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K-GROUPS OF BANACH ALGEBRAS AND STRONGLY IRREDUCIBLE DECOMPOSITIONS OF OPERATORS

YANG CAO, JUNSHENG FANG and CHUNLAN JIANG

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Abstract. A bounded linear operator T on the Hilbert space $\mathcal{H}$ is called strongly irreducible if T does not commute with any nontrivial idempotent operator. One says that T has a finite (SI) decomposition if T can be written as the direct sum of finitely many strongly irreducible operators. In this paper, we use the K_0-group of the commutant of operators to characterize operators with unique finite (SI) decomposition up to similarity. Also we show that the K_0-group of $H^\infty(\Omega)$ is isomorphic to the integers, where Ω is simply connected.

Keywords: K_0*-group,* (SI) *decomposition, commutant of operators.*

MSC (2000): 47A05, 47A55.

0. INTRODUCTION

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ the collection of all bounded linear operators on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ is called *strongly irreducible* if T does not commute with any nontrivial idempotent operator. If idempotent operator is replaced by self-adjoint idempotent, then T is said to be *irreducible* (see [10], [15], [12]). Strong irreducibility is preserved by similarity. This is quite different from irreducible operators.

When $\mathcal{H}$ is finite dimensional, classical matrix theory gives two important theorems.

Schur Theorem. *Each $n \times n$ matrix can be uniquely written as an orthogonal direct sum of irreducible matrices up to unitary equivalence.*

Jordan Standard Theorem. *Each $n \times n$ matrix can be uniquely written as a direct sum of strongly irreducible matrices up to similarity.*

Obviously, an $n \times n$ matrix A is strongly irreducible if and only if A is similar to an $n \times n$ Jordan block. When $\mathcal{H}$ is an infinitely dimensional complex and separable Hilbert space, a natural question is raised: Can we establish analogues of the Jordan Standard Theorem and Schur Theorem in $\mathcal{L}(\mathcal{H})$?

Behncke ([2]) proved the following:

THEOREM B. *Let $T \in \mathcal{L}(\mathcal{H})$ be pure essentially normal. Then T can be uniquely expressed as a direct sum of countably many irreducible operators up to unitary equivalence.*

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *pure essentially normal* if $T^*T - TT^*$ is compact and has no nontrivial self-adjoint idempotent P commuting with T such that $A = T|_{P\mathcal{H}}$ is normal, i.e. $A^*A = AA^*$. By the *Berger-Shaw Theorem*, every n-rationally multicyclic hyponormal operator is essentially normal. Thus, every n-rationally multicyclic hyponormal operator can be uniquely written as a direct sum of countably many irreducible operators up to unitary equivalence. An operator T is called *hyponormal* if $T^*T - TT^*$ is a positive operator.

Let Ω be a bounded and connected open subset of the complex plane $\mathbb{C}$ and n a positive integer. Let $\mathcal{B}_n(\Omega)$ denote the set of operators B in $\mathcal{L}(\mathcal{H})$ satisfying:

(i) $\Omega \subset \sigma(B) = \{\omega \in \mathbb{C} : B - \omega \text{ is not invertible}\}$,
(ii) $\operatorname{ran}(B - \omega) = \mathcal{H}$ for every ω in Ω,
(iii) $\bigvee\{\ker(B - \omega) : \omega \in \Omega\} = \mathcal{H}$,
(iv) $\dim\ker(B - \omega) = n$ for every $\omega \in \Omega$.

We call an operator in $\mathcal{B}_n(\Omega)$ a *Cowen-Douglas operator* (see [6]). G.L. Yu and C.Q. Yan independently proved the following.

THEOREM YY. ([20], [21]) *Let $B \in \mathcal{B}_2(\Omega)$. Then B has a unique irreducible decomposition up to the unitary equivalence.*

L.J. Gray proved the following.

THEOREM G. ([11]) *Let $T \in \mathcal{L}(\mathcal{H})$ be nilpotent i.e. there exists a natural number n such that $T^n = 0$. Then T can be uniquely written as a direct sum of countably many Jordan blocks up to similarity if and only if $\operatorname{ran} T^j$ is closed for all $j = 1, 2, \ldots$.*

K. Davidson and D.A. Herrero obtained the following:

THEOREM DH. ([7]) *Let $T \in \mathcal{L}(\mathcal{H})$ be biquasitriangular and $\varepsilon > 0$. Then there exists a compact operator K with $\|K\| < \varepsilon$ such that $T + K$ is quasisimilar to an orthogonal direct sum of countably many Jordan blocks.*

An operator is called *biquasitriangular* if $\operatorname{ind}(T - \lambda) = 0$ for $\lambda \in \rho_{\mathrm{s-F}}(T)$, where $\rho_{\mathrm{s-F}}(T) \stackrel{\triangle}{=} \{\lambda : T - \lambda \text{ is semi-Fredholm}\}$.

We say an operator A is *quasisimilar* to an operator B if there exist two injective operators with dense range, X and Y, satisfying $XA = BX$ and $AY = YB$.

In the last ten years, a lot of work on strongly irreducible operators has been done by the functional analysis seminar of Jilin University. D.A. Herrero, C.L. Jiang, Z.Y. Wang and C.K. Fong confirmed Ze Jian Jiang's Conjecture: A strongly irreducible operator is a suitable analogue of Jordan blocks in $\mathcal{L}(\mathcal{H})$ (see [13], [16], [17], [18]). D.A. Herrero and C.L. Jiang obtained the following result.

Theorem HJ. *Let $T \in \mathcal{L}(\mathcal{H})$ and $\varepsilon > 0$. Then there exists an operator A which can be written as a topological direct sum of finitely many strongly irreducible operators such that $\|A - T\| < \varepsilon$.*

Theorem HJ shows that the class of the operators which can be written as a topological direct sum of finitely many strongly irreducible operators is dense in $\mathcal{L}(\mathcal{H})$.

The next theorem, given by Ze Jian Jiang, shows that a lot of operators cannot be expressed as a topological direct sum of countably many strongly irreducible operators.

Theorem J. ([17]) *Let $T \in \mathcal{L}(\mathcal{H})$ be normal and $\sigma_{\mathrm{p}}(T)$, the point spectrum of T, be empty. Then T can not be written as a topological direct sum of countably many strongly irreducible operators.*

The main purpose of this paper is to discuss the following question: For an operator T in $\mathcal{L}(\mathcal{H})$, when can T be uniquely expressed as a topological direct sum of finitely many strongly irreducible operators up to similarity?

In what follows, $T \in$ (SI) means that T is a strongly irreducible operator and $T \in$ (IR) means that T is irreducible.

Definition 0.1. Let $T \in \mathcal{L}(\mathcal{H})$. $\mathcal{P} = \{P_i\}_{i=1}^{n}$, $(n < \infty)$, the set of idempotent elements of $\mathcal{L}(\mathcal{H})$, is called a *unit finite decomposition of* T if the following are satisfied:

(i) $P_i \in \mathcal{A}'(T)$, that is the commutant of T, $i = 1, 2, \dots, n$;

(ii) $P_iP_j = 0$, $i \neq j$;

(iii) $\sum\limits_{i=1}^{n} P_i = I$, where I denotes identity operator on $\mathcal{H}$.

