quad \beta(a) = \sum_{t \in T} \dim \operatorname{Coker} \pi_t(a).$$

Observe that the occurring sums are actually finite thanks to Theorem 3.10.

Proof. We will verify the first assertion only. Let $a \in \mathcal{A}$ be a Fredholm element, and let $b \in \mathcal{A}$ and $k \in C(\mathcal{A})$ be elements such that

$$ba^*a = e + k \quad \text{and} \quad \operatorname{rank} k = \alpha(a).$$

Write k as $\sum k_t$ with $k_t \in I_t$ (which can be done uniquely). By Proposition 3.11, $ba^*a = e + \sum k_t$ and $\alpha(a) = \sum \operatorname{rank} k_t$. The separation property Proposition 3.9 (ii) implies that $\pi_t(b)\pi_t(a^*)\pi_t(a) = e_t + \pi_t(k_t)$ with e_t referring to the identity operator on H_t. Hence, $\operatorname{Ker} \pi_t(a) = \alpha(\pi_t(a)) \leqslant \operatorname{rank} \pi_t(k_t)$, whence

$$\sum \operatorname{Ker} \pi_t(a) \leqslant \sum \operatorname{rank} \pi_t(k_t) = \sum \operatorname{rank} k_t = \operatorname{rank} k = \alpha(a).$$

For the reverse inequality, let P_t stand for the orthogonal projection from H_t onto $\operatorname{Ker} \pi_t(a)$, and write p_t for the (uniquely determined) element in I_t such that $\pi_t(p_t) = P_t$. Consider the element $\widehat{a} := a^*a + \sum p_t$ (again this sum contains only a finite number of non-zero terms due to Theorem 3.10). Since a is a Fredholm element modulo $C(\mathcal{A})$, also $\widehat{a}$ is a $C(\mathcal{A})$-Fredholm element. Moreover, the operators

$$\pi_t(\widehat{a}) = \pi_t(a^*)\pi_t(a) + P_t$$

are invertible for every $t \in T$. The lifting theorem implies that $\widehat{a}$ is an invertible element of $\mathcal{A}$. In particular, there exists a $b \in \mathcal{A}$ such that $b\widehat{a} = e$ respectively

$$ba^*a = e - b\sum p_t = e - \sum bp_t.$$

Hence,

$$\begin{aligned} \alpha(a) &\leqslant \operatorname{rank} \sum bp_t \leqslant \sum \operatorname{rank} bp_t \leqslant \sum \operatorname{rank} p_t \\ &= \sum \operatorname{rank} P_t = \sum \dim \operatorname{Im} P_t = \sum \operatorname{Ker} \pi_t(a), \end{aligned}$$

yielding finally the desired kernel dimension identity. ∎

Let us mention a consequence of the kernel dimension identity. Clearly, the Fredholm element a determines its α-number uniquely, but the finite rank element k in $ba^*a = e + k$ is not determined uniquely by a (it depends on the choice of b also). Thus, it is a priori not evident whether a uniquely determines the ranks of the elements k_t (only their sum $\operatorname{rank} k$ is determined by a). The preceding theorem states that a also determines the "local" ranks $\operatorname{rank} k_t$ uniquely since $\operatorname{rank} k_t = \operatorname{Ker} \pi_t(a)$.

COROLLARY 3.14. *Let $a \in \mathcal{A}$ be a Fredholm element modulo $C(\mathcal{A})$. Then*

$$\operatorname{ind} a = \sum_{t \in T} \operatorname{ind} \pi_t(a).$$

With this corollary it becomes obvious that the introduced functionals α, β and ind for $C(\mathcal{A})$-Fredholm elements satisfy all the common properties one knows for the functionals dim Ker, dim Coker and ind in case of Fredholm operators.

4. FREDHOLM SEQUENCES IN STANDARD ALGEBRAS

In this section we return to Fredholm sequences (i.e. sequences which are invertible modulo the ideal $J(\mathcal{F})$ of the centrally compact sequences in $\mathcal{F}$). The main goal is to single out a class of subalgebras $\mathcal{A}$ of $\mathcal{F}$ such that, for sequences belonging to $\mathcal{A}$, their Fredholmness coincides with the invertibility modulo an ideal of compact elements as considered in the previous section. This makes the lifting theorem and its consequences (the $\dim \operatorname{Ker}$ identity) available to the determination of the α-number of Fredholm sequences. Let us agree upon reserving the notation $\alpha(A_n)$ for the α-number of a Fredholm sequence (A_n) (i.e. modulo $J(\mathcal{F})$). If we want to consider α-numbers with respect to other ideals of compact or essentially compact sequences, we will mention this explicitely.

ALGEBRAS WITH CENTER c. Let $\mathcal{A}$ be a unital C^*-subalgebra of the algebra $\mathcal{F}$, and throughout what follows suppose that the ideal $\mathcal{G} = C(\mathcal{F})$ belongs to $\mathcal{A}$. Then $\mathcal{G}$ is a closed ideal of $\mathcal{A}$, and it is easy to check that $C(\mathcal{A})$ coincides with $\mathcal{G}$.

LEMMA 4.1. *If $\mathcal{G} \subseteq \mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{A}$ is unital, then the center of $\mathcal{A}$ is isomorphic to a subalgebra of ℓ^∞ which contains c.*

Proof. Let I_n denote the $n \times n$ identity matrix, and let $(a_n) \subset \mathbb{C}$ be a sequence in c with limit a. Then

$$(a_n I_n) = a(I_n) + ((a_n - a)I_n).$$

The sequence $a(I_n)$ belongs to $\mathcal{A}$ since this algebra is unital, and the sequence $((a_n - a)I_n)$ tends to zero in the norm whence $((a_n - a)I_n) \in \mathcal{G} \subseteq \mathcal{A}$. Clearly, $(a_n I_n)$ belongs to the center of $\mathcal{A}$.

Conversely, let the sequence (C_n) be in the center of $\mathcal{A}$. Since $\mathcal{G} \subseteq \mathcal{A}$, every matrix C_n commutes with every other matrix in $\mathbb{C}^{n \times n}$, hence, $C_n = a_n I_n$ with a sequence $(a_n) \in \ell^\infty$. ∎

We say that *the center of the algebra $\mathcal{A} \subseteq \mathcal{F}$ is c* if this center consists exactly of the sequences $(c_n I_n)$ with $(c_n) \in c$. Here are two instances of algebras $\mathcal{A}$ with center c.

Suppose there is a separable infinite-dimensional Hilbert space H as well as a sequence (P_n) of orthogonal projections P_n from H onto an n-dimensional subspace of H such that (P_n) converges strongly to the identity operator on H. Assume further that all sequences (A_n) in $\mathcal{A}$ possess the following property: If the matrix A_n is identified with an operator on $\operatorname{Im} P_n$, then the strong limit $\operatorname{s-lim} A_n P_n$ exists. If, in particular, $A_n = c_n I_n$ with complex numbers c_n, then this strong convergence implies that $(c_n) \in c$.

