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SOME CLASSES OF q-DEFORMED OPERATORS

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Abstract. Motivated by recent developments in the theory of quantum groups, some classes of q-deformed operators (q-normal, q-quasinormal, q-hyponormal operators) are introduced and investigated systematically, where q is a positive deformation parameter. It turns out that many basic properties of these q-deformed operators are different from that of the corresponding undeformed operators (i.e., normal, quasinormal, hyponormal operators).

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1. INTRODUCTION

In the theory of quantum groups one often meets, in different situations, elements x obeying the formal algebraic relation

$$xx^* = qx^*x, \tag{1.1}$$

where q is a positive real number. For instance, the q-deformed quantum plane $\mathbb{C}_q^1$ is a $*$-algebra with one generator x and defining relation (1.1). Another example is the q-Heisenberg algebra in [33] which contains a generator x satisfying (1.1). More generally, most quantum groups and quantum spaces contain the equation $xy = qyx$ as defining relation for certain generators x and y of the corresponding algebras. Further information can be found in [16], [17], [18] and the references cited in them.

The class of Hilbert space operators x fulfilling the relation (1.1) seems to be important and of interest in itself. Such an operator is called q-normal. As pointed out in [21] (see also [22]), a non-zero q-normal operator is always unbounded. The study of pairs of a selfadjoint operator x and a unitary operator y such that $xy = qyx$ can be always reduced to that of q-normal operators. On the other hand, many variants of q-oscillators occur in quantum groups theory ([17], Chapter 5). One possible variety is based on the relation $xx^* - qx^*x = 1$. Under some domain assumption, all (irreducible) closed, densely defined operators satisfying the above relation have been described in [8]. All such operators are in fact weighted (backward) shifts and in particular their adjoints are deformed hyponormal weighted shifts with deformation parameter q^{-1} (see Corollary 4.2 below).

The purpose of this paper is to begin a systematic investigation of q-normal operators and various other related classes such as q-quasinormal and q-hyponormal operators in the context of operator theory in Hilbert space. Among others, we show that a non-zero q-quasinormal operator with $0 < q < 1$ must be unbounded and, in contrast to the unbounded case, every bounded non-zero q-quasinormal operator (and hence, with $q > 1$) has spectrum consisting only of zero. Thus, the behavior of q-quasinormal operators depends essentially on the parameter q. It is also shown that every non-zero q-normal operator T has sufficient large spectrum and there exists an intertwining relation between T and qT. This causes such an operator to have some special properties distinguished from undeformed operators (e.g., [20], [38]).

The paper is organized as follows. In Section 2, the concepts of q-normality and q-quasinormality are introduced and the q-quasinormality is characterized in terms of the spectral measure of the absolute value in the polar decomposition.

In Section 3, q-hyponormality is introduced by inspired from the above mention. It is shown that there exists a unique contraction attached to (and uniquely determined by) each q-hyponormal operator. The relation between this attached contraction and the partial isometry in the polar decomposition is established for a class of q-quasinormal operators. Though the results and methods in this section (and also in some parts in another section) are valid even for undeformed corresponding operators (hyponormal, normal and quasinormal operators), we shall focus our attention on q-deformed operators.

In Section 4, bilateral and unilateral (possibly unbounded) weighted shifts are treated as typical examples of q-deformed operators and their spectrum are given in the case where they are q-normal and q-quasinormal.

In Section 5, we will see q-normality and q-quasinormality are characterized by the property of their corresponding attached contractions. It is shown that every q-quasinormal operator has a structure of the Brown type (for a bounded quasinormal operator [7]) and, as a direct application, it has a q-normal extension in a larger Hilbert space.

In Section 6, the relations between T and qT for q-normal and q-quasinormal operators T are given by their attached partial isometries. As a consequence, it is proved a non-zero q-quasinormal operator is unbounded provided that $0 < q < 1$, and it is shown in Section 9 every bounded q-quasinormal operator is nilpotent.

We investigate, in Section 7, the spectrum of a q-normal operator and show that its spectrum is sufficient large in the sense of the planar Lebesgue measure. Moreover, a list of the spectra is given.

In Sections 8 and 9, we shall present typical properties of q-normal operators, which are, roughly speaking, very different from those of the undeformed operators. We first discuss in Section 8 about Cartesian decompositions of q-deformed operators and show the real and imaginary parts are determined by the attached contraction. Furthermore, it is shown that both parts of a non-zero q-normal operator are always unbounded, closed. Secondly related to [14], it is shown the real and imaginary parts of q-quasinormal weighted shifts have their equal deficiency indices $(1,1)$. It is known the powers of a closed, densely defined operator are neither closable nor densely defined in general. It is, however, shown in Section 9 that the powers of q-normal and q-quasinormal operators are q-normal and q-quasinormal, respectively. We also show the powers of the real and imaginary parts of a q-normal operator are represented by the attached partial isometries and the sets of their analytic elements coincide.

2. q-normal and q-quasinormal operators

In this paper, all operators are assumed to be linear. For an operator T in a Hilbert space $\mathcal{H}$, the domain, the range and the kernel of T are denoted by $\mathcal{D}(T), \mathcal{R}(T)$ and $\ker T$, respectively. The usual inner product of $\mathcal{H}$ is denoted by $\langle\cdot\,,\cdot\rangle$. If $\mathcal{M}$ is a subspace of $\mathcal{H}$, $\overline{\mathcal{M}}$ and $\mathcal{M}^{\perp}$ denote its closure and its orthogonal complement, respectively and $T|\mathcal{M}$ denotes the restriction of T to $\mathcal{M}$.

Let S and T be operators in $\mathcal{H}$. Then the relation $S \subset T$ means that $\mathcal{D}(S) \subseteq \mathcal{D}(T)$ and $S\eta = T\eta$ for all $\eta \in \mathcal{D}(S)$. If T is closable, then we denote its closure by $\widetilde{T}$.

We shall denote by $\mathbb{C}, \mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$ the set of complex numbers, the set of real numbers, the set of integers and the set of positive integers, respectively. For a set $\mathfrak{M} \subseteq \mathbb{C}$ and $\gamma \in \mathfrak{M}$, the complex conjugate of γ is denoted by $\overline{\gamma}$ and $\overline{\mathfrak{M}}$ means the set $\{\overline{\gamma} : \gamma \in \mathfrak{M}\}$.

Let us begin with the concept of q-deformed normality.

Definition 2.1. Let q be a positive number with $q \neq 1$. Let T be a closed, densely defined operator in $\mathcal{H}$. If T satisfies

$$TT^* = qT^*T, \tag{2.1}$$

then T is called a *deformed normal operator* with deformation parameter q.

We remark, as is easily seen, that the condition (2.1) is equivalent to

$$|T^*| = \sqrt{q}\,|T|. \tag{2.2}$$

Lemma 2.2. *Let T be a non-zero closed, densely defined operator in $\mathcal{H}$ and let $T = U|T|$ be the polar decomposition. Let γ be a complex number. Then the relation*

$$U|T| \subset \gamma|T|U$$

is equivalent to

$$U|T| = \gamma|T|U.$$

If this is the case, γ is a positive number.

Proof. Suppose that $U|T| \subset \gamma|T|U$. Since $T^* = |T|U^* \supset \overline{\gamma}U^*|T|$, we have $\mathcal{D}(T) \subseteq \mathcal{D}(T^*)$. Take $\eta \in \mathcal{H}$ such that $U\eta \in \mathcal{D}(|T|) = \mathcal{D}(T)$. Then, $U^*U\eta \in U^*\mathcal{D}(T^*)$. Using the general property of U^*; that is, $U^*\mathcal{D}(T^*) \subseteq \mathcal{D}(T)$, we have $U^*U\eta \in \mathcal{D}(T)$. Since $1 - U^*U$ is the orthogonal projection onto $\mathcal{R}(|T|)^{\perp} = \ker|T| = \ker T$, we have

$$\eta = U^*U\eta + (1 - U^*U)\eta \in \mathcal{D}(T).$$

This means that $U|T| = \gamma|T|U$. We lastly show that γ is positive. Since $U|T| = \gamma|T|U$, we have

$$\gamma U^*|T|U = U^*U|T| = |T|.$$

Since T is non-zero, it follows that

$$\langle |T|\xi, \xi\rangle > 0.$$

for some $\xi \in \mathcal{D}(T)$. Hence γ is positive. ∎

Definition 2.3. Let q be a positive number with $q \neq 1$. Let T be a closed, densely defined operator in $\mathcal{H}$ with polar decomposition $T = U|T|$. If T satisfies the relation

$$U|T| \subset \sqrt{q}\,|T|U,$$

then T is called a *deformed quasinormal operator* with deformation parameter q.

For a deformed normal (respectively deformed quasinormal) operator T with deformation parameter q, we will simply say T is *q-normal* (respectively *q-quasinormal*).

Let T be a closed, densely defined operator in $\mathcal{H}$. In the same way as in [38], Theorem 5.40, p. 124, it is shown that the q-normality of T is equivalent to the conditions:

$$\mathcal{D}(T) = \mathcal{D}(T^*) \quad \text{and} \quad \|T^*\eta\| = \sqrt{q}\,\|T\eta\|, \quad \eta \in \mathcal{D}(T).$$

We next show a q-normal operator is q-quasinormal. In fact, if T is q-normal, one can easily check that

$$U|T|U^* = \sqrt{q}\,|T|.$$

Since $(1 - U^*U)$ is the orthogonal projection onto $\ker|T|$, for $x \in \mathcal{D}(|T|) = \mathcal{D}(T)$ we have

$$U^*Ux \in \mathcal{D}(|T|) \quad \text{and} \quad U|T|x = U|T|U^*Ux.$$

Hence $U|T| \subset \sqrt{q}\,|T|U$, so that T is q-quasinormal. If T is q-quasinormal, then we obtained $\mathcal{D}(T) \subseteq \mathcal{D}(T^*)$ in the proof of Lemma 2.2. Using the relation $T^* \supset \sqrt{q}\,U^*|T|$, we have

$$\|T^*x\| = \sqrt{q}\,\|U^*|T|x\| \leqslant \sqrt{q}\,\|Tx\|,$$

for all $x \in \mathcal{D}(T)$. Thus, we proved the following:

Proposition 2.4. *Let T be a closed, densely defined operator in a Hilbert space $\mathcal{H}$. Then the following statements hold:*

(i) *If T is q-normal, then T is q-quasinormal.*

(ii) *T is q-normal if and only if*

$$\mathcal{D}(T) = \mathcal{D}(T^*), \quad \textit{and} \quad \|T^*\eta\| = \sqrt{q}\,\|T\eta\|, \quad \eta \in \mathcal{D}(T).$$

(iii) *If T is q-quasinormal, then*

$$\mathcal{D}(T) \subseteq \mathcal{D}(T^*), \quad \textit{and} \quad \|T^*\eta\| \leqslant \sqrt{q}\,\|T\eta\|, \quad \eta \in \mathcal{D}(T).$$

Next we shall characterize the q-quasinormality in terms of the spectral measure of the absolute value of the polar decomposition.

Theorem 2.5. *Let T be a closed, densely defined operator in a Hilbert space $\mathcal{H}$ and let $T = U|T|$ be the polar decomposition. The following conditions are equivalent:*

(i) *T is q-quasinormal.*

(ii) *For all $t \in \mathbb{R}$,*

$$U\mathrm{e}^{\mathrm{i}t|T|} = \mathrm{e}^{\mathrm{i}\sqrt{q}\,t|T|}U, \quad \mathrm{i} = \sqrt{-1}.$$

(iii) *For all $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$,*

$$U(\lambda - |T|)^{-1} = (\lambda - \sqrt{q}\,|T|)^{-1}U.$$

(iv) *For all Borel sets $\mathfrak{M}$,*

$$E(\sqrt{q}^{\,-1}\mathfrak{M})U = UE(\mathfrak{M}),$$

where $E(\,\cdot\,)$ is the spectral measure of $|T|$.

Especially, every q-quasinormal operator T satisfies the relation

$$Uf(|T|) = f(\sqrt{q}\,|T|)U$$

for any Borel function f.

Proof. If T is q-quasinormal, then we obtain $U|T|^n = q^{\frac{n}{2}}|T|^nU$ for all $n \in \mathbb{N}$. If η is an analytic vector of $|T|$, then it is easily seen that $U\eta$ is also analytic for $\sqrt{q}\,|T|$ and

$$U\mathrm{e}^{\mathrm{i}t|T|}\eta = \mathrm{e}^{\mathrm{i}\sqrt{q}\,t|T|}U\eta$$

for all $t \in \mathbb{R}$. Since the set of analytic elements of $|T|$ is dense in $\mathcal{H}$, we obtain $U\mathrm{e}^{\mathrm{i}t|T|} = \mathrm{e}^{\mathrm{i}\sqrt{q}\,t|T|}U$ for all $t \in \mathbb{R}$. Conversely, assume that condition (ii) holds. For each $\eta \in \mathcal{D}(|T|)$,

$$\frac{1}{t}(\mathrm{e}^{\mathrm{i}\sqrt{q}\,t|T|}U\eta - U\eta) = U\Big(\frac{\mathrm{e}^{\mathrm{i}t|T|}\eta - \eta}{t}\Big) \longrightarrow \mathrm{i}U|T|\eta$$

as $t \to 0$. Hence, $U\eta \in \mathcal{D}(\sqrt{q}\,|T|) = \mathcal{D}(|T|)$, and $\sqrt{q}\,|T|U\eta = U|T|\eta$, so that $U|T| \subset \sqrt{q}\,|T|U$. Therefore T is q-quasinormal. Assume that condition (ii) holds. We check condition (iii) for $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda < 0$. For $\eta \in \mathcal{H}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda < 0$, by [30], VIII.9, p. 287, and condition (ii) we have

$$U(\lambda - |T|)^{-1}\eta = \mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}t\lambda}U\mathrm{e}^{\mathrm{i}t|T|}\eta\,\mathrm{d}t = (\lambda - \sqrt{q}\,|T|)^{-1}U\eta.$$

We next show the implication (iii) $\Rightarrow$ (iv). Let $E(\,\cdot\,)$ and $F(\,\cdot\,)$ be the spectral measures of $|T|$ and $\sqrt{q}\,|T|$, respectively. For $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$ and $\eta, \xi \in \mathcal{H}$

$$\langle(\lambda - \sqrt{q}\,|T|)^{-1}U\eta, \xi\rangle = \int \frac{1}{\lambda - \nu}\,\mathrm{d}\langle F_\nu U\eta, \xi\rangle$$

and

$$\langle U(\lambda - |T|)^{-1}\eta, \xi\rangle = \int \frac{1}{\lambda - \nu}\,\mathrm{d}\langle E_\nu \eta, U^*\xi\rangle.$$

Therefore, in view of [36], Lemma 5.2, condition (iii) implies $\langle F(\mathfrak{M})U\eta, \xi\rangle = \langle E(\mathfrak{M})\eta, U^*\xi\rangle$ for all Borel sets $\mathfrak{M}$. Since $F(\mathfrak{M}) = E((\sqrt{q}\,)^{-1}\mathfrak{M})$ (for instance,

[11], XII, 2.9), condition (iv) holds. Suppose that condition (iv) holds. By the change of measure principle, we have

$$\langle \mathrm{e}^{\mathrm{i}t|T|}\eta, U^*\xi\rangle = \int \mathrm{e}^{\mathrm{i}t\nu}\,\mathrm{d}\langle E_\nu\eta, U^*\xi\rangle = \int \mathrm{e}^{\mathrm{i}\sqrt{q}\,t\nu}\,\mathrm{d}\langle E_\nu U\eta, \xi\rangle = \langle \mathrm{e}^{\mathrm{i}\sqrt{q}\,t|T|}U\eta, \xi\rangle,$$

which implies condition (ii). Finally, take a Borel function f. Then, by condition (iv), we have

$$\langle Uf(|T|)\eta, \xi\rangle = \int f(\nu)\,\mathrm{d}\langle E_\nu\eta, U^*\xi\rangle = \int f(\nu)\,\mathrm{d}\langle F_\nu U\eta, \xi\rangle = \langle f(\sqrt{q}\,|T|)U\eta, \xi\rangle.$$

This completes the proof. ∎

3. q-HYPONORMAL OPERATORS AND THEIR ATTACHED CONTRACTIONS

In this section we shall introduce a certain contraction uniquely determined by each deformed operator, which will be a useful tool for the study of deformed operators.

