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A FACTORIZATION THEOREM FOR THE TRANSFER FUNCTION ASSOCIATED WITH A 2×2 OPERATOR MATRIX HAVING UNBOUNDED COUPLINGS

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ABSTRACT. We construct operators which factorize the transfer function associated with a self-adjoint 2×2 operator matrix whose diagonal entries may have overlapping spectra and whose off-diagonal entries are unbounded operators. We prove completeness and basis properties of the eigenvectors of the transfer function corresponding to the real point spectrum of the 2×2 operator matrix. We also discuss some properties of the root vectors of the analytically continued transfer function.

KEYWORDS: *Operator matrix, operator pencil, Herglotz function, resonance.*
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1. INTRODUCTION

This note is a continuation of the papers [22] and [23]. As in [22] and [23], we consider the 2×2 operator matrices

$$\mathbf{H}_0 = \begin{pmatrix} A_0 & T_{01} \\ T_{10} & A_1 \end{pmatrix} \tag{1.1}$$

acting in the orthogonal sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of separable Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$. The entry $A_0 : \mathcal{H}_0 \to \mathcal{H}_0$ is assumed to be a (not necessarily bounded) selfadjoint operator with the domain $\mathcal{D}(A_0)$. We assume that A_0 is semibounded from below,

$$A_0 \geqslant \alpha_0 \tag{1.2}$$

for some $\alpha_0 \in \mathbb{R}$. We suppose that the entry A_1 is a bounded selfadjoint operator in $\mathcal{H}_1$. In contrast to [22] and [23], in the present paper we consider unbounded coupling operators T_{ij}, $i, j = 0, 1$, $i \neq j$. Regarding these couplings we assume that

(i) T_{01} is a densely defined closable operator from $\mathcal{D}(T_{01}) \subset \mathcal{H}_1$ to $\mathcal{H}_0$;

(ii) T_{10} is the adjoint operator of T_{01} (that is $T_{10} = T_{01}^*$) and

$$\mathcal{D}(T_{10}) \supset \mathcal{D}(|A_0|^{1/2}). \tag{1.3}$$

Since $\mathcal{D}(|A_0|^{1/2}) \supset \mathcal{D}(A_0)$, these assumptions mean that $\mathbf{H}_0$ is a densely defined, symmetric and, hence, closable operator on the domain $\mathcal{D}(\mathbf{H}_0) = \mathcal{D}(A_0) \oplus \mathcal{D}(T_{01})$.

The assumptions (i) and (ii) are similar to those used in the works by V.M. Adamyan, H. Langer, R. Mennicken and J. Saurer ([2]) and by R. Mennicken and A.A. Shkalikov ([24]). It follows from the results of [2] and [24] that under such assumptions the closure $\mathbf{H} = \overline{\mathbf{H}}_0$ is a selfadjoint operator in $\mathcal{H}$ (see Section 2). In applications arising from physical problems (see [9], [16], [17], [18], [19] and references cited therein), one typically deals just with the case where $\mathbf{H}_0$ is a selfadjoint operator in a Hilbert space or a symmetric operator admitting a selfadjoint closure.

Many results regarding selfadjoint 2×2 operator matrices $\mathbf{H} = \overline{\mathbf{H}}_0$ (see [1], [2], [18], [19], [24], [26] and references cited therein) are related to the problem of the existence of invariant subspaces $\mathcal{G}_i$, $i = 0, 1$, for $\mathbf{H}$ admitting so-called *graph representations*

$$\begin{aligned} \mathcal{G}_0 &= \left\{ u \in \mathcal{H} : u = \begin{pmatrix} u_0 \\ Q_{10} u_0 \end{pmatrix},\ u_0 \in \mathcal{H}_0 \right\}, \\ \mathcal{G}_1 &= \left\{ u \in \mathcal{H} : u = \begin{pmatrix} Q_{01} u_1 \\ u_1 \end{pmatrix},\ u_1 \in \mathcal{H}_1 \right\}, \end{aligned} \tag{1.4}$$

with bounded $Q_{ji} : \mathcal{H}_i \to \mathcal{H}_j$ such that $Q_{ij} = -Q_{ji}^*$ and $\mathcal{H} = \mathcal{G}_0 \oplus \mathcal{G}_1$. The point is that if such invariant subspaces exist, then the restrictions of $\mathbf{H}$ to $\mathcal{G}_i$, $i = 0, 1$, are similar to some operators H_i which are explicitly written in terms of the operators Q_{ij} and act in the corresponding component Hilbert spaces $\mathcal{H}_i$. In particular, in the case of bounded T_{ij} the operators H_i read

$$H_i = A_i + T_{ij} Q_{ji}. \tag{1.5}$$

A replacement of the initial inner products in the spaces $\mathcal{H}_i$ by appropriate equivalent new inner products turns the operators H_i into selfadjoint ones (see [1], [2], [24] and [26]). Thus, in case of the existence of invariant subspaces of the form (1.4), the study of the operator matrix $\mathbf{H}$ is reduced to the study of the, in general, more simple operators H_i.

The existence of the above mentioned invariant subspaces for $\mathbf{H}$ with selfadjoint entries A_0 and A_1 and with bounded couplings $T_{01} = T_{10}^*$ having small norm has been proved by V.A. Malyshev and R.A. Minlos ([18] and [19]), under the condition that the spectra $\sigma(A_0)$ and $\sigma(A_1)$ are separated

$$\operatorname{dist}\{\sigma(A_0), \sigma(A_1)\} > 0. \tag{1.6}$$

A similar result was obtained by A.K. Motovilov [26] (see also [25]) for the Hilbert-Schmidt class entries T_{ij} whose Hilbert-Schmidt norm satisfies the condition

$$\|T_{ij}\|_2 < \frac{1}{2} \operatorname{dist}\{\sigma(A_0), \sigma(A_1)\}. \tag{1.7}$$

In [1], V.M. Adamyan and H. Langer proved the existence of invariant subspaces admitting a graph representation for arbitrary bounded entries T_{ij}, however assuming, instead of the condition (1.6), the essentially different assumption that the spectrum of one of the entries A_i, $i = 0, 1$, is situated strictly below the spectrum of the other one, say

$$\max \sigma(A_1) < \min \sigma(A_0). \tag{1.8}$$

The result of [1] was extended by V.M. Adamyan, H. Langer, R. Mennicken and J. Saurer ([2]) to the case where

$$\max \sigma(A_1) \leqslant \min \sigma(A_0) \tag{1.9}$$

and where the couplings T_{ij} were allowed to be unbounded operators satisfying the assumptions (i) and (ii). The condition (1.9) was somewhat weakened by R. Mennicken and A.A. Shkalikov ([24]) in the case of a bounded entry A_1 and unbounded entries T_{ij} of the same type as in [2].

In the spectral theory of the operator matrices (1.1) an important role is played by the transfer functions which, in the case of bounded T_{ij}, are defined as (see, e.g., [23], [24]):

$$M_i(z) := A_i - z - T_{ij}(A_j - z)^{-1}T_{ji}, \quad i, j = 0, 1,\ j \neq i. \tag{1.10}$$

For a definition of the transfer function $M_1(z)$ in the case of unbounded entries T_{ij} satisfying the assumptions (i) and (ii) see Section 2 (see also [2], [24]). The particular role of the functions $M_i(z)$ can be understood already from the fact that the resolvent of the operator $\mathbf{H}$ can be expressed explicitly in terms of $\big[M_0(z)\big]^{-1}$ or $\big[M_1(z)\big]^{-1}$ (see, e.g., [23]). Therefore, in studying the spectral properties of the transfer functions one studies at the same time the spectral properties of the operator matrix $\mathbf{H}$. In [1], [2], [18], [19], [22], [23], [24], [25], [26] and [27] these functions were used as a main tool for a spectral analysis of operator matrices of the form (1.1). Notice that for $T_{01} = T_{10}^*$ and selfadjoint A_0 and A_1 the operator-valued functions $-M_i$ defined on $\mathbb{C}_+ \cup \mathbb{C}_-$ belong to the class of operator-valued Herglotz functions (see, e.g., [4], [8], [14], [15], and [28]).

It was proved in [26] that under the condition (1.7) the operators (1.5) can be constructed as operator roots of the corresponding transfer functions $M_i(z)$. More precisely, each operator H_i, $i = 0, 1$, can be determined as a solution of the nonlinear equation

$$M_i(H_i) = 0, \tag{1.11}$$

where $M_i(Y) := A_i - Y + V_i(Y)$. Here $V_i(Y)$ is an operator-valued function on the space of bounded linear operators in $\mathcal{H}_i$ which has been constructed in such a way (see [26]) that

$$V_i(Y)f = -T_{ij}(A_j - z)^{-1}T_{ji}f$$

and, thus, $M_i(Y)f = M_i(z)f$ for any eigenvector f corresponding to an eigenvalue z of the argument operator Y.

In [22] and [23] the approach of [26] in the construction of the operators (1.5) was carried out in a case which is different from the spectral situations considered in [1], [2], [18], [18], [19], [24], [25], [26] and [27]. Namely, the papers [22] and [23] consider the case where the spectrum of one of the main-diagonal entries, say A_1, is partly or totally embedded into the continuous spectrum of the other one, A_0. This is done under the assumptions that the couplings T_{ij} are bounded operators satisfying a certain smallness condition and such that the transfer function

$M_1(z)$ admits analytic continuation, as an operator-valued function, through the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(A_0)$ of the entry A_0. In [22] and [23], the nonreal discrete spectrum of the continued transfer function M_1 is interpreted as resonances since points of this spectrum correspond to the poles of the analytic continuation of the resolvent $(\mathbf{H}-z)^{-1}$ located on the unphysical sheets. In the present work we follow this interpretation. For more details concerning unphysical sheets, resonances and the history of the subject see, e.g., [29].

It has been proved in [22] and [23] that, having found an operator root H_1 of the analytically continued transfer function M_1, one can factorize it in a certain domain surrounding the spectrum of A_1 and partly lying on unphysical sheets in such a way that

$$M_1(z) = W_1(z)(H_1 - z) \tag{1.12}$$

where W_1 is a holomorphic function whose values represent bounded and boundedly invertible operators in $\mathcal{H}_1$. The factorization formula (1.12) implies that the spectrum of M_1 coincides in this domain with the spectrum of the operator H_1. Thus, in contrast to the case where $\sigma(A_0)\cap\sigma(A_1)=\emptyset$, the results of [22] and [23] imply that for $\sigma(A_0)\cap\sigma(A_1)\neq\emptyset$ the operator roots of M_1 may have in general a nonreal spectrum, in particular a discrete nonreal spectrum which corresponds to resonances of $\mathbf{H}$. In this case no similarity transform in $\mathcal{H}_1$ can turn these operator roots of M_1 into selfadjoint operators.

Note that under the conditions on the entries T_{ij} assumed in [22] and [23], the factorization formula (1.12) holds true for the solutions of (1.11) also for the case where $\sigma(A_0)\cap\sigma(A_1)=\emptyset$. The factorization formula of type (1.12) for the transfer function was obtained in [24] by an application of the general factorization results of A.S. Markus and V.I. Matsaev ([21]) and A.I. Virozub and V.I. Matsaev ([30]).

In the present paper, like in the papers [22], [23], we work under the assumption that $\sigma(A_0)\cap\sigma(A_1)\neq\emptyset$ and that the transfer function M_1 admits analytic continuation through $\sigma_{\mathrm{ac}}(A_0)$ onto unphysical sheets. Section 2 includes a detailed description of the conditions making such a continuation of M_1 possible in the case of unbounded entries T_{ij} satisfying the conditions (i) and (ii). In Section 3 we introduce the basic nonlinear equation (3.5) giving a rigorous sense to the equation $M_1(H_1)=0$ in the considered spectral situation. We explicitly show that eigenvalues and accompanying eigenvectors of a solution H_1 of (3.5) are automatically eigenvalues and corresponding eigenvectors for the analytically continued transfer function M_1. We prove solvability of the basic equation (3.5) under the hypothesis (3.10). In Section 4 we prove a factorization formula of the form (1.12) for the analytically continued transfer function M_1 (see Theorem 4.1). Further, on the basis of the factorization theorem, we describe in that section some relations between different solutions of (3.5) and some relations between their spectra. Section 5 pays special attention to the real point spectrum of the solutions of (3.5) and, thereby, to this part of the spectrum of the transfer function as well. It is found in this section that the real eigenvalues are the same for all considered solutions of (3.5). Furthermore, we establish that the real isolated eigenvalues of these solutions correspond to the same algebraic eigenspaces which consist only of eigenvectors. We prove the basis property for these eigenvectors with respect to their closed linear span. Section 6 is devoted to a short discussion of the nonreal spectra of the solutions H_1 under some additional assumptions. Finally, in Section 7 we present a simple example.

2. ANALYTIC CONTINUATION OF THE TRANSFER FUNCTION M_1

We may assume without loss of generality that the lower bound α_0 for the entry A_0 in the assumption (1.2) is positive, $\alpha_0 > 0$ (otherwise we could simply make a shift of the origin of the spectral parameter axis). The condition (1.3) implies that the product $B_{10} = T_{10}A_0^{-1/2}$ represents a bounded linear operator between $\mathcal{H}_1$ and $\mathcal{H}_0$ and $T_{10}|\mathcal{D}(A_0^{1/2}) = B_{10}A_0^{1/2}$. This means that

$$B_{10}^* \supset A_0^{-1/2}T_{10}^* = A_0^{-1/2}\overline{T}_{01} \supset A_0^{-1/2}T_{01}. \tag{2.1}$$

In view of the assumption (i) we conclude that $A_0^{-1/2}T_{01}$ is a densely defined operator. Therefore, it follows from (2.1) that $A_0^{-1/2}T_{01}$ has a bounded extension to the whole space $\mathcal{H}_1$. This extension coincides with $B_{01} := B_{10}^*$ and $\overline{T}_{01} = T_{10}^* \subset A_0^{1/2}B_{01}$.

It was already mentioned that under the assumptions (i) and (ii) the operator matrix $\mathbf{H}_0$ is a densely defined symmetric operator in $\mathcal{H}$ and, hence, closable. Lemma 1.4, Theorem 1.1 and Proposition 1.5 of [2] (cf. Section 2 of [24]) imply that the closure $\mathbf{H} = \overline{\mathbf{H}}_0$ represents a selfadjoint operator in $\mathcal{H}$. For z in the resolvent set $\varrho(A_0)$ of A_0, the operator function

$$A_1 - z - T_{10}(A_0 - z)^{-1}T_{01}$$

is defined on $\mathcal{D}(T_{01})$ and has a bounded extension to the whole space $\mathcal{H}_1$. We denote this extension by

$$M_1(z) := A_1 - z - \overline{T_{10}(A_0 - z)^{-1}T_{01}} \tag{2.2}$$

and call it the *transfer function* associated to the operator $\mathbf{H}$. For $z \in \varrho(A_0)$, the operator $\mathbf{H}$ has the representation

$$\mathbf{H} = z\mathbf{I} + \begin{pmatrix} I_0 & 0 \\ G_{10}(z) & I_1 \end{pmatrix} \begin{pmatrix} A_0 - z & 0 \\ 0 & M_1(z) \end{pmatrix} \begin{pmatrix} I_0 & [G_{10}(\overline{z})]^* \\ 0 & I_1 \end{pmatrix} \tag{2.3}$$

where $\mathbf{I}$ stands for the identity operator in $\mathcal{H}$, I_j for the identity operators in $\mathcal{H}_j$, $j = 0, 1$, and

$$G_{10}(z) = T_{10}(A_0 - z)^{-1} = B_{10}A_0^{-1/2}\big[I_0 + z(A_0 - z)^{-1}\big]. \tag{2.4}$$

The domain of $\mathbf{H}$ reads

$$\mathcal{D}(\mathbf{H}) = \{x = (x_0, x_1) : x_0 \in \mathcal{H}_0, x_1 \in \mathcal{H}_1, x_0 + [G_{10}(\overline{z})]^*x_1 \in \mathcal{D}(A_0)\}. \tag{2.5}$$

Note that $x_0 + [G_{10}(\overline{z})]^*x_1 \in \mathcal{D}(A_0)$ implies $x_0 \in \mathcal{D}(A_0^{1/2}) \subset \mathcal{D}(T_{10})$. From (2.3) one finds that for $(x_0, x_1) \in \mathcal{D}(\mathbf{H})$

$$\mathbf{H}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} A_0\big(x_0 + [G_{10}(\overline{z})]^*x_1\big) - z[G_{10}(\overline{z})]^*x_1 \\ T_{10}x_0 + A_1x_1 \end{pmatrix}. \tag{2.6}$$

The set (2.5) and the representation (2.6) (for fixed x_0, x_1) do not depend on the choice of $z \in \varrho(A_0)$ in the description above. In particular, choosing $z = 0$ we find

$$\begin{cases} \mathcal{D}(\mathbf{H}) = (x_0, x_1) \in \mathcal{H} : x_0 + A_0^{-1/2}B_{01}x_1 \in \mathcal{D}(A_0), \\ \mathbf{H}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} A_0\big(x_0 + A_0^{-1/2}B_{01}x_1\big) \\ T_{10}x_0 + A_1x_1 \end{pmatrix}. \end{cases} \tag{2.7}$$

By definition

$$\begin{aligned} M_1(z)|\mathcal{D}(T_{01}) &= A_1 - z + T_{10}(z - A_0)^{-1}T_{01} \\ &= A_1 - z + B_{10}A_0(z - A_0)^{-1}B_{01} = \widetilde{A}_1 - z + V_1(z) \end{aligned}$$

where

$$\widetilde{A}_1 := A_1 - B_{10}B_{01}, \quad V_1(z) := zB_{10}(z - A_0)^{-1}B_{01}. \tag{2.8}$$

$\widetilde{A}_1$ is a bounded selfadjoint operator in $\mathcal{H}_1$, $V_1(z)$ a bounded operator function on $\varrho(A_0)$. It follows that

$$M_1(z) = \widetilde{A}_1 - z + V_1(z).$$

The transfer function M_1, considered on $\varrho(A_0)$, represents a particular case of a holomorphic operator-valued function. In the present work we use the standard definition of holomorphy of an operator-valued function with respect to the operator norm topology (see, e.g., [2]). One can extend the usual definitions of the spectrum and its components to operator-valued functions. The function M_1 is holomorphic at least in the resolvent set $\varrho(A_0)$ of the entry A_0. Since the inverse transfer function $\big[M_1(z)\big]^{-1}$ coincides with the right lower block component $\mathbf{R}_{11}(z)$ of the resolvent $\mathbf{R}(z) = (\mathbf{H} - z)^{-1}$, it is holomorphic at least in the resolvent set $\varrho(\mathbf{H})$.

Let E_0 be the spectral measure for the entry A_0, $A_0 = \int\limits_{\sigma(A_0)} \lambda\, \mathrm{d}E_0(\lambda)$, $\sigma(A_0) \subset \mathbb{R}$. Then the function $V_1(z)$ can be written

$$V_1(z) = B_{10} \int\limits_{\sigma(A_0)} \frac{z}{z-\mu}\, \mathrm{d}E_0(\mu)B_{01}.$$

Thus, it is convenient to introduce the quantities

$$\operatorname{Var}_\theta(B) := \sup_{\{\delta_k,\mu_k\in\delta_k\}} \sum_k (1 + |\mu_k|)^{-\theta} \|B_{10}E_0(\delta_k)B_{01}\|, \tag{2.9}$$

where θ is some real number and $\{\delta_k\}$ stands for a finite or countable complete system of Borel subsets of $\sigma(A_0)$ such that $\delta_k \cap \delta_l = \emptyset$, if $k \neq l$, and $\bigcup\limits_k \delta_k = \sigma(A_0)$. The points μ_k are arbitrarily chosen points of δ_k. The number $\operatorname{Var}_\theta(B)$ is called *weighted variation* of the operators B_{ij} with respect to the spectral measure E_0. Along with the "total" weighted variation $\operatorname{Var}_\theta(B)$, we use the "truncated" variations

$$\operatorname{Var}_\theta(B)|\Delta := \sup_{\{\delta_k,\mu_k\in\delta_k\cap\Delta\}} \sum_k (1 + |\mu_k|)^{-\theta} \|B_{10}E_0(\delta_k \cap \Delta)B_{01}\|,$$

where Δ is a certain Borel subset of $\sigma(A_0)$; $\operatorname{Var}_\theta(B)|\Delta \leqslant \operatorname{Var}_\theta(B)$.