FRACTAL ALGEBRAS. Another class of algebras with center c is constituted by the fractal algebras. This class has been introduced and studied in [11] and [9]. Here are the definition as well as an important property of fractal algebras as stated in [11] and [9].

Given a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, let $\mathcal{F}_\eta$ refer to the C^*-algebra of all bounded sequences (A_n) with $A_n \in \mathbb{C}^{\eta(n) \times \eta(n)}$, and write $\mathcal{G}_\eta$ for the ideal of all sequences $(A_n) \in \mathcal{F}_\eta$ which tend to zero in the norm. Further, let R_η stand for the restriction mapping $R_\eta : \mathcal{F} \to \mathcal{F}_\eta$, $(A_n) \mapsto (A_{\eta(n)})$. This mapping is a $*$-homomorphism from $\mathcal{F}$ onto $\mathcal{F}_\eta$ which moreover maps $\mathcal{G}$ onto $\mathcal{G}_\eta$. Given a C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$, let $\mathcal{A}_\eta$ denote the image of $\mathcal{A}$ under R_η which is a C^*-algebra again.

DEFINITION 4.2. Let $\mathcal{A}$ be a C^*-subalgebra of the sequence algebra $\mathcal{F}$.

(a) A $*$-homomorphism $W : \mathcal{A} \to \mathcal{B}$ of $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ is *fractal* if, for every strongly monotonically increasing sequence η, there is a $*$-homomorphism $W_\eta : \mathcal{A}_\eta \to \mathcal{B}$ such that $W = W_\eta R_\eta$.

(b) The algebra $\mathcal{A}$ is *fractal* if the canonical homomorphism π from $\mathcal{A}$ onto $\mathcal{A}/(\mathcal{A} \cap \mathcal{G})$ is fractal.

Thus, given a subsequence $(a_{\eta(n)})$ of a sequence (a_n) which belongs to a fractal algebra $\mathcal{A}$, it is possible to reconstruct the original sequence (a_n) from this subsequence modulo sequences in $\mathcal{A} \cap \mathcal{G}$. This assumption is very natural for sequences arising from discretization procedures. It is, for example, not hard to show that the algebra of the polynomial collocation method examined in the introduction is fractal. On the other hand, the algebra $\mathcal{F}$ of all bounded sequences fails to be fractal.

THEOREM 4.3. *A C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ is fractal if and only if the following implication holds for every element $(a_n) \in \mathcal{A}$ and every strongly monotonically increasing sequence η:*

$$R_\eta(a_n) \in \mathcal{G}_\eta \;\Rightarrow\; (a_n) \in \mathcal{A} \cap \mathcal{G}.$$

This property also implies that the center of a fractal unital C^*-subalgebra of $\mathcal{F}$ which contains the ideal $\mathcal{G}$ is c.

FREDHOLM INVERSE CLOSED SUBALGEBRAS. If $\mathcal{A}$ is a unital subalgebra of $\mathcal{F}$ which contains $\mathcal{G}$ and has center c, then every central rank one sequence in $\mathcal{A}$ is also a central rank one sequence in $\mathcal{F}$, hence, $J(\mathcal{A}) \subseteq J(\mathcal{F})$.

DEFINITION 4.4. The subalgebra $\mathcal{A}$ of $\mathcal{F}$ is called *Fredholm inverse closed* if

$$J(\mathcal{A}) = \mathcal{A} \cap J(\mathcal{F}). \tag{4.1}$$

Thus, if $\mathcal{A}$ is Fredholm inverse closed, and if $(A_n) \in \mathcal{A}$ is a Fredholm sequence (i.e. invertible modulo $J(\mathcal{F})$), then (A_n) is invertible modulo $\mathcal{A} \cap J(\mathcal{F})$ (due to the common inverse closedness of $(\mathcal{A} + J(\mathcal{F}))/J(\mathcal{F}) \cong \mathcal{A}/(\mathcal{A} \cap J(\mathcal{F}))$) and, hence, (A_n) is invertible modulo $J(\mathcal{A})$.

It will be pointed out that, for Fredholm sequences (A_n) which belong to a fractal and Fredholm inverse closed subalgebra $\mathcal{A}$ of $\mathcal{F}$, their α-number can be determined by an analogue of the kernel dimension identity in Theorem 3.13.

SEQUENCES OF ESSENTIAL RANK ONE. Let $\mathcal{A}$ be a unital C^*-subalgebra of $\mathcal{F}$ which contains the ideal $\mathcal{G}$ and has center c. A central rank one sequence of $\mathcal{A}$ is said to be of *essential rank one* if it does not belong to the ideal $\mathcal{G}$. For every essential rank one sequence (K_n), let $J(K_n)$ refer to the smallest closed ideal of $\mathcal{A}$ which contains the sequence (K_n) and the ideal $\mathcal{G}$.

Since $\mathcal{A}$ has center c, it is obvious that, for every essential rank one sequence (K_n) of $\mathcal{A}$, the coset $(K_n) + \mathcal{G}$ is a rank one element of the quotient algebra $\mathcal{A}/\mathcal{G}$ and that

$$J(K_n)/\mathcal{G} = I((K_n) + \mathcal{G})$$

(with the notation I as in the previous section). This shows that Fredholmness of a sequence $(A_n) \in \mathcal{A}$ is closely related to the common Fredholm theory (as described in the previous section) in the algebra $\mathcal{A}/\mathcal{G}$. In particular, if T refers to the set of all equivalence classes of rank one elements in $\mathcal{A}/\mathcal{G}$, then there exists a $t \in T$ such that

$$J(K_n)/\mathcal{G} = I_t. \tag{4.2}$$

Let T_{ess} stand for the set of all $t \in T$ for which there is an essential rank one sequence $(K_n) \in \mathcal{A}$ such that (4.2) holds. Further, write $G_{\text{ess}}(\mathcal{A}/\mathcal{G})$ for the smallest closed ideal of $\mathcal{A}/\mathcal{G}$ which contains all ideals I_t with $T \in T_{\text{ess}}$. It is easy to check that the lifting theorems (Theorems 3.8 and 3.10) and their consequences (in particular, Theorem 3.13) remain valid if the set T and the invertibility modulo the ideal $C(\mathcal{A}/\mathcal{G})$ are replaced by the set T_{ess} and by invertibility modulo $G_{\text{ess}}(\mathcal{A}/\mathcal{G})$. In particular, the coset $(A_n) + \mathcal{G}$ is invertible in $\mathcal{A}/\mathcal{G}$ if and only if the operators $\pi_t((A_n) + \mathcal{G})$ are invertible for all $t \in T_{\text{ess}}$ and if $(A_n) + \mathcal{G}$ is invertible modulo $G_{\text{ess}}(\mathcal{A}/\mathcal{G})$. Moreover, if the coset $(A_n) + \mathcal{G}$ is invertible modulo $G_{\text{ess}}(\mathcal{A}/\mathcal{G})$, then

$$\alpha_{G_{\text{ess}}(\mathcal{A}/\mathcal{G})}((A_n) + \mathcal{G}) = \sum_{t \in T_{\text{ess}}} \dim \operatorname{Ker} \pi_t((A_n) + \mathcal{G}).$$