Taking account of Proposition 2.4, we introduce the concepts corresponding to hyponormality and formal normality of unbounded operators (see, for example, [15], [20], [23] and [25] and the references cited in them).

Let q be a positive number with $q \neq 1$. A densely defined operator T is called *q-hyponormal* (or a *deformed hyponormal* operator with deformation parameter q) if it satisfies

$$\mathcal{D}(T) \subseteq \mathcal{D}(T^*) \quad \text{and} \quad \|T^*\eta\| \leqslant \sqrt{q}\,\|T\eta\|$$

for all $\eta \in \mathcal{D}(T)$. If a q-hyponormal operator T satisfies

$$\|T^*\eta\| = \sqrt{q}\,\|T\eta\|$$

for all $\eta \in \mathcal{D}(T)$, then T is said to be *q-formally normal*. By Proposition 2.4, every q-quasinormal operator is q-hyponormal. One can also check that a q-hyponormal operator T is closable and that its closure $\widetilde{T}$ is also q-hyponormal. Such an operator with deformation parameter q is sometimes called a *q-deformed operator* as a generic term.

LEMMA 3.1. *Let T be a densely defined operator in $\mathcal{H}$. Then T is q-hyponormal if and only if there is a contraction K such that*

$$T^* \supset \sqrt{q}KT.$$

In this case, the contraction K is taken such that

$$\mathcal{R}(K^*) \subseteq \overline{\mathcal{R}(T)}, \quad \text{or equivalently } \ker K \supseteq \ker T^*. \tag{3.1}$$

Moreover, K is uniquely determined under this condition (3.1).

Proof. Suppose that T is q-hyponormal. Define an operator K_0 from $\mathcal{R}(T)$ to $\mathcal{R}(T^*)$ by

$$K_0T\eta = \frac{1}{\sqrt{q}}T^*\eta$$

for $\eta \in \mathcal{D}(\mathcal{T})$. Then K_0 is a contraction on $\mathcal{R}(T)$, so that K_0 continuously extends $\widetilde{K}_0$ on $\overline{\mathcal{R}(T)}$. Put $K = \widetilde{K}_0$ on $\overline{\mathcal{R}(T)}$ and $K = 0$ on $\mathcal{R}(T)^\perp$. Then K is a contraction such that $T^* \supset \sqrt{q}\,KT$ and $\mathcal{R}(K^*) \subseteq \overline{\mathcal{R}(T)}$. The converse and the uniqueness of K are easily shown. ∎

Corollary 3.2. *Let T be a q-hyponormal operator in $\mathcal{H}$. Then $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$. In particular, if T is q-normal, then*

$$\mathcal{R}(T) = \mathcal{R}(T^*).$$

Proof. The first relation follows from $T \subset \sqrt{q}\, T^* K^*$. If T is q-normal, then T^* is q^{-1}-normal. Therefore, $\mathcal{R}(T) = \mathcal{R}(T^*)$. ∎

Definition 3.3. For each q-hyponormal operator T, we denote by K_T the contraction K in Lemma 3.1 uniquely determined by condition (3.1). K_T is called the *attached contraction to T*.

Proposition 3.4. *Let T be a q-hyponormal operator in $\mathcal{H}$. Then T is a q-formally normal if and only if K_T is a partial isometry with initial domain $\overline{\mathcal{R}(T)}$.*

Proof. If T is q-formally normal,

$$\|K_T T\eta\| = \frac{1}{\sqrt{q}}\|T^*\eta\| = \|T\eta\|,$$

for $\eta \in \mathcal{D}(\mathcal{T})$. Hence, by [13], p. 63, K_T is a partial isometry with initial domain $\overline{\mathcal{R}(T)}$. The converse is clear. ∎

The following theorem contains a characterization of q-quasinormality in terms of K_T.

Theorem 3.5. *Let T be a closed q-hyponormal operator in $\mathcal{H}$ and let $T = U|T|$ be the polar decomposition. Then T is a q-quasinormal if and only if*

$$K_T = (U^*)^2.$$

Proof. Suppose that T is q-quasinormal. Since U^*U is the orthogonal projection onto $\overline{\mathcal{R}(|T|)}$, we have

$$T^* = |T|U^* \supset \sqrt{q}\, U^*|T| = \sqrt{q}\, (U^*)^2 T.$$

Since UU^* is the orthogonal projection onto $\overline{\mathcal{R}(T)}$, it follows that $U^* = 0$ on $\mathcal{R}(T)^{\perp}$. Hence, by the uniqueness of K_T, $K_T = (U^*)^2$. Assume, conversely $K_T = (U^*)^2$. By the definition and the same way as mentioned above, we have $|T|U^* \supset \sqrt{q}\, U^*|T|$. Hence, $U|T| \subset \sqrt{q}\, |T|U$. Thus T is q-quasinormal. ∎

Lemma 3.6. *Let T be a q-formally normal operator. Then T satisfies the relation*

$$T^* \supset \frac{1}{\sqrt{q}} T K_T.$$

In particular, if T is q-normal, then the equality holds in this relation.

Proof. Take any η in $\mathcal{D}(TK_T)$. Noticing $T^* \supset \sqrt{q}\, K_T T$ and $(K_T)^* K_T$ is the orthogonal projection onto $\overline{\mathcal{R}(T)}$ by Proposition 3.4, we have

$$\langle T\xi, \eta\rangle = \langle K_T T\xi, K_T\eta\rangle = \Big\langle \frac{1}{\sqrt{q}} T^*\xi, K_T\eta\Big\rangle = \Big\langle \xi, \frac{1}{\sqrt{q}} TK_T\eta\Big\rangle$$

for all $\xi \in \mathcal{D}(T)$. Hence, $\eta \in \mathcal{D}(T^*)$ and $T^*\eta = \big(\frac{1}{\sqrt{q}}\big) TK_T\eta$. If T is q-normal, then by Theorem 3.5 $K_T = (U^*)^2$, where $T = U|T|$. By the property of the polar decomposition, $U^*\mathcal{D}(T^*) \subseteq \mathcal{D}(T) = \mathcal{D}(T^*)$. Thus, $\mathcal{D}(TK_T) = \mathcal{D}(T^*)$. Hence, $T^* = \frac{1}{\sqrt{q}} TK_T$. ∎

PROPOSITION 3.7. *Let T be a q-quasinormal operator in a Hilbert space $\mathcal{H}$. Then K_T is a partial isometry with final domain $\overline{\mathcal{R}(T^*)}$.*

Proof. Let $T = U|T|$ be the polar decomposition of T and put $P \equiv UU^*$ and $Q \equiv U^*U$. By the relation $\ker T \subseteq \ker T^*$, we have $P \leqslant Q$. By Theorem 3.5,

$$\big((K_T)^*K_T\big)^2 = (UPU^*)^2 = UPQPU^* = UPU^* = (K_T)^*K_T.$$

Therefore, K_T is a partial isometry. On the other hand, we have

$$K_T(K_T)^* = U^*QU, \quad \big(K_T(K_T)^*\big)^2 = U^*QPQU = U^*PU.$$

Since K_T is a partial isometry, $K_T(K_T)^*$ is also a projection, and hence $U^*QU = U^*PU$. Since $PU = UQ$, it follows that

$$K_T(K_T)^* = Q.$$

This shows that the final domain of K_T is equal to $\overline{\mathcal{R}(T^*)}$. ∎

COROLLARY 3.8. *Let T be a q-normal operator in $\mathcal{H}$. Then the following statements hold:*

(i) *K_T is a partial isometry such that the initial domain is $\overline{\mathcal{R}(T)}$ and the final domain is $\overline{\mathcal{R}(T^*)}(=\overline{\mathcal{R}(T)})$.*

(ii) $(K_T)^* = K_{T^*}$.

Proof. Statement (i) is a direct consequence of Proposition 3.4 and Proposition 3.7. Since T^* is q^{-1}-normal and $T^* = U^*|T^*|$ as the polar decomposition of T^*, it follows from Theorem 3.5

$$K_{T^*} = U^2 = (K_T)^*. \quad ∎$$

The following lemma follows immediately from Definition 3.3.

LEMMA 3.9. *Let T be a q-hyponormal operator. Then, the following relations are valid:*

$$K_{-T} = K_T, \quad K_{\widetilde{T}} = K_T \quad \textit{and} \quad K_{\mathrm{i}T} = -K_T.$$

The following proposition is a q-deformed version of Stampfli's theorem.

PROPOSITION 3.10. *Let T be a q-hyponormal operator in $\mathcal{H}$. If T has dense range, then T is injective and the inverse T^{-1} is a q-hyponormal operator satisfying*

$$K_{T^{-1}} = (K_T)^*.$$

Proof. In view of Corollary 3.2, $\mathcal{R}(T) \subseteq (\ker T)^\perp$, so that T is injective. Take $\eta \in \mathcal{R}(T)$. Since $T \subset \sqrt{q}\,T^*(K_T)^*$, we have $\eta = \sqrt{q}\,T^*(K_T)^*T^{-1}\eta$. Hence, $\eta \in \mathcal{R}(T^*)$ and $(T^*)^{-1}\eta = \sqrt{q}\,(K_T)^*T^{-1}\eta$. This means that

$$(T^{-1})^* = (T^*)^{-1} \supset \sqrt{q}\,(K_T)^*T^{-1}.$$

Since $(K_T)^*$ is a contraction and by Lemma 3.1, T^{-1} is q-hyponormal. Since $\mathcal{R}(T^{-1})$ is dense in $\mathcal{H}$ and by the uniqueness of $K_{T^{-1}}$, we have $K_{T^{-1}} = (K_T)^*$. ∎

4. WEIGHTED SHIFTS

In this section we shall treat (possibly unbounded) weighted shifts as typical examples of q-deformed operators and determine their spectra. Let S_{u} be a closed, densely defined operator in a separable Hilbert space $\mathcal{H}$. If there are an orthonormal basis $\{e_n\}$, $n \geqslant 0$, and a sequence $\{w_n\}$, $w_n \neq 0$, $n \geqslant 0$, of complex numbers such that

$$\mathcal{D}(S_{\mathrm{u}}) = \Big\{ \sum_{n=0}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=0}^{\infty} |\alpha_n|^2 |w_n|^2 < \infty \Big\}$$

and

$$S_{\mathrm{u}} e_n = w_n e_{n+1}$$

for all $n \geqslant 0$, then S_{u} is called a *unilateral* (*injective*) *weighted shift* with weights $\{w_n\}$ (with respect to $\{e_n\}$). Let $\mathcal{D}_{\{e_n\}}$ denote the linear span of a basis $\{e_n\}$. One can easily check that $\mathcal{D} \equiv \mathcal{D}_{\{e_n\}}$ is a core for S_{u}, that is, $(\widetilde{S_{\mathrm{u}}|\mathcal{D}}) = S_{\mathrm{u}}$. The adjoint S_{u}^* is given by

$$\mathcal{D}(S_{\mathrm{u}}^*) = \Big\{ \sum_{n=0}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=0}^{\infty} |\alpha_n|^2 |w_{n-1}|^2 < \infty \Big\},$$

and

$$S_{\mathrm{u}}^* \Big(\sum_{n=0}^{\infty} \alpha_n e_n \Big) = \sum_{n=0}^{\infty} \alpha_n \overline{w}_{n-1} e_{n-1},$$

where $w_{-1} = 0$ and $e_{-1} = 0$. Analogously, a bilateral (injective) weighted shift S_{b} in $\mathcal{H}$ is defined in an obvious way. For the unilateral weighted shift S_{u}, if $S_{\mathrm{u}} = U|S_{\mathrm{u}}|$ is the polar decomposition, then

$$U e_n = \frac{w_n}{|w_n|} e_{n+1} \quad \text{and} \quad |S_{\mathrm{u}}| e_n = |w_n| e_n.$$

Information about bounded weighted shifts can be founded in [10] and [34].

Proposition 4.1. *The following statements hold:*

(i) *A unilateral weighted shift S_{u} in $\mathcal{H}$ with weights $\{w_n\}$ is q-quasinormal if and only if*

$$|w_n| = \Big(\frac{1}{\sqrt{q}} \Big)^n |w_0| \tag{4.1}$$

for all $n \geqslant 0$. In particular, a unilateral weighted shift cannot be q-normal.

(ii) *A bilateral weighted shift S_{b} in $\mathcal{H}$ with weights $\{w_n\}$ is q-normal if and only if the equation* (4.1) *is valid for all $n \in \mathbb{Z}$.*

(iii) *A weighted shift S_{u} (respectively S_{b}) is q-hyponormal if and only if*

$$|w_{n+1}| \geqslant \frac{1}{\sqrt{q}} |w_n| \tag{4.2}$$

for all $n \geqslant 0$ (respectively $n \in \mathbb{Z}$).

Proof. Let S_{u} be a unilateral shift with the weights $\{w_n\}$ and let $S_{\mathrm{u}} = U|S_{\mathrm{u}}|$ be the polar decomposition of S_{u}. If S_{u} is q-quasinormal, then $U|S_{\mathrm{u}}|e_n = \sqrt{q}\,|S_{\mathrm{u}}|Ue_n$, and so

$$\sqrt{q}\,|w_{n+1}| = |w_n|$$

for all $n \geqslant 0$. Hence $|w_n| = q^{-\frac{n}{2}}|w_0|$, $n \geqslant 0$. Conversely, suppose that the equality (4.1) is valid for $n \geqslant 0$. It is clear $U|S_{\mathrm{u}}|e_n = \sqrt{q}\,|S_{\mathrm{u}}|Ue_n$ for each n, and so $U|S_{\mathrm{u}}| = \sqrt{q}\,|S_{\mathrm{u}}|U$ on $\mathcal{D}_{\{e_n\}}$. For each $\eta \in \mathcal{D}(|S_{\mathrm{u}}|) = \mathcal{D}(S_{\mathrm{u}})$, there is a sequence $\{\eta_n\}$ in $\mathcal{D}_{\{e_n\}}$ such that $\eta_n \to \eta$ and $S_{\mathrm{u}}\eta_n \to S_{\mathrm{u}}\eta$, as $n \to \infty$. Since $S_{\mathrm{u}}\eta_n = U|S_{\mathrm{u}}|\eta_n = \sqrt{q}\,|S_{\mathrm{u}}|U\eta_n$, we have $\sqrt{q}\,|S_{\mathrm{u}}|U\eta_n \to S_{\mathrm{u}}\eta$ and $U\eta_n \to U\eta$, as $n \to \infty$. Since $|S_{\mathrm{u}}|$ is closed, $U\eta \in \mathcal{D}(|S_{\mathrm{u}}|)$ and $\sqrt{q}\,|S_{\mathrm{u}}|U\eta = S_{\mathrm{u}}\eta$. Hence $U|S_{\mathrm{u}}| \subset \sqrt{q}\,|S_{\mathrm{u}}|U$. If S_{u} is q-normal, then we have $qS_{\mathrm{u}}^*S_{\mathrm{u}}e_0 = S_{\mathrm{u}}S_{\mathrm{u}}^*e_0 = 0$. Hence $w_n = w_0 = 0$ for all n. Thus statement (i) holds. Next, let S_{b} be a bilateral weighted shift with the weights $\{w_n\}$. Then

$$S_{\mathrm{b}}S_{\mathrm{b}}^*e_n = |w_{n-1}|^2e_n \quad \text{and} \quad S_{\mathrm{b}}^*S_{\mathrm{b}}e_n = |w_n|^2e_n$$

for all $n \in \mathbb{Z}$. Hence, if S_{b} is q-normal, we have $|w_n| = q^{-\frac{n}{2}}|w_0|$, for all $n \in \mathbb{Z}$. Conversely, assume that the equality (4.1) is valid for all $n \in \mathbb{Z}$. Then it is easily seen that

$$\mathcal{D}(S_{\mathrm{b}}) = \mathcal{D}(S_{\mathrm{b}}^*).$$

By our assumption, $S_{\mathrm{b}}S_{\mathrm{b}}^*e_n = qS_{\mathrm{b}}^*S_{\mathrm{b}}e_n$ for all $n \in \mathbb{Z}$. Hence,

$$\|S_{\mathrm{b}}^*\xi\| = \sqrt{q}\ \|S_{\mathrm{b}}\xi\|$$

for all $\xi \in \mathcal{D}_{\{e_n\}}$. For each $\eta \in \mathcal{D}(S_{\mathrm{b}})$, there is a sequence $\{\xi_n\}$ in $\mathcal{D}_{\{e_n\}}$ such that $\xi_n \to \eta$ and $S_{\mathrm{b}}\xi_n \to S_{\mathrm{b}}\eta$, as $n \to \infty$. Since S_{b}^* is closed, it follows that the sequence$\{S_{\mathrm{b}}^*\xi_n\}$ converges $S_{\mathrm{b}}^*\eta$. Hence, $\|S_{\mathrm{b}}^*\eta\| = \sqrt{q}\ \|S_{\mathrm{b}}\eta\|$.