Note that in contrast to [22] and [23], where the variation (2.9) was considered in case of $\theta = 0$, we now will mainly consider $\theta = 1$. Of course, the consideration of the variation $\operatorname{Var}_\theta(B)$ for $\theta \neq 0$ only makes sense when the entry A_0 is an unbounded operator.

We assume that the spectrum of the operator $\widetilde{A}_1$ may only intersect the continuous spectrum of A_0 and this intersection is realized on finitely many pairwise nonintersecting open intervals $\Delta_k^0 = (\mu_k^{(1)}, \mu_k^{(2)}) \subset \sigma_c(A_0)$, $\mu_k^{(1)} < \mu_k^{(2)}$, $k = 1, 2, \ldots, m$, $m < \infty$. Therefore, we assume that $\Delta_k^0 \cap \sigma(\widetilde{A}_1) \neq \emptyset$ for all $k = 1, 2, \ldots, m$ and $\sigma(\widetilde{A}_1) \cap \sigma'(A_0) = \emptyset$ where $\sigma'(A_0) = \sigma(A_0) \setminus \bigcup_{k=1}^{m} \Delta_k^0$. For the case considered in this work, $\mu_1^{(1)} > 0$ and $\mu_m^{(2)} \leqslant +\infty$. We shall suppose that the product $K_B(\mu) := B_{10} E^0(\mu) B_{01}$, where $E^0(\mu)$ stands for the spectral function of A_0, $E^0(\mu) = E_0\big((-\infty, \mu)\big)$, is differentiable in $\mu \in \Delta_k^0$, $k = 1, 2, \ldots, m$, in the operator norm topology. The derivative $K_B'(\mu)$ is non-negative,

$$K_B'(\mu) \geqslant 0, \tag{2.10}$$

since $K_B(\mu)$ is a non-decreasing function. Obviously,

$$\mathrm{Var}_\theta(B)|\Delta_k^0 = \int\limits_{\Delta_k^0} (1 + |\mu|)^{-\theta} \|K_B'(\mu)\| \, \mathrm{d}\mu.$$

Further, we suppose that the function $K_B'(\mu)$ is continuous within the intervals Δ_k^0 and, moreover, that it admits analytic continuation from each of these intervals to a simply connected domain situated, say, in $\mathbb{C}^-$. For the interval Δ_k^0, let this domain be called D_k^-. We assume that the boundary of each domain D_k^- includes the entire spectral interval Δ_k^0 and the domains D_k^- and D_j^- for different k and j do not intersect each other. Since $K_B'(\mu)$ is a selfadjoint operator for $\mu \in \Delta_k^0$ and $\Delta_k^0 \subset \mathbb{R}$, the function $K_B'(\mu)$ admits an analytic continuation from Δ_k^0 into the domain D_k^+, symmetric to D_k^- with respect to the real axis, $D_k^+ = \{z : \overline{z} \in D_k^-\}$. For the continuation into D_k^+ we will use the same notation $K_B'(\mu)$. The selfadjointness of $K_B'(\mu)$ for $\mu \in \Delta_k^0$ implies $[K_B'(\mu)]^* = K_B'(\overline{\mu})$, $\mu \in D_k^{\pm}$. Also, we shall always suppose that the $K_B'(\mu)$ satisfies the following condition at the (finite) end points $\mu_k^{(1)}$, $\mu_k^{(2)}$ of the spectral intervals Δ_k^0:

$$\|K_B'(\mu)\| \ \leqslant \ C|\mu - \mu_k^{(i)}|^{\gamma}, \quad i = 1, 2, \, \mu \in D_k^{\pm},$$

with some $C > 0$ and $\gamma \in (-1, 0]$.

Let $l = (l_1, l_2, \ldots, l_m)$ be a multi-index having the components $l_k = +1$ or $l_k = -1$, $k = 1, 2, \ldots, m$. In what follows we consider the domains $D_l = \bigcup_{k=1}^{m} D_k^{l_k}$. Let $\Gamma_k^{l_k}$ be a rectifiable Jordan curve in $D_k^{l_k}$ resulting from continuous deformation of the interval Δ_k^0, the (finite) end points of this interval being fixed (see Figure 1).

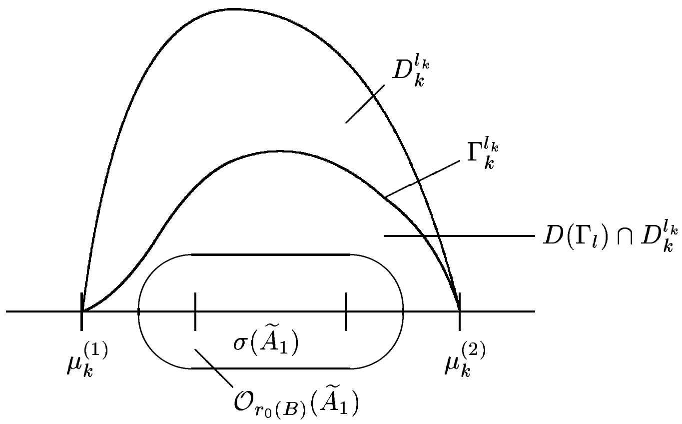

Figure 1: Domain $D_k^{l_k}$, contour $\Gamma_k^{l_k}$ (both for $l_k = +1$), domain $D(\Gamma_l)\cap D_k^{l_k}$ and location of the domain $\mathcal{O}_{r_0(B)}(\widetilde{A}_1)$ in the neighbourhood of the interval $[\mu_k^{(1)}, \mu_k^{(2)}]$.

With the exception of the end points, the closure $\overline{\Gamma}_k^{l_k}$ of the contour $\Gamma_k^{l_k}$ should have no other common points with the set $\sigma_c(A_0)$. By Γ_l, $l = (l_1, l_2, \ldots, l_m)$, we shall denote the union of the contours $\Gamma_k^{l_k}$, $\Gamma_l = \bigcup\limits_{k=1}^{m} \Gamma_k^{l_k}$. Finally, by $D(\Gamma_l)$ we will denote the subdomain of D_l bounded by the set $\bigcup\limits_{k=1}^{m} \Delta_k^0$ and contour Γ_l.

As mentioned above, in the following we deal with the variation $\mathrm{Var}_1(B)$. We extend the definition of this variation to the set $\sigma'(A_0) \bigcup \Gamma_l$ by introducing the modified variation

$$\mathrm{Var}_1(B, \Gamma_l) := \mathrm{Var}_1(B)|\sigma'(A_0) + \int\limits_{\Gamma_l} (1 + |\mu|)^{-1} \|K_B'(\mu)\| \, |\mathrm{d}\mu|, \tag{2.11}$$

where $|\mathrm{d}\mu|$ denotes the Lebesgue measure on Γ_l. We suppose that the operators B_{ij} are such that there exists a contour (exist contours) Γ_l on which the value $\mathrm{Var}_1(B, \Gamma_l)$ is finite, i.e., $\mathrm{Var}_1(B, \Gamma_l) < \infty$, including also the case of an unbounded set $\bigcup\limits_{k=1}^{m} \Delta_k^0$.

The contours Γ_l satisfying the condition $\mathrm{Var}_1(B, \Gamma_l) < \infty$ are said to be K_B-bounded contours.

Note that, in the case of unbounded A_0, the condition of boundedness of $\mathrm{Var}_1(B, \Gamma_l)$ is much weaker than the condition of boundedness of $\mathrm{Var}_0(B, \Gamma_l)$ used in [22] and [23].

LEMMA 2.1. *The analytic continuation of the transfer function $M_1(z)$, $z \in \mathbb{C} \setminus \sigma(A_0)$, through the spectral intervals Δ_k^0 into the subdomain $D(\Gamma_l) \subset D_l$ is given by*

$$M_1(z, \Gamma_l) := \widetilde{A}_1 - z + V_1(z, \Gamma_l) \tag{2.12}$$

where

$$\begin{aligned} V_1(z, \Gamma_l) &:= \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)\frac{z}{z-\mu} \\ &:= \int\limits_{\sigma'(A_0)} B_{10}\frac{z}{z-\mu}\,\mathrm{d}E_0(\mu)B_{01} + \int\limits_{\Gamma_l} K_B'(\mu)\frac{z}{z-\mu}\,\mathrm{d}\mu. \end{aligned} \tag{2.13}$$

For $z \in D_k^{l_k} \cap D(\Gamma_l)$ the function $M_1(z, \Gamma_l)$ may be written as

$$M_1(z, \Gamma_l) = M_1(z) + 2\pi \mathrm{i} l_k z K_B'(z). \tag{2.14}$$

Proof. Obviously, the function (2.13) is well defined due to the K_B-boundedness of the contour Γ_l and since for all $z \in \mathbb{C} \setminus [\sigma'(A_0) \cup \Gamma_l]$ there exist a number $c(z) > 0$ such that the estimate $\left|(z-\mu)^{-1}\right| < c(z)\big(1+|\mu|\big)^{-1}$ holds where μ runs through $\sigma'(A_0) \cup \Gamma_l$. Thus, the proof of this lemma is reduced to the observation that the function $M_1(z, \Gamma_l)$ is holomorphic for $z \in \mathbb{C} \setminus [\sigma'(A_0) \cup \Gamma_l]$ and coincides with $M_1(z)$ for $z \in \mathbb{C} \setminus [\sigma'(A_0) \cup \overline{D(\Gamma_l)}]$. The equation (2.14) is obtained from (2.13) using the Residue Theorem. ∎

The formula (2.14) shows that in general the transfer function M_1 has a Riemann surface with at least 2^m sheets. The sheet of the complex plane where the transfer function M_1 together with the resolvent $\mathbf{R}$ is initially considered is said to be the physical sheet. The remaining sheets of the Riemann surface of $M_1(z)$ are said to be unphysical sheets. In the present work we deal with the unphysical sheets neighboring the physical one, i.e., with the sheets connected through the intervals Δ_k^0 for some $k \in \{1, 2, \dots, m\}$ directly to the physical sheet.

REMARK 2.2. For $z \in \mathbb{C} \setminus \big(\sigma'(A_0) \cup \Gamma_l\big)$, the equation (2.13) defines values of the function $V_1(\,\cdot\,, \Gamma_l)$ in the space of bounded operators in $\mathcal{H}_1$. As mentioned above, the inverse transfer function $\big[M_1(z)\big]^{-1}$ coincides with the right lower block component $\mathbf{R}_{11}(z)$ of the resolvent $\mathbf{R}(z) = (\mathbf{H} - z)^{-1}$ and, thus, it is holomorphic in $\mathbb{C} \setminus \sigma(\mathbf{H}) \supset \mathbb{C} \setminus \mathbb{R}$. Since $M_1(z, \Gamma_l)$ coincides with $M_1(z)$ for all $z \in \mathbb{C} \setminus \big[\sigma'(A_0) \cup \overline{D(\Gamma_l)}\big]$, one concludes that $[M_1(z, \Gamma_l)]^{-1}$ exists as a bounded operator and is holomorphic in z at least for $z \in \mathbb{C} \setminus \big[\sigma(\mathbf{H}) \cup \overline{D(\Gamma_l)}\big]$.

REMARK 2.3. Note that in contrast to the papers of V. Adamyan and H. Langer ([1]), V. Adamyan, H. Langer, R. Mennicken and J. Saurer ([2]) and R. Mennicken and A.A. Shkalikov ([24]) we can also consider the case when the spectra of $\widetilde{A}_1$ and A_0 alternate and do not intersect or intersect as described above.

3. THE BASIC EQUATION

In the following we use integrals of the form

$$\int_{\sigma} X(\mu)\, \mathrm{d}E(\mu) Y(\mu)$$

where E is the spectral function of a selfadjoint operator and σ a part of the spectrum of this operator. For the definition and some properties of integrals of this form we refer the reader to Appendix B in [23], cf. also [2].

If an operator-valued function $F : \sigma'(A_0) \cup \Gamma \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$ is continuous on a K_B-bounded contour Γ and satisfies the condition

$$\sup_{\mu \in \sigma'(A_0) \cup \Gamma} (1 + |\mu|) \|F(\mu)\| < \infty$$

and a Lipschitz condition on $\sigma'(A_0)$, then the integral

$$\int_{\sigma'(A_0) \cup \Gamma} K_B(\mathrm{d}\mu) F(\mu) := \int_{\sigma'(A_0)} \mathrm{d}K_B(\mu) F(\mu) + \int_{\Gamma} K_B'(\mu) F(\mu)\, \mathrm{d}\mu \tag{3.1}$$

exists in the sense of the operator norm topology (see Lemma 7.2 in [2]) and

$$\left\| \int_{\sigma'(A_0) \cup \Gamma} K_B(\mathrm{d}\mu) F(\mu) \right\| \leqslant \mathrm{Var}_1(B, \Gamma) \sup_{\mu \in \sigma'(A_0) \cup \Gamma} (1 + |\mu|) \|F(\mu)\|. \tag{3.2}$$

In particular, if $F(z) = Y(Y - zI_1)^{-1}$, where Y stands for an arbitrary bounded operator in $\mathcal{H}_1$ such that the spectrum of Y is separated from the set $\sigma'(A_0) \cup \Gamma$, then one may define the operator

$$V_1(Y, \Gamma) := \int_{\sigma'(A_0) \cup \Gamma} K_B(\mathrm{d}\mu) Y (Y - \mu)^{-1}. \tag{3.3}$$

This operator is bounded, $V_1(Y, \Gamma) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$, and, because of (3.2), its norm admits the estimate

$$\|V_1(Y, \Gamma)\| \leqslant \mathrm{Var}_1(B, \Gamma) \|Y\| \sup_{\mu \in \sigma'(A_0) \cup \Gamma} (1 + |\mu|) \|(Y - \mu)^{-1}\|. \tag{3.4}$$

In what follows we consider the equation

$$Y = \widetilde{A}_1 + V_1(Y, \Gamma). \tag{3.5}$$

This equation possesses the following special property: If an operator H_1 is a solution of (3.5) and u_1 is an eigenvector of H_1, $H_1 u_1 = z u_1$, then

$$\begin{aligned} z u_1 = \widetilde{A}_1 u_1 + V_1(H_1, \Gamma) u_1 &= \widetilde{A}_1 u_1 + \int_{\sigma'(A_0) \cup \Gamma} K_B(\mathrm{d}\mu) H_1 (H_1 - \mu)^{-1} u_1 \\ &= \widetilde{A}_1 u_1 + \int_{\sigma'(A_0) \cup \Gamma} K_B(\mathrm{d}\mu) \frac{z}{z - \mu} u_1 = \widetilde{A}_1 u_1 + V_1(z, \Gamma) u_1. \end{aligned}$$

This means that any eigenvalue z of such an operator H_1 is automatically an eigenvalue for the analytically continued transfer function $M_1(z,\Gamma)$ and u_1 is a corresponding eigenvector. Thus, having found the solution(s) of the equation (3.5), one obtains an effective way of studying the spectral properties of the transfer function $M_1(z,\Gamma)$. It is convenient to rewrite the equation (3.5) in the form

$$X = V_1(\widetilde{A}_1 + X, \Gamma), \tag{3.6}$$

where $X := Y - \widetilde{A}_1$.

Let Γ be a K_B-bounded contour and the spectrum of the operator $\widetilde{A}_1$ be separated from the set $\sigma'(A_0) \cup \Gamma$, i.e.,

$$d_0(\Gamma) := \operatorname{dist}\{\sigma(\widetilde{A}_1), \sigma'(A_0) \cup \Gamma\} > 0. \tag{3.7}$$

Since $\widetilde{A}_1$ is bounded, it is obvious that the following quantity

$$\begin{aligned} \operatorname{Var}_{\tilde{A}_1}(B,\Gamma) := \sup_{\{\delta_k, \mu_k \in \delta_k \cap \sigma'(A_0)\}} \sum_k \frac{\|B_{10} E_0\big(\delta_k \cap \sigma'(A_0)\big) B_{01}\|}{\operatorname{dist}\{\mu_k, \sigma(\widetilde{A}_1)\}} \\ + \int\limits_{\Gamma} \frac{\|K'_B(\mu)\|}{\operatorname{dist}\{\mu, \sigma(\widetilde{A}_1)\}} |\,\mathrm{d}\mu| \end{aligned} \tag{3.8}$$

is finite,

$$\operatorname{Var}_{\tilde{A}_1}(B,\Gamma) \leqslant \operatorname{Var}_1(B,\Gamma) \sup_{\mu \in \sigma'(A_0)\cup\Gamma} (1 + |\mu|)\big[\operatorname{dist}\{\mu, \sigma(\widetilde{A}_1)\}\big]^{-1} < \infty, \tag{3.9}$$

where $\{\delta_k\}$ is as in the definition (2.9) of $\operatorname{Var}_\theta(B)$ and μ_k are arbitrarily chosen points of $\delta_k \cap \sigma'(A_0)$.

It is more convenient to make the subsequent estimations in terms of the variation $\operatorname{Var}_{\tilde{A}_1}(B,\Gamma)$, rather then in terms of the variation $\operatorname{Var}_1(B,\Gamma)$.