For $t \in T_{\text{ess}}$, abbreviate the homomorphism

$$\mathcal{A} \to L(H_t), \quad (A_n) \mapsto \pi_t((A_n) + \mathcal{G})$$

by W_t. The homomorphism W_t is an irreducible representation of the algebra $\mathcal{A}$ ([4], Proposition 2.11.2) which maps the ideal $\mathcal{G}$ to zero. With these notations and identifications we have

$$J(\mathcal{A})/\mathcal{G} = G_{\text{ess}}(\mathcal{A}/\mathcal{G}),$$

and the following version of the lifting theorem holds.

Theorem 4.5. *Let $\mathcal{A}$ be a unital and Fredholm inverse closed C^*-subalgebra of $\mathcal{F}$ which contains the ideal $\mathcal{G}$ and has center c. A sequence $(A_n) \in \mathcal{A}$ is stable if and only if (A_n) is a Fredholm sequence and if the operators $W_t(A_n)$ are invertible for all $t \in T_{\text{ess}}$.*

Fredholm sequences in fractal and Fredholm inverse closed algebras. Now we are going to formulate and prove the main result of this section: a relation between the α-number of a Fredholm sequence (A_n) and the kernel dimensions of the operators $W_t(A_n)$. This result is a generalization of the identity (1.5) which served in [12] as a basis to *define* Fredholm sequences and their α-number. In particular, it is not needed for this generalization that the underlying algebra is standard (cf. the introduction; the point is that the fourth axiom of the definition of a standard algebra is not required, and also the explicit form of the homomorphisms W_t as strong limits is not assumed in what follows).

THEOREM 4.6. *Let $\mathcal{A}$ be a unital, fractal and Fredholm inverse closed C^*-subalgebra of $\mathcal{F}$ which contains the ideal $\mathcal{G}$, and let $(A_n) \in \mathcal{A}$ be a Fredholm sequence. Then*

$$\alpha(A_n) = \sum_{t \in T_{\text{ess}}} \dim \operatorname{Ker} W_t(A_n). \tag{4.3}$$

The remainder of this section is devoted to the proof of this result. Let $(A_n) \in \mathcal{A}$ be a Fredholm sequence. Since $\mathcal{A}$ is Fredholm inverse closed, (A_n) is invertible modulo $J(\mathcal{A})$. Then, by Theorem 3.1, the operators $W_t(A_n)$ are Fredholm for every $t \in T_{\text{ess}}$ and, by Theorem 3.10, there are only finitely many operators $W_t(A_n)$ which are not invertible. Let $P_{\operatorname{Ker} W_t(A_n)}$ denote the orthogonal projection from H_t onto the kernel of $W_t(A_n)$ (only a finite number of these projections are not zero). Decompose each of these projections into a sum of $\operatorname{Ker} W_t(A_n)$ orthogonal projections of range dimension one:

$$P_{\operatorname{Ker} W_t(A_n)} = \sum_{i=1}^{\dim \operatorname{Ker} W_t(A_n)} P_{i,t}$$

such that $P_{i,t}P_{j,t} = P_{j,t}P_{i,t} = 0$ whenever $i \neq j$ (for example, by choosing an orthonormal basis in $\operatorname{Ker} W_t(A_n)$ and by defining $P_{i,t}$ as the orthogonal projection onto the ith element of this basis). Since π_t is an isomorphism between I_t and $K(H_t)$, every projection $P_{i,t}$ corresponds uniquely to a coset $p_{i,t} \in I_t$. Clearly, $p_{i,t}p_{j,t} = p_{j,t}p_{i,t} = 0$ if $i \neq j$. Since $I_s \cap I_t = \{0\}$ for all $s, t \in T$ with $s \neq t$, one has moreover

$$p_{i,t}p_{j,s} = p_{j,s}p_{i,t} = 0 \quad \text{whenever } (i,t) \neq (j,s). \tag{4.4}$$

The next results say that projections in $\mathcal{A}/\mathcal{G}$ can be lifted.

PROPOSITION 4.7. *Let $p \in \mathcal{F}/\mathcal{G}$ be a projection. Then there exists a sequence of projections $(P_n) \in \mathcal{F}$ such that $(P_n) + \mathcal{G} = p$.*

A proof of this well-known result is in [12].

In particular, Proposition 4.7 guarantees the existence of sequences $(P_n^{i,t}) \in \mathcal{F}$ of projections such that $(P_n^{i,t})$ belongs to the coset $p_{i,t}$. Observe that $(P_n^{i,t}) \in \mathcal{A}$ since $\mathcal{A}$ contains $\mathcal{G}$. Moreover, since $(P_n^{i,t}) + \mathcal{G}$ belongs to I_t and t is in T_{ess}, every sequence $(P_n^{i,t})$ belongs to an ideal of the form $J(K_n)$ with an essential rank one sequence (K_n). In particular, $(P_n^{i,t}) \in J(\mathcal{A})$.

We claim that the lifting of the projections $p_{i,t}$ to the sequences $(P_n^{i,t})$ can be done in such a way that all projections $P_n^{i,t}$ have one-dimensional range and that

$$P_n^{i,t}\, P_n^{j,s} = P_n^{j,s}\, P_n^{i,t} = 0 \quad \text{whenever } (i,t) \neq (j,s) \tag{4.5}$$

for all sufficiently large n.