Finally, we have to prove statement (iii). It is verified by the same way as in statement (ii). ∎

Corollary 4.2. *Let T be a (unilateral or bilateral) weighted shift with weights $\{w_n\}$, with respect to a basis $\{e_n\}$. If T satisfies*

$$T^*T - qTT^* = 1, \quad q > 0, q \neq 1.$$

on $\mathcal{D}_{\{e_n\}}$, then T is q^{-1}-hyponormal with $\mathcal{D}(T^*) = \mathcal{D}(T)$.

Proof. The relation implies that

$$|w_{n+1}|^2 - q|w_n|^2 = 1.$$

It follows that $\mathcal{D}(T^*) = \mathcal{D}(T)$. Clearly, $|w_{n+1}| > \sqrt{q}\,|w_n|$ for all n. Hence, by (4.2) in Proposition 4.1, T is q^{-1}-hyponormal. ∎

In what follows, we will often use the fact [10], Proposition 6.2: A weighted shift with weights $\{w_n\}$ is unitarily equivalent to the weighted shift with weights $\{|w_n|\}$.

We denote the spectrum of T by $\sigma(T)$. The point spectrum, the continuous spectrum and the residual spectrum are denoted by $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{c}}(T)$ and $\sigma_{\mathrm{r}}(T)$, respectively.

LEMMA 4.3. *If S_{u} is q-quasinormal with $0<q<1$, then*

$$\sigma_{\mathrm{p}}(S_{\mathrm{u}}^*)=\mathbb{C},$$

and the multiplicity of any eigenvalue of S_{u}^* is equal to 1.

Proof. Upon applying a unitary transformation, we can assume without loss of generality that all weights w_n are positive. It is clear that 0 is an eigenvalue of S_{u}^* with eigenvector e_0 and its multiplicity is 1. For $\lambda\in\mathbb{C}\setminus\{0\}$, take some x_0 and put

$$x_n\equiv\frac{\lambda^n}{w_0w_1\cdots w_{n-1}}x_0,\quad n\geqslant 1,\qquad\text{and}\qquad x_\lambda\equiv\sum_{n=0}^{\infty}x_ne_n.$$

Then we show that $x_\lambda\in\mathcal{D}(S_{\mathrm{u}}^*)\,(=\mathcal{D}(S_{\mathrm{u}}))$ and $S_{\mathrm{u}}^*x_\lambda=\lambda x_\lambda$. Since $w_n=q^{-\frac{n}{2}}w_0$, it follows that

$$\sum_{n=0}^{\infty}(w_n)^2|x_n|^2=|x_0w_0|^2\sum_{n=0}^{\infty}\Big|\frac{\lambda}{w_0}\Big|^{2n}q^{\frac{n(n-3)}{2}}.$$

Now we consider the series $\sum\limits_{n=0}^{\infty}a^nb^{n(n-3)}$ for constants $a>0$, and $0<b<1$. Then there is a constant δ, $0<\delta<1$, such that $a^{\frac{1}{n-3}}b<\delta$, $n\geqslant N$ for some large $N\in\mathbb{N}$. Since $a^nb^{n(n-3)}<\delta^{n^2-3n}<\delta^n$, $n\geqslant N$, the series $\sum\limits_{n=0}^{\infty}a^nb^{n(n-3)}$ is convergent. Thus, since $0<q<1$,

$$\sum_{n=0}^{\infty}|x_n|^2\leqslant\frac{1}{(w_0)^2}\sum_{n=0}^{\infty}(w_n)^2|x_n|^2<+\infty.$$

Hence, $x_\lambda\in\mathcal{D}(S_{\mathrm{u}}^*)$. Obviously, $S_{\mathrm{u}}^*x_\lambda=\lambda x_\lambda$, by the definition of x_λ. Therefore λ is an eigenvalue of S_{u}^* and we also showed that its multiplicity is equal to 1. ∎

LEMMA 4.4. *If S_{u} is q-quasinormal with $q>1$, then S_{u} is bounded and quasinilpotent.*

Proof. It is easily shown that S_{u} is bounded with $\|S_{\mathrm{u}}\|=|w_0|$. For $m\in\mathbb{N}$,

$$\Big\|(S_{\mathrm{u}})^m\Big(\sum_{n=0}^{\infty}\alpha_ne_n\Big)\Big\|^2=|w_0|^{2m}q^{-\frac{m^2-m}{2}}\sum_{n=0}^{\infty}|\alpha_n|^2q^{-mn}.$$

Since $q>1$, it follows that

$$\|(S_{\mathrm{u}})^m\|^{\frac{1}{m}}\leqslant|w_0|q^{-\frac{m-1}{4}}\to 0$$

as $m\to\infty$. Thus, S_{u} is quasinilpotent. ∎

Under the same assumption as in Lemma 4.3, $\sigma_{\mathrm{r}}(S_{\mathrm{u}})=\mathbb{C}$. Indeed, for each $\lambda\in\mathbb{C}$, it is clear that $S_{\mathrm{u}}+\lambda$ is one to one. By Lemma 4.3, $\overline{\mathcal{R}(S_{\mathrm{u}}+\lambda)}\neq\mathcal{H}$. On the other hand, if $q>1$ then the spectral radius of a q-quasinormal S_{u} is equal to 0 by Lemma 4.4, and hence $\sigma(S_{\mathrm{u}})=\{0\}$. Moreover, it is easily seen that

$0 \notin \sigma_{\mathrm{p}}(S_{\mathrm{u}})$ and $0 \notin \sigma_{\mathrm{c}}(S_{\mathrm{u}})$. Thus we obtain the following list for the spectrum of a q-quasinormal S_{u}:

	σ_{p}	σ_{c}	σ_{r}	σ
$S_{\mathrm{u}}(0<q<1)$	$\emptyset$	$\emptyset$	$\mathbb{C}$	$\mathbb{C}$
$S_{\mathrm{u}}(q>1)$	$\emptyset$	$\emptyset$	$\{0\}$	$\{0\}$

We now turn to the study of spectrum of a q-normal, bilateral weighted shift S_{b}. We first notice that the continuous spectrum of the q-normal S_{b} contains 0. In fact, S_{b} and S_{b}^* are injective and $0 \in \sigma(S_{\mathrm{b}})$ by Theorem 7.1, as we will see in Section 7. Therefore, $0 \in \sigma_{\mathrm{c}}(S_{\mathrm{b}})$.

LEMMA 4.5. *Suppose that S_{b} is q-normal with $q > 1$. Then we have*

$$\sigma_{\mathrm{p}}(S_{\mathrm{b}}) = \mathbb{C} \setminus \{0\},$$

and the multiplicity of any eigenvalue is equal to 1.

Proof. Without loss of generality, we can assume that all weights w_n are positive.

For $\lambda \in \mathbb{C} \setminus \{0\}$, take some x_0 and put

$$x_n \equiv \Big(\frac{w_0 w_1 \cdots w_{n-1}}{\lambda^n}\Big) x_0 \qquad \text{for } n \geqslant 1,$$

$$x_n \equiv \Big(\frac{1}{\lambda^n w_{-1} w_{-2} \cdots w_n}\Big) x_0 \qquad \text{for } n \leqslant -1,$$

and

$$x_\lambda \equiv \sum_{n=-\infty}^{\infty} x_n e_n.$$

Then we show that $x_\lambda \in \mathcal{D}(S_{\mathrm{b}})$ and $S_{\mathrm{b}} x_\lambda = \lambda x_\lambda$. Since $w_n = q^{-\frac{n}{2}} w_0$, we have

$$|x_n| = \Big(\frac{w_0}{|\lambda|}\Big)^n q^{-\frac{n(n-1)}{4}} |x_0|, \quad n \geqslant 1$$

and

$$w_{-n}|x_{-n}| = \Big(\frac{|\lambda|}{w_0}\Big)^n q^{-\frac{n(n-1)}{4}} w_0 |x_0|, \quad n \geqslant 1.$$

For $a > 0$ and $0 < b < 1$, the series $\sum\limits_{n=0}^{\infty} a^n b^{n^2-n}$ converges by the same reason as in the proof of Lemma 4.3. Hence, $\sum\limits_{n=0}^{\infty} |x_n|^2 < +\infty$ and $\sum\limits_{n=1}^{\infty} (w_{-n})^2 |x_{-n}|^2 < +\infty$. Moreover, since $q > 1$,

$$\sum_{n=0}^{\infty} (w_n)^2 |x_n|^2 \leqslant (w_0)^2 \sum_{n=0}^{\infty} |x_n|^2 < +\infty$$

and

$$\sum_{n=1}^{\infty}|x_{-n}|^2 \leqslant \frac{1}{(w_0)^2}\sum_{n=1}^{\infty}(w_{-n})^2|x_{-n}|^2 < +\infty.$$

Thus, $\sum_{n=-\infty}^{\infty}|x_n|^2 < +\infty$ and $\sum_{n=-\infty}^{\infty}(w_n)^2|x_n|^2 < +\infty$. Therefore, $x_\lambda \in \mathcal{D}(S_{\mathrm{b}})$. By the definition of x_λ, it is clear that $S_{\mathrm{b}}x_\lambda = \lambda x_\lambda$ and we have also showed that the multiplicity of λ is equal to 1. ∎

Lemma 4.6. *If S_{b} is q-normal with $0 < q < 1$, then*

$$\sigma_{\mathrm{p}}(S_{\mathrm{b}}) = \emptyset, \quad \textit{and} \quad \sigma_{\mathrm{p}}(S_{\mathrm{b}}^*) = \mathbb{C}\setminus\{0\}.$$

The multiplicity of any eigenvalue of S_{b}^* is equal to 1.

Proof. We can assume without loss of generality that all weights w_n are positive. Suppose that $\lambda \in \mathbb{C}\setminus\{0\}$ is in $\sigma_{\mathrm{p}}(S_{\mathrm{b}})$, Then there is a non-zero vector $y_\lambda \in \mathcal{D}(S_{\mathrm{b}})$ such that $S_{\mathrm{b}}y_\lambda = \lambda y_\lambda$. If $y_\lambda = \sum_{n=-\infty}^{\infty} y_n e_n$, then we have

$$y_n = \Big(\frac{w_0w_1\cdots w_{n-1}}{\lambda^n}\Big)y_0 \qquad \text{for } n \geqslant 1,$$

$$y_n = \Big(\frac{1}{\lambda^n w_{-1}w_{-2}\cdots w_n}\Big)y_0 \qquad \text{for } n \leqslant -1.$$

On the other hand, since $0 < q < 1$, it follows that

$$\sum_{n=0}^{\infty}|y_n|^2 = +\infty \quad \text{or} \quad \sum_{n=1}^{\infty}(w_{-n})^2|y_{-n}|^2 = +\infty,$$

depending on the value $(w_0)^{-1}|\lambda|$. This is a contradiction to $y_\lambda \in \mathcal{D}(S_{\mathrm{b}})$. Since $0 \in \sigma_{\mathrm{c}}(S_{\mathrm{b}})$, we have $\sigma_{\mathrm{p}}(S_{\mathrm{b}}) = \emptyset$. We next show that $\sigma_{\mathrm{p}}(S_{\mathrm{b}}^*) = \mathbb{C}\setminus\{0\}$. Fix $\lambda \in \mathbb{C}\setminus\{0\}$, and take some x_0. Define

$$x_n \equiv \Big(\frac{\lambda^n}{w_0w_1\cdots w_{n-1}}\Big)x_0 \qquad \text{for } n \geqslant 1,$$

$$x_n \equiv w_{-1}w_{-2}\cdots w_n\lambda^n x_0 \qquad \text{for } n \leqslant -1.$$

In view of the proof of Lemma 4.3, we have

$$\sum_{n=0}^{\infty}|x_n|^2 < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty}(w_n)^2|x_n|^2 < +\infty.$$

Since

$$|x_{-n}| = \Big(\frac{w_0}{|\lambda|}\Big)^n\Big(\frac{1}{q}\Big)^{-\frac{n(n+1)}{4}}|x_0| \quad \text{and} \quad w_{-n} < w_0,\ n \geqslant 1,$$

it follows from the proof of Lemma 4.5 that

$$\sum_{n=1}^{\infty}|x_{-n}|^2 < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty}(w_{-n})^2|x_{-n}|^2 < +\infty.$$

Define

$$x_\lambda \equiv \sum_{n=-\infty}^{\infty} x_n e_n.$$

Then $x_\lambda \in \mathcal{D}(S_{\mathrm{b}}) = \mathcal{D}(S_{\mathrm{b}}^*)$ and, by the construction of $\{x_n\}$, $S_{\mathrm{b}}^*x_\lambda = \lambda x_\lambda$. We also proved that the multiplicity of λ is equal to 1. ∎

We summarize the above in a list:

	σ_{p}	σ_{c}	σ_{r}	σ
$S_{\mathrm{b}}(0 < q < 1)$	$\emptyset$	$\{0\}$	$\mathbb{C} \setminus \{0\}$	$\mathbb{C}$
$S_{\mathrm{b}}(q > 1)$	$\mathbb{C} \setminus \{0\}$	$\{0\}$	$\emptyset$	$\mathbb{C}$

5. q-NORMAL EXTENSIONS AND REDUCING SUBSPACES

Let us begin this section by recalling the notion of a reducing subspace for an unbounded operator. For a densely defined operator T in $\mathcal{H}$ and a closed subspace $\mathcal{M}$ of $\mathcal{H}$, we say that $\mathcal{M}$ *reduces* T if the following two conditions are satisfied:

(i) $$P_{\mathcal{M}}\mathcal{D}(T) \subseteq \mathcal{D}(T),$$

where $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$;

(ii) $$T(\mathcal{M} \cap \mathcal{D}(T)) \subseteq \mathcal{M} \quad \text{and} \quad T(\mathcal{M}^{\perp} \cap \mathcal{D}(T)) \subseteq \mathcal{M}^{\perp}.$$

For a reducing subspace $\mathcal{M}$ of T, define the parts T_1 and T_2 of T on Hilbert spaces $P_{\mathcal{M}}\mathcal{H}$ and $(1 - P_{\mathcal{M}})\mathcal{H}$ by

$$T_1 \equiv T|\mathcal{M} \quad \text{and} \quad T_2 \equiv T|\mathcal{M}^{\perp}$$

respectively, and we write its decomposition as

$$T = T_1 \oplus T_2.$$

It then follows that $\mathcal{M}$ reduces also T^* and the following relations hold:

$$(T_1)^* = (T^*)_1 \quad \text{and} \quad (T_2)^* = (T^*)_2.$$

Moreover, if T is closed, then its part T_1 and T_2 are closed. For the above facts we refer to e.g., [5] and [26].