Theorem 3.1. *Let $\widetilde{A}_1$ be a bounded operator, the contour Γ be K_B-bounded and*

$$\operatorname{Var}_{\tilde{A}_1}(B,\Gamma) < 1, \quad \operatorname{Var}_{\tilde{A}_1}(B,\Gamma)\|\widetilde{A}_1\| < \frac{1}{4} d_0(\Gamma)[1 - \operatorname{Var}_{\tilde{A}_1}(B,\Gamma)]^2. \tag{3.10}$$

Let

$$\begin{aligned} r_{\min}(\Gamma) := \frac{1}{2} d_0(\Gamma)[1 - \operatorname{Var}_{\tilde{A}_1}(B,\Gamma)] \\ - \sqrt{\frac{1}{4} d_0^2(\Gamma)[1 - \operatorname{Var}_{\tilde{A}_1}(B,\Gamma)]^2 - d_0(\Gamma)\operatorname{Var}_{\tilde{A}_1}(B,\Gamma)\|\widetilde{A}_1\|} \end{aligned} \tag{3.11}$$

and

$$r_{\max}(\Gamma) := d_0(\Gamma) - \sqrt{\operatorname{Var}_{\tilde{A}_1}(B,\Gamma) d_0(\Gamma)[d_0(\Gamma) + \|\widetilde{A}_1\|]}\,. \tag{3.12}$$

Then the equation (3.6) *is uniquely solvable in any closed ball*

$$\mathcal{S}_1(r) := \big\{X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1) : \|X\| \leqslant r\big\}$$

where

$$r_{\min}(\Gamma) \leqslant r < r_{\max}(\Gamma). \tag{3.13}$$

The solution X of the equation (3.6) *is the same for any r satisfying* (3.13) *and in fact it belongs to the smallest ball $\mathcal{S}_1(r_{\min})$, $\|X\| \leqslant r_{\min}(\Gamma)$.*

Proof. Let $\Phi(X) = V_1(\widetilde{A}_1 + X, \Gamma)$ with $X \in \mathcal{S}_1(r)$. First, we search for a condition under which the function Φ is a contracting mapping of the ball $\mathcal{S}_1(r)$ into itself. Since in view of (3.13) and (3.12) the condition $0 \leqslant r < d_0$, $d_0 := d_0(\Gamma)$ holds, the spectrum of the operator $\widetilde{A}_1 + X$ does not intersect the set $\sigma'(A_0) \cup \Gamma$. This means that for all $\mu \in \sigma'(A_0) \cup \Gamma$ the resolvent $(\widetilde{A}_1 + X - \mu)^{-1}$ exists as a bounded operator in $\mathcal{H}_1$. Moreover,

$$\begin{aligned} \|(\widetilde{A}_1 + X - \mu)^{-1}\| &= \big\|\big(I_1 + (\widetilde{A}_1 - \mu)^{-1} X\big)^{-1} (\widetilde{A}_1 - \mu)^{-1}\big\| \\ &\leqslant \frac{1}{\operatorname{dist}\{\mu, \sigma(\widetilde{A}_1)\} - \|X\|}. \end{aligned} \tag{3.14}$$

Note also that

$$\frac{\operatorname{dist}\{\mu, \sigma(\widetilde{A}_1)\}}{\operatorname{dist}\{\mu, \sigma(\widetilde{A}_1)\} - \|X\|} \leqslant \frac{d_0}{d_0 - r}. \tag{3.15}$$

Thus, we can write

$$\begin{aligned} \|\Phi(X)\| \leqslant{} & \big(\|\widetilde{A}_1\| + \|X\|\big) \\ & \Bigg(\sup_{\{\delta_k, \mu_k \in \delta_k \cap \sigma'(A_0)\}} \sum_k \|B_{10} E_0\big(\delta_k \cap \sigma'(A_0)\big) B_{01}\| \, \|(\widetilde{A}_1 + X - \mu_k)^{-1}\| \\ & + \int_\Gamma \|K'_B(\mu)\| \, \|(\widetilde{A}_1 + X - \mu)^{-1}\| \, \mathrm{d}\mu \Bigg) \\ \leqslant{} & \big(\|\widetilde{A}_1\| + \|X\|\big) \\ & \Bigg(\sup_{\{\delta_k, \mu_k \in \delta_k \cap \sigma'(A_0)\}} \sum_k \frac{\|B_{10} E_0\big(\delta_k \cap \sigma'(A_0)\big) B_{01}\|}{\operatorname{dist}\{\mu_k, \sigma(\widetilde{A}_1)\}} \frac{\operatorname{dist}\{\mu_k, \sigma(\widetilde{A}_1)\}}{\operatorname{dist}\{\mu_k, \sigma(\widetilde{A}_1)\} - \|X\|} \\ & + \int_\Gamma \frac{\|K'_B(\mu)\|}{\operatorname{dist}\{\mu, \sigma(\widetilde{A}_1)\}} \frac{\operatorname{dist}\{\mu, \sigma(\widetilde{A}_1)\}}{\operatorname{dist}\{\mu, \sigma(\widetilde{A}_1)\} - \|X\|} \, \mathrm{d}\mu \Bigg) \\ \leqslant{} & \operatorname{Var}_{\widetilde{A}_1}(B, \Gamma)(\|\widetilde{A}_1\| + r) \frac{d_0}{d_0 - r}. \end{aligned}$$

At the same time we conclude

$$\begin{aligned} & \|\Phi(X) - \Phi(Y)\| \\ & = \Bigg\| \int_{\sigma'(A_0) \cup \Gamma} K_B(\mathrm{d}\mu) \Big[(\widetilde{A}_1 + X)(\widetilde{A}_1 + X - \mu)^{-1} - (\widetilde{A}_1 + Y)(\widetilde{A}_1 + Y - \mu)^{-1} \Big] \Bigg\| \\ & \leqslant \Bigg\| \int_{\sigma'(A_0) \cup \Gamma} K_B(\mathrm{d}\mu)(\widetilde{A}_1 + X) \Big[(\widetilde{A}_1 + X - \mu)^{-1} - (\widetilde{A}_1 + Y - \mu)^{-1} \Big] \Bigg\| \\ & \quad + \Bigg\| \int_{\sigma'(A_0) \cup \Gamma} K_B(\mathrm{d}\mu)(X - Y)(\widetilde{A}_1 + Y - \mu)^{-1} \Bigg\| \end{aligned}$$

for $X, Y \in \mathcal{S}_1(r)$. Since

$$(\widetilde{A}_1 + X - \mu)^{-1} - (\widetilde{A}_1 + Y - \mu)^{-1} = (\widetilde{A}_1 + X - \mu)^{-1}(Y - X)(\widetilde{A}_1 + Y - \mu)^{-1},$$

we obtain using the inequality (3.14) that

$$\|\Phi(X) - \Phi(Y)\| \leqslant d_0 \left(\frac{\mathrm{Var}_{\tilde{A}_1}(B,\Gamma)(\|\widetilde{A}_1\| + r)}{(d_0 - r)^2} + \frac{\mathrm{Var}_{\tilde{A}_1}(B,\Gamma)}{d_0 - r} \right) \|X - Y\|.$$

Thus, Banach's Fixed Point theorem can be applied to the ball $\mathcal{S}_1(r)$ if the conditions

$$\mathrm{Var}_{\tilde{A}_1}(B,\Gamma)(\|\widetilde{A}_1\| + r)\frac{d_0}{d_0 - r} \ \leqslant\ r \tag{3.16}$$

and

$$d_0 \mathrm{Var}_{\tilde{A}_1}(B,\Gamma) \left(\frac{\|\widetilde{A}_1\| + r}{(d_0 - r)^2} + \frac{1}{d_0 - r} \right) < 1 \tag{3.17}$$

are fulfilled. Under the conditions (3.10) the inequalities (3.16) and (3.17) considered together are equivalent to the condition (3.13). Thus, if this condition is fulfilled, then the equation (3.6) has a solution in any ball $\mathcal{S}_1(r)$ with r satisfying (3.13) and this solution is unique. This means that the solution is the same for all the radii satisfying (3.13). Moreover, it belongs to the ball $\mathcal{S}_1(r_{\min})$ with the radius $r_{\min}$ given by (3.11). ∎

The following statement follows immediately from the conditions (3.10).

REMARK 3.2. The values of $r_{\min}(\Gamma)$ and $r_{\max}(\Gamma)$ satisfy the estimates

$$r_{\min}(\Gamma) < \frac{1}{2} d_0(\Gamma)[1 - \mathrm{Var}_{\tilde{A}_1}(B,\Gamma)] < r_{\max}(\Gamma).$$

THEOREM 3.3. *Let the conditions of Theorem* 3.1 *be fulfilled for a K_B-bounded contour $\Gamma \subset D_l$ and let X be the solution of the equation* (3.6). *Then X coincides with the analogous solution $\widetilde{X}$ for any other K_B-bounded contour $\widetilde{\Gamma} \subset D_l$ satisfying the estimates*

$$\mathrm{Var}_{\tilde{A}_1}(B,\widetilde{\Gamma}) < 1 \quad \textit{and} \quad \mathrm{Var}_{\tilde{A}_1}(B,\widetilde{\Gamma})\|\widetilde{A}_1\| < \frac{1}{4}\widetilde{d}_0[1 - \mathrm{Var}_{\tilde{A}_1}(B,\widetilde{\Gamma})]^2$$

where $0 < \widetilde{d}_0 = \mathrm{dist}\{\sigma(\widetilde{A}_1), \sigma'(A_0) \cup \widetilde{\Gamma}\} \leqslant d_0(\Gamma)$. Moreover, this solution satisfies the inequality $\|X\| \leqslant r_0(B)$ where

$$r_0(B) := \inf\bigl\{ r_{\min}(\Gamma_l) : \mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l) < 1 \ \textit{and}\ \omega(B,\Gamma_l) > 0 \bigr\}$$

with $r_{\min}(\Gamma_l)$ given by (3.11) *and*

$$\omega(B,\Gamma_l) := d_0(\Gamma_l)\bigl[1 - \mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)\bigr]^2 - 4\|\widetilde{A}_1\|\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l).$$

The value of $r_0(B)$ does not depend on l.

Proof. The proof of this theorem is reduced to an appropriate continuous deformation of the integration paths (see the proof of Theorem 3.3 in [23]). The essential point consists in checking the independence of the radius $r_0(B)$ of the multi-index l. To this end, we consider an arbitrary K_B-bounded contour $\Gamma_l \subset D_l$, $\Gamma_l = \bigcup_{k=1}^{m} \Gamma_k^{l_k}$. Denote by $\Gamma_{l'}$ the contour resulting from Γ_l by replacing $\Gamma_k^{l_k}$ by the curves $\Gamma_k^{(-l_k)} = \{\mu : \overline{\mu} \in \Gamma_k^{l_k}\}$, symmetric to $\Gamma_k^{l_k}$ with respect to the real axis. Obviously, such replacements generate, additionally to Γ_l, $2^m - 1$ different contours $\Gamma_{l'}$ where $l' = (l'_1, l'_2, \ldots, l'_m)$ with $l'_k = \pm l_k$, $k = 1, 2, \ldots, m$. For any such contour the value of $\mathrm{Var}_{\tilde{A}_1}(B, \Gamma_{l'})$ is the same, namely $\mathrm{Var}_{\tilde{A}_1}(B, \Gamma_{l'}) = \mathrm{Var}_{\tilde{A}_1}(B, \Gamma_l)$, since the replacement of Γ_l with $\Gamma_{l'}$ does not change $\int_{\Gamma_l} \|K'_B(\mu)\| [\mathrm{dist}\{\mu, \sigma(\widetilde{A}_1)\}]^{-1} |\,\mathrm{d}\mu|$. But this means that $r_0(B)$ does not depend on l. ∎

So, for a given holomorphy domain D_l, the solutions X and H_1, $H_1 = \widetilde{A}_1 + X$, do not depend on the K_B-bounded contours $\Gamma_l \subset D_l$ satisfying the conditions (3.10). But when the index l changes, X and H_1 can also change. For this reason we shall supply them in the following, when it is necessary, with the index l writing $X^{(l)}$ and $H_1^{(l)}$, $H_1^{(l)} = \widetilde{A}_1 + X^{(l)}$. Therefore, Theorem 3.1 guarantees the existence of 2^m solutions $X^{(l)}$ of the basic equation (3.6) and hence 2^m corresponding solutions $H_1^{(l)}$ of the basic equation (3.5). Surely, the equations (3.5) and (3.6) are nonlinear equations and, outside the balls $\|X\| < r_{\max}(\Gamma_l)$, they may, in principle, have other solutions, different from the operators $X^{(l)}$ or $H_1^{(l)}$. In the following we only deal with the solutions $X^{(l)}$ or $H_1^{(l)}$ constructed above.

4. A FACTORIZATION THEOREM

As a next step, we prove a factorization theorem for the transfer function $M_1(z, \Gamma_l)$. This statement will play an important role when we study the spectral properties of the operators $H_1^{(l)}$.

THEOREM 4.1. *Let Γ_l be a K_B-bounded contour satisfying the conditions* (3.10). *Suppose $X^{(l)}$ is the solution of the basic equation* (3.6) *for $\Gamma = \Gamma_l$, $\|X^{(l)}\| \leqslant r_0(B)$, and $H_1^{(l)} = \widetilde{A}_1 + X^{(l)}$. Then, for $z \in \mathbb{C} \setminus (\sigma'(A_0) \cup \Gamma_l)$, the transfer function $M_1(z, \Gamma_l)$ admits the factorization*

$$M_1(z, \Gamma_l) = W_1(z, \Gamma_l)(H_1^{(l)} - z) \tag{4.1}$$

where $W_1(z, \Gamma_l)$ is a bounded operator in $\mathcal{H}_1$,

$$\begin{aligned} W_1(z, \Gamma_l) &= I_1 - \int_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)(H_1^{(l)} - \mu)^{-1} \\ &\quad + z \int_{\sigma'(A_0)\cup\Gamma_l} (z-\mu)^{-1} K_B(\mathrm{d}\mu)(H_1^{(l)} - \mu)^{-1} \\ &= I_1 + \int_{\sigma'(A_0)\cup\Gamma_l} \frac{\mu}{z-\mu} K_B(\mathrm{d}\mu)(H_1^{(l)} - \mu)^{-1}. \end{aligned} \tag{4.2}$$

If $\operatorname{dist}\{z,\sigma(\widetilde{A}_1)\} \leqslant d_0(\Gamma_l)[1-\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)]/2$, *then the operator* $W_1(z,\Gamma_l)$ *is boundedly invertible and*

$$\big\|[W_1(z,\Gamma_l)]^{-1}\big\| \leqslant \left(1-\frac{4\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)\big[d_0(\Gamma_l)+\|\widetilde{A}_1\|\big]}{d_0(\Gamma_l)\big[1+\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)\big]^2}\right)^{-1}<\infty. \tag{4.3}$$

Proof. First, we prove the formula (4.1). Note that, according to (3.3) and (3.6),

$$\widetilde{A}_1 = H_1^{(l)} - V_1(\widetilde{A}_1+X^{(l)},\Gamma_l) = H_1^{(l)} - \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)H_1^{(l)}(H_1^{(l)}-\mu)^{-1}. \tag{4.4}$$

Thus, in view of the representations (2.12) and (2.13), the function $M_1(z,\Gamma_l)$ can be written as

$$\begin{aligned}
M_1(z,\Gamma_l) &= \widetilde{A}_1 - z + V_1(z,\Gamma_l) = \widetilde{A}_1 - z + \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)\frac{z}{z-\mu}\\
&= H_1^{(l)} - z - \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)\left[H_1^{(l)}(H_1^{(l)}-\mu)^{-1}-\frac{z}{z-\mu}\right]\\
&= H_1^{(l)} - z - \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)(H_1^{(l)}-\mu)^{-1}\big(H_1^{(l)}-z\big)\\
&\quad + z\int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)\left[\frac{1}{z-\mu}-(H_1^{(l)}-\mu)^{-1}\right]\\
&= \big(H_1^{(l)}-z\big) - \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)(H_1^{(l)}-\mu)^{-1}\big(H_1^{(l)}-z\big)\\
&\quad + z\int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)\frac{(H_1^{(l)}-\mu)^{-1}}{z-\mu}\big(H_1^{(l)}-z\big)
\end{aligned}$$

which proves the equation (4.1). The boundedness of the operator $W_1(z,\Gamma_l)$ for $z\in\mathbb{C}\setminus(\sigma'(A_0)\cup\Gamma_l)$ is obvious.

Further, we prove that the factor $W_1(z,\Gamma_l)$ is a boundedly invertible operator if the condition $\operatorname{dist}\{z,\sigma(\widetilde{A}_1)\} \leqslant d_0(\Gamma_l)[1-\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)]/2$ holds. Indeed, due to (3.14) and the definition of $d_0(\Gamma_l)$, we have

$$\Bigg\|\int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)(H_1^{(l)}-\mu)^{-1}\Bigg\| \leqslant \operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)\frac{d_0(\Gamma_l)}{d_0(\Gamma_l)-\|X^{(l)}\|}. \tag{4.5}$$

The inequality $\|X^{(l)}\|\leqslant r_{\min}(\Gamma_l)$ and Remark 3.2 yield that

$$\|X^{(l)}\| < \frac{1}{2}d_0(\Gamma_l)[1-\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)].$$

Thus, (4.5) implies

$$\Bigl\| \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)(H_1^{(l)}-\mu)^{-1} \Bigr\| \leqslant \frac{2\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)}{1+\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)}. \tag{4.6}$$

Again, using inequality (3.14) and Remark 3.2, we find

$$\begin{aligned}
&\Bigl\| z \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)(H_1^{(l)}-\mu)^{-1}(z-\mu)^{-1} \Bigr\| \\
&\qquad \leqslant |z|\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l) \sup_{\mu\in\sigma'(A_0)\cup\Gamma_l} \Bigl[\mathrm{dist}\{\mu,\sigma(\widetilde{A}_1)\}\|(H_1^{(l)}-\mu)^{-1}\|\,|z-\mu|^{-1}\Bigr] \\
&\qquad \leqslant |z|\frac{2\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)}{1+\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)} \sup_{\mu\in\sigma'(A_0)\cup\Gamma_l} |z-\mu|^{-1}.
\end{aligned}$$

The inequality $\mathrm{dist}\{z,\sigma(\widetilde{A}_1)\} \leqslant d_0(\Gamma_l)[1-\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)]/2$ yields that

$$|z| \leqslant \|\widetilde{A}_1\| + \mathrm{dist}\bigl\{z,\sigma(\widetilde{A}_1)\bigr\} \leqslant \|\widetilde{A}_1\| + \frac{1}{2}d_0(\Gamma_l)[1-\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)]$$

and one obtains for $\mu\in\sigma'(A_0)\cup\Gamma_l$ that

$$\begin{aligned}
|z-\mu| &\geqslant \mathrm{dist}\{z,\sigma'(A_0)\cup\Gamma_l\} \\
&\geqslant \mathrm{dist}\{\sigma(\widetilde{A}_1),\sigma'(A_0)\cup\Gamma_l\} - \mathrm{dist}\{z,\sigma(\widetilde{A}_1)\} \\
&\geqslant d_0(\Gamma_l) - \frac{1}{2}d_0(\Gamma_l)[1-\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)].
\end{aligned}$$

Thus,

$$\sup_{\mu\in\sigma'(A_0)\cup\Gamma_l} |z-\mu|^{-1} \leqslant \frac{2}{d_0(\Gamma_l)[1+\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)]}.$$

Hence for $\mathrm{dist}\{z,\sigma(\widetilde{A}_1)\} \leqslant d_0(\Gamma_l)[1-\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)]/2$

$$\begin{aligned}
&\|W_1(z,\Gamma_l)-I_1\| \\
&\qquad \leqslant \frac{2\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)}{1+\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)} + \frac{4\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)\Bigl\{\|\widetilde{A}_1\| + \frac{1}{2}d_0(\Gamma_l)[1-\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)]\Bigr\}}{d_0(\Gamma_l)[1+\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)]^2}
\end{aligned}$$

and thus

$$\|W_1(z,\Gamma_l)-I_1\| \leqslant \frac{4\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)[d_0(\Gamma_l)+\|\widetilde{A}_1\|]}{d_0(\Gamma_l)[1+\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)]^2} < 1. \tag{4.7}$$

Note that the inequality

$$\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)[d_0(\Gamma_l)+\|\widetilde{A}_1\|] < \frac{1}{4}d_0(\Gamma_l)[1+\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)]^2$$

is equivalent to

$$\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)\|\widetilde{A}_1\| < \frac{1}{4}d_0(\Gamma_l)\left[1-\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)\right]^2$$

which holds by the second inequality in (3.10). We conclude from the estimate (4.7) that $W_1(z,\Gamma_l)$ is invertible and that

$$\begin{aligned}\big\|W_1(z,\Gamma_l)^{-1}\big\| &\leqslant \big(1-\|W_1(z,\Gamma_l)\|\big)^{-1}\\ &\leqslant \left(1-\frac{4\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma_l)[d_0(\Gamma_l)+\|\widetilde{A}_1\|]}{d_0(\Gamma_l)[1+\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma_l)]^2}\right)^{-1}<\infty. \quad\blacksquare\end{aligned}$$

It is easy to write some simple but useful relations between the operators $H_1^{(l)}$. In particular, we derive such relations between $H_1^{(l)}$ and $H_1^{(-l)}$, $(-l)=(-l_1,-l_2,\ldots,-l_m)$. According to our convention, $\Gamma_{(-l)}\subset D_{(-l)}$ is the contour which is obtained from Γ_l by replacing all the components $\Gamma_k^{l_k}$ by the conjugate contours $\Gamma_k^{(-l_k)}$.