PROPOSITION 4.8. *Let $\mathcal{A}$ be as in Theorem* 4.6 *and* $(P_n^{i,t})$ *the lifting of a rank one projection* $p_{i,t} \in \mathcal{A}/\mathcal{G}$. *Then the projections* $P_n^{i,t}$ *can be chosen in such a way that their range dimension is one.*

Proof. Let $(K_n) \in \mathcal{A}$ be an essential rank one sequence such that $(P_n^{i,t}) \in J(K_n)$. Then the operator $K := W_t(K_n)$ has rank one (see the proof of Theorem 3.1), and $P_{i,t} = W_t(P_n^{i,t})$ is a projection operator with one-dimensional range by construction of $(P_n^{i,t})$. Choose compact operators $E, F \in L(H_t)$ such that $EKF = P_{i,t}$, and let (E_n) and (F_n) be pre-images of E and F under the mapping W_t in $J(K_n)$, respectively. Then

$$E_n K_n F_n = P_n^{i,t} + G_n$$

with a sequence (G_n) tending to zero in the norm (recall that the mapping W_t is a homomorphism from $J(K_n)$ onto $K(H_t)$ with kernel $\mathcal{G}$ due to its definition and Proposition 3.6). Multiplying this equality from both sides by $P_n^{i,t}$ yields

$$P_n^{i,t} E_n K_n F_n P_n^{i,t} = P_n^{i,t} + P_n^{i,t} G_n P_n^{i,t}. \tag{4.6}$$

Consider the operators occuring in (4.6) as acting on $\operatorname{Im} P_n^{i,t}$. Then the right hand side of (4.6) is invertible for n large enough (say, for n such that $\|G_n\| < 1/2$), whereas the left hand side of (4.6) has range dimension at most one for every n. In case $\dim \operatorname{Im} P_n^{i,t} \geqslant 2$, this is a contradiction.

Assume that $\dim \operatorname{Im} P_n^{i,t} = 0$ for infinitely many n. Then the fractality of $\mathcal{A}$ implies via Theorem 4.3 that $(P_n^{i,t})$ belongs to $\mathcal{G}$ which contradicts the definition of $p_{i,t} \neq 0$. Hence, $\dim \operatorname{Im} P_n^{i,t} \leqslant 1$ for all n, and this range dimension is zero for at most finitely many n. Modifying the sequence $(P_n^{i,t})$ by adding a sequence which tends to zero one can obviously reach that all projections $P_n^{i,t}$ have range dimension one. ∎

PROPOSITION 4.9. *Let $\mathcal{A}$ be as in Theorem* 4.6 *and* $(P_n^{i,t})$ *the lifting of a rank one projection* $p_{i,t} \in \mathcal{A}/\mathcal{G}$ *such that* $\dim \operatorname{Im} P_n^{i,t} = 1$ *for all* n. *Then the sequences* $(P_n^{i,t})$ *can be modified by adding sequences in* $\mathcal{G}$ *in such a way that the modified sequences still consist of rank one projections and that the orthogonality condition* (4.5) *is satisfied for all sufficiently large* n.

Proof. We proceed by induction. For one lifted sequence there is nothing to prove. Assume that already k of the sequences $(P_n^{i,t})$ are modified such that (4.5) holds. Let (P_n) abbreviate the sum of these k sequences, and let (Q_n) be a further sequence of rank one projections Q_n. The orthogonality condition (4.5) implies that the operators P_n are projections for n large enough.

Consider the operators $(I_n - P_n)Q_n(I_n - P_n) =: \widehat{Q}_n$ with I_n referring to the $n \times n$ identity matrix again, and let p and q denote the cosets $(P_n) + \mathcal{G}$ and $(Q_n) + \mathcal{G}$, respectively. Since $\dim \operatorname{Im} Q_n = 1$, the $\widehat{Q}_n$ are operators with range dimension at most one, and

$$\widehat{Q}_n^2 = (I_n - P_n)Q_n(I_n - P_n)Q_n(I_n - P_n) = \mu_n(I_n - P_n)Q_n(I_n - P_n) = \mu_n \widehat{Q}_n \tag{4.7}$$

with an ℓ^∞-sequence (μ_n). Since the operators $\widehat{Q}_n$ are self-adjoint and non-negative, the numbers μ_n can be assumed to be real and non-negative. Further, since $pq = qp = 0$, one has $(e - p)q(e - p) = q$ and, consequently,

$$\|(I_n - P_n)Q_n(I_n - P_n) - Q_n\| \to 0 \quad \text{respectively} \quad \|(I_n - P_n)Q_n(I_n - P_n)\| \to 1$$

as $n \to \infty$. Together with (4.7), this shows that $\lim \mu_n = 1$. Hence, $\mu_n \neq 0$ for sufficiently large n, and the operators

$$\frac{1}{\mu_n}\widehat{Q}_n = \frac{1}{\mu_n}(I_n - P_n)Q_n(I_n - P_n)$$

are projections with rank one which are orthogonal to P_n. Modifying a finite number of entries of the sequence $((1/\mu_n)\widehat{Q}_n)$ one gets a sequence all entries of which are projections. ∎

Now we can finish the proof of Theorem 4.6. By the preceding propositions, we can assume that all sequences $(P_n^{i,t})$ consist of projections with range dimension one and that (4.5) holds for all large n. Let (P_n) stand for the sum of all these sequences. The orthogonality (4.5) ensures that the operators P_n are projections and that

$$\dim \operatorname{Im} P_n = \sum_{t \in T_{\text{ess}}} \operatorname{Ker} W_t(A_n)$$

for all sufficiently large n (the term on the right hand side is just the number of the different sequences $(P_n^{i,t})$). Furthermore,

$$(4.8) \qquad (A_n)^*(A_n) + (P_n) \quad \text{is a stable sequence, and } (A_n)^*(A_n)(P_n) \in \mathcal{G}.$$

Indeed, the sequence (A_n) is Fredholm by assumption and the sequence (P_n) belongs to the ideal $J(\mathcal{A})$ by construction. Thus, $(A_n)^*(A_n) + (P_n)$ is a Fredholm sequence. Further, all operators

$$W_t((A_n)^*(A_n) + (P_n)) = W_t(A_n)^* W_t(A_n) + P_{\operatorname{Ker} W_t(A_n)}$$

are invertible, and the lifting theorem (Theorem 4.5) implies stability of the sequence $(A_n)^*(A_n) + (P_n)$. Similarly, the sequence $(A_n)^*(A_n)(P_n)$ belongs both to the ideal $J(\mathcal{A})$ and to the kernels of all representations W_t with $t \in T_{\text{ess}}$. Again by the lifting theorem and due to the semi-simplicity of C^*-algebras, the intersection of $J(\mathcal{A})$ with all these kernels is the ideal $\mathcal{G}$ whence the second assertion of (4.8).

Sequences (A_n) and (P_n) as in (4.8) are called a *Moore-Penrose sequence* and the sequence of the associated *Moore-Penrose projection* in [12]. It is further shown in [12] that, for every Moore-Penrose sequence (in particular, for every Fredholm sequence), there is a sequence (Π_n) of projections such that every projection Π_n belongs to the C^*-algebra generated by $A_n^* A_n$ and by the identity matrix I_n and that (4.8) holds with Π_n in place of P_n. Moreover, since Moore-Penrose projections are uniquely determined modulo the ideal $\mathcal{G}$, $\|P_n - \Pi_n\| \to 0$.