If, in addition, the following is satisfied:

(iv) $T|_{P_i\mathcal{H}} \in$ (SI), $i = 1, 2, \dots, n$,

then we call $\mathcal{P}$ a *unit finite* (SI) *decomposition of* T and call the *cardinality of* $\mathcal{P}$ a (SI) cardinality of T.

It is clear that if T has a unit finite (SI) decomposition, then T can be written as a topological direct sum of finitely many (SI) operators.

Definition 0.2. For T in $\mathcal{L}(\mathcal{H})$, one says T has finite (SI) decomposition if for an arbitrary idempotent P in $\mathcal{A}'(T)$, $T|_{P\mathcal{H}}$ has a unit finite (SI) decomposition.

C.K. Fong and C.L. Jiang ([9]) proved that $\mathcal{B}_1(\Omega) \subset$ (SI). A simple computation shows that every Cowen-Douglas operator has finite (SI) decomposition.

Definition 0.3. Let T have finite (SI) decomposition. If, for any two unit finite (SI) decompositions of T, say $\mathcal{P}_1 = \{P_i\}_{i=1}^{n}$ and $\mathcal{P}_2 = \{Q_i\}_{i=1}^{m}$, the following are satisfied:

(i) $m = n$;

(ii) there exists an $X \in \mathrm{GL}(\mathcal{A}'(T)) = \{A : A$ is invertible in $\mathcal{A}'(T)\}$ and a permutation $\Pi \in S_n$ such that $XP_iX^{-1} = Q_{\Pi(i)}$ for $i = 1, 2, \dots, n$.

Then we say that T has unique finite (SI) decomposition up to similarity.

DEFINITION 0.1$'$. Let $T \in \mathcal{L}(\mathcal{H})$. We call $\mathcal{P} = \{P_i\}_{i=1}^n$, $n < \infty$, a *set of orthogonal projections of* $\mathcal{L}(\mathcal{H})$, a unit finite orthogonal decomposition of T, if the following are satisfied:

(i) $P_i \in \mathcal{A}'(T, T^*)$, that is the commutant of T and T^*, $i = 1, 2, \ldots, n$,

(ii) $P_i P_j = 0$, $i \neq j$;

(iii) $\sum\limits_{i=1}^n P_i = I$;

If, in addition, the following is satisfied:

(iv) $T|_{P_i\mathcal{H}}$ is irreducible, $i = 1, 2, \ldots, n$;

then we call $\mathcal{P}$ *unit finite* (IR) *decomposition of* T.

DEFINITION 0.2$'$. For T in $\mathcal{L}(\mathcal{H})$, one says T has *finite* (IR) *decomposition* if, for an arbitrary orthogonal projection $P \in \mathcal{A}'(T, T^*)$, $T|_{P\mathcal{H}}$ has a unit finite (IR) decomposition.

DEFINITION 0.3$'$. Let T have finite (IR) decomposition. If, for any two unit finite (IR) decompositions of T, say $\mathcal{P}_1 = \{P_i\}_{i=1}^n$ and $\mathcal{P}_2 = \{Q_i\}_{i=1}^m$, the following are satisfied:

(i) $m = n$;

(ii) there exists a unitary $U \in \mathcal{A}'(T, T^*)$, and a permutation $\Pi \in S_n$ such that $UP_iU^* = Q_{\Pi(i)}$ for $i = 1, 2, \ldots, n$.

Then we say T has *unique finite* (IR) *decomposition up to unitary equivalence*.

According to the above definitions, we can see that the (SI) decomposition of operator T is completely determined by the commutant of T.

K-theory has revolutionized the study of operator algebras in the last few years. As the primary component of subject of "non-commutative topology", K-theory has opened vast new vistas within the structure theory of C^*-algebras, and has also led to profound and unexpected applications of operator algebras to problems in geometry and topology. In this paper, we will use the K_0-group of the commutant to characterize operators with unique finite (SI) decomposition up to similarity and we will calculate the K_0-group of $H^\infty(\Omega)$ by using the uniqueness of (SI) decomposition of operators up to similarity.

In the following definitions, $\mathcal{A}$ always denotes a unital Banach algebra.

DEFINITION 0.4. Let e and f be idempotents in $\mathcal{A}$. We write $e \sim (a)f$ if there exist $x, y \in \mathcal{A}$ with $xy = e$, $yx = f$ (algebraic equivalence). We write $e \sim (\mathcal{A})f$ if there exists a $z \in \mathrm{GL}(\mathcal{A})$ with $zez^{-1} = f$.

Obviously, $\sim (a)$ and $\sim (\mathcal{A})$ are equivalence relations.

DEFINITION 0.5. $M_\infty(\mathcal{A})$ is the algebraic direct limit of $M_n(\mathcal{A})$ under the embedding $a \mapsto \mathrm{diag}(a, 0) = a \oplus 0$, where

$$M_n(A) \triangleq \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} : a_{ij} \in \mathcal{A} \right\}.$$

DEFINITION 0.6. $\mathrm{Proj}\,(A)$ is the set of algebraic equivalence classes of idempotents in $\mathcal{A}$ and $\bigvee(\mathcal{A}) = \mathrm{Proj}\,(M_\infty(\mathcal{A}))$.

There is a binary operation (orthogonal addition) on $\bigvee(A)$: if $[e],[f]\in\bigvee(A)$, choose $e'\in[e]$, $f'\in[f]$ with $e'f'=f'e'=0$ (this is always possible by "moving down the diagonal"), and define $[e]+[f]=[e'+f']$. Obviously, this operation is well defined and makes $\bigvee(A)$ into an abelian semigroup with identity.

Because of the classical results of K-theory, one obtains exactly the same semigroup starting with $\sim(\mathcal{A})$ instead of $\sim(a)$, since the two notions coincide on $M_\infty(\mathcal{A})$.

Note that $\bigvee(\mathcal{A})$ depends on $\mathcal{A}$ only up to stable isomorphism. If $M_\infty(\mathcal{A}_1)$ is isomorphic ($\cong$) to $M_\infty(\mathcal{A}_2)$, then $\bigvee(\mathcal{A}_1)\cong\bigvee(\mathcal{A}_2)$. In particular, $\bigvee(M_n(\mathcal{A}))\cong\bigvee(\mathcal{A})$.

DEFINITION 0.7. $\mathrm{K}_0(\mathcal{A})$ is the Grothendieck group of $\bigvee(\mathcal{A})$.

In Section 1, we will prove the following theorems.