We next recall the strong commutant of an operator. Let T be a closable, densely defined operator in $\mathcal{H}$. Define

$$\mathcal{C}^{\mathrm{s}}(T) \equiv \{X \in \mathcal{B}(\mathcal{H}) : XT \subseteq TX\}.$$

Here, $\mathcal{B}(\mathcal{H})$ means the algebra of all bounded operators on $\mathcal{H}$. Then $\mathcal{C}^{\mathrm{s}}(T)$ is a subalgebra of $\mathcal{B}(\mathcal{H})$. If T is a self-adjoint operator with spectral decomposition $T = \int \lambda \, \mathrm{d}E_T(\lambda)$, then $\mathcal{C}^{\mathrm{s}}(T)$ is a von Neumann algebra and moreover, the commutant of the von Neumann algebra $\mathcal{C}^{\mathrm{s}}(T)$ is generated by the spectral projections $\{E_T(\cdot)\}$ (e.g., [27]). $\mathcal{C}^{\mathrm{s}}(T)$ is called the *strong commutant* of T. Let $\mathcal{M}$ be a closed subspace in $\mathcal{H}$. It is well-known that $\mathcal{M}$ reduces T if and only if

$$P_{\mathcal{M}} \in \mathcal{C}^{\mathrm{s}}(T). \tag{5.1}$$

LEMMA 5.1. *Let T be a closed q-hyponormal operator. Then* $\ker T$ *reduces* T.

Proof. In view of Corollary 3.2, $\ker T \subseteq (\mathcal{R}(T))^{\perp}$. It follows that

$$P \in \mathcal{C}^{\mathrm{s}}(T),$$

where P denotes the orthogonal projection onto $\ker T$. Hence, condition (5.1) is satisfied. ∎

One can easily check that, for a q-hyponormal operator, a q-formally normal operator and a q-normal operator, their parts (that is, their restrictions to a reducing subspace) are q-hyponormal, q-formally normal and q-normal, respectively. Before indicating the q-quasinormality of parts of q-quasinormal operators, we present the following lemma. It seems to be of interest in itself from the view point of operator theory.

Lemma 5.2. *Let T be a closed, densely defined operator with polar decomposition $T = U|T|$. Let $\mathcal{M}$ be a reducing subspace of T. Then $\mathcal{M}$ reduces U and $|T|$.*

Proof. By (5.1), $P_{\mathcal{M}}T \subset TP_{\mathcal{M}}$. Hence, $P_{\mathcal{M}}T^*T \subset T^*P_{\mathcal{M}}T \subset T^*TP_{\mathcal{M}}$. On the other hand, it follows that

$$\mathcal{C}^{\mathrm{s}}(|T|) \supseteq \mathcal{C}^{\mathrm{s}}(T^*T).$$

Hence, $P_{\mathcal{M}} \in \mathcal{C}^{\mathrm{s}}(|T|)$, which implies that $\mathcal{M}$ reduces $|T|$. Since $P_{\mathcal{M}} \in \mathcal{C}^{\mathrm{s}}(T) \cap \mathcal{C}^{\mathrm{s}}(|T|)$, for all $\xi \in \mathcal{D}(T)$ we have

$$P_{\mathcal{M}}U(|T|\xi) = P_{\mathcal{M}}T\xi = U|T|P_{\mathcal{M}}\xi = UP_{\mathcal{M}}|T|\xi.$$

Hence, $UP_{\mathcal{M}} = P_{\mathcal{M}}U$ on $\overline{\mathcal{R}(|T|)}$. Take $\xi \in (\mathcal{R}(|T|))^{\perp} = \ker|T|$. It is clear that $P_{\mathcal{M}}U\xi = 0$. Since $P_{\mathcal{M}} \in \mathcal{C}^{\mathrm{s}}(|T|)$, we have $|T|P_{\mathcal{M}}\xi = 0$, so that

$$P_{\mathcal{M}}\xi \in \ker T = \ker U.$$

Therefore, $UP_{\mathcal{M}}\xi = P_{\mathcal{M}}U\xi = 0$. Hence, $UP_{\mathcal{M}} = P_{\mathcal{M}}U$ and so $\mathcal{M}$ reduces U. ∎

Corollary 5.3. *Let T be a q-quasinormal operator in $\mathcal{H}$. Suppose that $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ which reduces T. If T_1 and T_2 are the parts of T on $\mathcal{M}$ and $\mathcal{M}^{\perp}$, respectively, then T_1 and T_2 are q-quasinormal.*

Proof. It suffices to show that T_1 is q-quasinormal. Let $T = U|T|$ be the polar decomposition of T. By the above lemma, $\mathcal{M}$ reduces $|T|$ and U. Let $|T|_1$ and U_1 be the parts of $|T|$ and U on $\mathcal{M}$, respectively. If $T_1 = V_1|T_1|$ is the polar decomposition of T_1 in $\mathcal{M}$, it then follows from the uniqueness of the polar decomposition that

$$|T_1| = |T|_1 \quad \text{and} \quad V_1 = U_1.$$

Therefore, we have

$$V_1|T_1| = U_1|T|_1 = (U|T|)|\mathcal{M} = \sqrt{q}\,(|T|U)|\mathcal{M} = \sqrt{q}\,|T|_1U_1 = \sqrt{q}\,|T_1|V_1.$$

Thus T_1 is q-quasinormal. ∎

PROPOSITION 5.4. *If a closed subspace $\mathcal{M}$ reduces a q-hyponormal operator T, then $\mathcal{M}$ reduces K_T and*

$$(K_T)|\mathcal{M} = K_{(T|\mathcal{M})}.$$

Proof. By (5.1), $P_{\mathcal{M}} \in \mathcal{C}^{\mathrm{s}}(T) \cap \mathcal{C}^{\mathrm{s}}(T^*)$ and by the definition of K_T, we get

$$P_{\mathcal{M}} K_T T\xi = \frac{1}{\sqrt{q}} P_{\mathcal{M}} T^*\xi = \frac{1}{\sqrt{q}} T^* P_{\mathcal{M}}\xi = K_T T P_{\mathcal{M}}\xi = K_T P_{\mathcal{M}} T\xi$$

for all $\xi \in \mathcal{D}(T)$. Thus, $K_T P_{\mathcal{M}} = P_{\mathcal{M}} K_T$ on $\overline{\mathcal{R}(T)}$. Since $P_{\mathcal{M}} \in \mathcal{C}^{\mathrm{s}}(T^*)$, we obtain that $P_{\mathcal{M}}(\ker T^*) \subseteq \ker T^*$. On the other hand, K_T satisfies (3.1). Hence, $K_T P_{\mathcal{M}} = P_{\mathcal{M}} K_T = 0$ on $(\mathcal{R}(T))^{\perp}$. Therefore, $K_T P_{\mathcal{M}} = P_{\mathcal{M}} K_T$, and hence $\mathcal{M}$ reduces K_T. Since T is q-hyponormal, we have

$$(T|\mathcal{M})^* = (T^*)|\mathcal{M} \supset \sqrt{q}\,(K_T T)|\mathcal{M} = \sqrt{q}\,\big[(K_T)|\mathcal{M}\big]\big[T|\mathcal{M}\big].$$

For each $\xi \in \ker\big((T|\mathcal{M})^*\big)$, $T^* P_{\mathcal{M}}\xi = ((T^*)|\mathcal{M})\xi = (T|\mathcal{M})^*\xi = 0$. Hence, by (3.1),

$$((K_T)|\mathcal{M})\xi = 0$$

Thus,

$$\ker\big((K_T)|\mathcal{M}\big) \supseteq \ker\big((T|\mathcal{M})^*\big).$$

Hence, by the uniqueness of $K_{(T|\mathcal{M})}$, $(K_T)|\mathcal{M} = K_{(T|\mathcal{M})}$. ∎

PROPOSITION 5.5. *A q-quasinormal operator T in a Hilbert space $\mathcal{H}$ is q-normal if and only if*

$$\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^*)}, \quad \textit{namely, } \ker T = \ker T^*.$$

Especially, if T has polar decomposition $T = U|T|$ with unitary U, then T is q-normal.

Proof. Let $T = U|T|$ be the polar decomposition of T. By Lemma 5.1, $\ker T$ is a reducing subspace of T. Let T_2 be the part of T on $(\ker T)^{\perp}$. Then T_2 is q-quasinormal by virtue of Corollary 5.3, and $T = 0 \oplus T_2$ with respect to $\mathcal{H} = \ker T \oplus (\ker T)^{\perp}$. Suppose that $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^*)}$. Then, we clearly have

$$\overline{\mathcal{R}(T_2)} = \overline{\mathcal{R}({T_2}^*)} = (\ker T)^{\perp}.$$

Thus we may assume that $T = U|T|$ is injective and U is unitary. Since $U|T| = \sqrt{q}\,|T|U$, we have $|T|U^* = \sqrt{q}\,U^*|T|$. Hence, using general property of the polar decomposition, we get

$$|T^*| = U|T|U^* = \sqrt{q}\,UU^*|T| = \sqrt{q}\,|T|.$$

Therefore, T is q-normal. Since every q-normal operator satisfies the equality $\mathcal{R}(T) = \mathcal{R}(T^*)$ by Corollary 3.2, the assertion follows. ∎

Now we present the converse to statement (i) of Corollary 3.8 in terms of K_T.

Proposition 5.6. *Let T be a q-quasinormal operator in a Hilbert space $\mathcal{H}$. If the initial domain of K_T is $\overline{\mathcal{R}(T)}$, then T is q-normal.*

Proof. Let $T = U|T|$ be the polar decomposition of T and put $P \equiv UU^*$and $Q \equiv U^*U$. It follows from our assumption and Theorem 3.5 that

$$P = UPU^*.$$

By $P \leqslant Q$, we have $U^*PU = U^*(UPU^*)U = P$. On the other hand, $U^*PU = (U^*U)^2 = Q$. Therefore, $P = Q$ and, by Proposition 5.5, T is q-normal. ∎

Proposition 5.7. *Let T be a q-quasinormal operator in a Hilbert space $\mathcal{H}$. If K_T is normal, then T is q-normal.*

Proof. Using the same notation as in the above proof, we have

$$(K_T)^*K_T = K_T(K_T)^* \leqslant P.$$

Thus, $\overline{\mathcal{R}(T^*)} \subseteq \overline{\mathcal{R}(T)}$. Since $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$ and by Proposition 5.5, T is q-normal. ∎

The following theorem gives some information about the structure of q-quasinormal operators, which is similar to that in the case of a bounded quasinormal operator ([7]; see also [10], Chapter II, Theorem 3.2). Before describing the theorem, we present some terminology: Let $\mathcal{L}$ be a Hilbert space and put

$$\ell^2_+(\mathcal{L}) \equiv \bigoplus_{n=0}^{\infty} \mathcal{L}_n, \quad \mathcal{L}_n \equiv \mathcal{L},$$

that is, $\ell^2_+(\mathcal{L}) = \Big\{(x_n)_{n=0}^{\infty} : x_n \in \mathcal{L}, \sum\limits_{n=0}^{\infty} \|x_n\|^2 < +\infty\Big\}$. We will introduce a kind of weighted shift in $\ell^2_+(\mathcal{L})$. Let $\{w_n\}$ be a sequence of complex numbers and define an operator S on $\ell^2_+(\mathcal{L})$ by

$$\mathcal{D}(S) \equiv \big\{(x_n)_{n=0}^{\infty} \in \ell^2_+(\mathcal{L}) : x_n = 0 \text{ except for a finite number } n\big\},$$

$$S((x_n)_{n=0}^{\infty}) \equiv (y_n)_{n=0}^{\infty}, \quad y_n \equiv w_{n-1}x_{n-1}$$

where $n \geqslant 0, x_{-1} = w_{-1} = 0$. Clearly, S is closable. Its closure is called a *unilateral weighted shift* with weights $\{w_n\}$ defined by $\mathcal{L}$, and is denoted by $S_{\mathcal{L},w_n}$. For a densely defined operator T, define $T^{(\infty)}$ by

$$\mathcal{D}(T^{(\infty)}) \equiv \Big\{(x_n)_{n=0}^{\infty} \in \ell^2_+(\mathcal{L}) : \sum_{n=0}^{\infty} \|Tx_n\|^2 < +\infty\Big\},$$

$$T^{(\infty)}((x_n)_{n=0}^{\infty}) \equiv (y_n)_{n=0}^{\infty}, \quad y_n \equiv Tx_n.$$

The outline of the proof in the following theorem is essentially based on that of the classical result of [7] (see also the corresponding part in [10]). The proof of this theorem, however, contains some more delicate aspects which come from the unboundedness of the operators. It also needs more technical facts related to q-deformed operators.

Theorem 5.8. *Every q-quasinormal operator is unitarily equivalent to the direct sum*

$$\textit{some } q\textit{-normal operator} \oplus S_q.$$

Here, S_q is a q-quasinormal operator in $\ell^2_+(\mathcal{L})$ for some Hilbert space $\mathcal{L}$ such that

$$S_q = S_{\mathcal{L},w_n} \cdot K^{(\infty)}, \quad w_n = \Big(\frac{1}{\sqrt{q}}\Big)^n,$$

where K is an injective positive, selfadjoint operator in $\mathcal{L}$. Thus every q-quasinormal operator has a q-normal extension in a larger Hilbert space.