The following theorems can be proved in the same way as Theorem 4.4 and Theorem 4.7 in [23].

Theorem 4.2. *The spectrum $\sigma(H_1^{(l)})$ of the operator $H_1^{(l)}=\widetilde{A}_1+X^{(l)}$ belongs to the closed $r_0(B)$-neighbourhood*

$$\mathcal{O}_{r_0(B)}(\widetilde{A}_1):=\big\{z\in\mathbb{C}:\mathrm{dist}\{z,\sigma(\widetilde{A}_1)\}\leqslant r_0(B)\big\} \tag{4.8}$$

of the spectrum of $\widetilde{A}_1$ (see Figure 1). *The nonreal spectrum of $H_1^{(l)}$ is contained in $D_l\cap\mathcal{O}_{r_0(B)}(\widetilde{A}_1)$ while outside D_l the spectrum of $H_1^{(l)}$ is pure real. Moreover, the spectrum $\sigma(H_1^{(l)})$ coincides with a subset of the spectrum of the transfer function $M_1(z,\Gamma_l)$. More precisely, the spectrum of $M_1(z,\Gamma_l)$ in the set*

$$\mathcal{O}(\widetilde{A}_1,\Gamma_l):=\big\{z\in\mathbb{C}:\mathrm{dist}\{z,\sigma(\widetilde{A}_1)\}\leqslant d_0(\Gamma_l)[1-\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma_l)]/2\big\} \tag{4.9}$$

equals the spectrum of $H_1^{(l)}$, i.e.,

$$\sigma\big(M_1(\,\cdot\,,\Gamma_l)\big)\cap\mathcal{O}(\widetilde{A}_1,\Gamma_l)=\sigma(H_1^{(l)}). \tag{4.10}$$

In fact, such a statement separately holds for the point and continuous spectra.

Theorem 4.3. *Suppose that two different domains $D_{l'}$ and $D_{l''}$ include the same subdomain $D_k^{l_k}$ for some $k=1,2,\ldots,m$, i.e., $l'_k=l''_k=l_k$. Then the spectra of the operators $H_1^{(l')}$ and $H_1^{(l'')}$ in $D_k^{l_k}$ coincide.*

Lemma 4.4. *Let $\Gamma_l\subset D_l$ be a K_B-bounded contour for which the conditions of Theorem 3.1 are fulfilled. Then, for any $z\in\mathbb{C}\setminus(\sigma'(A_0)\cup\Gamma_l)$, the following relation holds:*

$$W_1(z,\Gamma_l)(H_1^{(l)}-z)=(H_1^{(-l)*}-z)[W_1(\overline{z},\Gamma_{(-l)})]^*. \tag{4.11}$$

Further, the spectrum of $H_1^{(-l)*}$ coincides with the spectrum of $H_1^{(l)}$.

Proof. Let $z\in\mathbb{C}\setminus\big(\sigma'(A_0)\cup\Gamma_l\big)$. By definition, $\overline{z}\in\mathbb{C}\setminus\big(\sigma'(A_0)\cup\Gamma_{(-l)}\big)$ and

$$M_1(z,\Gamma_l)^*=M_1(\overline{z},\Gamma_{(-l)}). \tag{4.12}$$

Therefore, the relation (4.11) follows from the factorizations $M_1(z,\Gamma_l)=W_1(z,\Gamma_l)\cdot(H_1^{(l)}-z)$ and $M_1(\overline{z},\Gamma_{(-l)})=W_1(\overline{z},\Gamma_{(-l)})(H_1^{(-l)}-\overline{z})$. By the relation (4.10), z

belongs to the spectrum of the operator $H_1^{(-l)*}$ if and only if $\overline{z} \in \mathcal{O}(\widetilde{A}_1, \Gamma_{(-l)})$ and $0 \in \sigma\big([M_1(\overline{z}, \Gamma_{(-l)})]^*\big)$. From (4.12) we conclude that $0 \in \sigma\big([M_1(\overline{z}, \Gamma_{(-l)})]^*\big)$ if and only if $0 \in \sigma\big(M_1(z, \Gamma_l)\big)$. Again by (4.10), the coincidence of the spectra of $H_1^{(l)}$ and $H_1^{(-l)*}$ follows. ∎

Let

$$\begin{aligned}\Omega^{(l)} &:= \int\limits_{\sigma'(A_0)\cup\Gamma_l} \mu(H_1^{(-l)*} - \mu)^{-1} K_B(\mathrm{d}\mu)(H_1^{(l)} - \mu)^{-1} \\ &:= \int\limits_{\sigma'(A_0)} \mu(H_1^{(-l)*} - \mu)^{-1}\, \mathrm{d}K_B(\mu)(H_1^{(l)} - \mu)^{-1} \\ &\qquad + \int\limits_{\Gamma_l} \mu(H_1^{(-l)*} - \mu)^{-1} K_B'(\mu)(H_1^{(l)} - \mu)^{-1}\, \mathrm{d}\mu\end{aligned} \tag{4.13}$$

where as above Γ_l denotes a K_B-bounded contour satisfying the conditions (3.10).

The operator $\Omega^{(l)}$ does not depend on the choice of such Γ_l. It follows that $\Omega^{(-l)} = \Omega^{(l)*}$ and

$$\|\Omega^{(l)}\| < 1. \tag{4.14}$$

Indeed, since $\mu(H_1^{(l)} - \mu)^{-1} = -I_1 + H_1^{(l)}(H_1^{(l)} - \mu)^{-1}$, one can rewrite $\Omega^{(l)}$ in the form

$$\begin{aligned}\Omega^{(l)} = &- \int\limits_{\sigma'(A_0)\cup\Gamma_l} (H_1^{(-l)*} - \mu)^{-1} K_B(\mathrm{d}\mu) \\ &+ \int\limits_{\sigma'(A_0)\cup\Gamma_l} (H_1^{(-l)*} - \mu)^{-1} K_B(\mathrm{d}\mu) H_1^{(l)}(H_1^{(l)} - \mu)^{-1}.\end{aligned}$$

By Lemma 4.4 we know that $\sigma\big(H_1^{(-l)*}\big) = \sigma\big(H_1^{(l)}\big)$. Following the proof of the estimate (4.6) we conclude that

$$\Bigg\| \int\limits_{\sigma'(A_0)\cup\Gamma_l} (H_1^{(-l)*} - \mu)^{-1} K_B(\mathrm{d}\mu) \Bigg\| \leqslant \frac{2\mathrm{Var}_{\tilde{A}_1}(B, \Gamma_l)}{1 + \mathrm{Var}_{\tilde{A}_1}(B, \Gamma_l)}. \tag{4.15}$$

The estimate (4.15) and the inequalities

$$\|H_1^{(l)}\| \leqslant \|\widetilde{A}_1\| + \|X^{(l)}\| \leqslant \|\widetilde{A}_1\| + \frac{1}{2} d_0(\Gamma_l)[1 - \mathrm{Var}_{\tilde{A}_1}(B, \Gamma_l)]$$

and

$$\begin{aligned}\sup_{\mu\in\sigma'(A_0)\cup\Gamma_l} \big\|(H_1^{(l)} - \mu)^{-1}\big\| &\leqslant \sup_{\mu\in\sigma'(A_0)\cup\Gamma_l} \big[\,\mathrm{dist}\{\mu, \sigma(\widetilde{A}_1)\} - \|X^{(l)}\|\big]^{-1} \\ &\leqslant \frac{2}{d_0(\Gamma_l)[1 + \mathrm{Var}_{\tilde{A}_1}(B, \Gamma_l)]}\end{aligned}$$

imply that

$$\left\| \int\limits_{\sigma'(A_0)\cup\Gamma_l} (H_1^{(-l)*}-\mu)^{-1}K_B(\mathrm{d}\mu)H_1^{(l)}(H_1^{(l)}-\mu)^{-1}\right\| \leqslant \frac{4\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)\big\{\|\widetilde{A}_1\|+\frac{1}{2}d_0(\Gamma_l)[1-\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)]\big\}}{d_0(\Gamma_l)\,\big[1+\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)\big]^2}. \tag{4.16}$$

Combining the estimates (4.15) and (4.16), one finds

$$\|\Omega^{(l)}\| \leqslant \frac{4\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)[d_0(\Gamma_l)+\|\widetilde{A}_1\|]}{d_0(\Gamma_l)[1+\mathrm{Var}_{\tilde{A}_1}(B,\Gamma_l)]^2}.$$

Due to the assumptions (3.10), the inequality (4.14) holds (see formula (4.7)).

The estimate (4.14) assures that the sum $I_1+\Omega^{(l)}$ represents a boundedly invertible operator in $\mathcal{H}_1$.

THEOREM 4.5. *The operators $\Omega^{(l)}$ possess the following properties* (cf. [20], [21], [22], [23], [24], and [30])*:*

$$-\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma}[M_1(z,\Gamma_l)]^{-1}\,\mathrm{d}z = (I_1+\Omega^{(l)})^{-1}, \tag{4.17}$$

$$-\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma} z[M_1(z,\Gamma_l)]^{-1}\,\mathrm{d}z = (I_1+\Omega^{(l)})^{-1}H_1^{(-l)*} = H_1^{(l)}(I_1+\Omega^{(l)})^{-1}, \tag{4.18}$$

where γ stands for an arbitrary rectifiable closed contour going around the spectrum of $H_1^{(l)}$ inside the set $\mathcal{O}(\widetilde{A}_1,\Gamma_l)$ in the positive direction. The integration along γ is understood in the strong sense.

Proof. First, we recall that due to the factorization Theorem 4.1 and the formula (4.11), the following factorization holds for $z\in\mathcal{O}(\widetilde{A}_1,\Gamma_l)\setminus\sigma(H_1^{(l)})$:

$$[M_1(z,\Gamma_l)]^{-1} = \big(H_1^{(l)}-z\big)^{-1}[W_1(z,\Gamma_l)]^{-1} = [W_1(\overline{z},\Gamma_{(-l)})]^{*-1}\big(H_1^{(-l)*}-z\big)^{-1}, \tag{4.19}$$

where $[W_1(z,\Gamma_l)]^{-1}$ and $[W_1(\overline{z},\Gamma_{(-l)})]^{*-1}$ are holomorphic functions with values in $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_1)$. By the resolvent equation

$$(H_1^{(l)}-\mu)^{-1}(H_1^{(l)}-z)^{-1} = [(H_1^{(l)}-\mu)^{-1}-(H_1^{(l)}-z)^{-1}](\mu-z)^{-1}$$

and due to (4.2)

$$[W_1(\overline{z},\Gamma_{(-l)})]^* = I_1 - \int\limits_{\sigma'(A_0)\cup\Gamma_l}(H_1^{(-l)*}-\mu)^{-1}K_B(\mathrm{d}\mu) + z\int\limits_{\sigma'(A_0)\cup\Gamma_l}(H_1^{(-l)*}-\mu)^{-1}(z-\mu)^{-1}K_B(\mathrm{d}\mu), \tag{4.20}$$

the product $\Omega^{(l)}(H_1^{(l)}-z)^{-1}$ can be written as

$$\Omega^{(l)}(H_1^{(l)}-z)^{-1} = F_1(z)+F_2(z) \tag{4.21}$$

where

$$F_1(z) := \int\limits_{\sigma'(A_0)\cup\Gamma_l} \mu(H_1^{(-l)*} - \mu)^{-1} K_B(\mathrm{d}\mu)(H_1^{(l)} - \mu)^{-1}(\mu - z)^{-1} \tag{4.22}$$

and

$$\begin{aligned} F_2(z) :=& \Bigg(-\int\limits_{\sigma'(A_0)\cup\Gamma_l} \frac{\mu}{\mu - z}(H_1^{(-l)*} - \mu)^{-1} K_B(\mathrm{d}\mu)\Bigg)(H_1^{(l)} - z)^{-1} \\ =& \left([W_1(\overline{z}, \Gamma_{(-l)})]^* - I_1\right)(H_1^{(l)} - z)^{-1}. \end{aligned} \tag{4.23}$$

Further, the formula (4.19) yields that

$$\begin{aligned} &(I_1 + \Omega^{(l)})[M_1(z, \Gamma_l)]^{-1} \\ &\quad = [M_1(z, \Gamma_l)]^{-1} + [F_1(z) + F_2(z)][W_1(z, \Gamma_l)]^{-1} \\ &\quad = [M_1(z, \Gamma_l)]^{-1} + F_1(z)[W_1(z, \Gamma_l)]^{-1} + ([W_1(\overline{z}, \Gamma_{(-l)})]^* - I_1)[M_1(z, \Gamma_l)]^{-1} \\ &\quad = F_1(z)[W_1(z, \Gamma_l)]^{-1} + (H_1^{(-l)*} - z)^{-1}. \end{aligned}$$

The function $F_1(z)$ is holomorphic inside the contour γ, $\gamma \subset \mathcal{O}(\widetilde{A}_1, \Gamma_l)$, since the argument μ of the integrand in the formula (4.22) belongs to $\sigma'(A_0) \cup \Gamma_l$ and thereby

$$|z - \mu| \geqslant \frac{d_0(\Gamma_l) + \mathrm{Var}_{\tilde{A}_1}(B, \Gamma_l)}{2} > 0.$$

Thus, the term $F_1(z)[W_1(z, \Gamma_l)]^{-1}$ does not contribute to the integral

$$-\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma}(I_1 + \Omega^{(l)})[M_1(z, \Gamma_l)]^{-1}\,\mathrm{d}z$$

while the resolvent $(H_1^{(-l)*} - z)^{-1}$ gives the identity I_1 which proves the equation (4.17).

Concerning the equation (4.18) we obtain

$$\begin{aligned} &-\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma}(I_1 + \Omega^{(l)})z[M_1(z, \Gamma_l)]^{-1}\,\mathrm{d}z \\ &\qquad = -\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma} zF_1(z)[W_1(z, \Gamma_l)]^{-1}\,\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma} z(H_1^{(-l)*} - z)^{-1}\,\mathrm{d}z. \end{aligned}$$

The first integral vanishes whereas the second integral equals $H_1^{(-l)*}$. The second equation of (4.18) can be checked in the same way. ∎

Note that the formulae (4.17) and (4.18) allow, in principle, to construct the operators $H_1^{(l)}$ and thus to solve the equation (3.6) by a contour integration of the inverse of the transfer function $M_1(z, \Gamma_l)$.

REMARK 4.6. The formula (4.18) implies that

$$H_1^{(l)*} = (I_1 + \Omega^{(-l)}) H_1^{(-l)} (I_1 + \Omega^{(-l)})^{-1}.$$

THEOREM 4.7. *Let λ be an isolated eigenvalue of the operator $H_1^{(l)}$ and, consequently, of the operator $H_1^{(-l)*}$ and of the transfer function $M_1(z, \Gamma_l)$ taken for a K_B-bounded contour Γ_l satisfying the conditions* (3.10). *By $\mathrm{P}_\lambda^{(l)}$ and $\mathrm{P}_\lambda^{(-l)*}$ we denote the eigenprojections of the operators $H_1^{(l)}$ and $H_1^{(-l)*}$, respectively, and by $P_\lambda^{(l)}$ the residue of $M_1(z, \Gamma_l)$ at $z = \lambda$,*

$$\mathrm{P}_\lambda^{(l)} := -\frac{1}{2\pi \mathrm{i}} \int_\gamma (H_1^{(l)} - z)^{-1} \,\mathrm{d}z, \tag{4.24}$$

$$\mathrm{P}_\lambda^{(-l)*} := -\frac{1}{2\pi \mathrm{i}} \int_\gamma (H_1^{(-l)*} - z)^{-1} \,\mathrm{d}z \tag{4.25}$$

and

$$P_\lambda^{(l)} := -\frac{1}{2\pi \mathrm{i}} \int_\gamma [M_1(z, \Gamma_l)]^{-1} \,\mathrm{d}z. \tag{4.26}$$

Here γ denotes an arbitrary rectifiable closed contour going around λ in the positive direction in a sufficiently small neighbourhood of this point such that $\gamma \cap \Gamma_l = \emptyset$ and no points of the spectrum of $M_1(\,\cdot\,, \Gamma_l)$, except the eigenvalue λ, lie inside γ. Then the following relations are valid:

$$P_\lambda^{(l)} = \mathrm{P}_\lambda^{(l)} (I_1 + \Omega^{(l)})^{-1} = (I_1 + \Omega^{(l)})^{-1} \mathrm{P}_\lambda^{(-l)*}. \tag{4.27}$$

Proof. The proof is carried out in the same way as the proof of the relation (4.17), only the path of integration being changed. ∎

5. PROPERTIES OF REAL EIGENVALUES

For arbitrary l let Γ_l be a K_B-bounded contour satisfying the conditions (3.10), and $H_1^{(l)} = \widetilde{A}_1 + X^{(l)}$ where $X^{(l)}$ is the solution of (3.6) mentioned in Theorem 3.1.

REMARK 5.1. $\sigma'(A_0) \cap \sigma(H_1^{(l)}) = \emptyset$ *and in particular* $\sigma'(A_0) \cap \sigma_p(H_1^{(l)}) = \emptyset$.

Proof. According to Theorem 4.2, the spectrum $\sigma(H_1^{(l)})$ belongs to the closed $r_0(B)$-neighbourhood $\mathcal{O}_{r_0(B)}(\widetilde{A}_1)$ of the spectrum of $\widetilde{A}_1$ whence

$$\sup_{\mu \in \sigma(H_1^{(l)})} \big\{\mu, \sigma(\widetilde{A}_1)\big\} \;\leqslant\; r_0(B). \tag{5.1}$$

The definitions of $d_0(\Gamma_l)$ and $r_{\min}(\Gamma_l)$ (see (3.7) and (3.11)) yield that

$$\begin{aligned} r_0(B) \leqslant r_{\min}(\Gamma_l) \leqslant \frac{1}{2} d_0(\Gamma_l)[1 - \mathrm{Var}_{\widetilde{A}_1}(B, \Gamma_l)] \leqslant \frac{1}{2} \operatorname{dist}\big\{\sigma'(A_0) \cup \Gamma_l, \sigma(\widetilde{A}_1)\big\} \\ \leqslant \frac{1}{2} \operatorname{dist}\big\{\sigma'(A_0), \sigma(\widetilde{A}_1)\big\}, \end{aligned}$$

so that the statements of the remark follow from the inequality (5.1). ∎

If λ is a real eigenvalue of $H_1^{(l)}$, then by Remark 5.1 λ belongs either to the resolvent set $\varrho(A_0)$ of the entry A_0 or it is embedded in the continuous spectrum of A_0 in $\bigcup_{k=1}^{m} \Delta_k^0$. The connection between the real eigenvalues of $H_1^{(l)}$ and $\mathbf{H}$ is described below in the Lemmas 5.2 and 5.5.

LEMMA 5.2. *Let* $\lambda \in \varrho(A_0) \cap \mathbb{R}$.

(i) *Suppose that* λ *is an eigenvalue of* $H_1^{(l)}$ *and* $\psi^{(1)} \in \mathcal{H}_1$ *is a corresponding eigenvector, i.e.* $H_1^{(l)}\psi^{(1)} = \lambda\psi^{(1)}$ *with* $\psi^{(1)} \neq 0$. *Let*

$$
\begin{aligned}
\psi^{(0)} &= -G_{10}(\lambda)^*\psi^{(1)} = -A_0^{-1/2}\big(I_0 + \lambda(A_0-\lambda)^{-1}\big)B_{01}\psi^{(1)} \\
&= -A_0^{1/2}(A_0-\lambda)^{-1}B_{01}\psi^{(1)}.
\end{aligned}
\tag{5.2}
$$

Then the vector $\Psi := (\psi^{(0)}, \psi^{(1)}) \in \mathcal{H}$ *is an eigenvector of* $\mathbf{H}$ *to the eigenvalue* λ, *i.e.* $\mathbf{H}\Psi = \lambda\Psi$.