The latter convergence implies that $\dim \operatorname{Im} P_n = \dim \operatorname{Im} \Pi_n$ for sufficiently large n, and it remains to verify that the range dimension of the Π_n for large n (which is independent of n and equal to the sum of the kernel dimensions of the operators $W_t(A_n)$ as we have already checked) coincides with the α-number of the sequence (A_n). The property $\Pi_n \in \operatorname{alg}(A_n^* A_n, I_n)$ ensures that the matrices $A_n^* A_n$ and Π_n can be diagonalized simultaneously:

$$\begin{aligned} U_n^* A_n^* A_n U_n &= \operatorname{diag}(a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)}), \\ U_n^* \Pi_n U_n &= \operatorname{diag}(p_1^{(n)}, p_2^{(n)}, \ldots, p_n^{(n)}) \end{aligned}$$

with $0 \leqslant a_1^{(n)} \leqslant a_2^{(n)} \leqslant \cdots \leqslant a_n^{(n)}$. Let $k = \alpha(A_n)$, i.e. $\liminf_{n\to\infty} a_k^{(n)} = 0$ and $\liminf_{n\to\infty} a_{k+1}^{(n)} > 0$. The stability of the sequence $(A_n)^*(A_n) + (\Pi_n)$ requires that

$$\liminf_{n\to\infty}(a_r^{(n)} + p_r^{(n)}) > 0 \quad \text{whence} \quad \lim_{n\to\infty} p_r^{(n)} = 1 \quad \text{for} \quad r \leqslant k,$$

and the condition $\|A_n^* A_n \Pi_n\| \to 0$ implies

$$\lim_{n\to\infty} a_r^{(n)} p_r^{(n)} = 0 \quad \text{whence} \quad \lim_{n\to\infty} p_r^{(n)} = 0 \quad \text{for } r > k$$

since the numbers $p_r^{(n)}$ can take the values 0 and 1 only. Here we also used the fact that the range dimension of the projections Π_n stabilizes as $n \to \infty$. This observation finishes the proof of the dim Ker-formula (4.3) and of Theorem 4.6. ∎

Remarks. 1. The following example shows that, without the hypothesis of fractality, one cannot expect that the ideals $J(K_n)/\mathcal{G}$ are isomorphic to $K(H)$ for essential rank one sequences (K_n). Let $k \in \mathcal{F}$ be the sequence $(P_1, P_2, P_3, \ldots)$ where every P_n is a projection from $\mathbb{C}^n$ onto a one-dimensional subspace of $\mathbb{C}^n$. This sequence has essential rank one. The ideal $J(k)$ contains the sequence

$$k' := (P_1, 0, P_3, 0, P_5, \ldots) = (P_1, P_2, P_3, P_4, P_5, \ldots)\,(I_1, 0, I_3, 0, I_5, \ldots)$$

which, on its hand, is essential rank one, too, and generates a proper ideal $J(k')$ of $J(k)$ which is strictly larger that $\mathcal{G}$. Since $K(H)$ has no proper non-zero ideals, the quotient $J(k)/\mathcal{G}$ cannot be isomorphic to $K(H)$ for some Hilbert space H.

2. It turns out that the Fredholm inverse closedness of the algebra $\mathcal{A}$ is also necessary for the dim Ker-formula. To be more precise, let $\mathcal{A} \subseteq \mathcal{F}$ be a unital algebra with center c which contains the ideal $\mathcal{G}$. Suppose that, for every Fredholm sequence $(A_n) \in \mathcal{A}$ (i.e. for every sequence in $\mathcal{A}$ which is invertible modulo $J(\mathcal{F})$), the operators $W_t(A_n)$ are Fredholm, only a finite number of these operators is not invertible, and the identity

$$\alpha(A_n) = \sum_{t\in T_{\text{ess}}} \dim \operatorname{Ker} W_t(A_n)$$

holds. Then, necessarily, $\mathcal{A}$ is Fredholm inverse closed. Indeed, let $(A_n) \in \mathcal{A}$ be a Fredholm sequence, let $P_{\operatorname{Ker} W_t(A_n)}$ denote the orthogonal projection onto the kernel of $W_t(A_n)$, and choose sequences (P_n^t) in $J(\mathcal{A})$ such that $W_t(P_n^t) = P_{\operatorname{Ker} W_t(A_n)}$ (which can be done due to Proposition 4.7). Then the sequence

$$(B_n) := \Big(A_n^* A_n + \sum_{t\in T_{\text{ess}}} P_n^t\Big)$$

is a Fredholm sequence, too (since $(A_n)^*(A_n)$ is Fredholm and $J(\mathcal{A}) \subseteq J(\mathcal{F})$). Furthermore,

$$\operatorname{Ker} W_t(B_n) = 0 \quad \text{for all } t \in T_{\text{ess}}.$$

From the dim Ker-formula we conclude that $\alpha(B_n) = 0$, hence (B_n) is a stable sequence. This implies the invertibility of the sequence $(A_n)^*(A_n)$ modulo $J(\mathcal{A})$. Similarly, one gets the invertibility of $(A_n)(A_n)^*$ modulo this ideal. But then, the sequence (A_n) itself is invertible modulo $J(\mathcal{A})$. Thus, every sequence in $\mathcal{A}$ which is invertible modulo $J(\mathcal{F})$ in $\mathcal{F}$ (and, hence, modulo $J(\mathcal{F})\cap\mathcal{A}$ in $\mathcal{A}$), is also invertible modulo $J(\mathcal{A})$. So, $\mathcal{A}$ is Fredholm inverse closed.

3. In case of a standard algebra $\mathcal{A}$, its Fredholm theory reduces to the theory sketched in the introduction. Indeed, it is clear from the definition that $\mathcal{A}$ is unital and contains the ideal $\mathcal{G}$. Further, all homomorphisms $W^t(A_n) = \text{s-lim}(E_n^t)^* A_n E_n^t$ are fractal, hence, $\mathcal{A}$ is a fractal algebra (see [11]). Finally, the fourth axiom ensures the Fredholm inverse closedness of $\mathcal{A}$ (actually, this axiom guarantees much more: It requires that a sequence (A_n) is stable if all operators $W^t(A_n)$ are invertible, whereas Fredholm inverse closedness essentially means that (A_n) is stable if all operators $W^t(A_n)$ are invertible *and* if (A_n) is a Fredholm sequence).

It is also easy to identify the ideals I_t and $J(\mathcal{A})$ in case of a standard algebra, namely

$$I_t = \{(E_n^t K (E_n^t)^*) + \mathcal{G} : K \in K(H^t)\}$$

and $J(\mathcal{A}) = \mathcal{J}$. The irreducible representations W_t are just the strong limit homomorphisms W^t, and the dim Ker-formula (4.3) reduces exactly to the identity (1.5).