THEOREM 1.1. *Let $T\in\mathcal{L}(\mathcal{H})$, and let $\mathcal{H}^{(n)}$ denote the direct sum of n copies of $\mathcal{H}$ and $A^{(n)}$ the operator $\bigoplus_1^n A$ acting on $\mathcal{H}^{(n)}$. Then the following are equivalent:*

(i) *T is similar to ($\sim$) $\bigoplus_{i=1}^k A_i^{(n_i)}$ with respect to the decomposition $\mathcal{H}=\bigoplus_{i=1}^k\mathcal{H}_i^{(n_i)}$, where $k,n_i<\infty$, $A_i\in$* (SI)*, $A_i\not\sim A_j$ for $i\neq j$, and for each natural number n, $T^{(n)}$ has unique finite* (SI) *decomposition up to similarity.*

(i′) *$\bigvee(\mathcal{A}'(T))\cong\mathbb{N}^k$ and this isomorphism h sends $[I]$ to $(n_1,n_2,\ldots,n_k)$, i.e., $h([I])=n_1e_1+n_2e_2+\cdots+n_ke_k$, where $0\neq n_i\in\mathbb{N}$, $i=1,2,3,\ldots,k$, $\{e_i\}_{i=1}^k$ are the generators of $\mathbb{N}^k$, and $\mathbb{N}=\{0,1,2,3,\ldots\}$.*

COROLLARY 1.2. *Let $T_1,T_2\in$* (SI)*, $T=T_1\oplus T_2$. If $\bigvee(\mathcal{A}'(T))\cong\mathbb{N}$, then $T_1\sim T_2$. Furthermore, if, for all natural number n, $T^{(n)}$ has unique finite* (SI) *decomposition up to similarity, then $T_1\sim T_2$ if and only if*

$$\mathrm{K}_0(\mathcal{A}'(T))\cong\mathbb{Z}\overset{\triangle}{=}\{0,\pm1,\pm2,\ldots\}.$$

THEOREM 1.3. *Let $T\in\mathcal{L}(\mathcal{H})$ and let T have a unit finite* (IR) *decomposition. Then the following are equivalent:*

(i) *$T\simeq$ (unitary equivalent) $\bigoplus_{i=1}^k A_i^{(n_i)}$, where $k,n_i<\infty$, $A_i\in$* (IR)*, $A_i\not\simeq A_j$, $(i\neq j)$.*

(ii) *$\mathrm{K}_0(\mathcal{A}'(T,T^*))\cong\mathbb{Z}^k$.*

COROLLARY 1.4. *Let $T_1,T_2\in\mathcal{L}(\mathcal{H})$ be irreducible. Then $T_1\simeq T_2$ if and only if $\mathrm{K}_0(\mathcal{A}'(T_1\oplus T_2,(T_1\oplus T_2)^*))\cong\mathbb{Z}$.*

COROLLARY 1.5. *Let $T\in\mathcal{L}(\mathcal{H})$ have a unit finite* (IR) *decomposition. Then T has unique finite* (IR) *decomposition up to unitary equivalence.*

Corollary 1.4 tells us the following fact: for two irreducible operators T_1 and T_2 in $\mathcal{L}(\mathcal{H})$, their unitary equivalence is completely determined by K_0-group of $\mathcal{A}'(T_1\oplus T_2,(T_1\oplus T_2)^*)$.

In Section 2, we will give applications of Theorem 1.1.

Let Ω be a bounded and simply connected domain and $H^\infty(\Omega)$ the bounded analytic functions on Ω. We can obtain

THEOREM 2.1. *$\bigvee(H^\infty(\Omega)) \cong \mathbb{N}$ and $\mathrm{K}_0(H^\infty(\Omega)) \cong \mathbb{Z}$.*

Let $\mathbb{D}$ be the unit disk. For φ in $H^\infty(\mathbb{D})$ and λ in $\mathbb{D}$, $\operatorname{inn}(\varphi - \varphi(\lambda))$ denotes the inner function in the inner-outer factorization of $(\varphi - \varphi(\lambda))$. Let T_φ denote the analytic Toeplitz operator with symbol φ.

THEOREM 2.2. *For φ in $H^\infty(\Omega)$, if there exists a λ in $\mathbb{D}$ such that $\operatorname{inn}(\varphi - \varphi(\lambda))$ is a finite Blaschke product, then T_φ has unique finite (SI) decomposition up to similarity.*

THEOREM 2.3. *Let $\varphi_1, \varphi_2, \dots, \varphi_n \in H^\infty(\mathbb{D})$ be univalent analytic functions. Then there exists a natural number k such that $\bigvee \mathcal{A}'\left(\bigoplus_{i=1}^{n} T_{\varphi_i}\right) \cong \mathbb{N}^k$. Furthermore, $T = \bigoplus_{i=1}^{n} T_{\varphi_i}$ has unique finite (SI) decomposition up to similarity.*

COROLLARY 2.4. *Let $\varphi_1, \varphi_2 \in H^\infty(\mathbb{D})$ be univalent analytic functions. Then $T_{\varphi_1} \sim T_{\varphi_2}$ if and only if $\mathrm{K}_0(\mathcal{A}'(T_{\varphi_1} \oplus T_{\varphi_2})) \cong \mathbb{Z}$.*

In Section 3, we will give a new proof of the Jordan Standard Theorem by using Theorem 1.1 and the K_0-group of the commutant. In this proof, we use only elementary matrices instead of determinants. We will see that $\bigvee(\mathcal{A}'(T)) \cong \mathbb{N}^k$ and $[I] = \sum_{i=1}^{k} n_i e_i$, as in Theorem 1.1, is exactly the minimum polynomial of T when T is an $n \times n$ matrix. This shows that Theorem 1.1 is a generalization of the Jordan Standard Theorem to infinite-dimensional Hilbert space.

For a unital Banach algebra $\mathcal{A}$, $\operatorname{Rad}\mathcal{A}$ denotes the Jacobson radical of $\mathcal{A}$.

THEOREM 3.1. *Let $A_1, \dots, A_k \in (\mathrm{SI}) \cap \mathcal{L}(\mathcal{H})$ satisfying*

$$\mathcal{A}'(A_i)/\operatorname{Rad}\mathcal{A}'(A_i) \cong \mathbb{C}, \quad i = 1, 2, \dots.$$

Then the following hold:

(i) *$A_i \sim A_j$ if and only if $\mathrm{K}_0(\mathcal{A}'(A_i \oplus A_j)) \cong \mathbb{Z}$.*

(ii) *Set $T = \sum_{i=1}^{k} A_i^{(n_i)}$ where $A_i \not\sim A_j$ for $i \neq j$. Then $\bigvee(\mathcal{A}'(T)) \cong \mathbb{N}^k$ and $\mathrm{K}_0(\mathcal{A}'(T)) \cong \mathbb{Z}^k$. Furthermore, T has unique (SI) decomposition up to similarity.*

The above arguments suggest the following.

CONJECTURE 1. Let $A_1, A_2 \in (\mathrm{SI})$. Then $A_1 \sim A_2$ if and only if

$$\mathrm{K}_0(\mathcal{A}'(A_1 \oplus \mathcal{A}_2)) \cong \mathbb{Z}.$$

CONJECTURE 2. Let $T \in \mathcal{L}(\mathcal{H})$ have unique finite (SI) decomposition up to similarity. Then $T^{(n)}$ does for each natural number n.