(ii) *Conversely, let* λ *be an eigenvalue of* $\mathbf{H}$ *and* $\Psi = (\psi^{(0)}, \psi^{(1)}) \in \mathcal{D}(\mathbf{H})$ *a corresponding eigenvector. Then* $\psi^{(0)}$ *and* $\psi^{(1)}$ *fulfill the relation* (5.2). *If* λ *belongs to the set* $\mathcal{O}(\widetilde{A}_1, \Gamma_l)$ *(see* (4.9)*), then* $\lambda \in \sigma_p(H_1^{(l)})$ *and* $\psi^{(1)}$ *is a corresponding eigenvector.*

Proof. (i) From (5.2) it immediately follows that $\Psi \in \mathcal{D}(\mathbf{H})$. By the Schur factorization (2.3), only the relation

$$M_1(\lambda)\psi^{(1)} = 0$$

remains to be proved. By Theorem 4.2, λ belongs to the set $\mathcal{O}_{r_0(B)}(\widetilde{A}_1)$. The factorization formula (4.1) yields that

$$M_1(\lambda,\Gamma_l)\psi^{(1)} = W_1(\lambda,\Gamma_l)(H_1^{(l)} - \lambda)\psi^{(1)} = 0.$$

Since λ is supposed to be real, $M_1(\lambda,\Gamma_l) = M_1(\lambda)$.

(ii) The Schur factorization (2.3) yields the relations

$$(A_0 - \lambda)\big(\psi^{(0)} + G_{10}(\lambda)^*\psi^{(1)}\big) = 0, \tag{5.3}$$

$$M_1(\lambda)\psi^{(1)} = 0. \tag{5.4}$$

Since $\lambda \in \varrho(A_0)$, (5.3) infers that $\psi^{(0)}$ and $\psi^{(1)}$ fulfill the relation (5.2). Since it is supposed that $\lambda \in \mathcal{O}(\widetilde{A}_1, \Gamma_l)$, $W_1(\lambda,\Gamma_l)$ is invertible and thus

$$\big(H_1^{(l)} - \lambda\big)\psi^{(1)} = 0$$

by the factorization (4.1). ∎

If an eigenvalue λ of $H_1^{(l)}$ belongs to $\Delta_k^0 = (\mu_k^{(1)}, \mu_k^{(2)})$ for some $k = 1, 2, \ldots, m$, then (see Remark 3.2)

$$
\begin{aligned}
|\lambda - \mu_k^{(i)}| &\geqslant \operatorname{dist}\{\mu_k^{(i)}, \sigma(\widetilde{A}_1)\} - \|X^{(l)}\| \geqslant \operatorname{dist}\{\sigma'(A_0) \cup \Gamma_l, \sigma(\widetilde{A}_1)\} - \|X^{(l)}\| \\
&\geqslant d_0(\Gamma_l) - r_{\min}(\Gamma_l) > \frac{1}{2} d_0(\Gamma_l)[1 + \operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)] > 0
\end{aligned}
$$

for $i = 1, 2$. Therefore, in this case λ is situated strictly inside the interval Δ_k^0. Recall that, according to our assumption, the entry A_0 has no point spectrum inside Δ_k^0. Since Δ_k^0 is a part of the continuous spectrum of A_0, the resolvent $(A_0-z)^{-1}$ exists for $z = \lambda \pm \mathrm{i}0$ being however an unbounded operator. Nevertheless a statement analogous to Lemma 5.2 holds in this case.

First let u_1 be an arbitrary element of $\mathcal{H}_1$. By our assumptions, the function

$$g_{u_1}(\mu) := \langle K_B(\mu)u_1, u_1\rangle = \left\|E^0(\mu)B_{01}u_1\right\|^2 \tag{5.5}$$

is continuously differentiable on $\Delta_k^0 = (\mu_k^{(1)}, \mu_k^{(2)})$. By (2.10), $g'_{u_1}(\mu) \geqslant 0$ for $\mu \in \Delta_k^0$ and g'_{u_1} admits an analytic continuation to the domain $D_k^+ \cup D_k^- \cup \Delta_k^0$.

PROPOSITION 5.3. *Let $\lambda \in \Delta_k^0 = (\mu_k^{(1)}, \mu_k^{(2)})$ for some $k \in \{1, 2, \ldots, m\}$. Suppose that λ is an eigenvalue of $H_1^{(l)}$ and $\psi^{(1)} \in \mathcal{H}_1$ a corresponding eigenvector, i.e., $H_1^{(l)}\psi^{(1)} = \lambda\psi^{(1)}$ with $\psi^{(1)} \neq 0$. Then the first derivative as well as the second derivative of the function*

$$g_{\psi^{(1)}}(\mu) = \langle K_B(\mu)\psi^{(1)}, \psi^{(1)}\rangle \;=\; \left\|E^0(\mu)B_{01}\psi^{(1)}\right\|^2$$

vanish at $\mu = \lambda$, which means that the function $g'_{\psi^{(1)}}(\mu)/(\mu-\lambda)^2$ is holomorphic in $D_k^+ \cup D_k^- \cup \Delta_k^0$.

Proof. The equation (3.5) for $Y = H_1^{(l)}$, $\Gamma = \Gamma_l$ and the definition (3.3) yield that

$$\begin{aligned} 0 &= \langle (H_1^{(l)} - \lambda)\psi^{(1)}, \psi^{(1)}\rangle \\ &= \langle (\widetilde{A}_1 - \lambda)\psi^{(1)}, \psi^{(1)}\rangle + \langle V_1(H_1^{(l)}, \Gamma_l)\psi^{(1)}, \psi^{(1)}\rangle \\ &= \langle (\widetilde{A}_1 - \lambda)\psi^{(1)}, \psi^{(1)}\rangle + \int\limits_{\sigma'(A_0)\cup\Gamma_l} \frac{\lambda}{\lambda-\mu}\, \mathrm{d}\langle K_B(\mu)\psi^{(1)}, \psi^{(1)}\rangle. \end{aligned}$$

Since the denominator of the integrand is non-zero for $\mu \in \sigma(A_0) \setminus \Delta_k^0$, we can deform the part $\Gamma_l \setminus \Gamma_k^{l_k}$ of the contour Γ_l back into the interval Δ_i^0, $i = 1, 2, \ldots, m$, $i \neq k$. As a result, we conclude that

$$\begin{aligned} \langle (\widetilde{A}_1 - \lambda)\psi^{(1)}, \psi^{(1)}\rangle &+ \int\limits_{\sigma(A_0)\setminus\Delta_k^0} \frac{\lambda}{\lambda-\mu}\, \mathrm{d}\langle K_B(\mu)\psi^{(1)}, \psi^{(1)}\rangle \\ &+ \int\limits_{\Gamma_k^{l_k}} \frac{\lambda}{\lambda-\mu}\langle K_B'(\mu)\psi^{(1)}, \psi^{(1)}\rangle\, \mathrm{d}\mu = 0. \end{aligned} \tag{5.6}$$

Obviously, the first two terms are real, and an imaginary component may appear in the left-hand side of the equation (5.6) only in the third term. To determine this component, one simply transforms the path of integration into the two intervals $[\mu_k^{(1)}, \lambda-\varepsilon]$ and $[\lambda+\varepsilon, \mu_k^{(2)}]$ and the semicircle $|\mu-\lambda| = \varepsilon$, $l_k \operatorname{Im}\mu \geqslant 0$, between them. Then, taking the limit $\varepsilon \downarrow 0$, one obtains

$$0 = \operatorname{Im}\int\limits_{\Gamma_k^{l_k}} \frac{\lambda}{\lambda-\mu}\langle K_B'(\mu)\psi^{(1)}, \psi^{(1)}\rangle\, \mathrm{d}\mu = l_k\pi\lambda\langle K_B'(\lambda)\psi^{(1)}, \psi^{(1)}\rangle$$

whence $g'_{\psi^{(1)}}(\lambda) = \langle K'_B(\lambda)\psi^{(1)}, \psi^{(1)}\rangle = 0$. Since the function $g'_{\psi^{(1)}}$ is nonnegative on Δ_k^0 and λ is an interior point of this interval, apart from $g'_{\psi^{(1)}}$ also $g''_{\psi^{(1)}}$ vanishes at $\mu = \lambda$ which completes the proof of Proposition 5.3. ∎

COROLLARY 5.4. *Under the assumptions of Proposition* 5.3, *the following alternative holds: Either*

(a) $E^0(\mu)B_{01}\psi^{(1)} = 0$ *for all* $\mu \leqslant \mu_k^{(2)}$ *or*

(b) $E^0(\mu)B_{01}\psi^{(1)} \neq 0$ *for all* $\mu \in \Delta_k^0$, *the function* $\|E^0(\mu)B_{01}\psi^{(1)}\|$ *is twice differentiable in* μ *on* Δ_k^0, *and its first derivative as well as its second derivative vanish at* $\mu = \lambda$.

Proof. First consider the case that $E^0(\lambda)B_{01}\psi^{(1)} = 0$. Since $\|E^0(\mu)B_{01}\psi^{(1)}\|$ is a non-decreasing function of the variable μ, we conclude that $\|E^0(\mu)B_{01}\psi^{(1)}\| = 0$ for all $\mu \leqslant \lambda$, whence the function $g'_{\psi^{(1)}}$ vanishes for $\mu \leqslant \lambda$. Since this function is holomorphic in $D_k^+ \cup D_k^- \cup \Delta_k^0$, it vanishes in this domain, which implies that condition (a) holds.

If $E^0(\lambda)B_{01}\psi^{(1)} \neq 0$, the function $\|E^0(\mu)B_{01}\psi^{(1)}\|$ is non-zero for any $\mu \in \Delta_k^0$. Since the function $\|E^0(\mu)B_{01}\psi^{(1)}\|^2$ is twice differentiable in Δ_k^0, it follows that also its square root $\|E^0(\mu)B_{01}\psi^{(1)}\|$ is twice differentiable in this interval. The further statements then immediately follow from Proposition 5.3. ∎

LEMMA 5.5. *Let* $\lambda \in \Delta_k^0 = (\mu_k^{(1)}, \mu_k^{(2)})$ *for some* $k \in \{1, 2, \ldots, m\}$.

(i) *Suppose that* λ *is an eigenvalue of* $H_1^{(l)}$ *and* $\psi^{(1)} \in \mathcal{H}_1$ *is a corresponding eigenvector, i.e.,* $H_1^{(l)}\psi^{(1)} = \lambda\psi^{(1)}$ *with* $\psi^{(1)} \neq 0$. *Then the limits*

$$G_{10}(\lambda \mp \mathrm{i}0)^*\psi^{(1)} = A_0^{-1/2}\big(I_0 + \lambda(A_0 - \lambda \pm \mathrm{i}0)^{-1}\big)B_{01}\psi^{(1)}$$

exist and coincide. Define

$$\psi^{(0)} := -G_{10}(\lambda \mp \mathrm{i}0)^*\psi^{(1)}. \tag{5.7}$$

Then $\Psi := (\psi^{(0)}, \psi^{(1)})$ *belongs to* $\mathcal{D}(\mathbf{H})$, $\mathbf{H}\Psi = \lambda\Psi$, *i.e.,* λ *is an eigenvalue of* $\mathbf{H}$ *and* Ψ *is a corresponding eigenvector.*

(ii) *Conversely, let* λ *be an eigenvalue of* $\mathbf{H}$ *and* $\Psi = (\psi^{(0)}, \psi^{(1)}) \in \mathcal{D}(\mathbf{H})$ *a corresponding eigenvector. Then the components* $\psi^{(0)}$ *and* $\psi^{(1)}$ *fulfill the relation* (5.7). *If* λ *belongs to the set* $\mathcal{O}(\widetilde{A}_1, \Gamma_l)$, *then* $\lambda \in \sigma_p\big(H_1^{(l)}\big)$ *and* $\psi^{(1)}$ *is a corresponding eigenvector.*

Proof. It is sufficient to prove the existence and the coincidence of the limits $(A_0-\lambda\pm\mathrm{i}0)^{-1}B_{01}\psi^{(1)}$. To this end, we consider an arbitrary sequence $\lambda_n = \lambda+\mathrm{i}\eta_n$, $n = 0, 1, 2, \ldots$ such that $\eta_n \neq 0$ and $\eta_n \to 0$ as $n \to \infty$. By the resolvent equation we obtain

$$\begin{aligned}(A_0-\lambda_n)^{-1}&B_{01}\psi^{(1)} - (A_0-\lambda_m)^{-1}B_{01}\psi^{(1)}\\ &= (\lambda_n-\lambda_m)\int\limits_{\mathbb{R}} \frac{\mathrm{d}E^0(\mu)B_{01}\psi^{(1)}}{(\mu-\lambda_n)(\mu-\lambda_m)} = \mathrm{i}(\eta_n-\eta_m)\int\limits_{\mathbb{R}} \frac{\mathrm{d}E^0(\mu)B_{01}\psi^{(1)}}{(\mu-\lambda_n)(\mu-\lambda_m)}.\end{aligned} \tag{5.8}$$

An easy estimate yields

$$\Big\|\int\limits_{\mathbb{R}} \frac{\mathrm{d}E^0(\mu)B_{01}\psi^{(1)}}{(\mu-\lambda_n)(\mu-\lambda_m)}\Big\|^2 = \int\limits_{\mathbb{R}} \frac{\mathrm{d}\langle E^0(\mu)B_{01}\psi^{(1)},B_{01}\psi^{(1)}\rangle}{|\mu-\lambda_n|^2|\mu-\lambda_m|^2}$$

$$= \int\limits_{\mathbb{R}\setminus\Delta_k^0} \frac{\mathrm{d}\langle E^0(\mu)B_{01}\psi^{(1)},B_{01}\psi^{(1)}\rangle}{|\mu-\lambda_n|^2|\mu-\lambda_m|^2} + \int\limits_{\Delta_k^0} \frac{g'_{\psi^{(1)}}(\mu)}{[(\mu-\lambda)^2+\eta_n^2][(\mu-\lambda)^2+\eta_m^2]}\,\mathrm{d}\mu.$$

Suppose that $m \geqslant n$. First, consider the case that $|\eta_n| \geqslant |\eta_m|$. Since, by Proposition 5.3, $g'_{\psi^{(1)}}(\mu)/(\mu-\lambda)^2$ is a continuous bounded function on a neighborhood of $\mu=\lambda$ in Δ_k, we conclude that

$$\|(A_0-\lambda_n)^{-1}B_{01}\psi^{(1)} - (A_0-\lambda_m)^{-1}B_{01}\psi^{(1)}\|^2$$

$$\leqslant |\eta_n|^2\left(c_1 + c_2\int\limits_{-\infty}^{\infty}\frac{1}{(\mu-\lambda)^2+|\eta_n|^2}\,\mathrm{d}\mu\right) = |\eta_n|(c_1|\eta_n|+c_2\pi),$$

where $c_1 \geqslant 0$ and $c_2 \geqslant 0$ are suitable constants. If $|\eta_m| \geqslant |\eta_n|$, then we obtain an estimate by $|\eta_m|(c_1|\eta_m|+c_2\pi)$. Thus, we have shown that

$$\Big\|(A_0-\lambda_n)^{-1}B_{01}\psi^{(1)} - (A_0-\lambda_m)^{-1}B_{01}\psi^{(1)}\Big\| = O\big(\sup_{m\geqslant n}|\eta_m|\big),$$

which implies the existence and coincidence of the limits $(A_0-\lambda\pm\mathrm{i}0)^{-1}B_{01}\psi^{(1)}$ and $G_{10}(\lambda\mp\mathrm{i}0)^*\psi^{(1)}$.

It also follows that the limits $M_1(\lambda\pm\mathrm{i}0)\psi^{(1)}$ are well-defined and $M_1(\lambda\pm\mathrm{i}0)\psi^{(1)} = M_1(\lambda,\Gamma_l)\psi^{(1)}$. For the sequence $(\lambda_n)_{n=0}^{\infty}$ considered above we define

$$\psi_n^{(0)} := -G_{10}(\overline{\lambda}_n)^*\psi^{(1)}.$$

By (2.5), the vector $\big(\psi_n^{(0)},\psi^{(1)}\big)$ belongs to $\mathcal{D}(\mathbf{H})$ and by (2.3),

$$\mathbf{H}\begin{pmatrix}\psi_n^{(0)}\\ \psi^{(1)}\end{pmatrix} = \lambda_n\begin{pmatrix}\psi_n^{(0)}\\ \psi^{(1)}\end{pmatrix} + \begin{pmatrix}0\\ M_1(\lambda_n)\psi^{(1)}\end{pmatrix}.$$

The closedness of the operator $\mathbf{H}$ yields that $(\psi^{(0)},\psi^{(1)}) \in \mathcal{D}(\mathbf{H})$ and

$$\mathbf{H}\begin{pmatrix}\psi^{(0)}\\ \psi^{(1)}\end{pmatrix} = \lambda\begin{pmatrix}\psi^{(0)}\\ \psi^{(1)}\end{pmatrix} + \begin{pmatrix}0\\ M_1(\lambda,\Gamma_l)\psi^{(1)}\end{pmatrix}.$$

By the factorization Theorem 4.1, $M_1(\lambda,\Gamma_l)\psi^{(1)} = 0$ which implies that λ is an eigenvalue of $\mathbf{H}$ and $(\psi^{(0)},\psi^{(1)})$ is a corresponding eigenvector.

Let us now prove the converse statement. Suppose that λ is an eigenvalue of $\mathbf{H}$ and $\Psi = (\widetilde{\psi}^{(0)},\psi^{(1)}) \in \mathcal{D}(\mathbf{H})$ is a corresponding eigenvector, i.e., $\mathbf{H}\Psi = \lambda\Psi$, $\Psi \neq 0$. The formula (2.7) yields

$$\big(A_0^{1/2} - \lambda A_0^{-1/2}\big)\widetilde{\psi}^{(0)} = -B_{01}\psi^{(1)}; \tag{5.9}$$

note that $\widetilde{\psi}^{(0)} \in \mathcal{D}(A_0^{1/2})$. Let $E_{\mathrm{ac}}^0(\mu)$ be the spectral function corresponding to the absolutely continuous spectrum of A_0, $E_{\mathrm{ac}}^0(\mu) = E_0^{\mathrm{ac}}\big((-\infty,\mu)\big)$. Let δ be any subinterval of Δ_k^0. Recall that the differentiability of $K_B(\mu)$ yields that $E_0(\delta)B_{01} = E_0^{\mathrm{ac}}(\delta)B_{01}$. Applying the projection $E_0^{\mathrm{ac}}(\delta)$ to both sides of the equation (5.9) leads to the relation

$$\int_\delta \frac{\mu-\lambda}{\sqrt{\mu}}\,\mathrm{d}E_{\mathrm{ac}}^0(\mu)\widetilde{\psi}^{(0)} = -\int_\delta \mathrm{d}E^0(\mu)B_{01}\psi^{(1)}.$$

Thus, taking the norm squares, one finds

$$\int_\delta \frac{(\mu-\lambda)^2}{\mu}\,\mathrm{d}\langle E_{\mathrm{ac}}^0(\mu)\widetilde{\psi}^{(0)},\widetilde{\psi}^{(0)}\rangle = \int_\delta \mathrm{d}\langle K_B(\mu)\psi^{(1)},\psi^{(1)}\rangle.$$

Since the function $\langle E_{\mathrm{ac}}^0(\mu)\widetilde{\psi}^{(0)},\widetilde{\psi}^{(0)}\rangle$ is absolutely continuous and hence almost everywhere differentiable, we obtain

$$g'_{\psi^{(1)}}(\mu) = \langle K'_B(\mu)\psi^{(1)},\psi^{(1)}\rangle = \frac{(\mu-\lambda)^2}{\mu}\frac{\mathrm{d}}{\mathrm{d}\mu}\langle E_{\mathrm{ac}}^0(\mu)\widetilde{\psi}^{(0)},\widetilde{\psi}^{(0)}\rangle$$

for almost all $\mu \in \Delta_k^0$. The derivative on the right-hand side of this equation is an element of $L_1\big(\sigma_{\mathrm{ac}}(A_0)\big)$. Thus the function $g'_{\psi^{(1)}}(\mu)/(\mu-\lambda)^2$ is integrable over any subinterval $\delta \subset \Delta_k^0$ which is only possible if $g'_{\psi^{(1)}}(\lambda) = \langle K'_B(\lambda)\psi^{(1)},\psi^{(1)}\rangle = 0$. As in the proof of Proposition 5.3, it follows that $g'_{\psi^{(1)}}(\mu)/(\mu-\lambda)^2$ is holomorphic in $D_k^+ \cup D_k^- \cup \Delta_k^0$.