5. EXAMPLES AND APPLICATIONS

We finish with a brief discussion of several examples and applications of Fredholm approximation sequences.

SPLITTING OF THE SINGULAR VALUES OF TOEPLITZ MATRICES. Given an $L^\infty(\mathbb{T})$ function a with kth Fourier coefficient a_k, let $T(a)$ denote the *Toeplitz operator generated by* a, i.e. the operator which is defined by its matrix representation $(a_{i-j})_{i,j=1}^\infty$ with respect to the standard basis of ℓ^2. Toeplitz operators with bounded generating function are bounded. Further write P_n for the projection operator on ℓ^2 which maps the sequence $(x_i)_{i=1}^\infty$ to $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$, and consider the sequence $(P_n T(a) P_n)_{n\geqslant 1}$ of the finite sections of $T(a)$. The operators $P_n T(a) P_n$ can be identified with finite Toeplitz matrices acting on $\mathbb{C}^n$.

Let $\mathcal{A}$ stand for the smallest closed subalgebra of the algebra $\mathcal{F}$ which contains all sequences $(P_n T(a) P_n)_{n\geqslant 1}$ with a a piecewise continuous function on $\mathbb{T}$. Finally, write R_n for the reflection operators on ℓ^2, $(x_i) \mapsto (x_n, x_{n-1}, \ldots, x_1, 0, 0, \ldots)$. It has been shown in [2] that $\mathcal{A}$ is a standard algebra with the set T consisting of two elements and with associated strong limits

$$W(A_n) := \text{s-lim}\, A_n \quad \text{and} \quad \widetilde{W} := \text{s-lim}\, R_n A_n R_n. \tag{5.1}$$

THEOREM 5.1. ([3]) *A sequence $(A_n) \in \mathcal{A}$ is stable if and only if the operators $W(A_n)$ and $\widetilde{W}(A_n)$ are invertible.*

In particular, Theorem 4.6 implies that

$$\alpha(A_n) = \operatorname{Ker} W(A_n) + \operatorname{Ker} \widetilde{W}(A_n) \tag{5.2}$$

for every Fredholm sequence $(A_n) \in \mathcal{A}$. In Figures 2 and 4, there are plotted the singular values of the Toeplitz matrices $P_n T(a) P_n$ and $P_n T(b) P_n$ with n between 1 and 150 for

$$a(t) = 5t^{-3} + t^{-2} + 3t^{-1} + 1 + 4t + 7t^2 + t^3 \quad \text{and} \quad b(t) = 0.7t + t^5$$

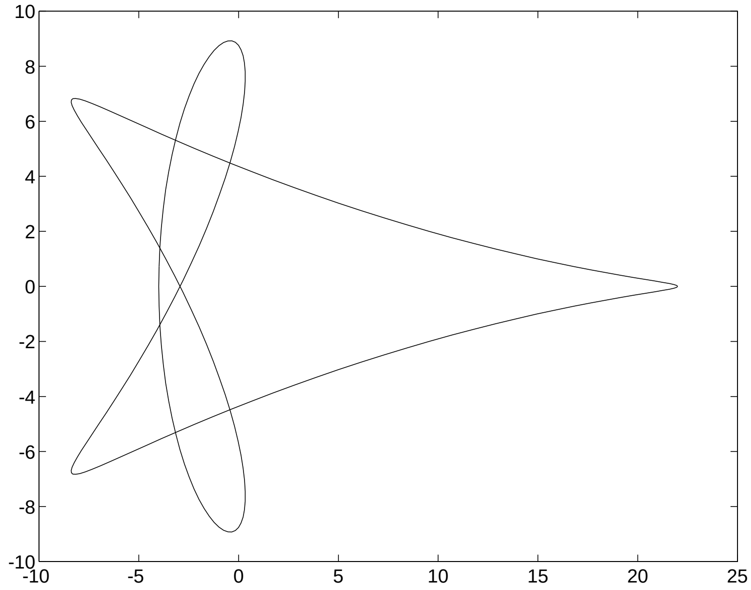

Figure 1. Image of the unit circle under the generating function a.

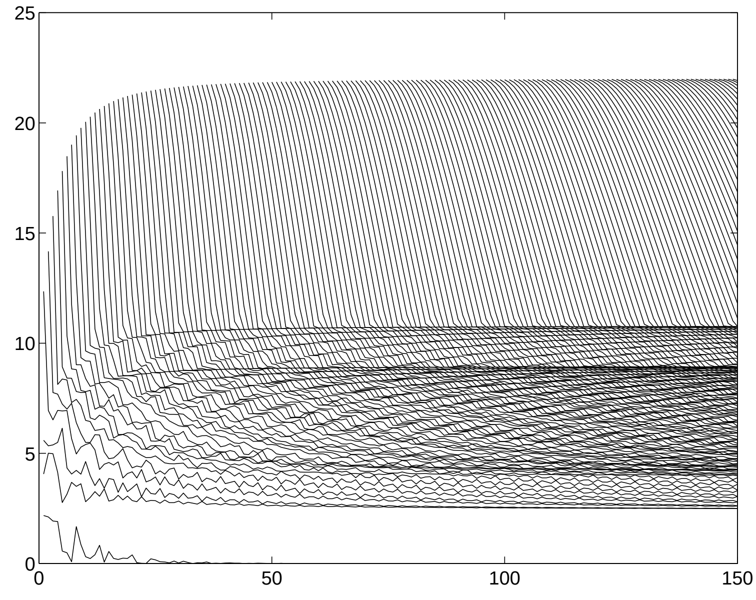

Figure 2. Singular values of $P_nT(a)P_n$ for n between 1 and 150.

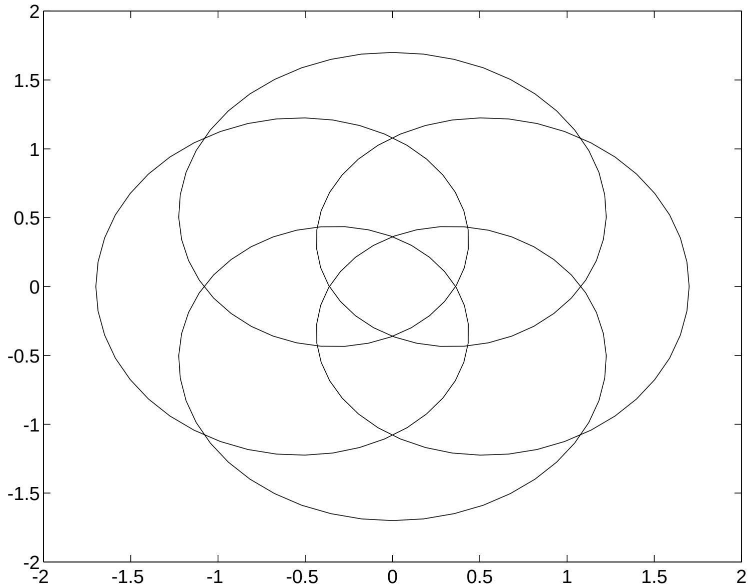

Figure 3. Image of the unit circle under the generating function b.