1. THE PROOF OF THEOREM 1.1 AND THEOREM 1.3

1.1. Several auxiliary lemmas.

LEMMA 1.6. *Let $A, B \in \mathcal{L}(\mathcal{H})$ and φ be an isomorphism from $\mathcal{A}'(A)$ to $\mathcal{A}'(B)$. Then $\{P_i\}_{i=1}^n$ is a unit* (SI) *decomposition of A if and only if $\{\varphi(P_i)\}_{i=1}^n$ is a unit* (SI) *decomposition of B. In particular, if $A \sim B$ then $\mathcal{A}'(A) \cong \mathcal{A}'(B)$.*

Proof. Since φ is an isomorphism, $0 = \varphi(P_iP_j) = \varphi(P_i)\varphi(P_j)$, $(i \neq j)$ and $\sum_{i=1}^n \varphi(P_i) = I$. We need only to prove that $B|_{\varphi(P_i)\mathcal{H}} \in$ (SI). Otherwise, there exist two non-zero idempotents Q_1 and Q_2 in $\mathcal{A}'(B)$ so that $Q_2Q_1 = Q_1Q_2 = 0$ and $Q_1 + Q_2 = \varphi(P_i)$. Note that $\varphi^{-1}(Q_1), \varphi^{-1}(Q_2)$ are two non-zero idempotents in $\mathcal{A}'(A)$ and $P_i = \varphi^{-1}(Q_1) + \varphi^{-1}(Q_2)$. This contradicts $A|_{P_i\mathcal{H}} \in$ (SI).

If A is similar to B, then there exists an invertible operator X satisfying $XAX^{-1} = B$. Define a mapping φ below: $\varphi(T) = XTX^{-1}$, $\forall T \in \mathcal{A}'(A)$. It is clear that φ is an isomorphism from $\mathcal{A}'(A)$ to $\mathcal{A}'(B)$. ∎

LEMMA 1.7. *Let $T \in \mathcal{L}(\mathcal{H})$ and $P_1, P_2 \in \mathcal{A}'(T)$ be idempotent operators. If $P_1 \sim (\mathcal{A}'(T))P_2$ then $T|_{P_1\mathcal{H}} \sim T|_{P_2\mathcal{H}}$.*

Proof. Since $P_1 \sim (\mathcal{A}'(T))P_2$, there exists an $X \in \mathrm{GL}(\mathcal{A}'(T))$ such that $XP_1X^{-1} = P_2$. Therefore $X\mathrm{ran}\,P_1 = \mathrm{ran}\,P_2$, $X\mathrm{ran}\,(I - P_1) = \mathrm{ran}\,(I - P_2)$. Set $X_1 = X|_{\mathrm{ran}\,P_1}$, $X_2 = X|_{\mathrm{ran}\,(I-P_1)}$. Then $X = X_1 \dot{+} X_2$, where $\dot{+}$ denotes the topological direct sum, and $X_1 \in \mathrm{GL}(\mathcal{L}(P_1\mathcal{H}, P_2\mathcal{H}))$, $X_2 \in \mathrm{GL}(\mathcal{L}((I - P_1)\mathcal{H}, (I - P_2)\mathcal{H}))$. Note that

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{matrix} P_1\mathcal{H} \\ (I-P_1)\mathcal{H} \end{matrix} = \begin{bmatrix} T_1' & 0 \\ 0 & T_2' \end{bmatrix} \begin{matrix} P_2\mathcal{H} \\ (I-P_2)\mathcal{H} \end{matrix},$$

where $T_1 = T|_{P_1\mathcal{H}}$, $T_2 = T|_{(I-P_1)\mathcal{H}}$, $T_1' = T|_{P_2\mathcal{H}}$, and $T_2' = T|_{(I-P_2)\mathcal{H}}$. A simple computation shows that

$$\begin{bmatrix} T_1' & 0 \\ 0 & T_2' \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}.$$

Thus $T|_{P_1\mathcal{H}} \sim T|_{P_2\mathcal{H}}$. ∎

LEMMA 1.8. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $\{P_i\}_{i=1}^n$ and $\{Q_i\}_{i=1}^n$ be two unit* (SI) *decompositions of T. If there exist $X_1, \ldots, X_n \in \mathrm{GL}(\mathcal{L}(P_i\mathcal{H}, Q_i\mathcal{H}))$ satisfying*

$$X_i(T|_{P_i\mathcal{H}})X_i^{-1} = T|_{Q_i\mathcal{H}}, \quad i = 1, \ldots, n,$$

then $X = X_1 \dot{+} X_2 \dot{+} \cdots \dot{+} X_n \in \mathrm{GL}(\mathcal{A}'(T))$.

Proof. Since $\mathcal{H} = \mathrm{ran}\,P_1 \dot{+} \mathrm{ran}\,P_2 \dot{+} \cdots \dot{+} \mathrm{ran}\,P_n = \mathrm{ran}\,Q_1 \dot{+} \mathrm{ran}\,Q_2 \dot{+} \cdots \dot{+} \mathrm{ran}\,Q_n$,

$$T = \begin{bmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_n \end{bmatrix} \begin{matrix} P_1\mathcal{H} \\ \vdots \\ P_n\mathcal{H} \end{matrix} = \begin{bmatrix} T_1' & & 0 \\ & \ddots & \\ 0 & & T_n' \end{bmatrix} \begin{matrix} Q_1\mathcal{H} \\ \vdots \\ Q_n\mathcal{H} \end{matrix}$$

where $T_i = T|_{P_i\mathcal{H}}$, $T_i' = T|_{Q_i\mathcal{H}}$, $i = 1, 2, 3, \ldots, n$. Clearly, $XT = TX$ and X is invertible. ∎

Lemma 1.9. *Suppose that* $\{P_1,\dots,P_m,P_{m+1},\dots,P_n\}$ *and* $\{Q_1,\dots,Q_m,$ $Q_{m+1},\dots,Q_n\}$ *are two sets of idempotent operators in* $\mathcal{A}'(T)$, $T\in\mathcal{L}(\mathcal{H})$. *If there exist* $X,Y\in\mathrm{GL}(\mathcal{A}'(T))$ *and a permutation* $\Pi\in S_n$ *satisfying*

(i) $XP_iX^{-1}=Q_i$, $1\leqslant i\leqslant m$;

(ii) $YP_iY^{-1}=Q_{\Pi(i)}$, $1\leqslant i\leqslant n$;

then $\forall Q_r$, $m<r'\leqslant n$, *there exists a* $P_{r'}$, $m<r\leqslant n$, *and* Z_r, *a finite product of* Y *and* X, *such that* $Z_rQ_rZ_r^{-1}=P_{r'}$. *Moreover,* $\{P_{r'}\}$ *is exactly a rearrangement of* $\{P_r\}_{r=m+1}^n$.