As in the proof of part (i), we conclude that the limits $G_{10}(\lambda \mp \mathrm{i}0)^*\psi^{(1)}$ and $M_1(\lambda \pm \mathrm{i}0)\psi^{(1)}$ exist and define

$$\psi^{(0)} := -G_{10}(\lambda \mp \mathrm{i}0)^*\psi^{(1)}. \tag{5.10}$$

Choose a sequence $\lambda_n = \lambda + \mathrm{i}\eta_n$, $\eta_n \neq 0$, $\eta_n \to 0$ as $n \to \infty$, as above. Consider the relation

$$\begin{aligned}(\mathbf{H}-\lambda_n)&\begin{pmatrix}\widetilde{\psi}^{(0)}\\ \psi^{(1)}\end{pmatrix}\\ &=\begin{pmatrix} I_0 & 0\\ G_{10}(\lambda_n) & I_1\end{pmatrix}\begin{pmatrix} A_0-\lambda_n & 0\\ 0 & M_1(\lambda_n)\end{pmatrix}\begin{pmatrix} I_0 & G_{10}(\overline{\lambda}_n)^*\\ 0 & I_1\end{pmatrix}\begin{pmatrix}\widetilde{\psi}^{(0)}\\ \psi^{(1)}\end{pmatrix}.\end{aligned} \tag{5.11}$$

Since the left-hand side of this equation converges to 0 as $n \to \infty$, we obtain with respect to the first component that $(A_0-\lambda_n)\big(\widetilde{\psi}^{(0)} + G_{10}(\overline{\lambda}_n)^*\psi^{(1)}\big) \longrightarrow 0$ as $n \to \infty$. The closedness of A_0 yields that $\widetilde{\psi}^{(0)} - \psi^{(0)} = \widetilde{\psi}^{(0)} + G_{10}(\lambda \mp \mathrm{i}0)^*\psi^{(1)}$ belongs to $\mathcal{D}(A_0)$ and $(A_0-\lambda)(\widetilde{\psi}^{(0)} - \psi^{(0)}) = 0$. The continuity of the spectrum of A_0 within the intervals Δ_k^0 implies that $\widetilde{\psi}^{(0)} = \psi^{(0)}$. Further, it then follows from the second component in (5.11) that $M_1(\lambda \pm \mathrm{i}0)\psi^{(1)} = 0$. This implies that $M_1(\lambda,\Gamma_l)\psi^{(1)} = 0$ for any K_B-bounded contour $\Gamma_l \subset D_l$ satisfying the conditions (3.10). Applying Theorem 4.1 yields $H_1^{(l)}\psi^{(1)} = \lambda\psi^{(1)}$. Obviously $\psi^{(1)} \neq 0$ because otherwise $\widetilde{\psi}^{(0)} = \psi^{(0)}$ would vanish by (5.10). ∎

Corollary 5.6. *The statements of Lemmas* 5.2 *and* 5.5 *imply that* $\sigma_{\mathrm{p}}(H_1^{(l)}) \cap \mathbb{R} \subset \sigma_{\mathrm{p}}(\mathbf{H})$. *It also follows from these lemmas that any eigenvector* $\psi^{(1)}$ *corresponding to an eigenvalue* $\lambda \in \sigma_{\mathrm{p}}(H_1^{(l)}) \cap \mathbb{R}$ *of the operator* $H_1^{(l)} = \widetilde{A}_1 + X^{(l)}$ *for a certain* $l = (l_1, l_2, \ldots, l_m)$ *is such an eigenvector,* $H_1^{(l')}\psi^{(1)} = \lambda\psi^{(1)}$, *for the remaining* $2^m - 1$ *operators* $H_1^{(l')} = \widetilde{A}_1 + X^{(l')}$ *for* $l' = (l'_1, l'_2, \ldots, l'_m)$ *with arbitrary* $l'_k = \pm 1$, $k = 1, 2, \ldots, m$. *Thus, the set* $\sigma_{\mathrm{p}}(H_1^{(l)}) \cap \mathbb{R}$ *is the same for all the* 2^m *operators* $H_1^{(l)}$.

Lemma 5.7. *If some* $\lambda \in \mathbb{R}$ *is an isolated eigenvalue of the operator* $H_1^{(l')} = \widetilde{A}_1 + X^{(l')}$ *for some* $l' = (l'_1, l'_2, \ldots, l'_m)$, *then this* λ *is also such an eigenvalue for the remaining* $2^m - 1$ *operators* $H_1^{(l)} = \widetilde{A}_1 + X^{(l)}$ *for* $l = (l_1, l_2, \ldots, l_m)$ *with arbitrary* $l_k = \pm 1$, $k = 1, 2, \ldots, m$. *Moreover, the resolvents for all the* 2^m *operators* $H_1^{(l)}$ *have a pole of the first order at* $z = \lambda$ *allowing the decomposition*

$$(H_1^{(l)} - z)^{-1} = \frac{\mathrm{P}_\lambda^{(l)}}{\lambda - z} + \widetilde{R}_\lambda^{(l)}(z) \tag{5.12}$$

with $\widetilde{R}_\lambda^{(l)}(z)$ *holomorphic in a neighbourhood of* λ. *In the factorization formula* (4.27) *the operator* $P_\lambda^{(l)}$ *reads as*

$$P_\lambda^{(l)} = \lim_{\substack{z \to \lambda \\ z \notin \sigma(\mathbf{H})}} (\lambda - z)\mathbf{R}_{11}(z) \tag{5.13}$$

and thus does not depend on l.

Proof. As the factorization formula (4.1) is fulfilled for $M_1(z, \Gamma_l)$, any isolated real eigenvalue of $H_1^{(l)}$ is at the same time such an eigenvalue of $M_1(\cdot, \Gamma_l)$.

If λ is an isolated point of the spectrum of $\mathbf{H}$, then the validity of the statements of the lemma are immediately clear. Indeed, in this case there is a pointed open neighbourhood of λ where $M_1(z, \Gamma_l)$ coincides with $M_1(z)$. But $M_1^{-1}(z) = \mathbf{R}_{11}(z)$ where $\mathbf{R}_{11}(z)$ is the 11-block of the resolvent $\mathbf{R}(z) = (\mathbf{H} - z)^{-1}$. Since $\mathbf{H}$ is a selfadjoint operator, the resolvent $\mathbf{R}(z)$ and, consequently, the block $\mathbf{R}_{11}(z)$ can have a pole of at most first order of the isolated eigenvalue $z = \lambda$.

Consider the case that λ is not an isolated point of the spectrum of $\mathbf{H}$. Denote by $\mathcal{O}(\lambda)$ an open circle centered at λ such that $\mathcal{O}(\lambda) \subset \mathcal{O}_{r_0(B)}(\widetilde{A}_1)$ and $\sigma\big(H_1^{(l)}\big) \cap \mathcal{O}(\lambda) = \{\lambda\}$. Due to the factorization (4.1), there are also no singularities of $M_1^{-1}(\,\cdot\,, \Gamma_l)$ in $\mathcal{O}(\lambda) \setminus \{\lambda\}$. The function $M_1^{-1}(\,\cdot\,, \Gamma_l)$ represents an analytic continuation of the block component $\mathbf{R}_{11}(z)$ of the resolvent $\mathbf{R}(z)$ to the domain $D(\Gamma_l)$ and in particular to $\mathcal{O}(\lambda) \cap D(\Gamma_l)$. Recalling again that the operator $\mathbf{H}$ is selfadjoint and applying the spectral theorem we get

$$\mathbf{R}_{11}(z) = \int\limits_{\sigma(\mathbf{H})} \frac{1}{\mu - z}\, \mathrm{d}\mathbf{E}^{11}(\mu). \tag{5.14}$$

Here $\mathbf{E}^{11}(\mu)$ denotes the 11-component of the (right-continuous) spectral function $\mathbf{E}(\mu)$ of $\mathbf{H}$, $\mathbf{E}^{11}(\mu) = P_{\mathcal{H}_1}\mathbf{E}(\mu)|\mathcal{H}_1$ where $P_{\mathcal{H}_1}$ denotes the orthogonal projection onto $\mathcal{H}_1$ in $\mathcal{H}$.

Consider the quadratic form $\langle M_1^{-1}(z)f_1, f_1\rangle$ where f_1 is an arbitrary element of $\mathcal{H}_1$. Since $M_1^{-1}(z)$ admits an analytic continuation from $\mathcal{O}(\lambda)\setminus\overline{D(\Gamma_l)}$ to $\mathcal{O}(\lambda)\setminus\{\lambda\}$, the same holds for the scalar function $\langle M_1^{-1}(z)f_1, f_1\rangle$. The function $\omega_{f_1}(\mu) = \langle \mathbf{E}^{11}(\mu)f_1, f_1\rangle$ is a non-decreasing non-negative function having bounded variation on $\mathbb{R}$ and the form $\langle M_1^{-1}(z)f_1, f_1\rangle$ has the Stieltjes integral representation

$$\langle M_1^{-1}(z)f_1, f_1\rangle = \int\limits_{\sigma(\mathbf{H})} \frac{1}{\mu - z}\, \mathrm{d}\omega_{f_1}(\mu). \tag{5.15}$$

Theorem 1.2 from [12] implies that the function ω_{f_1} is real-analytic in the open intervals $\Delta_\mathrm{l} = \mathcal{O}(\lambda)\cap(-\infty,\lambda)$ and $\Delta_\mathrm{r} = \mathcal{O}(\lambda)\cap(\lambda,+\infty)$ and its derivative $\omega'_{f_1}(\mu)$ admits analytic continuation both from Δ_l and Δ_r to the whole set $\mathcal{O}(\lambda)\setminus\{\lambda\}$. Moreover, the result of this continuation does not depend on whether one starts from Δ_l or from Δ_r since the function $\big[M_1(z,\Gamma_l)\big]^{-1}$ is single-valued on $\mathcal{O}(\lambda)\setminus\{\lambda\}$. Thus, the continued derivative, for which we keep the same notation $\omega'_{f_1}(\mu)$, is a single-valued function on $\mathcal{O}(\lambda)\setminus\{\lambda\}$, too.

The real-analyticity of the function ω_{f_1} in the set $\Delta_\mathrm{l}\cup\Delta_\mathrm{r}$ implies that it can be represented in $\Delta_\mathrm{l}\cup\{\lambda\}\cup\Delta_\mathrm{r}$ as a sum of two terms:

$$\omega_{f_1}(\mu) = \omega_{f_1}^{\mathrm{ac}}(\mu) + \omega_{f_1}^{\mathrm{j}}(\mu) \tag{5.16}$$

where the first term, $\omega_{f_1}^{\mathrm{ac}}$, is an absolutely continuous function while the second term, $\omega_{f_1}^{\mathrm{j}}$, is a jump function having in $\Delta_\mathrm{l}\cup\{\lambda\}\cup\Delta_\mathrm{r}$ the only discontinuity point $\mu=\lambda$. Moreover, $\omega_{f_1}^{\mathrm{j}}(\lambda+0) - \omega_{f_1}^{\mathrm{j}}(\lambda-0) = \omega_{f_1}(\lambda+0) - \omega_{f_1}(\lambda-0)$. (In fact $\omega_{f_1}(\lambda+0) = \omega_{f_1}(\lambda)$ since we assume that the spectral function $\mathbf{E}(\mu)$ is right-continuous.) Also we know that $\frac{\mathrm{d}}{\mathrm{d}\mu}\omega_{f_1}^{\mathrm{j}}(\mu) = 0$ for $\mu\neq\lambda$. For the derivative $\frac{\mathrm{d}}{\mathrm{d}\mu}\omega_{f_1}^{\mathrm{ac}}$ we have $\frac{\mathrm{d}}{\mathrm{d}\mu}\omega_{f_1}^{\mathrm{ac}}(\mu) = \omega'_{f_1}(\mu)$ for any $\mu\in\Delta_\mathrm{l}\cup\Delta_\mathrm{r}$ and, hence, this derivative admits analytic continuation as ω'_{f_1} to $\mathcal{O}(\lambda)\setminus\{\lambda\}$.

Let $[a,b]$ be a real interval in $\mathcal{O}(\lambda)$ such that $\lambda\in(a,b)$. Surely, $[a,\lambda)\subset\Delta_\mathrm{l}$ and $(\lambda,b]\subset\Delta_\mathrm{r}$. With $\omega_{f_1}^{\mathrm{ac}}$ and $\omega_{f_1}^{\mathrm{j}}$ we can rewrite the Stieltjes integral (5.15) in the form

$$\begin{aligned}&\langle M_1^{-1}(z)f_1, f_1\rangle \\ &\quad = \int\limits_{\sigma(\mathbf{H})\setminus[a,b]} \frac{1}{\mu-z}\,\mathrm{d}\omega_{f_1}(\mu) + \int\limits_a^b \frac{1}{\mu-z}\frac{\mathrm{d}}{\mathrm{d}\mu}\omega_{f_1}^{\mathrm{ac}}(\mu)\,\mathrm{d}\mu + \frac{\omega_{f_1}(\lambda)-\omega_{f_1}(\lambda-0)}{\lambda-z}\end{aligned}$$

where the second term is understood as the usual Lebesgue integral.

For arbitrary $z\in\mathcal{O}(\lambda)\setminus\overline{D(\Gamma_l)}$ we denote by $F_{f_1}(\mu,z)$ an antiderivative for the $L_1(a,b)$-function $(\mu-z)^{-1}\frac{\mathrm{d}}{\mathrm{d}\mu}\omega_{f_1}^{\mathrm{ac}}(\mu)$. Then

$$\int\limits_a^b \frac{1}{\mu-z}\frac{\mathrm{d}}{\mathrm{d}\mu}\omega_{f_1}^{\mathrm{ac}}(\mu)\,\mathrm{d}\mu = F_{f_1}(b,z) - F_{f_1}(a,z).$$

Since for $z \in \mathcal{O}(\lambda)\backslash\overline{D(\Gamma_l)}$ the product $(\mu-z)^{-1}\frac{\mathrm{d}}{\mathrm{d}\mu}\omega^{\mathrm{ac}}_{f_1}(\mu)$ is a holomorphic function of the variable μ in $\mathcal{O}(\lambda)\cap D(\Gamma_l)$, we can also write

$$F_{f_1}(b,z)-F_{f_1}(a,z)=\int\limits_{\gamma(a,b)}\frac{1}{\mu-z}\frac{\mathrm{d}}{\mathrm{d}\mu}\omega^{\mathrm{ac}}_{f_1}(\mu)\,\mathrm{d}\mu$$

where $\gamma(a,b)$ is a rectifiable Jordan curve lying in $\mathcal{O}(\lambda)\cap D(\Gamma_l)$ and having end points in a and b. It is assumed that the contour $\gamma(a,b)$ is obtained by continuous deformation of the interval (a,b) and, therefore, inherits the corresponding orientation "from the left to right". Thus, finally we get

$$\begin{aligned}&\langle M_1^{-1}(z)f_1,f_1\rangle\\ &\quad=\int\limits_{\sigma(\mathbf{H})\backslash[a,b]}\frac{1}{\mu-z}\,\mathrm{d}\omega_{f_1}(\mu)+\int\limits_{\gamma(a,b)}\frac{1}{\mu-z}\frac{\mathrm{d}}{\mathrm{d}\mu}\omega^{\mathrm{ac}}_{f_1}(\mu)\,\mathrm{d}\mu+\frac{\omega_{f_1}(\lambda)-\omega_{f_1}(\lambda-0)}{\lambda-z}.\end{aligned}\tag{5.17}$$

Obviously, there is an open neighborhood $\widetilde{\mathcal{O}}(\lambda)$ of the point λ lying in $\mathcal{O}(\lambda)$ such that $\widetilde{\mathcal{O}}(\lambda)\cap\gamma(a,b)=\emptyset$. Then the first two terms in the right hand side of (5.17) are holomorphic functions while the third term generates a pole of the first order of the function $\langle M_1^{-1}(z)f_1,f_1\rangle$ at $z=\lambda$. It follows that the function $\langle(z-\lambda)M_1^{-1}(z)f_1,f_1\rangle$ admits an analytic continuation to $\mathcal{O}(\lambda)$ for any $f_1\in\mathcal{H}_1$. We conclude by the polar formulae that the same holds true for the function $\langle(z-\lambda)M_1^{-1}(z)f_1,f_2\rangle$ for arbitrary $f_1,f_2\in\mathcal{H}_1$. By twice applying Banach-Steinhaus Theorem we obtain that $(z-\lambda)M_1^{-1}(z)$ has an analytic continuation to $\mathcal{O}(\lambda)$. Finally, it follows that $(z-\lambda)\mathbf{R}_{11}(z)$ has an analytic continuation to $\mathcal{O}(\lambda)$ and, consequently, $\mathbf{R}_{11}(z)$ can have a pole of at most first order at $z=\lambda$. ∎

Let $\sigma_{\mathrm{pri}}(H_1^{(l)})$ be the set of all real isolated eigenvalues of the operator $H_1^{(l)}$. According to Lemma 5.7 (cf. Corollary 5.6), the set $\sigma_{\mathrm{pri}}(H_1^{(l)})$ is the same for all $l=(l_1,l_2,\dots,l_m)$, $l_k=\pm1$, $k=1,2,\dots,m$. Moreover, this set coincides with the part $\sigma_{\mathrm{pri}}\big(M_1(\cdot,\Gamma_l)\big)$ of the set of the real isolated eigenvalues of the transfer function $M_1(z,\Gamma_l)$ belonging to $\mathcal{O}(\widetilde{A}_1,\Gamma_l)$ for any K_B-bounded contour Γ_l satisfying the conditions (3.10),

$$\sigma_{\mathrm{pri}}(H_1^{(l)})=\sigma_{\mathrm{pri}}(M_1(\,\cdot\,,\Gamma_l))\cap\mathcal{O}(\widetilde{A}_1,\Gamma_l).$$

Since in the remainder of this section we will consider different eigenvalues $\lambda\in\sigma_{\mathrm{pri}}(H_1^{(l)})$, we will use a more specific notation, $\psi^{(1)}_{\lambda,j}$, $j=1,2,\dots,m_\lambda$, for the corresponding eigenvectors of the operator $H_1^{(l)}$. The notation m_λ, $m_\lambda\leqslant\infty$, stands for the multiplicity of the eigenvalue λ. Recall that every $\psi^{(1)}_{\lambda,j}$ is an eigenvector simultaneously for all the operators $H_1^{(l)}$ and $M_1(\lambda\pm\mathrm{i}0,\Gamma_l)$, $l=(l_1,l_2,\dots,l_m)$ with $l_k=\pm1$, $k=1,2,\dots,m$ (see Lemmas 5.2 and 5.5). Since, according to Lemma 5.7, the resolvent $(H_1^{(l)}-z)^{-1}$ has a pole of the first order at $z=\lambda\in\sigma_{\mathrm{pri}}(H_1^{(l)})$, the multiplicity m_λ is, in the considered case, both the geometric and algebraic multiplicity of λ (which means that every element of the subspace $\mathrm{P}^{(l)}_\lambda\mathcal{H}_1$

is an eigenvector of $H_1^{(l)}$ since $(H_1^{(l)} - \lambda)\mathrm{P}_\lambda^{(l)} = 0)$. The corresponding eigenvectors of the total matrix $\mathbf{H}$ will be denoted by $\Psi_{\lambda,j}$, $\Psi_{\lambda,j} = (\psi_{\lambda,j}^{(0)}, \psi_{\lambda,j}^{(1)})$. It will be supposed that the $\psi_{\lambda,j}^{(1)}$ are chosen in such a way that the vectors $\Psi_{\lambda,j}$ are orthonormal, $\langle \Psi_{\lambda,j}, \Psi_{\lambda',j'} \rangle = \delta_{\lambda\lambda'}\delta_{jj'}$.