Proof. Let T be a q-quasinormal operator in a Hilbert space $\mathcal{H}$. Considering the restriction of T to the reducing subspace $(\ker T)^{\perp}$ (Lemma 5.1), we assume without loss of generality that T is injective . Let $T = U|T|$ be the polar decomposition. Then U is an isometry. We consider the Wold decomposition of U (e.g., [3], [10]). Define

$$\mathcal{H}_1 \equiv \bigcap_{n=0}^{\infty} U^n\mathcal{H}, \quad \text{and} \quad \mathcal{H}_2 \equiv {\mathcal{H}_1}^{\perp}.$$

Then $\mathcal{H}_1$ reduces U, the part $U|\mathcal{H}_1$ is unitary on $\mathcal{H}_1$ and $U|\mathcal{H}_2$ is an isometry such that $\bigcap_{n=0}^{\infty}(U|\mathcal{H}_2)^n\mathcal{H}_2 = \{0\}$. Let Q be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_1$. We will show that

$$Q \in \mathcal{C}^{\mathrm{s}}(|T|) \cap \mathcal{C}^{\mathrm{s}}(T).$$

Since $|T|$ is selfadjoint, it follows that $\mathcal{C}^{\mathrm{s}}(|T|)$ coincides with the commutant of the von Neumann algebra generated by $\{\mathrm{e}^{\mathrm{i}t|T|} : t \in \mathbb{R}\}$. In view of Theorem 2.5,

$$\mathrm{e}^{\mathrm{i}t|T|}U^n\eta = U^n\big(\mathrm{e}^{\mathrm{i}(\sqrt{q}\,)^{-n}t|T|}\big)\eta,$$

for all $\eta \in \mathcal{H}$ and $t \in \mathbb{R}$. Hence, $\mathrm{e}^{\mathrm{i}t|T|}\mathcal{H}_1 \subseteq \mathcal{H}_1$, so that $Q\mathrm{e}^{\mathrm{i}t|T|}Q = \mathrm{e}^{\mathrm{i}t|T|}Q$, for all $t \in \mathbb{R}$. Therefore, $Q \in \mathcal{C}^{\mathrm{s}}(|T|)$. Using this relation, we have

$$QT = UQ|T| \subset U|T|Q = TQ.$$

Thus $Q \in \mathcal{C}^{\mathrm{s}}(T)$, and hence $\mathcal{H}_1$ reduces T and $|T|$. Let T_1 and T_2 be the parts of T on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then we have

$$T = T_1 \bigoplus T_2 \quad \text{with respect to } \mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2.$$

Suppose that $T_1 = U_1|T_1|$ and $T_2 = U_2|T_2|$ are the polar decompositions of T_1 and T_2. By the uniqueness of the polar decomposition, we conclude that

$$U_1 = U|\mathcal{H}_1, \quad U_2 = U|\mathcal{H}_2, \qquad \text{and} \qquad |T_1| = |T|\,|\mathcal{H}_1, \quad |T_2| = |T|\,|\mathcal{H}_2.$$

By Corollary 5.3, T_1 and T_2 are q-quasinormal. Since U_1 is unitary, it follows from Proposition 5.5 that T_1 is q-normal in $\mathcal{H}_1$. Clearly, U_2 is an isometry on $\mathcal{H}_2$ such that $\bigcap_{n=0}^{\infty}(U_2)^n\mathcal{H}_2 = \{0\}$. Put

$$\mathcal{L} \equiv \mathcal{H}_2 \cap (U_2\mathcal{H}_2)^{\perp}.$$

Then $\mathcal{H}_2 = \bigoplus_{n=0}^{\infty} (U_2)^n \mathcal{L}$, and $\ell_+^2(\mathcal{L}) \equiv \bigoplus_{n=0}^{\infty} \mathcal{L}_n$, $\mathcal{L}_n \equiv \mathcal{L}$ is carried onto $\mathcal{H}_2$ by the unitary transformation Γ defined by

$$\Gamma\big((x_n)_{n=0}^{\infty}\big) \equiv \sum_{n=0}^{\infty} (U_2)^n x_n$$

for $(x_n)_{n=0}^{\infty} \in \ell_+^2(\mathcal{L})$. Define operators on $\ell_+^2(\mathcal{L})$ corresponding to T_2 and U_2 by

$$(T_2)' = \Gamma^{-1} T_2 \Gamma \quad \text{and} \quad (U_2)' = \Gamma^{-1} U_2 \Gamma.$$

It is clear that $(U_2)'$ is a unilateral shift on $\ell_+^2(\mathcal{L})$. Take $(x_n)_{n=0}^{\infty}$ in $\mathcal{D}((T_2)')$. Then, it follows from the q-quasinormality of T_2 that

$$\Gamma\big(|(T_2)'|\big((x_n)_{n=0}^{\infty}\big)\big) = \sum_{n=0}^{\infty} q^{-\frac{n}{2}} (U_2)^n |T_2| x_n.$$

Therefore,

$$|(T_2)'|\big((x_n)_{n=0}^{\infty}\big) = \big(|T_2|x_0,\, q^{-\frac{1}{2}}|T_2|x_1,\, q^{-1}|T_2|x_2,\, q^{-\frac{3}{2}}|T_2|x_3, \dots\big),$$

so that,

$$(T_2)'\big((x_n)_{n=0}^{\infty}\big) = \big(0,\, |T_2|x_0,\, q^{-\frac{1}{2}}|T_2|x_1,\, q^{-1}|T_2|x_2,\, \dots\big).$$

In view of the above form of $(T_2)'$, one can easily check that every q-quasinormal operator is unitarily equivalent to the desired form in the statement of the theorem. The unilateral shift U_2 on $\ell_+^2(\mathcal{L})$ is naturally extended to the bilateral shift U_0 on

$$\ell^2(\mathcal{L}) \equiv \bigoplus_{n=-\infty}^{\infty} \mathcal{L}_n, \quad \mathcal{L}_n \equiv \mathcal{L}.$$

Define T_0 and L on $\ell^2(\mathcal{L})$ by

$$T_0\big((x_n)_{n=-\infty}^{\infty}\big) \equiv \big(\dots,\, q^{\frac{3}{2}}|T_2|x_{-3},\, \overbrace{q|T_2|x_{-2}}^{-1},\, \overbrace{q^{\frac{1}{2}}|T_2|x_{-1}}^{0},\, \overbrace{|T_2|x_0}^{1},\, q^{-\frac{1}{2}}|T_2|x_1, \dots\big)$$

and

$$L\big((x_n)_{n=-\infty}^{\infty}\big) \equiv \big(\dots,\, q|T_2|x_{-2},\, \overbrace{q^{\frac{1}{2}}|T_2|x_{-1}}^{-1},\, \overbrace{|T_2|x_0}^{0},\, \overbrace{q^{-\frac{1}{2}}|T_2|x_1}^{1},\, q^{-1}|T_2|x_2,\, \dots\big)$$

for $(x_n)_{n=0}^{\infty} \in \ell^2(\mathcal{L})$. Then T_0 on $\ell^2(\mathcal{L})$ extends $(T_2)'$ and $|(T_2)'|$ is also extended to a positive selfadjoint operator L in $\ell^2(\mathcal{L})$. Clearly, $L = |T_0|$, so that $T_0 = U_0|T_0|$ is the polar decomposition of T_0. By a simple computation, we obtain $U_0|T_0| = \sqrt{q}\,|T_0|U_0$. Since U_0 is unitary, T_0 is q-normal by Proposition 5.5. ∎

REMARK 5.9. The q-quasinormal part S_q in the above theorem can be represented as follows:

$$S_q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ K & 0 & 0 & 0 & 0 & \cdots \\ 0 & q^{-\frac{1}{2}}K & 0 & 0 & 0 & \cdots \\ 0 & 0 & q^{-1}K & 0 & 0 & \cdots \\ 0 & 0 & 0 & q^{-\frac{3}{2}}K & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

in $\ell_+^2(\mathcal{L})$.

6. INTERTWINING RELATIONS

We begin this section with an illustration of the attached contraction K_S to a unilateral weighted shift $S \equiv S_{\mathrm{u}}$ with weights $\{w_n\}$. It is easy to see that, if S is q-hyponormal, the contraction K_S is given by

$$K_S e_0 = K_S e_1 = 0 \quad \text{and} \quad K_S e_n = \Big(\frac{1}{\sqrt{q}}\Big)\frac{\overline{w}_{n-2}}{w_{n-1}} e_{n-2}, \quad n \geqslant 2.$$

Since S_{u} is unitarily equivalent to the unilateral weighted shift with weights $\{|w_n|\}$, we assume that all $w_n > 0$. Then if S is q-quasinormal, we have $K_S e_0 = K_S e_1 = 0$ and $K_S e_n = e_{n-2}$, $n \geqslant 2$. Moreover, it follows that

$$(K_S)^* e_n = e_{n+2}, \quad n \geqslant 0.$$

As we have seen in Lemma 4.4 of Section 4, every q-quasinormal, unilateral weighted shift S_{u} is bounded for $q > 1$. Related to this, we will show in the following there is no non-zero bounded q-quasinormal operator for $0 < q < 1$.

THEOREM 6.1. *Let T be a q-quasinormal operator in a Hilbert space $\mathcal{H}$. Then, there is a coisometry K such that*

$$KTK^* = \frac{1}{q}T.$$

Especially for $0 < q < 1$, every non-zero q-quasinormal operator must be unbounded.

Proof. Let $T = U|T|$ be the polar decomposition of T. Then by Theorem 3.5 we have

$$(K_T)^* T = \sqrt{q}\, U^2 |T| U = qU|T|U^2 = qT(K_T)^*.$$

By virtue of Corollary 3.2 and Proposition 3.7, $K_T(K_T)^*$ is the orthogonal projection onto $\overline{\mathcal{R}(T^*)}$ containing the range of T and hence the above equality implies the relation $K_T T (K_T)^* = q^{-1}T$. In view of Section 5, $\ker T$ reduces T and we have $T = 0 \oplus T_2$ with respect to $\mathcal{H} = \ker T \oplus (\ker T)^\perp$. Since T_2 is injective (and hence $\overline{\mathcal{R}((T_2)^*)} = (\ker T)^\perp$) and q-quasinormal on $(\ker T)^\perp$, $(K_{T_2})^*$ is isometry. Noticing that K_{T_2} coincides with the part of K_T on $(\ker T)^\perp$ by Proposition 5.4, define

$$K \equiv \iota \oplus K_{T_2},$$

where ι denotes the identity on $\ker T$. Then K^* is an isometry on $\mathcal{H}$. It follows that

$$KTK^* = \frac{1}{q}T.$$

We next consider the linear transformation on $\mathcal{B}(\mathcal{H})$ defined by $\Gamma(T) = KTK^*$, $T \in \mathcal{B}(\mathcal{H})$. It then follows that $\sigma(\Gamma) \subseteq \{\mu \in \mathbb{C} : |\mu| \leqslant 1\}$. This means that the operator equation

$$\Gamma(X) = \frac{1}{q}X$$

has no non-trivial solution X in $\mathcal{B}(\mathcal{H})$ for $0 < q < 1$. Therefore, every non-zero q-quasinormal operator with $0 < q < 1$ is always unbounded. ∎

COROLLARY 6.2. *Every non-zero q-normal operator is always unbounded. A q-normal operator T satisfies the relation*

$$(K_T)^* T K_T = qT, \quad \text{or equivalently } TK_T = qK_T T. \tag{6.1}$$

In particular, T is unitarily equivalent to qT.

Proof. The first assertion is a direct consequence of Theorem 6.1. Keep the same notation as in the above proof. Since T is q-normal, $(K_T)^* K_T$ is the orthogonal projection onto $(\ker T)^{\perp} (= \overline{\mathcal{R}(T)})$ and $\ker T$ reduces K_T by Proposition 5.4. It follows from the equality in Theorem 6.1 that $TK_T = qK_T T$. Since T_2 is injective and q-normal, K_{T_2} is unitary. Therefore, $K = \iota \oplus K_{T_2}$ is also unitary on $\mathcal{H}$. Since $(K_T)^* T K_T = qT$, we obtain $K^* T K = qT$. ∎

REMARKS 6.3. (1) The first part of Corollary 6.2 was proved in [21] as stated in Introduction.

(2) A non-zero q-quasinormal operator T is not always unitarily equivalent to qT. In fact, a q-quasinormal, unilateral weighted shift with $q > 1$ is such an example. Furthermore, using the observation made at the beginning of this section, one can easily check that any q-quasinormal, unilateral weighted shift can not satisfy condition (6.1) of Corollary 6.2.

(3) There is no non-zero bounded solution for the operator equation $K^* X K = qX$ where K is unitary and $q \in \mathbb{R}$, $q \neq 1$. In the case that K is isometric, however, the operator equation $K^* X K = qX$ can be solved with some bounded solution depending on the parameter q. When $\mathcal{H}$ is the classical Hardy space $\mathcal{H}^2(\mathbb{T})$ and K is the unilateral shift on $\mathcal{H}$, Shunhua Sun in [37] gave explicit bounded solutions depending on the parameter q.

7. SPECTRUM

Throughout this section we assume q-normal operators are non-zero. We shall present some general aspects of the spectrum of a q-normal operator.

THEOREM 7.1. *The spectrum of a q-quasinormal operator must contain $\{0\}$. In particular, for every q-normal operator T the spectrum has the property that*

$$|\sigma(T)|_2 = \infty,$$

where $|\cdot|_2$ denotes the planar Lebesgue measure.

Proof. Let T be a q-quasinormal operator in a Hilbert space $\mathcal{H}$ with polar decomposition $T = U|T|$. Assume $0 \notin \sigma(T)$. Then U is unitary, and hence T is q-normal by Proposition 5.5. Since $TT^* = qT^*T$, it follows that

$$T^{-1}(T^*)^{-1} = q(T^*)^{-1}T^{-1},$$

so that T^{-1} is also q-normal. Therefore, T^{-1} is unbounded by Corollary 6.2, which is a contradiction to $0 \notin \sigma(T)$. Therefore, $\sigma(T) \ni 0$. Next suppose that T is q-normal in $\mathcal{H}$. It suffices to show $|\sigma(T)|_2 = \infty$ in the case where $\sigma(T) \neq \mathbb{C}$. Since

$\sigma(T)$ contains 0, there is a number $\lambda \notin \sigma(T)$ with $\lambda \neq 0$. Since T is unbounded, it follows that

$$\sigma\big((\lambda - T)^{-1}\big) = \Big\{\frac{1}{\lambda - \nu} : \nu \in \sigma(T)\Big\} \cup \{0\}. \tag{7.1}$$

We recall that a densely defined operator T is called *hyponormal* if $\mathcal{D}(T) \subseteq \mathcal{D}(T^*)$ and $\|T^*\eta\| \leqslant \|T\eta\|$ for $\eta \in \mathcal{D}(T)$ (see, e.g., [15], [20] and [25] for more information).

It is obvious that T is hyponormal if $0 < q < 1$ and T^* is also hyponormal if $q > 1$. Replacing T by T^*, we may assume that $0 < q < 1$. Then, it is easy to see that $(\lambda - T)^{-1}$ is a bounded hyponormal operator ([28],[35]). If $|\sigma(T)|_2 = 0$, then by the equality (7.1) we have

$$\big|\sigma\big((\lambda - T)^{-1}\big)\big|_2 = 0.$$

Since the bounded operator $(\lambda - T)^{-1}$ is hyponormal, $(\lambda - T)^{-1}$ is normal by Putnam's Inequality ([29]). It follows that T is normal, which is a contradiction to $q \neq 1$. Therefore, $|\sigma(T)|_2 > 0$. Moreover, by Corollary 6.2, the spectrum of a q-normal operator T has the property that $q^n\sigma(T) = \sigma(T)$ for all $n \in \mathbb{Z}$. Therefore, we obtain

$$|\sigma(T)|_2 = \infty. \qquad \blacksquare$$

REMARK 7.2. Applying the same methods as above, it is verified that for every non-zero q-quasinormal operator T with $0 < q < 1$ one has

$$|\sigma(T)|_2 > 0.$$

On the contrary, in case $q > 1$ there exists a q-quasinormal (bounded) operator T with $\sigma(T) = \{0\}$. Indeed, as we have seen in Section 4 (Lemma 4.4), the q-quasinormal weighted shift S_{u} is such an example. It will be shown later (in Section 9) that every bounded, q-quasinormal operator is quasinilpotent and hence its spectrum coincides with $\{0\}$.

LEMMA 7.3. *Let T be a q-normal operator. Then, for all $\lambda \in \mathbb{C} \setminus \{0\}$,*
(i) *if $q > 1$, $\mathcal{R}(T - \lambda) = \mathcal{H}$;*
(ii) *if $0 < q < 1$, $\mathcal{R}(T^* - \lambda) = \mathcal{H}$.*

Proof. For any $\lambda \in \mathbb{C}$, it follows that

$$\|T^*\eta - \overline{\lambda}\eta\|^2 = \|T\eta - \lambda\eta\|^2 + (q-1)\|T\eta\|^2 \tag{7.2}$$

for all $\eta \in \mathcal{D}(T)$.