Proof. Given Q_r, $m<r\leqslant n$, it follows from Property (ii) that there exists a P_{j_1}, $1\leqslant j_1\leqslant n$ satisfying $YQ_rY^{-1}=P_{j_1}$. If $m<j_1\leqslant n$, then set $Z_r=Y$ and $P_{r'}=P_{j_1}$. If $1\leqslant j_1\leqslant m$, then it follows from Property (i) that there exists a Q_{j_1}, $j_1\neq r$, such that $XYQ_rY^{-1}X^{-1}=Q_{j_1}$. By Property (ii), $YQ_{j_1}Y^{-1}=P_{j_2}$. If $m<j_2\leqslant n$, then set $Z_r=YXY$, $P_{r'}=P_{j_2}$. If $1\leqslant j_2\leqslant m$, it is clear that $j_1\neq j_2$. Otherwise $Q_{j_1}=Y^{-1}P_{j_2}Y=Y^{-1}P_{j_1}Y=Q_r$, which is a contradiction. Using Property (ii) again, we can find P_{j_3} so that $YQ_{j_2}Y^{-1}=P_{j_3}$. Similarly, $j_3\notin\{j_1,j_2\}$. If $m<j_3\leqslant n$, then set $Z_r=YXYXY$, $P_{r'}=P_{j_3}$. Otherwise, we can continue the above choice procedure. Since n is a natural number, after s steps, $s\leqslant m+1$, we will force $P_{j_s}\in\{P_{m+1},\dots,P_n\}$. Set

$$P_{r'}=P_{j_s},\quad Z_r=YXY\cdots XY\quad (X\text{ appears } s \text{ times}),$$

then $Z_rQ_rZ_r^{-1}=P_{j_s}$. We assert that if $r_1\neq r_2$, with $r_1,r_2\in\{m+1,\dots,n\}$, then $j_{s_1}\neq j_{s_2}$. Otherwise, there exists $Z_{r_1}=YXY\cdots YXY$ (X appears s_1 times) and $Z_{r_2}=YXY\cdots YXY$ (X appears s_2 times) such that

$$Z_{r_1}Q_{r_1}Z_{r_1}^{-1}=Z_{r_2}Q_{r_2}Z_{r_2}^{-1}.$$

Without loss of generality, assume that $j_{s_1}\geqslant j_{s_2}$. If $j_{s_1}>j_{s_2}$, then it follows $Z_{r_2}^{-1}Z_{r_1}Q_{r_1}Z_{r_1}^{-1}Z_{r_2}=Q_{r_2}\in\{Q_i\}_{i=m+1}^n$. Note that

$$Z_{r_2}^{-1}Z_{r_1}=XY\cdots XY\quad (X\text{ appears } j_{s_1}-j_{s_2}\text{ times}).$$

Set

$$R=YXY\cdots XY\quad (X\text{ appears } j_{s_1}-j_{s_2}-1\text{ times}).$$

By this choice process, we can deduce that $RQ_{r_1}R^{-1}\in\{P_i\}_{i=1}^m$. Thus we have $XRQ_{r_1}R^{-1}X^{-1}\in\{Q_i\}_{i=1}^m$. But

$$XRQ_{r_1}R^{-1}X^{-1}=Z_{r_2}^{-1}Z_{r_1}Q_{r_1}Z_{r_1}^{-1}Z_{r_2}=Q_{r_2}\in\{Q_i\}_{i=1}^n,$$

which is impossible. If $j_{s_1}=j_{s_2}$, it is not difficult to check that $Q_{r_1}=Q_{r_2}$. This contradicts our assumption that $r_1\neq r_2$. ∎

Similarly to the proof of Lemma 1.9, we immediately can prove

Lemma 1.10. *Let* $T\in\mathcal{L}(\mathcal{H})$ *and let* $\{P_1,\dots,P_{m_1},\dots,P_{m_{k-1}-1},\dots,P_{m_k},$ $P_{m_k+1},\dots,P_n\}$ *and* $\{Q_1,\dots,Q_{m_1},\dots,Q_{m_{k-1}-1},\dots,Q_{m_k},Q_{m_k+1},\dots,Q_n\}$ *be two sets of idempotent operators in* $\mathcal{A}'(T)$. *If there exist* $X_1,X_2,\dots,X_k,Y\in\mathrm{GL}(\mathcal{A}'(T))$ *and a permutation* $\Pi\in S_n$ *satisfying*

$$X_iP_jX_i^{-1}=Q_j,\quad m_i+1\leqslant j\leqslant m_{i+1},\ i=0,1,\dots,k-1,m_0=0,$$

and

$$Y^{-1}P_jY=Q_{\Pi(i)},\quad 1\leqslant i\leqslant n,$$

then for each r, $m_k < r < n$, there exists a Z_r, a finite product of $\{Y, X_1, \ldots, X_k\}$, so that $\{Z_r Q_r Z_r^{-1}\}_{r=m_k+1}^n$ is exactly a rearrangement of $\{P_r\}_{r=m_k+1}^n$.

LEMMA 1.11. *Suppose that $\{P_1, \ldots, P_m, P_{m+1}, \ldots, P_n\}$ and $\{Q_1, \ldots Q_m, Q_{m+1}, \ldots, Q_n\}$ are two unit decompositions of T, $T \in \mathcal{L}(\mathcal{H})$. If the following properties are satisfied:*

(i) *for each P_i, there exists an $X_i \in \mathrm{GL}(P_i\mathcal{H}, Q_i\mathcal{H})$ satisfying $X_i T|_{P_i\mathcal{H}} X_i^{-1} = T|_{Q_i\mathcal{H}}$, $1 \leqslant i \leqslant m$;*

(ii) *there exists a $Y \in \mathrm{GL}(\mathcal{A}'(T))$ and a permutation $\Pi \in S_n$ satisfying $Y^{-1} P_i Y = Q_{\Pi(i)}$;*

then given Q_r, $r \in \{m+1, \ldots, n\}$, there exist $r' \in \{m+1, \ldots, n\}$ and $Z_r \in \mathrm{GL}(Q_r\mathcal{H}, P_{r'}\mathcal{H})$ satisfying $Z_r(T|_{Q_r\mathcal{H}})Z_r^{-1} = T|_{P_{r'}\mathcal{H}}$. Furthermore, if $r_1 \neq r_2$, then $r_1' \neq r_2'$.