Let $\mathcal{H}_1^{(\mathrm{pri})}, \mathcal{H}_1^{(\mathrm{pri})} \subset \mathcal{H}_1$, be the closed span of the eigenvectors $\psi_{\lambda,j}^{(1)}$ of $H_1^{(l)}$ corresponding to the spectrum $\sigma_{\mathrm{pri}}(H_1^{(l)})$,

$$\mathcal{H}_1^{(\mathrm{pri})} = \overline{\bigvee \{\psi_{\lambda,j}^{(1)}, \lambda \in \sigma_{\mathrm{pri}}(H_1^{(l)}), j = 1, 2, \dots, m_\lambda\}}.$$

Then the following statement holds.

THEOREM 5.8. *The system*

$$\psi_{\lambda,j}^{(1)}, \quad \lambda \in \sigma_{\mathrm{pri}}(H_1^{(l)}),\ j = 1, 2, \dots, m_\lambda, \tag{5.18}$$

forms a Riesz basis of the subspace $\mathcal{H}_1^{(\mathrm{pri})}$.

We first prove an auxiliary assertion.

LEMMA 5.9. *For any* $l = (l_1, l_2, \dots, l_m)$, $l_k = \pm 1$, $k = 1, 2, \dots, m$, *the operator* $\Omega^{(l)}$ *defined by the equation* (4.13) *is non-negative on the subspace* $\mathcal{H}_1^{(\mathrm{pri})}$.

Proof. It is sufficient to prove the assertion for a dense subset of $\mathcal{H}_1^{(\mathrm{pri})}$, say, for elements $u_1 \in \mathcal{H}_1^{(\mathrm{pri})}$ of the form

$$u_1 = \sum_{(\lambda,j)\in\mathcal{I}} c_{\lambda,j}\psi_{\lambda,j}^{(1)}, \quad c_{\lambda,j} \in \mathbb{C},$$

where $\mathcal{I}$ runs through the finite subsets of the set of all possible pairs (λ, j) with $\lambda \in \sigma_{\mathrm{pri}}(H_1^{(l)})$, $j = 1, 2, \dots, m_\lambda$. Since $\psi_{\lambda,j}^{(1)}$ and $\psi_{\lambda',j'}^{(1)}$ are eigenfunction for both $H_1^{(l)}$ and $H_1^{(-l)}$, we obtain that

$$\langle \Omega^{(l)} u_1, u_1 \rangle = \sum_{(\lambda,j)\in\mathcal{I}} \sum_{(\lambda',j')\in\mathcal{I}} c_{\lambda,j}\overline{c}_{\lambda',j'}\Omega_{\lambda,j;\lambda',j'}^{(l)}$$

with

$$\begin{aligned} \Omega_{\lambda,j;\lambda',j'}^{(l)} = &\int\limits_{\sigma'(A_0)} \frac{\mu}{(\mu-\lambda)(\mu-\lambda')}\, \mathrm{d}\langle K_B(\mu)\psi_{\lambda,j}^{(1)}, \psi_{\lambda',j'}^{(1)} \rangle \\ &+ \int\limits_{\Gamma_l} \frac{\mu}{(\mu-\lambda)(\mu-\lambda')} \langle K_B'(\mu)\psi_{\lambda,j}^{(1)}, \psi_{\lambda',j'}^{(1)} \rangle\, \mathrm{d}\mu. \end{aligned} \tag{5.19}$$

We will show that

$$\Omega_{\lambda,j;\lambda',j'}^{(l)} = \langle \psi_{\lambda,j}^{(0)}, \psi_{\lambda',j'}^{(0)} \rangle, \quad \text{independent of } l \tag{5.20}$$

and, hence, $\langle \Omega^{(l)} u_1, u_1 \rangle = \|u_0\|^2 \geqslant 0$ with $u_0 = \sum\limits_{(\lambda,j)\in\mathcal{I}} c_{\lambda,j}\psi_{\lambda,j}^{(0)}$. Thus, the operator $\Omega^{(l)}$ is non-negative on a dense subset of $\mathcal{H}_1^{(\mathrm{pri})}$ and consequently non-negative on the whole subspace $\mathcal{H}_1^{(\mathrm{pri})}$.

To prove (5.20), let us first suppose that λ and λ' belong to $\varrho(A_0)$. The relation (5.2) yields that

$$\psi^{(0)}_{\lambda,j} = -A_0^{1/2}(A_0-\lambda)^{-1}B_{01}\psi^{(1)}_{\lambda,j}, \quad \psi^{(0)}_{\lambda',j'} = -A_0^{1/2}(A_0-\lambda')^{-1}B_{01}\psi^{(1)}_{\lambda',j'}$$

and hence

$$\begin{aligned}\langle\psi^{(0)}_{\lambda,j},\psi^{(0)}_{\lambda',j'}\rangle &= \langle B^*_{01}(A_0-\lambda')^{-1}A_0(A_0-\lambda)^{-1}B_{01}\psi^{(1)}_{\lambda,j},\psi^{(1)}_{\lambda',j'}\rangle\\ &= \int\limits_{\sigma(A_0)} \frac{\mu}{(\mu-\lambda)(\mu-\lambda')}\,\mathrm{d}\langle B_{10}E_0(\mu)B_{01}\psi^{(1)}_{\lambda,j},\psi^{(1)}_{\lambda',j'}\rangle,\end{aligned}$$

which shows the validity of (5.20) since one can deform all the subcontours $\Gamma_k^{l_k}$ of Γ_l back to the corresponding intervals Δ_k^0 on which $K_B(\mu)=B_{10}E_0(\mu)B_{01}$.

Now consider the case that $\lambda\in\Delta_k^0$ for some $k\in\{1,2,\ldots,m\}$ and $\lambda'\in\varrho(A_0)$. As in the proof of Lemma 5.5, we choose a sequence $\lambda_n=\lambda+\mathrm{i}\eta_n$, $n\in\mathbb{N}$, such that $\eta_n\to 0$ $(n\to\infty)$ and $\eta_n<0$ if $l_k=+1$ and $\eta_n>0$ if $l_k=-1$. Then

$$\psi^{(0)}_{\lambda,j} = -\lim_{n\to\infty} A_0^{1/2}(A_0-\lambda_n)^{-1}B_{01}\psi^{(1)}_{\lambda,j}$$

and

$$\langle\psi^{(0)}_{\lambda,j},\psi^{(0)}_{\lambda',j'}\rangle = \lim_{n\to\infty}\langle B^*_{01}(A_0-\lambda')^{-1}A_0(A_0-\lambda_n)^{-1}B_{01}\psi^{(1)}_{\lambda,j},\psi^{(1)}_{\lambda',j'}\rangle.$$

Similar as before, we conclude that

$$\begin{aligned}\langle\psi^{(0)}_{\lambda,j},\psi^{(0)}_{\lambda',j'}\rangle &= \lim_{n\to\infty}\Bigg(\int\limits_{\sigma'(A_0)} \frac{\mu}{(\mu-\lambda_n)(\mu-\lambda')}\,\mathrm{d}\langle K_B(\mu)\psi^{(1)}_{\lambda,j},\psi^{(1)}_{\lambda',j'}\rangle\\ &\qquad + \int\limits_{\Gamma_l}\frac{\mu}{(\mu-\lambda_n)(\mu-\lambda')}\langle K'_B(\mu)\psi^{(1)}_{\lambda,j},\psi^{(1)}_{\lambda',j'}\rangle\,\mathrm{d}\mu\Bigg)\\ &= \int\limits_{\sigma'(A_0)} \frac{\mu}{(\mu-\lambda)(\mu-\lambda')}\,\mathrm{d}\langle K_B(\mu)\psi^{(1)}_{\lambda,j},\psi^{(1)}_{\lambda',j'}\rangle\\ &\qquad + \int\limits_{\Gamma_l}\frac{\mu}{(\mu-\lambda)(\mu-\lambda')}\langle K'_B(\mu)\psi^{(1)}_{\lambda,j},\psi^{(1)}_{\lambda',j'}\rangle\,\mathrm{d}\mu\\ &= \Omega^{(l)}_{\lambda,j;\lambda',j'}.\end{aligned}$$

The remaining cases $\lambda\in\varrho(A_0)$, $\lambda'\in\Delta^0_{k'}$ for some $k'\in\{1,2,\ldots,m\}$ and $\lambda\in\Delta^0_k$, $\lambda'\in\Delta^0_{k'}$ for some k, $k'\in\{1,2,\ldots,m\}$ can be treated analogously. ∎

Thus, one can introduce a new inner product in $\mathcal{H}_1^{(\mathrm{pri})}$,

$$[u_1,v_1]_{\mathcal{H}_1^{(\mathrm{pri})}} := \langle (I_1+\Omega^{(l)})u_1,v_1\rangle, \quad u_1,v_1\in\mathcal{H}_1^{(\mathrm{pri})}, \tag{5.21}$$

which is topologically equivalent to the initial inner product $\langle\cdot\,,\,\cdot\rangle$, since, in view of the estimate (4.14), the operator $I_1+\Omega^{(l)}$ is boundedly invertible. (One can even check that the restriction of $H_1^{(l)}$ on $\mathcal{H}_1^{(\mathrm{pri})}$ does not depend on l and is an operator in $\mathcal{H}_1^{(\mathrm{pri})}$ which is selfadjoint with respect to the inner product $[\,\cdot\,,\,\cdot\,]_{\mathcal{H}_1^{(\mathrm{pri})}}$.)

Proof of Theorem 5.8. We show that the system (5.18) is an orthonormal system with respect to the inner product $[\,\cdot\,,\,\cdot\,]_{\mathcal{H}_1^{(\mathrm{pri})}}$. Indeed, according to the equations (5.19) and (5.20) we have $\langle \Omega^{(l)}\psi^{(1)}_{\lambda,j}, \psi^{(1)}_{\lambda',j'}\rangle = \Omega^{(l)}_{\lambda,j;\lambda',j'} = \langle \psi^{(0)}_{\lambda,j}, \psi^{(0)}_{\lambda',j'}\rangle$. Thus,

$$[\psi^{(1)}_{\lambda,j}, \psi^{(1)}_{\lambda',j'}]_{\mathcal{H}_1^{(\mathrm{pri})}} = \langle \psi^{(1)}_{\lambda,j}, \psi^{(1)}_{\lambda',j'}\rangle + \langle \psi^{(0)}_{\lambda,j}, \psi^{(0)}_{\lambda',j'}\rangle = \langle \Psi_{\lambda,j}, \Psi_{\lambda',j'}\rangle = \delta_{\lambda\lambda'}\delta_{jj'}.$$

By definition, the system (5.18) is complete in $\mathcal{H}_1^{(\mathrm{pri})}$ and the inner product $[\,\cdot\,,\,\cdot\,]_{\mathcal{H}_1^{(\mathrm{pri})}}$ is topologically equivalent to the initial inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. According to a theorem of N.K. Bari (Theorem VI.2.1 of [11]), this means that the system (5.18) is a Riesz basis. ∎

6. SOME PROPERTIES OF COMPLEX EIGENVALUES

For arbitrary l, let $\Gamma_l \subset D_l$ be a K_B-bounded contour satisfying the conditions (3.10). Suppose $X^{(l)}$ is the solution of the equation (3.6) (see Theorem 3.1) and $H_1^{(l)} = \widetilde{A}_1 + X^{(l)}$. From Theorem 4.2 we know that the spectrum of $H_1^{(l)}$ outside D_l is pure real,

$$\sigma\big(H_1^{(l)}\big) \setminus \mathbb{R} \subset D_l \cap \mathcal{O}_{r_0(B)}(\widetilde{A}_1) \tag{6.1}$$

and

$$\sigma\big(M_1(\,\cdot\,,\Gamma_l)\big) \cap \mathcal{O}(\widetilde{A}_1,\Gamma_l) = \sigma\big(H_1^{(l)}\big). \tag{6.2}$$

Proposition 6.1. *Let $\Gamma_l \subset D_l$ be a K_B-bounded contour satisfying the conditions* (3.10). *Suppose $X^{(l)}$ is the solution of the basic equation* (3.6) *and $H_1^{(l)} = \widetilde{A}_1 + X^{(l)}$. Further, we assume that the operators $K_B(\mu)$ and $K'_B(\mu)$ are compact for all μ to be considered. Then the operator $X^{(l)}$ is compact and the nonreal* (*resonance*) *spectrum of the operator $H_1^{(l)}$ is discrete and may have accumulation points only in the essential spectrum of $\widetilde{A}_1$.*

Proof. By hypothesis, for any $\mu \in \sigma'(A_0)\cup\Gamma_l$, the operator $K_B(\mu)$ is compact while due to conditions (3.10) the product $H_1^{(l)}(H_1^{(l)}-\mu)^{-1}$ is a bounded operator. This implies that any finite sum in the definition of the integral which defines the operator $X^{(l)}$, $X^{(l)} = \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)H_1^{(l)}(H_1^{(l)}-\mu)^{-1}$, is a compact operator. Under the K_B-boundedness condition for the contour Γ_l, the sums converge to $X^{(l)}$ with respect to the operator norm topology (see [23], Appendix B). Thus, $X^{(l)}$ is a compact operator.

It is well known that a compact perturbation does not change the essential spectrum of a closed operator (see, e.g., [13], Theorem IV.5.35 or, for the case where the main operator is selfadjoint, see [11], Chapter I, Theorem 5.2). Thus, the essential spectrum $\sigma_{\rm ess}(H_1^{(l)})$ of $H_1^{(l)}$ coincides with the essential spectrum $\sigma_{\rm ess}(\widetilde{A}_1)$ of $\widetilde{A}_1$. Since $\sigma_{\rm ess}(\widetilde{A}_1) \subset \mathbb{R}$, the nonreal spectrum of $H_1^{(l)}$ is discrete. This also yields that possible accumulation points of the nonreal (resonance) spectrum of $H_1^{(l)}$ belong to $\sigma_{\rm ess}(\widetilde{A}_1)$. ∎

In the remaining part of this section we work under the assumption that the operators $K_B(\mu)$ and $K'_B(\mu)$ are compact for all μ to be considered. Let $\lambda \in \sigma(H_1^{(l)}) \setminus \mathbb{R}$. Then, by Proposition 6.1, λ is a discrete eigenvalue of the operator $H_1^{(l)}$ and, by Theorem 4.2, it is simultaneously a discrete eigenvalue of the transfer function $M_1(\,\cdot\,,\Gamma_l)$.

Recall some definitions related to isolated eigenvalues (see, e.g., [13], Section I.5 and Section III.5) bearing in mind the above discrete eigenvalue λ of $H_1^{(l)}$. The subspace $\mathcal{M}_\lambda^{(l)} = \mathrm{P}_\lambda^{(l)}\mathcal{H}_1$, where $\mathrm{P}_\lambda^{(l)}$ stands for the Riesz projection (4.24), is called the *algebraic eigenspace* for the eigenvalue λ. Since this λ is a discrete eigenvalue and, thus, of finite type, the algebraic multiplicity $m_\lambda = \dim \mathcal{M}_\lambda^{(l)}$ is finite, $m_\lambda < \infty$. Any nonzero vector of $\mathcal{M}_\lambda^{(l)}$ is called *root vector* of $H_1^{(l)}$ corresponding to the eigenvalue λ. By $N_\lambda^{(l)}$ we denote the eigennilpotent associated with λ,

$$N_\lambda^{(l)} = (H_1^{(l)} - \lambda)\mathrm{P}_\lambda^{(l)} = \mathrm{P}_\lambda^{(l)}(H_1^{(l)} - \lambda). \tag{6.3}$$

The kernel $\mathcal{G}_\lambda^{(l)} = \mathrm{Ker}(H_1^{(l)} - \lambda)$ is called the *geometric eigenspace* for the eigenvalue λ. Any nonzero $u \in \mathcal{G}_\lambda^{(l)}$ is an eigenvector of $H_1^{(l)}$, $H_1^{(l)}u = \lambda u$. The dimension $g_\lambda = \dim \mathcal{G}_\lambda^{(l)}$ is called the *geometric multiplicity* of λ. From $\mathcal{G}_\lambda^{(l)} \subset \mathcal{M}_\lambda^{(l)}$ it follows that $g_\lambda \leqslant m_\lambda$. There is an open neighborhood $\widetilde{\mathcal{O}}(\lambda)$ of λ such that for any $z \in \widetilde{\mathcal{O}}(\lambda) \setminus \{\lambda\}$ the following representation holds:

$$\big(H_1^{(l)} - z\big)^{-1} = -\frac{\mathrm{P}_\lambda^{(l)}}{z-\lambda} - \sum_{k=1}^{n_\lambda - 1} \frac{[N_\lambda^{(l)}]^k}{(z-\lambda)^{k+1}} + R_{1,\lambda}^{(l)}(z), \tag{6.4}$$

where $R_{1,\lambda}^{(l)}(z)$ is a holomorphic operator-valued function in $\widetilde{\mathcal{O}}(\lambda)$. This function satisfies the relations:

$$R_{1,\lambda}^{(l)}(z)\mathrm{P}_\lambda^{(l)} = \mathrm{P}_\lambda^{(l)} R_{1,\lambda}^{(l)}(z) = 0 \tag{6.5}$$

for any $z \in \widetilde{\mathcal{O}}(\lambda)$. The number n_λ represents the pole order of the resolvent $\big(H_1^{(l)} - z\big)^{-1}$ at $z = \lambda$. Indication of l in the notation n_λ is omitted since for a given λ the pole order does not depend on l, according to the formula (4.19) and Theorem 4.3. (Similarly to the pole order n_λ, the algebraic and geometric multiplicities m_λ and g_λ do not depend on l, either.) Since λ is a finite-type eigenvalue of $H_1^{(l)}$, the value of n_λ is finite and, moreover, $n_\lambda \leqslant m_\lambda$.

Note that the eigenprojection $\mathrm{P}_\lambda^{(l)}$ and eigennilpotent $N_\lambda^{(l)}$ possess the following standard properties (see [13], Section III.1):

$$\begin{aligned} &\mathrm{P}_\lambda^{(l)}\mathrm{P}_\lambda^{(l)} = \mathrm{P}_\lambda^{(l)}, \quad \mathrm{P}_\lambda^{(l)} N_\lambda^{(l)} = N_\lambda^{(l)}\mathrm{P}_\lambda^{(l)} = N_\lambda^{(l)}, \\ &\big[N_\lambda^{(l)}\big]^{n_\lambda - 1} \neq 0, \quad \big[N_\lambda^{(l)}\big]^{n_\lambda} = 0. \end{aligned} \tag{6.6}$$

From (6.5) and (6.6) we infer that

$$\big(H_1^{(l)}-z\big)^{-1}\mathrm{P}_\lambda^{(l)}=-\frac{\mathrm{P}_\lambda^{(l)}}{z-\lambda}-\sum_{k=1}^{n_\lambda-1}\frac{[N_\lambda^{(l)}]^k}{(z-\lambda)^{k+1}} \tag{6.7}$$

for any $z\in\widetilde{\mathcal{O}}(\lambda)$. From the uniqueness of the analytic continuation it follows that the equality (6.7) holds in fact for any $z\in\mathbb{C}\setminus\sigma(H_1^{(l)})$.