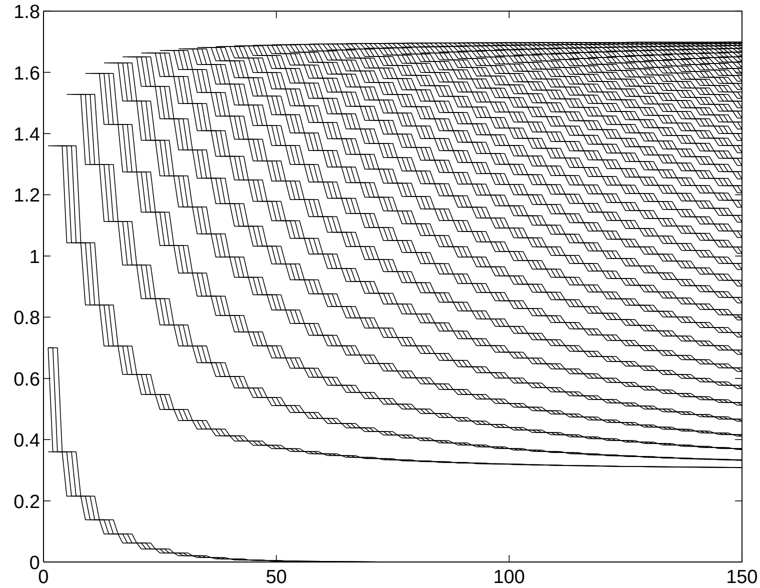

Figure 4. Singular values of $P_nT(b)P_n$ for n between 1 and 150.

respectively. The generating functions a and b have winding numbers 1 and 4 (Figures 1 and 3), and Figures 2 and 4 show exactly the predicted splitting of the singular values. These computations were done by F. Meyer using standard matlab.

Thus, the singular value splitting is an effect which can be observed numerically. On the other hand, the generating functions a and b are polynomials which might be the reason for the excellent convergence in Figures 2 and 4. There is an example (due to Tyrtyshnikov; [14]) of a piecewise continuous function c such that the smallest singular value of $P_n T(c) P_n$ decays to zero as $(\ln n)^{-1}$.

Numerical determination of the kernel dimension. In case of a Fredholm Toeplitz operator $T(a)$ with continuous generating function, the kernel dimension of $T(a)$ is simply the maximum of 0 and of the negative winding number of the curve $a(\mathbb{T})$ around the origin (where this curve is provided with the orientation which is naturally inherited from the counterclockwise orientation of the unit circle). A similar simple geometric argument applies to the determination of $\operatorname{Ker} T(a)$ if a is piecewise continuous. The geometric interpretation of the kernel dimension of Toeplitz operators is mainly a consequence of Coburn's theorem (see [3], Theorem 1.10).

In contrast to this, the determination of the kernel dimension of a compactly perturbed Toeplitz operator $T(a) + K$ can prove to be a serious problem even in case of a nice generating function a. An application of identity (5.2) to the sequence (A_n) where $A_n = P_n(T(a) + K)P_n$ with a piecewise continuous and K compact yields

$$\alpha(A_n) = \operatorname{Ker}(T(a) + K) + \operatorname{Ker} T(\widetilde{a})$$

where again $\widetilde{a}(t) := a(1/t)$. The kernel dimension of $T(\widetilde{a})$ can be determined via the winding number. Thus, if one is able to observe the α-number of (A_n) numerically (by consideration of the singular values of A_n for large n as in Figures 2 and 4 above), then this identity yields the desired kernel dimension of $T(a) + K$.

Sufficient stability conditions. Let $\mathcal{T}$ stand for the smallest closed subalgebra of $L(\ell^2)$ which contains all Toeplitz operators $T(a)$ with piecewise continuous generating function a. Again we consider the finite section method $(P_n A P_n)$, but now for operators A in $\mathcal{T}$. Accordingly, let $\mathcal{B}$ refer to the smallest closed subalgebra of $\mathcal{F}$ which contains all sequences $(P_n A P_n)$ with $A \in \mathcal{T}$. It is not hard to prove that the strong limits $W(A_n)$ and $\widetilde{W}(A_n)$ (defined as in (5.1)) exist for every sequence $(A_n) \in \mathcal{B}$. Thus, the invertibility of the operators $W(A_n)$ and $\widetilde{W}(A_n)$ is a necessary condition for the stability of the sequence (A_n). Assume the invertibility of these operators is also sufficient for the stability of (A_n). Then (and under the preliminary assumption that $\mathcal{B}$ is a standard algebra) the index identity

$$0 = \operatorname{ind}(A_n) = \operatorname{ind} W(A_n) + \operatorname{ind} \widetilde{W}(A_n)$$

should hold for every Fredholm sequence $(A_n) \in \mathcal{B}$, i.e. for every sequence (A_n) for which $W(A_n)$ and $\widetilde{W}(A_n)$ are Fredholm operators.

There are simple examples showing that this identity cannot be true for arbitrary Fredholm sequences in $\mathcal{B}$. Indeed, let

$$a(\mathrm{e}^{\mathrm{i}x}) := \begin{cases} 1 & \text{if } x \in (0, 2\pi/3), \\ \mathrm{e}^{5\mathrm{i}\pi/6} & \text{if } x \in (2\pi/3, 4\pi/3), \\ \mathrm{e}^{7\mathrm{i}\pi/6} & \text{if } x \in (4\pi/3, 2\pi). \end{cases}$$

Then

$$\widetilde{a}(\mathrm{e}^{\mathrm{i}x})^2 := \begin{cases} \mathrm{e}^{\mathrm{i}\pi/3} & \text{if } x \in (0, 2\pi/3), \\ \mathrm{e}^{-\mathrm{i}\pi/3} & \text{if } x \in (2\pi/3, 4\pi/3), \\ 1 & \text{if } x \in (4\pi/3, 2\pi), \end{cases}$$

hence, for $A_n = P_n T(a)^2 P_n$, the operator $W(A_n) = T(a)^2$ has index -2 whereas the index of $\widetilde{W}(A_n) = T(\widetilde{a}^2)$ is 0.