Proof. Given $r \in \{m+1, \ldots, n\}$, there exists $P_{j_1} \in \{P_i\}_{i=1}^n$ satisfying $YQ_rY^{-1} = P_{j_1}$, by Property (ii). If $m < j_1 \leqslant n$ set $Z_r = Y|_{Q_r\mathcal{H}}$. Otherwise, by $T|_{(YQ_rY^{-1})\mathcal{H}} = T|_{P_{j_1}\mathcal{H}}$ and Property (i), we have $X_{j_1} T|_{P_{j_1}} \mathcal{H} X_{j_1}^{-1} = T_{Q_{j_1}\mathcal{H}}$. Using Property (ii) again, we can find $j_2 \in \{1, \ldots, n\}$ satisfying $YQ_{j_1}Y^{-1} = P_{j_2}$. Obviously, $j_2 \neq j_1$. If $j_2 \in \{m+1, \ldots, n\}$, set $Z_r = Y|_{Q_j\mathcal{H}} X_{j_1} Y|_{Q_r\mathcal{H}}$, $P_{r'} = P_{j_2}$. Then $Z_r(T|_{Q_r\mathcal{H}})Z_r^{-1} = T|_{P_{r'}\mathcal{H}}$. Otherwise, similarly to the proof of Lemma 1.9, after s steps we can find $P_{j_s} \notin \{P_k\}_{k=m+1}^n$ such that

$$Z_r T|_{Q_r\mathcal{H}} Z_r^{-1} = T|_{P_{r'}\mathcal{H}}$$

where $P_{r'} = P_{j_s}$ and

$$Z_r = (Y|_{Q_{j_{s-1}}\mathcal{H}}) X_{j_s-1} \cdots (Y|_{Q_{j_1}\mathcal{H}}) X_{j_1} (Y|_{Q_r\mathcal{H}}).$$

Again similarly to the proof of Lemma 1.9, we can deduce that $r_1' = r_2'$ if $r_1 \neq r_2$. ∎

LEMMA 1.12. *Let $T \in \mathcal{L}(\mathcal{H})$ and suppose T has unique finite* (SI) *decomposition up to similarity. Then for an arbitrary idempotent P in $\mathcal{A}'(T)$, $T|_{P\mathcal{H}}$ has unique* (SI) *decomposition up to similarity.*

Proof. Since T has unique finite (SI) decomposition up to similarity, $T|_{P\mathcal{H}}$ has finite (SI) decomposition and all the (SI) cardinalities of $T_{P\mathcal{H}}$ must be the same.

Let $\{P_i\}_{i=1}^m$ and $\{Q_i\}_{i=1}^m$ be two unit (SI) decompositions of $T|_{P\mathcal{H}}$ and let $\{P_i\}_{i=m+1}^n$ be a unit (SI) decomposition of $T|_{(I-P)\mathcal{H}}$. Then $\{\{P_i\}_{i=1}^m, \{P_i\}_{i=m+1}^n\}$ and $\{\{Q_i\}_{i=1}^m, \{P_i\}_{i=m+1}^n\}$ are two unit (SI) decomposition of T. By uniqueness, we can find a $Y \in \mathrm{GL}(\mathcal{A}'(T))$ such that

$$\{YP_iY^{-1}\} = \{Q_1, \ldots, Q_m, P_{m+1}, \ldots, P_n\}.$$

By Lemma 1.11, we can find Z_i in $\mathrm{GL}(\mathcal{L}(Q_i\mathcal{H}, P_i\mathcal{H}))$ and a permutation $\Pi \in S_n$ satisfying

$$Z_i(T|_{Q_i\mathcal{H}})Z_i^{-1} = T|_{P_{\Pi(i)}\mathcal{H}}, \quad 1 \leqslant i \leqslant m.$$

Set $Z_k = I|_{P_k\mathcal{H}}$ for $k \geqslant m+1$ and set $Z = Z_1 \dot{+} \cdots \dot{+} Z_n$. By Lemma 1.8, $Z \in \mathrm{GL}(\mathcal{A}'(T))$ and $Z|_{P\mathcal{H}} \in \mathrm{GL}(\mathcal{A}'(T)|_{P\mathcal{H}})$. Note that $(Z|_{P\mathcal{H}})Q_i(Z|_{P\mathcal{H}})^{-1} = P_{\Pi(i)}$, $1 \leqslant i \leqslant m$. ∎

LEMMA 1.13. *Let $T \in \mathcal{L}(\mathcal{H})$ and suppose T has unique finite* (SI) *decomposition up to similarity. Then the following are equivalent:*
(i) $P \sim (\mathcal{A}'(T))Q$;
(ii) $T|_{P\mathcal{H}} \sim T|_{Q\mathcal{H}}$;
where P and Q are idempotents of $\mathcal{A}'(T)$.

Proof. (i) $\Rightarrow$ (ii) This is a straightforward consequence of Lemma 1.7.

(ii) $\Rightarrow$ (i) By Lemma 1.12, $T|_{P\mathcal{H}}, T|_{Q\mathcal{H}}, T|_{(I-P)\mathcal{H}}$, and $T|_{(I-Q)\mathcal{H}}$ have unique finite (SI) decomposition up to similarity. Since $T|_{P\mathcal{H}} \sim T|_{Q\mathcal{H}}$, there exists $X \in \mathrm{GL}(\mathcal{L}(P\mathcal{H}, Q\mathcal{H}))$ satisfying $X(T|_{P\mathcal{H}})X^{-1} = T|_{Q\mathcal{H}}$. Thus, if $\{P_1, \dots, P_m\}$ is a unit (SI) decomposition of $T|_{P\mathcal{H}}$, then $\{XP_1X^{-1}, \dots, XP_mX^{-1}\}$ is a unit (SI) decomposition of $T|_{Q\mathcal{H}}$.

Let $\{P_{m+1}, \dots, P_n\}$ and $\{Q_{m+1}, \dots, Q_n\}$ be arbitrary (SI) decompositions of $T|_{(I-P)\mathcal{H}}$ and $T|_{(I-Q)\mathcal{H}}$, respectively. Then $\{P_i\}_{i=1}^n$ and $\{\{XP_iX^{-1}\}_{i=1}^m, \{Q_i\}_{i=m+1}^n\}$ are two unit (SI) decompositions of T. By uniqueness, there exists $Y \in \mathrm{GL}(\mathcal{A}'(T))$ such that $\{Y^{-1}P_iY\}_{i=1}^n$ are an exact rearrangement of $\{\{XP_iX^{-1}\}_{i=1}^m, \{Q_i\}_{i=m+1}^n\}$. Applying Lemma 1.11 for each $r \in \{m+1, \dots, n\}$, we can find $P_{r'}$, $r' \in \{m+1, \dots, n\}$ and $Z_r \in \mathrm{GL}(\mathcal{L}(Q_r\mathcal{H}, P_{r'}\mathcal{H}))$ so that $Z_r(T|_{Q_r\mathcal{H}})Z_r^{-1} = T|_{P_{r'}\mathcal{H}}$ and $r_1' = r_2'$ if $r_1 = r_2$. Set

$$Z_r = X^{-1}|_{XP_kX^{-1}\mathcal{H}}, \quad k \leqslant m.$$

Then

$$Z = Z_1 \dot{+} \cdots \dot{+} Z_n \in \mathrm{GL}(\mathcal{A}'(T)).$$

Noting that $ZPZ^{-1} = Q$ and using Lemma 1.8, we can deduce that $P \sim (\mathcal{A}'(T))Q$. ∎

LEMMA 1.14. *Let $T \in \mathcal{L}(\mathcal{H})$ and let P and Q be idempotents in $\mathcal{A}'(T)$. If $T|_{P\mathcal{H}}$ is not similar to $T|_{Q\mathcal{H}}$, then for each natural number n, $P \oplus 0_{\mathcal{H}^{(n)}}$ is not similar to $Q \oplus 0_{\mathcal{H}^{(n)}}$ in $\mathcal{A}'(T^{(n+1)})$.*