Lemma 6.2. *Let the assumptions of Proposition* 6.1 *be fulfilled and* $\lambda\in\sigma\big(H_1^{(l)}\big)\setminus\mathbb{R}$. *Then the eigenprojection* $\mathrm{P}_\lambda^{(l)}$ *and the eigennilpotent* $N_\lambda^{(l)}$ *of the operator* $H_1^{(l)}$ *satisfy the equations*

$$M_1(\lambda,\Gamma_l)\mathrm{P}_\lambda^{(l)}=N_\lambda^{(l)}-\sum_{k=1}^{n_\lambda-1}\frac{1}{k!}V_1^{(k)}(\lambda,\Gamma_l)[N_\lambda^{(l)}]^k, \tag{6.8}$$

$$M_1(\lambda,\Gamma_l)[N_\lambda^{(l)}]^{n_\lambda-p}=[N_\lambda^{(l)}]^{n_\lambda-p+1}-\sum_{k=1}^{p-1}\frac{1}{k!}V_1^{(k)}(\lambda,\Gamma_l)[N_\lambda^{(l)}]^{n_\lambda-p+k} \tag{6.9}$$

for $p=1,\dots,n_\lambda-1$, *where* $V_1^{(k)}(z,\Gamma_l)$ *denotes the* k*-th derivative of the function* $V_1(z,\Gamma_l)$, *defined by the equation* (2.13), *at* $z=\lambda$

$$V_1^{(k)}(\lambda,\Gamma_l)=k!(-1)^k\int\limits_{\sigma'(A_0)\cup\Gamma_l}K_B(\mathrm{d}\mu)\left[\frac{\lambda}{(\lambda-\mu)^{k+1}}-\frac{1}{(\lambda-\mu)^k}\right],\quad k\geqslant 1. \tag{6.10}$$

Proof. The equations (3.5) and (3.6) yield (see (4.4)) that

$$\widetilde{A}_1=H_1^{(l)}-V_1(H_1^{(l)},\Gamma_l). \tag{6.11}$$

Multiplying both parts of (6.11) by $\mathrm{P}_\lambda^{(l)}$ from the right and taking into account the equality

$$H_1^{(l)}\mathrm{P}_\lambda^{(l)}=\lambda\mathrm{P}_\lambda^{(l)}+N_\lambda^{(l)} \tag{6.12}$$

yields the relation

$$\widetilde{A}_1\mathrm{P}_\lambda^{(l)}=\lambda\mathrm{P}_\lambda^{(l)}+N_\lambda^{(l)}-V_1(H_1^{(l)},\Gamma_l)\mathrm{P}_\lambda^{(l)}. \tag{6.13}$$

The definition (3.3) and the equalities (6.12), (6.6) and (6.7) imply that

$$\begin{aligned}V_1(H_1^{(l)},\Gamma_l)\mathrm{P}_\lambda^{(l)}&=\int\limits_{\sigma'(A_0)\cup\Gamma_l}K_B(\mathrm{d}\mu)\big(H_1^{(l)}-\mu\big)^{-1}H_1^{(l)}\mathrm{P}_\lambda^{(l)}\\&=\int\limits_{\sigma'(A_0)\cup\Gamma_l}K_B(\mathrm{d}\mu)\big(H_1^{(l)}-\mu\big)^{-1}\big(\lambda\mathrm{P}_\lambda^{(l)}+N_\lambda^{(l)}\big)\\&=\int\limits_{\sigma'(A_0)\cup\Gamma_l}K_B(\mathrm{d}\mu)\big(H_1^{(l)}-\mu\big)^{-1}\mathrm{P}_\lambda^{(l)}\big(\lambda\mathrm{P}_\lambda^{(l)}+N_\lambda^{(l)}\big)\end{aligned}$$

$$
\begin{aligned}
&= - \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)\Big(\frac{\mathrm{P}_\lambda^{(l)}}{\mu-\lambda} + \sum_{k=1}^{n_\lambda-1} \frac{[N_\lambda^{(l)}]^k}{(\mu-\lambda)^{k+1}}\Big)\big(\lambda \mathrm{P}_\lambda^{(l)} + N_\lambda^{(l)}\big) \\
&= - \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)\frac{\lambda}{\mu-\lambda}\mathrm{P}_\lambda^{(l)} \qquad (6.14)\\
&\quad - \sum_{k=1}^{n_\lambda-1} \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)\left[\frac{\lambda}{(\mu-\lambda)^{k+1}} + \frac{1}{(\mu-\lambda)^k}\right][N_\lambda^{(l)}]^k.
\end{aligned}
$$

The equation (2.12) together with (6.11) and (6.14) yields

$$
\begin{aligned}
M_1(\lambda,\Gamma_l)\mathrm{P}_\lambda^{(l)} &= \widetilde{A}_1\mathrm{P}_\lambda^{(l)} - \lambda\mathrm{P}_\lambda^{(l)} + \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)\frac{\lambda}{\lambda-\mu}\mathrm{P}_\lambda^{(l)} \\
&= N_\lambda^{(l)} + \sum_{k=1}^{n_\lambda-1} \int\limits_{\sigma'(A_0)\cup\Gamma_l} K_B(\mathrm{d}\mu)\Big[\frac{\lambda}{(\mu-\lambda)^{k+1}} + \frac{1}{(\mu-\lambda)^k}\Big][N_\lambda^{(l)}]^k.
\end{aligned}
$$

But according to (6.10) this is the equation (6.8) which we wanted to prove.

By multiplying both parts of (6.8) from the right by $[N_\lambda^{(l)}]^{n_\lambda-p}$ for $p = 1,\dots,n_\lambda-1$ and using the properties of the eigenprojection and eigennilpotent (6.6) we obtain the formula (6.9). ∎

Let $\lambda_1,\dots,\lambda_s \in \sigma(H_1^{(l)})\setminus\mathbb{R}$ be a finite set of pairwise different nonreal eigenvalues of the operator $H_1^{(l)}$. By $\mathrm{P}^{(l)}_{\lambda_1,\dots,\lambda_s}$ we denote the Riesz projection corresponding to this set, by $\mathrm{P}^{(l)}_{\lambda_j}$ the eigenprojection and by $N^{(l)}_{\lambda_j}$ the eigennilpotent associated with the individual eigenvalue λ_j, $j=1,\dots,s$. Since by Proposition 6.1 the eigenvalues λ_j are eigenvalues of finite type, we infer that $\dim \mathrm{P}^{(l)}_{\lambda_1,\dots,\lambda_s}\mathcal{H}_1 < \infty$. It is obvious from the definition that

$$
\mathrm{P}^{(l)}_{\lambda_1,\dots,\lambda_s} = \mathrm{P}^{(l)}_{\lambda_1} + \cdots + \mathrm{P}^{(l)}_{\lambda_s}. \tag{6.15}
$$

Further, we denote by n_{λ_j} the pole order of λ_j, $j=1,\dots,s$. Recall that the eigenprojections and eigennilpotents associated with different eigenvalues fulfill the following properties, in addition to the properties (6.6):

$$
\mathrm{P}^{(l)}_{\lambda_j}\mathrm{P}^{(l)}_{\lambda_m} = 0, \quad N^{(l)}_{\lambda_j}\mathrm{P}^{(l)}_{\lambda_m} = \mathrm{P}^{(l)}_{\lambda_m}N^{(l)}_{\lambda_j} = 0 \quad \text{and} \quad N^{(l)}_{\lambda_j}N^{(l)}_{\lambda_m} = 0, \quad j \neq m. \tag{6.16}
$$

Since the subspaces $\mathrm{P}^{(l)}_{\lambda_1,\dots,\lambda_s}\mathcal{H}_1$ and $(I_1 - \mathrm{P}^{(l)}_{\lambda_1,\dots,\lambda_s})\mathcal{H}_1$ are invariant under the operator $H_1^{(l)}$, we conclude that the following representation for the resolvent of the restriction $H_1^{(l)}|\mathrm{P}^{(l)}_{\lambda_1,\dots,\lambda_s}\mathcal{H}_1$ holds:

$$
\big(H_1^{(l)}|\mathrm{P}^{(l)}_{\lambda_1,\dots,\lambda_s}\mathcal{H}_1 - z\big)^{-1} = \sum_{j=1}^{s}\Big\{ -\frac{\mathrm{P}^{(l)}_{\lambda_j}}{z-\lambda_j} - \sum_{k=1}^{n_{\lambda_j}-1}\frac{[N^{(l)}_{\lambda_j}]^k}{(z-\lambda_j)^{k+1}}\Big\}. \tag{6.17}
$$

At the same time

$$H_1^{(l)}|\mathrm{P}^{(l)}_{\lambda_1,\ldots,\lambda_s}\mathcal{H}_1=\sum_{j=1}^{s}\big(\lambda_j\mathrm{P}^{(l)}_{\lambda_j}+N^{(l)}_{\lambda_j}\big). \tag{6.18}$$

The equations (6.15), (6.18) and

$$M_1(z,\Gamma_l)=\widetilde{A}_1-z+V_1(z,\Gamma_l)=H_1^{(l)}-z-V_1(H_1^{(l)},\Gamma_l)+V_1(z,\Gamma_l)$$

imply that

$$M_1(z,\Gamma_l)|\mathrm{P}^{(l)}_{\lambda_1,\ldots,\lambda_s}\mathcal{H}_1=\sum_{j=1}^{s}\Big[\lambda_j\mathrm{P}^{(l)}_{\lambda_j}+N^{(l)}_{\lambda_j}-z\mathrm{P}^{(l)}_{\lambda_j}+V_1(z,\Gamma_l)\mathrm{P}^{(l)}_{\lambda_j}-V_1(H_1^{(l)},\Gamma_l)\mathrm{P}^{(l)}_{\lambda_j}\Big].$$

Bearing in mind the equalities (6.10) and (6.14), we finally arrive at the following statement:

Remark 6.3. *The transfer function $M_1(\,\cdot\,,\Gamma_l)$ restricted to the space $\mathrm{P}^{(l)}_{\lambda_1,\ldots,\lambda_s}\mathcal{H}_1$ admits the representation*

$$\begin{aligned}M_1(z,\Gamma_l)|\mathrm{P}^{(l)}_{\lambda_1,\ldots,\lambda_s}\mathcal{H}_1=\sum_{j=1}^{s}\Big\{&(\lambda_j-z)\mathrm{P}^{(l)}_{\lambda_j}+N^{(l)}_{\lambda_j}+V_1(z,\Gamma_l)\mathrm{P}^{(l)}_{\lambda_j}\\&-V_1(\lambda_j^{(l)},\Gamma_l)\mathrm{P}^{(l)}_{\lambda_j}-\sum_{k=1}^{n_{\lambda_j}-1}\frac{1}{k!}V_1^{(k)}(\lambda_j^{(l)},\Gamma_l)[N^{(l)}_{\lambda_j}]^k\Big\}.\end{aligned}$$

7. AN EXAMPLE

Let $\mathcal{H}_0=\mathcal{H}_1=L_2(\mathbb{R})$ and $A_0=D^2+\lambda_0 I_0$ where $D=\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ and λ_0 is some positive number. It is assumed that the domain $\mathcal{D}(D)$ is the Sobolev space $W_2^1(\mathbb{R})$ and the domain $\mathcal{D}(A_0)$ is the Sobolev space $W_2^2(\mathbb{R})$. The spectrum of A_0 is absolutely continuous and fills the semi-axis $[\lambda_0,+\infty)$. By the operator A_1 we understand the multiplication by a bounded real-valued function a_1, $A_1f_1=a_1f_1$, $f_1\in\mathcal{H}_1$.

The operator T_{01} is defined by $\mathcal{D}(T_{01})=W_2^1(\mathbb{R})$ and

$$T_{01}=DQ$$

where Q is the multiplication by a bounded not necessarily real-valued function $q\in W_2^1(\mathbb{R})$, $Qf=qf$, $f\in L_2(\mathbb{R})$. Hence T_{01} is a densely defined closable operator. We set $T_{10}=T_{01}^*$. Since for $f\in\mathcal{D}(D)$ and $g\in\mathcal{D}(T_{01})$

$$\langle f,T_{01}g\rangle=\int_{\mathbb{R}}f(x)\overline{\mathrm{i}(qg)'(x)}\,\mathrm{d}x=\int_{\mathbb{R}}\mathrm{i}\overline{q(x)}f'(x)\overline{g(x)}\,\mathrm{d}x=\langle Q^*Df,g\rangle,$$

we conclude that $\mathcal{D}(T_{10})\supset\mathcal{D}(D)=W_2^1(\mathbb{R})=\mathcal{D}(A_0^{1/2})$ (The proof of the statement $W_2^1(\mathbb{R})=\mathcal{D}(A_0^{1/2})$ follows from the second representation theorem for quadratic forms, see [13], Theorem VI.2.23, and is similar to the proof of Proposition 2.4 in

[7].) and $T_{10}f = Q^*Df$, $f \in W_2^1(\mathbb{R})$. Moreover, we assume that the function q is exponentially decreasing at infinity, so that the estimate

$$|q(x)| \leqslant c\exp(-\alpha|x|), \quad x \in \mathbb{R}, \tag{7.1}$$

holds with some $c \geqslant 0$ and $\alpha > 0$.

For this example the operator B_{10} is given by

$$B_{10} = Q^*D(D^2+\lambda_0)^{-1/2} = Q^* \int\limits_{\mathbb{R}} \frac{\mu}{(\mu^2+\lambda_0)^{1/2}}\, \mathrm{d}E_D(\mu),$$

where E_D denotes the spectral function of the selfadjoint operator D. Thus

$$B_{01} = B_{10}^* = \int\limits_{\mathbb{R}} \frac{\mu}{(\mu^2+\lambda_0)^{1/2}}\, \mathrm{d}E_D(\mu)Q$$

and

$$\begin{aligned} \widetilde{A}_1 &= A_1 - B_{10}B_{01} \\ &= A_1 - Q^* \int\limits_{\mathbb{R}} \frac{\mu}{(\mu^2+\lambda_0)^{1/2}}\, \mathrm{d}E_D(\mu) \int\limits_{\mathbb{R}} \frac{\tilde{\mu}}{(\tilde{\mu}^2+\lambda_0)^{1/2}}\, \mathrm{d}E_D(\tilde{\mu})Q \\ &= A_1 - Q^* \int\limits_{\mathbb{R}} \frac{\mu^2}{\mu^2+\lambda_0}\, \mathrm{d}E_D(\mu)Q \\ &= A_1 - Q^*Q + \lambda_0 Q^*(D^2+\lambda_0)^{-1}Q. \end{aligned}$$

The operator $A_1 - Q^*Q$ is the multiplication by the function

$$\widetilde{a}_1(x) = a_1(x) - |q(x)|^2,$$

while the term $Q^*(D^2+\lambda_0 I_0)^{-1}Q$ is a compact (even Hilbert-Schmidt) operator in $L_2(\mathbb{R})$. Indeed, the inverse operator $A_0^{-1} = (D^2+\lambda_0)^{-1}$ is the integral operator whose kernel reads

$$A_0^{-1}(x,x') = \frac{1}{2\sqrt{\lambda_0}} \exp\bigl(-\sqrt{\lambda_0}|x-x'|\bigr).$$

Thus, the double integral $\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |(Q^*A_0^{-1}Q)(x,x')|^2\, \mathrm{d}x\, \mathrm{d}x'$ is convergent. Obviously,

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |(Q^*A_0^{-1}Q)(x,x')|^2\, \mathrm{d}x\, \mathrm{d}x' \leqslant \frac{1}{4\lambda_0}\|q\|^4_{L_2(\mathbb{R})}.$$

Thus, the essential spectrum of $\widetilde{A}_1$ coincides with the range of the function $\widetilde{a}_1$. In the following, we assume that either all the spectrum $\sigma(\widetilde{A}_1)$ is embedded into the interval $(\lambda_0, +\infty)$ or there is a gap in $\sigma(\widetilde{A}_1)$ and the number λ_0 belongs to this gap, i.e., there is some $\widetilde{c} > 0$ such that $\operatorname{dist}\{\lambda_0, \sigma(\widetilde{A}_1)\} \geqslant \widetilde{c}$.

It is easy to check that the spectral function $E^0(\mu)$ of the operator $A_0 = D^2 + \lambda_0 I_0$ is given by the integral operator whose kernel reads

$$E^0(\mu;x,x') = \begin{cases} 0 & \text{if } \mu < \lambda_0, \\ \frac{1}{\sqrt{2\pi}} \int\limits_{\lambda_0}^{\mu} \frac{\cos[(\nu-\lambda_0)^{1/2}(x-x')]}{(\nu-\lambda_0)^{1/2}}\, \mathrm{d}\nu & \text{if } \mu \geqslant \lambda_0. \end{cases}$$

Thus, the derivative $K_B'(\mu)$ is also an integral operator in $L_2(\mathbb{R})$. Its kernel $K_B'(\mu;x,x')$ is only nontrivial for $\mu>\lambda_0$ and, moreover, for these μ

$$K_B'(\mu;x,x')=\frac{(\mu-\lambda_0)^{1/2}}{\sqrt{2\pi}\mu}\cos[(\mu-\lambda_0)^{1/2}(x-x')]\overline{q(x)}q(x').$$

Obviously, this kernel is degenerate for $\mu>\lambda_0$,

$$K_B'(\mu;x,x')=\frac{(\mu-\lambda_0)^{1/2}}{2\sqrt{2\pi}\mu}[\overline{q_-(\mu,x)}q_-(\mu,x')+\overline{q_+(\mu,x)}q_+(\mu,x')] \tag{7.2}$$

where $q_\pm(\mu,x)=\mathrm{e}^{\pm\mathrm{i}(\mu-\lambda_0)^{1/2}x}q(x)$. From the assumption (7.1) on q, we conclude that in the domain $\pm\operatorname{Im}\sqrt{\mu-\lambda_0}<\alpha$, i.e., inside the parabola

$$\operatorname{Re}\mu>\lambda_0-\alpha^2+\frac{1}{4\alpha^2}(\operatorname{Im}\mu)^2, \tag{7.3}$$

the functions $q_\pm(\mu,\,\cdot\,)$ are elements of $L_2(\mathbb{R})$. The function $K_B'(\mu)$ admits an analytic continuation into this domain (cut along the interval $\lambda_0-\alpha^2<\mu\leqslant\lambda_0$) as a holomorphic function with values in $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_1)$ and the equation (7.2) implies that

$$\|K_B'(\mu)\|\leqslant\frac{|\mu-\lambda_0|^{1/2}}{2\sqrt{2\pi}|\mu|}[\|q_-(\mu,\,\cdot\,)\|^2+\|q_+(\mu,\,\cdot\,)\|^2].$$

Obviously, for $\mu\geqslant\lambda_0$ we have $\|q_\pm(\mu,\cdot)\|=\|q\|$. Since $\widetilde{A}_1$ is bounded, one can always choose a K_B-bounded contour Γ lying in the domain (7.3). Indeed, for the K_B-boundedness of the contour Γ it is sufficient to have its infinite part presented by an appropriate semi-infinite real interval. Thus, if the function q is sufficiently small in the sense that the conditions (3.10) hold, one can apply all the statements of the Sections 3–6 to the corresponding transfer function $M_1(z,\Gamma)$.

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