Assume $q > 1$ and $\lambda \neq 0$. Then $\mathcal{R}(T^* - \lambda)$ is closed. In fact, take any sequence $\{\eta_n\}$ in $\mathcal{D}(T^*) = \mathcal{D}(T)$ such that the sequence $\{(T^* - \lambda)\eta_n\}$ converges to some γ in $\mathcal{H}$. In virtue of the equality (7.2), both $\{(T - \overline{\lambda})\eta_n\}$ and $\{T\eta_n\}$ are Cauchy sequences. Since λ is non-zero, there is some $\eta_0 \in \mathcal{H}$ such that $\eta_n \to \eta_0$, and so the sequence $\{T^*\eta_n\}$ converges. Hence, $\eta_0 \in \mathcal{D}(T^*)$ and $\gamma = (T^* - \lambda)\eta_0$. Using (7.2) once again, it follows that

$$\ker(T^* - \lambda) = \{0\}.$$

On the other hand, according to [5], Theorem 10, p. 72, $\mathcal{R}(T - \lambda)$ is closed. Hence, $\mathcal{R}(T - \lambda) = \mathcal{H}$.

If $0 < q < 1$, then T^* is q^{-1}-normal. Therefore, the above argument for the case $q > 1$ can apply to that in the case of T^*. Thus we have $\mathcal{R}(T^* - \lambda) = \mathcal{H}$. $\blacksquare$

LEMMA 7.4. *Let T be a q-normal operator. Then,*

$$\sigma_{\mathrm{c}}(T) \subseteq \{0\}.$$

Moreover,

(i) *if $q > 1$, $\sigma_{\mathrm{r}}(T) = \emptyset$;*

(ii) *if $0 < q < 1$, $\sigma_{\mathrm{p}}(T) \subseteq \{0\}$, $\sigma_{\mathrm{r}}(T) \neq \emptyset$ and $0 \notin \sigma_{\mathrm{r}}(T)$.*

Proof. Take any $\lambda \in \mathbb{C} \setminus \{0\}$. It is clear by the above lemma that $\lambda \notin \sigma_{\mathrm{c}}(T)$ if $q > 1$. Since T^* is q^{-1}-normal and by $\sigma_{\mathrm{c}}(T^*) = \overline{\sigma_{\mathrm{c}}(T)}$, we have $\lambda \notin \sigma_{\mathrm{c}}(T)$ if $0 < q < 1$. Thus $\sigma_{\mathrm{c}}(T) \subseteq \{0\}$. If $0 \in \sigma_{\mathrm{r}}(T)$, then T is injective, and so $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^*)} = (\ker T)^{\perp} = \mathcal{H}$. This is a contradiction. Hence, $0 \notin \sigma_{\mathrm{r}}(T)$.

Assume that $q > 1$. Then, by the above lemma, we have $\lambda \notin \sigma_{\mathrm{r}}(T)$ for any $\lambda \in \mathbb{C} \setminus \{0\}$. Therefore, $\sigma_{\mathrm{r}}(T) = \emptyset$.

Next suppose that $0 < q < 1$. For any $\lambda \in \mathbb{C} \setminus \{0\}$, since $T - \lambda$ is injective, we have

$$\sigma_{\mathrm{p}}(T) \subseteq \{0\}.$$

Since $\sigma(T)$ properly contains $\{0\}$ by Theorem 7.1, $\sigma_{\mathrm{r}}(T) \neq \emptyset$. ∎

The following proposition indicates a certain property of the point spectrum of a q-normal operator.

PROPOSITION 7.5. *Let T be a q-quasinormal operator with polar decomposition $T = U|T|$. Suppose that λ is a non-zero eigenvalue of T with eigenvector ξ. Then $(\sqrt{q}\,)^{-n}\lambda$ is an eigenvalue with eigenvector $U^n\xi$ for each $n \in \mathbb{N}$. In addition, if T is q-normal, then $(\sqrt{q}\,)^{n}\lambda$ is an eigenvalue with eigenvector $(U^*)^n\xi$ for each $n \in \mathbb{N}$.*

Proof. Since ξ is an eigenvector corresponding to λ,

$$|T|U\xi = (\sqrt{q}\,)^{-1}T\xi = (\sqrt{q}\,)^{-1}\lambda\xi$$

and

$$TU\xi = U(|T|U)\xi = (\sqrt{q}\,)^{-1}\lambda U\xi.$$

Since λ is non-zero, $U\xi \neq 0$, and hence $(\sqrt{q}\,)^{-1}\lambda$ is an eigenvalue with eigenvector $U\xi$. Furthermore, we have

$$|T|U^2\xi = (\sqrt{q}\,)^{-1}TU\xi = (\sqrt{q}\,)^{-2}\lambda U\xi$$

and

$$TU^2\xi = U(|T|U^2)\xi = (\sqrt{q}\,)^{-2}\lambda U^2\xi.$$

Since $\lambda \neq 0$ and $U\xi \neq 0$, we have $U^2\xi \neq 0$, and so $(\sqrt{q}\,)^{-2}\lambda$ is an eigenvalue with eigenvector $U^2\xi$. Repeating this argument, we obtain $(\sqrt{q}\,)^{-n}\lambda$ is an eigenvalue with eigenvector $U^n\xi$ for each $n \in \mathbb{N}$.

If T is q-normal, then $\ker T^* = \ker T (= \ker |T|)$ by Corollary 3.2, and hence $1 - UU^*$ is the orthogonal projection onto $\ker |T|$. Therefore, we have

$$TU^*\xi = \sqrt{q}\,|T|UU^*\xi = \sqrt{q}\,|T|\xi = \sqrt{q}\,U^*T\xi = \sqrt{q}\,\lambda U^*\xi.$$

Since $TU^*\xi = \sqrt{q}\,|T|\xi$, it follows that $U^*\xi \neq 0$. Analogously, we have

$$T(U^*)^2\xi = \sqrt{q}\,|T|U^*\xi = (\sqrt{q}\,)^2\lambda(U^*)^2\xi,$$

and $(U^*)^2\xi \neq 0$. Repeating the argument mentioned above, the proposition follows. ∎

Now we are in a position of describing the whole spectrum of a q-normal operator.

We will make a list of the spectrum of a q-normal operator.

I. The case: $q > 1$

1. T is one-to-one

	σ_{p}	σ_{c}	σ_{r}
T	$\mathfrak{M}_1$	$\{0\}$	$\emptyset$
T^*	$\emptyset$	$\{0\}$	$\overline{\mathfrak{M}_1}$

Here, $\mathfrak{M}_1$ is a subset of $\mathbb{C}$ with $\left|\mathfrak{M}_1\right|_2 = \infty$.

2. T is not one-to-one

	σ_{p}	σ_{c}	σ_{r}
T	$\mathfrak{M}_2$	$\emptyset$	$\emptyset$
T^*	$\{0\}$	$\emptyset$	$\overline{\mathfrak{M}_2} \setminus \{0\}$

Here, $\mathfrak{M}_2$ is a subset of $\mathbb{C}$ such that $\mathfrak{M}_2 \ni 0$ and $\left|\mathfrak{M}_2\right|_2 = \infty$.

II. The case: $0 < q < 1$

1. T is one-to-one

	σ_{p}	σ_{c}	σ_{r}
T	$\emptyset$	$\{0\}$	$\mathfrak{N}_1$
T^*	$\overline{\mathfrak{N}_1}$	$\{0\}$	$\emptyset$

Here, $\mathfrak{N}_1$ is a subset of $\mathbb{C}$ with $\left|\mathfrak{N}_1\right|_2 = \infty$.

2. T is not one-to-one

	σ_{p}	σ_{c}	σ_{r}
T	$\{0\}$	$\emptyset$	$\mathfrak{N}_2$
T^*	$\overline{\mathfrak{N}_2} \cup \{0\}$	$\emptyset$	$\emptyset$

Here, $\mathfrak{N}_2$ is a subset of $\mathbb{C}$ with $\left|\mathfrak{N}_2\right|_2 = \infty$.

8. CARTESIAN DECOMPOSITIONS

We first recall a Cartesian decomposition for an unbounded operator (e.g., [24]). For a densely defined operator T in $\mathcal{H}$ with $\mathcal{D}(T) \subseteq \mathcal{D}(T^*)$, there exists uniquely a pair of symmetric operators T_1 and T_2 such that

$$T = T_1 + \mathrm{i}T_2, \quad \mathrm{i} = \sqrt{-1} \quad \text{and} \quad \mathcal{D}(T_1) = \mathcal{D}(T_2),$$

which is called the *Cartesian decomposition* of T. In fact, the Cartesian decomposition $T = T_1 + \mathrm{i}T_2$ is given by

$$T_1 = \frac{1}{2}(T + T^*) \quad \text{and} \quad T_2 = \frac{1}{2\mathrm{i}}(T - T^*).$$

Symmetric operators T_1 and T_2 are called the *real* and *imaginary part* of T and are denoted by $\mathrm{Re}\,(T)$ and $\mathrm{Im}\,(T)$, respectively.

Proposition 8.1. *Let T be a closed, q-hyponormal operator in a Hilbert space $\mathcal{H}$. Then* $\mathrm{Re}\,(T)$ *and* $\mathrm{Im}\,(T)$ *are given by*

$$\mathrm{Re}\,(T) = \frac{1}{2}(1 + \sqrt{q}\,K_T)T \quad \textit{and} \quad \mathrm{Im}\,(T) = \frac{1}{2\mathrm{i}}(1 - \sqrt{q}\,K_T)T. \tag{8.1}$$

If $0 < q < 1$, then they are necessarily closed. Moreover, there is a bounded operator L on $\mathcal{H}$ with $\mathrm{i} \notin \sigma(L)$ such that

$$\mathrm{Im}\,(T) = L \cdot \mathrm{Re}\,(T) \quad \textit{and} \quad K_T = \frac{1}{\sqrt{q}}(\mathrm{i} + L)(\mathrm{i} - L)^{-1}.$$

Proof. In view of the definition of K_T, we have $\mathrm{Re}\,(T) \supset \frac{1}{2}(1 + \sqrt{q}\,K_T)T$ and $\mathrm{Im}\,(T) \supset \frac{1}{2\mathrm{i}}(1 - \sqrt{q}\,K_T)T$. Since $\mathcal{D}(\mathrm{Re}\,(T)) = \mathcal{D}(\mathrm{Im}\,(T)) = \mathcal{D}(T)$, the relation (8.1) follows. Assume that $0 < q < 1$. Since K_T is a contraction ,

$$\Big\| 1 - \frac{1}{2}(1 \pm \sqrt{q}\,K_T) \Big\| \leqslant \frac{1}{2}(1 + \sqrt{q}\,) < 1. \tag{8.2}$$

Hence, the bounded operators $\frac{1}{2}(1 \pm \sqrt{q}\,K_T)$ are boundedly invertible. Because of the closedness of T, the symmetric operators $\mathrm{Re}\,(T)$ and $\mathrm{Im}\,(T)$ are closed. Define a bounded operator L on $\mathcal{H}$ by

$$L \equiv \mathrm{i}\Big\{1 - \Big(\frac{1 + \sqrt{q}\,K_T}{2}\Big)^{-1}\Big\}. \tag{8.3}$$

Since $\frac{1}{2}(1 + \sqrt{q}\,K_T)$ has a bounded inverse, $\Big(\frac{1+\sqrt{q}\,K_T}{2}\Big)^{-1}\mathrm{Re}\,(T) = T$, and hence

$$\mathrm{Im}\,(T) = \Big(\frac{1 - \sqrt{q}\,K_T}{2\mathrm{i}}\Big)\Big(\frac{1 + \sqrt{q}\,K_T}{2}\Big)^{-1}\mathrm{Re}\,(T) = L \cdot \mathrm{Re}\,(T). \tag{8.4}$$

Moreover, it follows from the definition (8.3) that

$$\mathrm{i} \notin \sigma(L) \quad \text{and} \quad K_T = \frac{1}{\sqrt{q}}(\mathrm{i} + L)(\mathrm{i} - L)^{-1}. \quad \blacksquare$$

Theorem 8.2. *Let T be a non-zero q-normal operator in $\mathcal{H}$. Then the following statements hold:*

(i) *The real and imaginary parts* $\mathrm{Re}\,(T)$ *and* $\mathrm{Im}\,(T)$ *are necessarily closed and unbounded.*

(ii)

$$\ker(\mathrm{Re}\,(T)) = \ker(\mathrm{Im}\,(T)) = \ker(\mathrm{Re}\,(T)^*) = \ker(\mathrm{Im}\,(T)^*) = \ker T, \tag{8.5}$$

that is,

$$\overline{\mathcal{R}(\mathrm{Re}\,(T))} = \overline{\mathcal{R}(\mathrm{Im}\,(T))} = \overline{\mathcal{R}\big(\mathrm{Re}\,(T)^*\big)} = \overline{\mathcal{R}\big(\mathrm{Im}\,(T)^*\big)} = \overline{\mathcal{R}(T)}.$$

(iii) $\sigma(\mathrm{Re}\,(T)) \cap \sigma(\mathrm{Im}\,(T)) \ni 0$.

Proof. In view of Proposition 8.1, if $0 < q < 1$ then $\mathrm{Re}\,(T)$ and $\mathrm{Im}\,(T)$ are closed. In case $q > 1$, T^* is q^{-1}-normal and it is easy to see that its Cartesian decomposition is given by

$$T^* = \mathrm{Re}\,(T) + \mathrm{i}(-\mathrm{Im}\,(T)).$$

It follows that the symmetric operators $\mathrm{Re}\,(T)$ and $\mathrm{Im}\,(T)$ are closed. Since T is unbounded and $\mathcal{D}(\mathrm{Re}\,(T)) = \mathcal{D}(\mathrm{Im}\,(T)) = \mathcal{D}(T)$, both $\mathrm{Re}(T)$ and $\mathrm{Im}\,(T)$ are unbounded. Thus, statement (i) holds. Substituting T^* for T as mentioned above, we have only to show statement (ii) in the case $0 < q < 1$. Thus we assume that $0 < q < 1$, and use the same notation as in the proof of Proposition 8.1. Since the relations (8.1) and (8.2) are valid, we have $\ker(\mathrm{Re}\,(T)) = \ker(\mathrm{Im}\,(T)) = \ker T$. By (8.2), $\Big(\frac{1+\sqrt{q}\,K_T}{2}\Big)^{-1}$ can be expanded as Neumann series, that is, $\Big(\frac{1+\sqrt{q}\,K_T}{2}\Big)^{-1} = \sum\limits_{n=0}^{\infty}\Big(\frac{1-\sqrt{q}\,K_T}{2}\Big)^{n}$, so that

$$\mathrm{i}L = \sum_{n=1}^{\infty}\Big(\frac{1-\sqrt{q}\,K_T}{2}\Big)^{n}.$$

In view of (6.1) in Corollary 6.2, for all $n \in \mathbb{N}$ $\mathrm{Re}\,(T)(K_T)^n = q^n (K_T)^n \mathrm{Re}\,(T)$. It follows that

$$\Big(\frac{1-\sqrt{q}\,K_T}{2}\Big)^{n}\mathrm{Re}\,(T) \subset \mathrm{Re}\,(T)\Big(\frac{1-q^{-1}\sqrt{q}\,K_T}{2}\Big)^{n}$$

for all $n \in \mathbb{N}$. Therefore,

$$\mathrm{Im}\,(T) = L\cdot\mathrm{Re}\,(T) \subset \frac{1}{\mathrm{i}}\sum_{n=1}^{\infty}\mathrm{Re}\,(T)\Big(\frac{1-q^{-1}\sqrt{q}\,K_T}{2}\Big)^{n}.$$

This implies that $\mathcal{R}(\mathrm{Im}\,(T)) \subseteq \overline{\mathcal{R}(\mathrm{Re}(T))}$. By (8.2), $\Big(\frac{1-\sqrt{q}\,K_T}{2}\Big)^{-1}$ has also a bounded inverse. One can easily check that L is boundedly invertible such that

$$L^{-1} = \mathrm{i}\Big\{\Big(\frac{1-\sqrt{q}\,K_T}{2}\Big)^{-1} - 1\Big\} \quad\text{and}\quad \mathrm{Re}\,(T) = L^{-1}\cdot\mathrm{Im}\,(T).$$

Repeating the argument mentioned above, we have $\overline{\mathcal{R}(\mathrm{Im}\,(T))} = \overline{\mathcal{R}(\mathrm{Re}\,(T))}$, or equivalently $\ker(\mathrm{Re}\,(T)^*) = \ker(\mathrm{Im}\,(T)^*)$. To prove the rest of statement (ii), we notice that

$$\mathcal{D}(\mathrm{Re}\,(T)^*) \cap \mathcal{D}(\mathrm{Im}\,(T)^*) = \mathcal{D}(T). \tag{8.6}$$

In fact, the above relation follows from the equations:

$$\mathcal{D}(\mathrm{Re}\,(T)^*) = \{\eta : \eta + \sqrt{q}\,(K_T)^*\eta \in \mathcal{D}(T)\},$$

and

$$\mathcal{D}(\mathrm{Im}\,(T)^*) = \{\eta : \eta - \sqrt{q}\,(K_T)^*\eta \in \mathcal{D}(T)\}.$$

Combining the relation (8.6) and the equations

$$\ker(\mathrm{Re}\,(T)) = \ker(\mathrm{Im}\,(T)) \quad \text{and} \quad \ker(\mathrm{Re}\,(T)^*) = \ker(\mathrm{Im}\,(T)^*),$$

it follows that

$$\ker(\mathrm{Re}\,(T)) = \ker(\mathrm{Re}\,(T)^*).$$

Thus statement (ii) is valid.