Proof. If not, there exists $n \in \mathbb{N}$ and $X \in \mathrm{GL}(\mathcal{A}'(T^{(n+1)}))$ satisfying

$$X(P \oplus 0_{\mathcal{H}^{(n)}})X^{-1} = (Q \oplus 0_{\mathcal{H}^{(n)}}).$$

According to Lemma 1.7,

$$T^{(n+1)}|_{(P \oplus 0_{\mathcal{H}^{(n)}})\mathcal{H}^{(n+1)}} \sim T^{(n+1)}|_{(Q \oplus 0_{\mathcal{H}^{(n)}})\mathcal{H}^{(n+1)}}.$$

Note that $T^{(n+1)}|_{(P \oplus 0_{\mathcal{H}^{(n)}})\mathcal{H}^{(n+1)}} \simeq T|_{P\mathcal{H}}$ and $T^{(n+1)}|_{(Q \oplus 0_{\mathcal{H}^{(n)}})\mathcal{H}^{(n+1)}} \simeq T|_{Q\mathcal{H}}$. Thus $T|_{P\mathcal{H}} \sim T|_{Q\mathcal{H}}$. This contradicts $T|_{P\mathcal{H}} \not\sim T|_{Q\mathcal{H}}$. ∎

LEMMA 1.15. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $T^{(n)}$ have unique finite* (SI) *decomposition up to similarity for each natural number n. Then $P \sim (\mathcal{A}'(T))Q$ if and only if $[P] = [Q]$ in $\bigvee(\mathcal{A}'(T))$.*

Proof. The "if" part is clear.

Let P, Q be two idempotent elements of $\mathcal{A}'(T)$. If $[P] = [Q]$ in $\bigvee(\mathcal{A}'(T))$, then there exists a natural number k satisfying

$$P \oplus 0_{\mathcal{H}^{(k)}} \sim (\mathcal{A}'(T^{(n+1)}))Q \oplus 0_{\mathcal{H}^{(k)}}.$$

By Lemma 1.7,

$$T|_{P\mathcal{H}} \sim T|_{Q\mathcal{H}}.$$

Furthermore, $P \sim (\mathcal{A}'(T))Q$ follows from Lemma 1.13. ∎

1.2. THE PROOF OF THEOREM 1.1. (i) $\Rightarrow$ (ii) Let P_i be the orthogonal projection onto $\mathcal{H}_i$. Let E be an idempotent in $M_n(\mathcal{A}'(T)) = \mathcal{A}'(T^{(n)})$. Since $T^{(n)}$ has unique finite (SI) decomposition, $T^{(n)}|_{E\mathcal{H}^{(n)}}$ and $T^{(n)}|_{(I-E)\mathcal{H}^{(n)}}$ have finite (SI) decompositions.

If $\{Q_1, \dots, Q_a\}$ is an (SI) decomposition of $T^{(n)}|_{E\mathcal{H}^{(n)}}$ and $\{Q_{a+1}, \dots, Q_b\}$ is an (SI) decomposition of $T^{(n)}|_{(I-E)\mathcal{H}^{(n)}}$, then $\{Q_1, \dots, Q_b\}$ is an (SI) decomposition of $T^{(n)}$. Since we also have an (SI) decomposition of $T^{(n)}$ using nn_i copies of each of the projections P_i uniqueness implies that there is $X \in \mathrm{GL}(\mathcal{A}'(T^{(n)}))$ so that conjugation by X carries Q_j to a copy of one of the P_i, with appropriate multiplicity conditions.

In particular, $XDX^{-1} = X(Q_1 + \cdots + Q_a)X^{-1}$ equals a sum of copies of the P_i. That is, there are integers m_i, $0 \leqslant m_i \leqslant nn_i$, so that

$$XEX^{-1} = \sum_{i=1}^{k} P_i^{(m_i)}.$$

Define a map $h : \bigvee(\mathcal{A}'(T)) \to \mathbb{N}^k$ by

$$h([E]) = (m_1, \dots, m_k).$$

To see that h is well-defined, we observe that if $[E] = [F]$ then $F \sim E \sim \sum_{i=1}^{k} P_i^{(m_i)}$ by using Lemma 1.15. If F can be similar at most to one projection of the form $\sum_{i=1}^{k} P_i^{(m_i)}$, it follows that if $h([F]) = h([E])$, then $F \sim E$, so h is one-to-one. For any k-tuple $(m_1, \dots, m_k)$ of nonnegative integers, we can find n so that $m_i \leqslant nn_i$ for all i and then h sends $\sum_{i=1}^{k} P_i^{(m_i)}$ to $(m_1, \dots, m_k)$, showing that h is onto. Thus, $\bigvee(\mathcal{A}'(T)) \cong \mathbb{N}^k$ and by our construction, $h([I]) = (n_1, \dots, n_k)$.

(ii) $\Rightarrow$ (i) Suppose $\bigvee(\mathcal{A}'(T)) \cong \mathbb{N}^k$ and h is the isomorphism. Then there exists a natural number r and $Q_1, \dots, Q_k$, k idempotents of $\mathcal{A}'(T^{(r)})$, satisfying $h([Q_i]) = e_i$, $1 \leqslant i \leqslant k$.

Since $\bigvee(\mathcal{A}'(T^{(n)})) \cong \bigvee(\mathcal{A}'(T))$, we need only prove that T has unique finite (SI) decomposition up to similarity. At first, we will prove the following:

(a) For an arbitrary idempotent P in $\mathcal{A}'(T)$, if $T|_{P\mathcal{H}} \in$ (SI), then there exists i, $1 \leqslant i \leqslant k$, satisfying $h([P]) = e_i$.

Let $h([P]) = \sum_{i=1}^{k} \lambda_i e_i = \sum_{i=1}^{k} \lambda_i h([Q_i])$, $\lambda_i \in \mathbb{N}$, set $w = r \sum_{i=1}^{k} \lambda_i$, then we can find a natural number $n > w$ satisfying

$$P \oplus 0_{\mathcal{H}^{(n-1)}} \sim (\mathcal{A}'(T^{(n)})) \sum_{i=1}^{k} Q_i^{(\lambda_i)} \oplus 0_{\mathcal{H}^{(n-w)}}.$$

By Lemma 1.7

$$T^{(n)}|_{(P \oplus 0_{\mathcal{H}^{(n-1)}})\mathcal{H}^{(n)}} \sim T^{(n)}|_{\left(\sum_{i=1}^{k} Q_i^{(\lambda_i)} \oplus 0_{\mathcal{H}^{(n-w)}}\right)\mathcal{H}^{(n)}}.$$

So
$$T|_{P\mathcal{H}} \sim T^{(w)}|_{\sum\limits_{i=1}^{k} Q_i^{(\lambda_i)}\mathcal{H}^{(w)}}.$$

Note that $T|_{P\mathcal{H}} \in$ (SI) but the righthand side of this similarity is in (SI) only if one λ_i is 1 and the rest are zero. Thus, there exists i, $1 \leqslant i \leqslant k$, $h([P]) = e_i$.