Thus, the index identity *predicts* that the invertibility of the operators $W(A_n)$ and $\widetilde{W}(A_n)$ cannot be sufficient for the stability of a sequence $(A_n) \in \mathcal{B}$ in general. (In a similar way, the kernel dimension identity implies that the invertibility of the operator $W(A_n)$ is not sufficient for the stability of a sequence $(A_n) \in \mathcal{A}$ in general, although it is sufficient for sequences of the form $(P_n T(a) P_n)$.)

A detailed analysis (performed by Werbitzky, Rathsfeld, Böttcher, Silbermann and the author) yields the following stability result for sequences in $\mathcal{B}$ where, besides the invertibility of $W(A_n)$ and $\widetilde{W}(A_n)$, certain *local stability conditions* occur. Let S_n refer to the subspace of the Hilbert space $L^2([0,1])$ which is spanned by the functions

$$\varphi_{k,n}(x) := \begin{cases} 1 & \text{if } x \in (k/n, (k+1)/n), \\ 0 & \text{if } x \in (0, k/n) \cup ((k+1)/n, 1); \end{cases}$$

with $k = 0, \ldots, n-1$, write E_n for the operators

$$E_n : \operatorname{Im} P_n \to S_n, \quad (x_1, \ldots, x_n, 0, \ldots) \mapsto \sum_{k=1}^{n} x_k \varphi_{k,n},$$

and define $E_{-n} : S_n \to \operatorname{Im} P_n$ by $E_{-n} := (E_n)^{-1}$. Finally, for $\tau \in \mathbb{T}$, let Y_τ stand for the operator

$$Y_\tau : \ell^2 \to \ell^2, \quad (x_k)_{k=1}^{\infty} \mapsto (\tau^{-k} x_k)_{k=1}^{\infty}.$$

One can show that, for every sequence $(A_n) \in \mathcal{B}$ and for every $\tau \in \mathbb{T}$, the strong limit

$$W^\tau(A_n) := \operatorname*{s-lim}_{n\to\infty} E_n Y_{\tau^{-1}} A_n Y_\tau E_{-n}$$

exists and that it defines a bounded linear operator $W^\tau(A_n)$ on $L^2([0,\,1])$.

THEOREM 5.2. *A sequence $(A_n) \in \mathcal{B}$ is stable if and only if the operators $W(A_n)$, $\widetilde{W}(A_n)$ and $W^\tau(A_n)$ are invertible for every $\tau \in \mathbb{T}$.*

For a proof see, e.g., [5], Theorem 4.1. This proof also shows that $\mathcal{B}$ is a standard algebra.

GLOBAL VERSUS LOCAL STABILITY CONDITIONS. For a more refined version of Theorem 5.2, let X stand for a closed subset of the unit circle $\mathbb{T}$ and denote by PC_X the C^*-algebra of all piecewise continuous functions on $\mathbb{T}$ which are continuous at the points of $\mathbb{T} \setminus X$. Accordingly, let $\mathcal{T}_X$ stand for the smallest closed subalgebra of $L(\ell^2)$ which contains all Toeplitz operators $T(a)$ with generating function $a \in \mathrm{PC}_X$, and let $\mathcal{B}_X$ refer to the smallest closed subalgebra of $\mathcal{F}$ which contains all sequences $(P_n A P_n)$ with $A \in \mathcal{T}_X$. A closer look at the proof of Theorem 5.2 reveals the following.

Theorem 5.3. *A sequence $(A_n) \in \mathcal{B}_X$ is stable if and only if the operators $W(A_n)$, $\widetilde{W}(A_n)$ and $W^\tau(A_n)$ are invertible for every $\tau \in X \subseteq \mathbb{T}$.*

Of particular interest is the case when X is a singleton, say $X = \{1\}$. For $A \in \mathcal{T}_{\{1\}}$, Theorem 5.3 says that the sequence $(P_n A P_n)$ is stable if and only if the three operators $W(P_n A P_n) = A$, $\widetilde{W}(P_n A P_n)$ and $W^1(P_n A P_n)$ are invertible. Actually, the invertibility of $W^1(P_n A P_n)$ proves to be redundant in this special setting which is again a consequence of the index identity.

Theorem 5.4. *Let $A \in \mathcal{T}_{\{1\}}$. Then the sequence $(P_n A P_n)$ is stable if and only if the operators $W(P_n A P_n) = A$ and $\widetilde{W}(P_n A P_n)$ are invertible.*

Proof. The index identity, specified to the setting of Theorem 5.3 with $X = \{1\}$, yields

$$\operatorname{ind} W(A_n) + \operatorname{ind} \widetilde{W}(A_n) + \operatorname{ind} W^1(A_n) = 0$$

for every Fredholm sequence $(A_n) \in \mathcal{T}_{\{1\}}$. Let, in particular, $A_n = P_n A P_n$ with $A \in \mathcal{T}_{\{1\}}$, and suppose $W(P_n A P_n)$ and $\widetilde{W}(P_n A P_n)$ are invertible. It is not hard to check (using the Gohberg/Krupnik symbol calculus) that then $W^1(P_n A P_n)$ is a Fredholm operator. Hence, $(P_n A P_n)$ is a Fredholm sequence, and the index identity yields $\operatorname{ind} W^1(P_n A P_n) = 0$. Further, the special form of the sequence $(A_n) = (P_n A P_n)$ implies that $W^1(P_n A P_n)$ is a Mellin operator, which is subject to Coburn's theorem. Hence, $W^1(P_n A P_n)$ is invertible. ∎

Regularization of non-stable sequences. Let us finally mention a further potential application of Fredholm sequences to the regularization of non-stable approximation sequences which has been discussed in detail in [12].

Let (A_n) be an approximation sequence for an invertible operator A (i.e. the A_n converge to A in the $*$-strong operator topology). Assume that this sequence is not stable but Fredholm with α-number k (which can be determined numerically or via the $\dim \operatorname{Ker}$-formula in case of a standard algebra, say). Finally, let $A_n = U_n \Sigma_n V_n^*$ refer to the singular value decomposition of A_n with $\Sigma_n := \operatorname{diag}(\sigma_1^{(n)}, \ldots, \sigma_n^{(n)})$ and with $0 \leqslant \sigma_1^{(n)} \leqslant \cdots \leqslant \sigma_n^{(n)}$ denoting the singular values of A_n. For $n > k$, set $\Sigma_n' := \operatorname{diag}(0, \ldots, 0, \sigma_{k+1}^{(n)}, \ldots, \sigma_n^{(n)})$ and $A_n' := U_n \Sigma_n' V_n^*$. Then (A_n') is also an approximation sequence for A, and this sequence is Moore-Penrose stable in the sense that the Moore-Penrose inverses of the matrices A_n' are uniformly bounded. Consequently, the Moore-Penrose inverses of the A_n' converge strongly to the inverse of A. In other words, the least square solutions of the equations $A_n' x_n = P_n y$ converge to the solution of $Ax = y$.

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