Finally, we will show statement (iii). If $0 < q < 1$, as was seen in the proof of Proposition 8.1, $\frac{1}{2}(1 + \sqrt{q}\,K_T)$ is boundedly invertible and $\left(\frac{1+\sqrt{q}\,K_T}{2}\right)^{-1}\mathrm{Re}\,(T) = T$. Therefore, by Theorem 7.1 we have $\sigma(\mathrm{Re}\,(T)) \ni 0$. Similarly, we have $\sigma(\mathrm{Im}\,(T)) \ni 0$. In case $q > 1$, we obtain $\sigma(\mathrm{Re}\,(T)) \cap \sigma(\mathrm{Im}\,(T)) \ni 0$, by replacing T by T^*. ∎

Let us recall that the deficiency indices $(n_+(T), n_-(T))$ of a symmetric operator T in $\mathcal{H}$ are defined by

$$n_\pm(T) \equiv \dim\ker(T^* \pm \mathrm{i}).$$

If T has equal deficiency indices, then T has a selfadjoint extension in $\mathcal{H}$.

In [33] (see also [14] and [32]), a variant of a q-deformed Heisenberg algebra was introduced and studied. It was shown in [14] that the imaginary part of a q-normal, bilateral weighted shift ($q > 1$), which derives from the generators, has equal deficiency indices (1,1). For a unilateral weighted shift, we present the following proposition.

PROPOSITION 8.3. *Let S_u be a unilateral weighted shift with weights $\{w_n\}$ in a Hilbert space $\mathcal{H}$. Suppose that S_u is q-quasinormal and $0 < q < 1$. Then both $\mathrm{Re}\,(S_\mathrm{u})$ and $\mathrm{Im}\,(S_\mathrm{u})$ are closed symmetric operators and have equal deficiency indices such that*

$$\Big(n_+(\mathrm{Re}\,(S_\mathrm{u})), n_-(\mathrm{Re}\,(S_\mathrm{u}))\Big) = \Big(n_+(\mathrm{Im}\,(S_\mathrm{u})), n_-(\mathrm{Im}\,(S_\mathrm{u}))\Big) = (1,1).$$

Proof. Upon applying a unitary transformation (see the remark before Lemma 4.3), we can assume without loss of generality that all weights w_n are positive. Define a transformation J by

$$J\Big(\sum_{n=0}^{\infty} \alpha_n e_n\Big) \equiv \sum_{n=0}^{\infty} \overline{\alpha}_n e_n.$$

Then, it is clear that J is a conjugation on $\mathcal{H}$ such that $JS_\mathrm{u} \subset S_\mathrm{u}J$.

Put $S_1 \equiv \mathrm{Re}\,(S_\mathrm{u})$ and $S_2 \equiv \mathrm{Im}\,(S_\mathrm{u})$. It then follows that $JS_1 \subset S_1J$ and $JS_2 \subset S_2J$. Namely, both S_1, S_2 are real with respect to J, and hence the deficiency indices of each S_i, $i = 1, 2$ equal each other. It is easy to see that

$$\mathcal{D}((S_1)^*) = \Big\{\sum_{n=0}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=0}^{\infty} |w_n\alpha_{n+1} + w_{n-1}\alpha_{n-1}|^2 < +\infty\Big\},$$

$$(S_1)^*\Big(\sum_{n=0}^{\infty}\alpha_n e_n\Big)=\frac{1}{2}\sum_{n=0}^{\infty}(w_n\alpha_{n+1}+w_{n-1}\alpha_{n-1})e_n,$$

where $w_{-1}=0$, $\alpha_{-1}=0$. Suppose that a non-zero vector $\sum_{n=0}^{\infty} x_n e_n$ belongs to $\ker((S_1)^*-\mathrm{i})$. Clearly, the following recurrence relation holds:

$$w_{n-1}x_{n-1}+w_n x_{n+1}=2\mathrm{i}x_n,\quad w_0x_1=2\mathrm{i}x_0.$$

Therefore, by the same argument as in [4], VII, there is a sequence $\{p_n\}$ of complex numbers such that $x_n=x_0p_n$, $p_0=1$ for all $n\geqslant 0$ and $w_{n-1}p_{n-1}+w_np_{n+1}=2\mathrm{i}p_n$. On the other hand, since $w_n=q^{-\frac{n}{2}}w_0$ and $0<q<1$, we have

$$w_{n-1}w_{n+1}=(w_n)^2,\quad \text{and}\quad \sum_{n=0}^{\infty}\frac{1}{w_n}<+\infty.$$

Applying the above relation to [4], VII, Theorem 1.5, we have $\sum_{n=0}^{\infty}|p_n|^2<+\infty$. Hence, $\sum_{n=0}^{\infty}\|x_n\|^2<+\infty$. This shows that

$$\dim\ker(S_1^*-\mathrm{i})=1.$$

Hence, S_1 has equal deficiency indices and $n_+(S_1)=n_-(S_1)=1$.

Repeating the argument mentioned above and by a slight modification of the proof of [4], VII, Theorem 1.5 (see also [1], p. 27), we obtain analogously

$$\dim\ker(S_2^*-\mathrm{i})=1.$$

Furthermore, we have proved in Proposition 8.1 that S_1 and S_2 are closed. ∎

9. POWERS OF q-NORMAL OPERATORS

In this section we will investigate powers of q-normal operators. In general the powers of a closed, densely defined operator are neither closable nor densely defined.

PROPOSITION 9.1. *Let T be a q-quasinormal operator in a Hilbert space. Then T^n, $n\in\mathbb{N}$ is $q^{(n^2)}$-quasinormal with*

$$|T^n|=\Big(\frac{1}{\sqrt{q}}\Big)^{\frac{n(n-1)}{2}}|T|^n.$$

Proof. Let $T=U|T|$ be the polar decomposition of T. Then, for $n\in\mathbb{N}$,

$$\begin{aligned}T^n&=U(|T|U)^{n-1}|T|=\Big(\frac{1}{\sqrt{q}}\Big)^{n-1}U(U|T|)^{n-1}|T|\\&=\Big(\frac{1}{\sqrt{q}}\Big)^{n-1}U^2(|T|U)^{n-2}|T|^2=\Big(\frac{1}{\sqrt{q}}\Big)^{\frac{n(n-1)}{2}}U^n|T|^n.\end{aligned}\tag{9.1}$$

Thus, T^n is densely defined with $\mathcal{D}(T^n) = \mathcal{D}(|T|^n)$. On the other hand,

$$(9.2)\qquad \begin{aligned} T^n &= (\sqrt{q}\,)^n |T|(U|T|)^{n-1}U = (\sqrt{q}\,)^{(n+(n-1))}|T|^2(U|T|)^{n-2}U^2 \\ &= (\sqrt{q}\,)^{\frac{n(n+1)}{2}}|T|^n U^n. \end{aligned}$$

Hence, by virtue of the closedness of $|T|^n$, T^n is closed. For each $n \in \mathbb{N}$, let $T^n = V_n|T^n|$ be the polar decomposition of T^n. By $U^2 = (K_T)^*$ (by Theorem 3.5) and Proposition 3.7, U^2 is a partial isometry with initial domain $\overline{\mathcal{R}(T^*)}$; that is,

$$(U^*)^2U^2 = U^*U.$$

Hence, by induction, U^n, $n \in \mathbb{N}$, is also a partial isometry with initial domain $\overline{\mathcal{R}(T^*)}$. For $\eta \in \ker(T^n)$, noticing

$$\ker(|T|^n) = \ker(|T|),$$

$U^n\eta \in \ker T$ by (9.2). By Corollary 3.2, $\ker T \subseteq \ker T^*$, and hence $U^n\eta \in (\mathcal{R}(T))^\perp$. Clearly, $U^n\eta \in \overline{\mathcal{R}(T)}$, so that $U^n\eta = 0$. Hence,

$$\ker(T^n) \subseteq \ker(U^n).$$

The converse inclusion follows immediately from (9.2). Therefore,

$$\ker(T^n) = \ker(U^n).$$

Combining the argument mentioned above, it follows from the uniqueness of the polar decomposition that

$$(9.3)\qquad V_n = U^n, \quad |T^n| = \Big(\frac{1}{\sqrt{q}}\Big)^{\frac{n(n-1)}{2}}|T|^n.$$

Using (9.3), we have

$$\begin{aligned} V_n|T^n| &= (\sqrt{q}\,)^{\frac{n(n+1)}{2}}|T|^nU^n = \big(\sqrt{q}\,\big)^{\frac{n(n+1)}{2}+\frac{n(n-1)}{2}}|T^n|U^n \\ &= \sqrt{q^{(n^2)}}|T^n|V_n, \end{aligned}$$

which implies the proposition. ∎

The following corollary was already noted in Remark 7.2.

Corollary 9.2. *Every bounded, q-quasinormal operator T is quasinilpotent, so that $\sigma(T) = \{0\}$.*

Proof. Let T be a non-zero bounded, q-quasinormal operator. Then, by virtue of Theorem 6.1, we have $q > 1$. By the equation (9.3),

$$\|T^n\| = \||T^n|\| \leqslant \Big(\frac{1}{\sqrt{q}}\Big)^{\frac{n(n-1)}{2}}\|T\|^n.$$

Since $q > 1$,

$$\|T^n\|^{\frac{1}{n}} \leqslant \Big(\frac{1}{\sqrt{q}}\Big)^{\frac{n-1}{2}}\|T\| \to 0$$

as $n \to \infty$. Hence, T is quasinilpotent and hence $\sigma(T) = \{0\}$. ∎

THEOREM 9.3. *Let T be a q-normal operator in a Hilbert space. Then T^n, $n \in \mathbb{N}$ is $q^{(n^2)}$-normal with*

$$K_{T^n} = (K_T)^n, \quad \text{and} \quad (T^n)^* = (T^*)^n.$$

Proof. We first notice that

$$U^*|T| = \Big(\frac{1}{\sqrt{q}}\Big)|T|U^*.$$

Indeed, T is also q-quasinormal, which implies $|T|U^* \supset \sqrt{q}\,U^*|T|$. Since the left side of this relation is nothing but T^*, we have $|T|U^* = \sqrt{q}\,U^*|T|$. For $n \in \mathbb{N}$, repeating this relation, we get

$$\begin{aligned}(T^*)^n &= (|T|U^*)^n = |T|(U^*|T|)^{n-1}U^* \\ &= \Big(\frac{1}{\sqrt{q}}\Big)^{n-1}|T|^2(U^*|T|)^{n-2}(U^*)^2 = \Big(\frac{1}{\sqrt{q}}\Big)^{\frac{n(n-1)}{2}}|T|^n(U^*)^n.\end{aligned}$$

Hence, by (9.1) we obtain $(T^n)^* = (T^*)^n$.

Recall that T^* is q^{-1}-normal. Substituting T^* for T in (9.3), we obtain

$$|(T^n)^*| = |(T^*)^n| = \big(\sqrt{q}\big)^{\frac{n(n-1)}{2}}|T^*|^n.$$

Combining the equality (2.2) mentioned after Definition 2.1 and (9.3), we have

$$|(T^n)^*| = \big(\sqrt{q}\big)^{\frac{n(n+1)}{2}}|T|^n = \sqrt{q^{(n^2)}}|T^n|.$$

This means that T^n is $q^{(n^2)}$-normal. On the other hand, using the equation (6.1) repeatedly, we get

$$\begin{aligned}(T^n)^* &= (T^*)^n = (\sqrt{q}\,K_T T)^n \\ &= (\sqrt{q}\,)^n K_T(TK_T)^{n-1}T = (\sqrt{q}\,)^{n^2}(K_T)^n T^n.\end{aligned} \tag{9.4}$$

Hence, by Lemma 3.1 we have $K_{T^n} = (K_T)^n$. ∎

We next consider the powers of the real and imaginary parts of q-normal operators.

LEMMA 9.4. *Let T be a q-normal operator with $0 < q < 1$. Then, the bounded operators $2^{-1}\big(1 \pm q^n\sqrt{q}\,K_T\big)$ are boundedly invertible such that*

$$\Big(\frac{1 \pm q^n\sqrt{q}\,K_T}{2}\Big)^{-1}\mathcal{D}(T) \subseteq \mathcal{D}(T)$$

and, for all $\eta \in \mathcal{D}(T)$,

$$T\Big(\frac{1 \pm q^n\sqrt{q}\,K_T}{2}\Big)^{-1}\eta = \Big(\frac{1 \pm q^{(n+1)}\sqrt{q}\,K_T}{2}\Big)^{-1}T\eta,$$

where $n \in \mathbb{N} \cup \{0\}$.

Proof. For each $n \in \mathbb{N} \cup \{0\}$, by our assumption we have

$$\big\|1 - 2^{-1}\big(1 \pm q^n\sqrt{q}\,K_T\big)\big\| < 1.$$

Hence,

$$\Big(\frac{1 \pm q^n\sqrt{q}\,K_T}{2}\Big)^{-1} = \sum_{k=0}^{\infty} \frac{1}{2^k}\big(1 \mp q^n\sqrt{q}\,K_T\big)^k \quad \text{(uniform convergence)}.$$

Using $T(K_T)^k = q^k(K_T)^k T$ by (6.1) in Corollary 6.2,

$$T\big(1 \mp q^n\sqrt{q}\,K_T\big)^k \supset \big(1 \mp q^{(n+1)}\sqrt{q}\,K_T\big)^k T$$

for all $k \in \mathbb{N}$.

Take $\eta \in \mathcal{D}(T)$. Then,

$$\begin{aligned}\sum_{k=0}^{\infty} T\Big(\frac{1}{2^k}\big(1 \mp q^n\sqrt{q}\,K_T\big)^k\Big)\eta &= \sum_{k=0}^{\infty} \frac{1}{2^k}\big(1 \mp q^{(n+1)}\sqrt{q}\,K_T\big)^k T\eta \\ &= \Big(\frac{1 \pm q^{(n+1)}\sqrt{q}\,K_T}{2}\Big)^{-1} T\eta.\end{aligned}$$

Thus, the series $\sum_{k=0}^{\infty} T\Big(\frac{1}{2^k}\big(1 \mp q^n\sqrt{q}K_T\big)^k\Big)\eta$ converges. Since T is closed,

$$\Big(\frac{1 \pm q^n\sqrt{q}\,K_T}{2}\Big)^{-1}\eta \in \mathcal{D}(T),$$

and

$$T\Big(\frac{1 \pm q^n\sqrt{q}\,K_T}{2}\Big)^{-1}\eta = \Big(\frac{1 \pm q^{(n+1)}\sqrt{q}\,K_T}{2}\Big)^{-1} T\eta. \qquad \blacksquare$$

THEOREM 9.5. *Let T be a q-normal operator in a Hilbert space $\mathcal{H}$ and let $T = \mathrm{Re}\,(T) + \mathrm{i}\,\mathrm{Im}\,(T)$ be its Cartesian decomposition. Then, for each $n \in \mathbb{N}$ the powers $(\mathrm{Re}\,(T))^n$ and $(\mathrm{Im}\,(T))^n$ are closed symmetric operators in $\mathcal{H}$ such that*

$$\mathcal{D}((\mathrm{Re}\,(T))^n) = \mathcal{D}((\mathrm{Im}\,(T))^n) = \mathcal{D}(T^n).$$

Furthermore, the powers of $\mathrm{Re}\,(T)$ and $\mathrm{Im}\,(T)$ are given by

$$(\mathrm{Re}\,(T))^n = \Big(\prod_{k=0}^{n-1} \frac{1 + q^k\sqrt{q}\,K_T}{2}\Big)T^n, \tag{9.5}$$

and

$$(\mathrm{Im}\,(T))^n = \Big(\prod_{k=0}^{n-1} \frac{1 - q^k\sqrt{q}\,K_T}{2\mathrm{i}}\Big)T^n. \tag{9.6}$$

Proof. Using Corollary 6.2 repeatedly,

$$\begin{aligned}(\mathrm{Re}\,(T))^n &= \Big(\frac{1 + \sqrt{q}\,K_T}{2}\Big)\Big[T\Big(\frac{1 + \sqrt{q}\,K_T}{2}\Big)\Big]^{n-1} T \\ &\supset \Big(\frac{1 + \sqrt{q}\,K_T}{2}\Big)\Big(\frac{1 + q\sqrt{q}\,K_T}{2}\Big)\Big[T\Big(\frac{1 + q\sqrt{q}\,K_T}{2}\Big)\Big]^{n-2} T^2 \\ &\supset \Big(\prod_{k=0}^{n-1} \frac{1 + q^k\sqrt{q}\,K_T}{2}\Big)T^n.\end{aligned} \tag{9.7}$$

We will show that $\mathcal{D}((\mathrm{Re}\,(T))^n) = \mathcal{D}(T^n)$. We first prove it in the case $0 < q < 1$. Clearly, $\mathcal{D}(\mathrm{Re}\,(T)) = \mathcal{D}(T)$. Assume that $\mathcal{D}((\mathrm{Re}\,(T))^{n-1}) = \mathcal{D}(T^{n-1})$ and take $\eta \in \mathcal{D}((\mathrm{Re}\,(T))^n)$. Then, by the hypothesis $0 < q < 1$ and the relation (9.7),

$$(T^{n-1})\eta = \Big(\prod_{k=0}^{n-2}\Big(\frac{1 + q^k\sqrt{q}\,K_T}{2}\Big)^{-1}\Big)(\mathrm{Re}\,(T))^{n-1}\eta.$$

Since $(\mathrm{Re}\,(T))^{n-1}\eta \in \mathcal{D}(\mathrm{Re}\,(T)) = \mathcal{D}(T)$ and by the above lemma, $T^{n-1}\eta \in \mathcal{D}(T)$. Thus, by induction, we have $\mathcal{D}((\mathrm{Re}\,(T))^n) = \mathcal{D}(T^n)$, and hence, the equality (9.5) holds.

The Cartesian decomposition of $\mathrm{i}T$ is given by $\mathrm{i}T = (-\mathrm{Im}\,(T)) + \mathrm{i}\mathrm{Re}\,(T)$ and $\mathrm{i}T$ is q-normal with $K_{\mathrm{i}T} = -K_T$ by Lemma 3.9. Therefore, applying the above argument to the operator $\mathrm{i}T$ we have

$$\mathcal{D}((\mathrm{Im}\,(T))^n) = \mathcal{D}((\mathrm{i}T)^n) = \mathcal{D}(T^n)$$

and

$$(-\mathrm{Im}\,(T))^n = \Big(\prod_{k=0}^{n-1}\frac{1 + q^k\sqrt{q}\,K_{\mathrm{i}T}}{2}\Big)(\mathrm{i}T)^n,$$

so that the equality (9.6) is valid. Moreover, by Lemma 9.4, the bounded operators $\prod_{k=0}^{n-1}\frac{1+q^k\sqrt{q}\,K_T}{2}$ and $\prod_{k=0}^{n-1}\frac{1-q^k\sqrt{q}\,K_T}{2\mathrm{i}}$ are boundedly invertible. Hence the closedness of T^n implies that $(\mathrm{Re}\,(T))^n$ and $(\mathrm{Im}\,(T))^n$ are closed. It is clear, by induction, that they are symmetric.

Now suppose that $q > 1$. Then T^* is q^{-1}-normal and

$$\mathrm{Re}\,(T^*) = \mathrm{Re}\,(T) \quad \text{and} \quad \mathrm{Im}\,(T^*) = -\mathrm{Im}\,(T). \tag{9.8}$$

Applying the argument mentioned above to T^*, it follows that $(\mathrm{Re}\,(T^*))^n$ and $(\mathrm{Im}\,(T^*))^n$ are closed symmetric operators and

$$(\mathrm{Re}\,(T))^n = (\mathrm{Re}\,(T^*))^n = \Big(\prod_{k=0}^{n-1}\frac{1 + (q^{-1})^k\sqrt{q^{-1}}K_{T^*}}{2}\Big)(T^*)^n \tag{9.9}$$

and

$$(\mathrm{Im}\,(T))^n = (-1)^n\Big(\prod_{k=0}^{n-1}\frac{1 - (q^{-1})^k\sqrt{q^{-1}}K_{T^*}}{2\mathrm{i}}\Big)(T^*)^n.$$

By (9.8), $(\mathrm{Re}\,(T))^n$ and $(\mathrm{Im}\,(T))^n$ are closed symmetric. Moreover, $\mathcal{D}((T^*)^n) = \mathcal{D}((\mathrm{Re}\,(T^*))^n) = \mathcal{D}((\mathrm{Re}\,(T))^n)$. On the other hand, T^n is $q^{(n^2)}$-normal by Theorem 9.3. Consequently,

$$\mathcal{D}(T^n) = \mathcal{D}((T^n)^*) = \mathcal{D}((T^*)^n). \tag{9.10}$$

Hence, $\mathcal{D}((\mathrm{Re}\,(T))^n) = \mathcal{D}(T^n)$. By Corollary 3.8 and Theorem 9.3, the equality (9.9) implies that

$$((\mathrm{Re}\,(T))^n)^* = T^n\Big(\prod_{k=0}^{n-1}\frac{1 + (q^{-1})^k\sqrt{q^{-1}}K_T}{2}\Big).$$

Since $T^n K_T = q^n K_T T^n$ by Corollary 6.2,

$$((\mathrm{Re}\,(T))^n)^* \supset \Big(\prod_{k=0}^{n-1} \frac{1+q^k\sqrt{q}\,K_T}{2}\Big)T^n.$$

Since $(\mathrm{Re}\,(T))^n$ is symmetric and $\mathcal{D}((\mathrm{Re}\,(T))^n) = \mathcal{D}(T^n)$, the equality (9.5) follows. Analogously, one can easily check that $\mathcal{D}((\mathrm{Im}\,(T))^n) = \mathcal{D}(T^n)$ and the equality (9.6) is valid. ∎

Let T be a densely defined operator in $\mathcal{H}$. Set $\mathcal{D}_\infty(T) \equiv \bigcap_{n\in\mathbb{N}} \mathcal{D}(T^n)$. A vector ξ in $\mathcal{D}_\infty(T)$ is called an *analytic vector* of T if there is a positive number γ such that

$$\sum_{n=1}^{\infty} \frac{\|T^n\xi\|}{n!}\gamma^n < +\infty,$$

and ξ is called a *quasi-analytic vector* of T if

$$\sum_{n=1}^{\infty} \|S^n\xi\|^{-\frac{1}{n}} = +\infty.$$

The sets of all analytic vectors of T and of all quasi-analytic vectors of T are denoted by $\mathcal{A}(T)$ and by $\mathcal{Q}(T)$, respectively.

We remark that a q-normal operator T has the property that $\mathcal{D}_\infty(T) = \mathcal{D}_\infty(T^*)$.

Proposition 9.6. *Let T be a q-normal operator in $\mathcal{H}$. It then follows that:*

(i) *If $0 < q < 1$, then $\mathcal{A}(T) \subseteq \mathcal{A}(T^*)$ and $\mathcal{Q}(T) \subseteq \mathcal{Q}(T^*)$;*

$$\mathcal{A}(\mathrm{Re}\,(T)) = \mathcal{A}(\mathrm{Im}\,(T)) = \mathcal{A}(T), \quad \mathcal{Q}(\mathrm{Re}\,(T)) = \mathcal{Q}(\mathrm{Im}\,(T)) = \mathcal{Q}(T).$$

(ii) *If $q > 1$, then $\mathcal{A}(T) \supseteq \mathcal{A}(T^*)$ and $\mathcal{Q}(T) \supseteq \mathcal{Q}(T^*)$;*

$$\mathcal{A}(\mathrm{Re}\,(T)) = \mathcal{A}(\mathrm{Im}\,(T)) = \mathcal{A}(T^*), \quad \mathcal{Q}(\mathrm{Re}\,(T)) = \mathcal{Q}(\mathrm{Im}\,(T)) = \mathcal{Q}(T^*).$$

Thus, if $\mathrm{Re}\,(T)$ is selfadjoint then so is $\mathrm{Im}\,(T)$, that is, $\mathrm{Re}\,(T)$ and $\mathrm{Im}\,(T)$ are selfadjoint simultaneously.

Proof. Since $\ker T$ is a reducing subspace of T and the restriction of T on $(\ker T)^\perp$ is also q-normal, we may assume that T is injective and hence K_T is unitary.

Assume that $0 < q < 1$. In view of (9.4) of the proof in Theorem 9.3, it is clear that $\mathcal{A}(T) \subseteq \mathcal{A}(T^*)$ and $\mathcal{Q}(T) \subseteq \mathcal{Q}(T^*)$. Since K_T is unitary, it follows that

$$\Big\|\Big(\prod_{k=0}^{n-1} \frac{1+q^k\sqrt{q}\,K_T}{2}\Big)\xi\Big\| \geqslant \Big(\prod_{k=0}^{n-1} \frac{1-q^k\sqrt{q}}{2}\Big)\|\xi\|$$

for all $\xi \in \mathcal{H}$. Since $0 < q < 1$, we have

$$\Big\|\Big(\prod_{k=0}^{n-1} \frac{1+q^k\sqrt{q}\,K_T}{2}\Big)\xi\Big\| \geqslant \Big(\frac{1-\sqrt{q}}{2}\Big)^n\|\xi\|$$

for all $\xi \in \mathcal{H}$. Hence, by (9.5) in Theorem 9.5 we obtain

$$\|(\mathrm{Re}\,(T))^n\eta\| \geqslant \Big(\frac{1-\sqrt{q}}{2}\Big)^n \|T^n\eta\|$$

for all $\eta \in \mathcal{D}(T^n) = \mathcal{D}((\mathrm{Re}\,(T))^n) = \mathcal{D}((\mathrm{Im}\,(T))^n)$ and all $n \in \mathbb{N}$. Analogously,

$$\|(\mathrm{Im}\,(T))^n\eta\| \geqslant \Big(\frac{1-\sqrt{q}}{2}\Big)^n \|T^n\eta\|$$

for all $\eta \in \mathcal{D}(T^n)$ and all $n \in \mathbb{N}$. On the other hand, since $0 < q < 1$, we have

$$\Big\|\Big(\prod_{k=0}^{n-1} \frac{1+q^k\sqrt{q}\,K_T}{2}\Big)\Big\| < 1 \quad \text{and} \quad \Big\|\Big(\prod_{k=0}^{n-1} \frac{1-q^k\sqrt{q}\,K_T}{2\mathrm{i}}\Big)\Big\| < 1.$$

Therefore, for all $\eta \in \mathcal{D}(T^n)$ and all $n \in \mathbb{N}$, we have

$$\|T^n\eta\| \geqslant \|(T_i)^n\eta\| \geqslant \Big(\frac{1-\sqrt{q}}{2}\Big)^n \|T^n\eta\|, \quad i = 1, 2,$$

where $T_1 = \mathrm{Re}\,(T)$ and $T_2 = \mathrm{Im}\,(T)$. Statement (i) follows immediately from this relation.

Next suppose that $q > 1$. Replacing T by T^* and applying the above argument to T^*, it is easy to check that statement (ii) holds.

By virtue of Nelson's theorem, if $\mathrm{Re}\,(T)$ is selfadjoint then so is $\mathrm{Im}\,(T)$. ∎

REMARK 9.7. $\mathcal{A}(T)$ and $\mathcal{A}(T^*)$ (respectively $\mathcal{Q}(T)$ and $\mathcal{Q}(T^*)$) do not coincide in general, as we will see in the proof of the following proposition.

PROPOSITION 9.8. *The following statements are valid:*

(i) *Let T be a q-quasinormal operator with $q > 1$. Then the set of analytic vectors is a core for T.*

(ii) *There is a q-normal operator with $0 < q < 1$ for which the set of analytic vectors is not dense.*

Proof. Let $T = U|T|$ be the polar decomposition. By $q > 1$ and (9.1), we have

$$\mathcal{A}(T) \supseteq \mathcal{A}(|T|).$$

This implies the assertion (i).

Let T be a q-normal operator with $0 < q < 1$. By the above proposition, if T has a dense set of analytic vectors, then the closed symmetric operators $\mathrm{Re}\,(T)$ and $\mathrm{Im}\,(T)$ have dense sets of analytic vectors, respectively. By Nelson's theorem, they are selfadjoint. Hence, the real and imaginary parts of T^* are also selfadjoint. Now, let $0 < q < 1$ and let S_{b} be the bilateral weighted shift with the weights $w_n = \big(\sqrt{q}\big)^n$, $n \in \mathbb{Z}$. Then S_{b} is q^{-1}-normal and its imaginary part has deficiency indices $(1, 1)$ by [14] as mentioned just before Proposition 8.3. Define $T = (S_{\mathrm{b}})^*$. Then T is q-normal and has no dense set of analytic vectors. In fact, otherwise, the imaginary part of S_{b} would be selfadjoint by the above observation. This is a contradiction. ∎

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