(b) For arbitrary idempotents P and Q in $\mathcal{A}'(T^{(n)})$, if $h([P]) = h([Q])$, then $T|_{P\mathcal{H}} \sim T|_{Q\mathcal{H}}$. The proof is similar to the proof of (a), so we omit it.

Let $(P_1, \dots, P_m)$ be a unit decomposition of T let $h([P_i]) = \sum\limits_{j=1}^{k} \lambda_{ij} e_j$, where $\lambda_{ij} \in \mathbb{N}$. Then $h([I]) = h\Big(\Big[\sum\limits_{i=1}^{m} P_i\Big]\Big) = \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{k} \lambda_{ij} e_j$. Note that $h([I]) = \sum\limits_{i=1}^{k} n_i e_i$, so that $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{k} \lambda_{ij} = \sum\limits_{i=1}^{k} n_i$, so $m \leqslant \sum\limits_{i=1}^{k} n_i$. This shows that T has finite (SI) decomposition.

Furthermore, let $(P_1, \dots, P_t)$ be a unit (SI) decomposition of T, then
$$h\Big(\sum_{i=1}^{t} [P_i]\Big) = h([I]) = \sum_{i=1}^{k} n_i e_i.$$

By (a), $t = \sum\limits_{i=1}^{k} n_i$, and for each i, $1 \leqslant i \leqslant k$, there exist $P_{i_1}, \dots, P_{i_{n_i}} \in \{P_1, \dots, P_t\}$ satisfying $h([P_{i_1}]) = \cdots = h([P_{i_{n_i}}]) = e_i$. By (b), $T|_{P_{i_j}\mathcal{H}} \sim T|_{P_{i_k}\mathcal{H}}$, $\forall 1 \leqslant j,k \leqslant n_i$. Letting $A_i = T|_{P_{i_1}\mathcal{H}}$, it is clear that
$$T \sim \sum_{i=1}^{k} A_i^{(n_i)}.$$

Suppose $(P_1', \dots, P_s')$ be another unit (SI) decomposition of T, then in the same way we know $r = \sum\limits_{i=1}^{k} n_i$, and for each i, $1 \leqslant i \leqslant k$, there exist n_i idempotents in $\{P_1', \dots, P_s'\}$ and h sends each of them to e_i. By (b) again, if $h([P_i]) = h([P_j])$, $1 \leqslant i,j \leqslant \sum\limits_{i=1}^{k} n_i$, then $T|_{P_i\mathcal{H}} \sim T|_{P_j'\mathcal{H}}$. By Lemma 1.8, T has unique finite (SI) decomposition up to similarity.

This completes the proof of Theorem 1.1. ∎

1.3. THE PROOF OF COROLLARY 1.2. The first part of Corollary 1.2 comes from Theorem 1.1.

Note that $T^{(n)}$ has unique finite (SI) decomposition up to similarity. That $T_1 \sim T_2$ if and only if $\bigvee(\mathcal{A}'(T_1 \oplus T_2)) \cong \mathbb{N}$ follows from Theorem 1.1. Thus, if $T_1 \sim T_2$ then $\mathrm{K}_0(\mathcal{A}'(T_1 \oplus T_2)) \cong \mathbb{Z}$ by using $\bigvee(\mathcal{A}'(T_1 \oplus T_2)) \cong \mathbb{N}$. Also, if $\mathrm{K}_0(\mathcal{A}'(T_1 \oplus T_2)) \cong \mathbb{Z}$, then $\bigvee(\mathcal{A}'(T_1 \oplus T_2)) \cong \mathbb{N}^k$, $k \leqslant 2$, by using Theorem 1.1. Since $\mathrm{K}_0(\mathcal{A}'(T_1 \oplus T_2)) \cong \mathbb{Z}$, $\bigvee(\mathcal{A}'(T_1 \oplus T_2)) \cong \mathbb{N}$.

This shows that $T_1 \sim T_2$, completing the proof of Corollary 1.2. ∎

1.4. THE PROOF OF THEOREM 1.3.

LEMMA 1.16. *Let $T \in \mathcal{L}(\mathcal{H})$. Then the following are equivalent:*

(i) $T \simeq \sum_{i=1}^{k} A_i^{(n_i)}$ *with respect to the decomposition* $\mathcal{H} = \sum_{i=1}^{k} \mathcal{H}_i^{(n_i)}$ *where* $A_i \not\simeq A_j$, $i \neq j$, *and* $n_i, k < \infty$;

(ii) $\bigvee(\mathcal{A}'(T, T^*) \cong \mathbb{N}^k$.

Proof. (i) $\Rightarrow$ (ii). If $A_i \in$ (IR), it is easy to see $\mathcal{A}'(A_i, A_i^*) \cong \mathbb{C}$. For arbitrary $A_i, A_j \in$ (IR), $i \neq j$, then $\ker \tau_{A_i,A_j} \cap \ker \tau_{A_i,A_j} = \{0\}$, where the Rosenblum operator $\tau_{A_i,A_j} \in L(L(H_j, H_i))$ is defined by $\tau_{A_i,A_j}(X) = A_iX - XA_j$ for every $X \in L(H_j, H_i)$ (see [8]). If $0 \neq A \in \ker \tau_{A_i,A_j} \cap \tau_{A_i,A_j}$, then $AA^*A_i = AA_jA^* = A_iAA^*$. Since $A_i \in$ (IR), $AA^* = \lambda I$ and $\lambda = 0$. Similarly, $A^*A = \mu I$ and $\mu \neq 0$. It is easy to see that $\lambda = \mu$. This shows $A/\lambda^{1/2}$ is a unitary operator and $A_i \simeq A_j$. It is a contradiction. So $\mathcal{A}'(T, T^*) \cong \sum_{i=1}^{k} M_{n_i}(\mathbb{C})$, $\bigvee(\mathcal{A}'(T, T^*)) \cong \mathbb{N}^k$.

(ii) $\Rightarrow$ (i) is similar to the proof of Theorem 1.1. ∎

Now we are in position to prove Theorem 1.3. If $T \simeq \sum_{i=1}^{k} A_i^{(n_i)}$ and $A_i \not\simeq A_j$ for $i \neq j$ then $\mathrm{K}_0(\mathcal{A}'(T, T^*)) \cong \mathbb{Z}^k$ by Lemma 1.16. Also, if $\mathrm{K}_0(\mathcal{A}'(T, T^*)) \cong \mathbb{Z}^k$, then since T has one unit finite (IR) decomposition, $\bigvee(\mathcal{A}'(T, T^*)) \cong \mathbb{N}^{k'}$ follows from Lemma 1.16. Thus $k' = k$ and $T \simeq \sum_{i=1}^{k} A_i^{(n_i)}$, $A_i \not\simeq A_j$ for $i \neq j$. The proof of Theorem 1.3 is now complete. ∎

Corollary 1.4 and Corollary 1.5 are straightforward consequences of Theorem 1.3.

2. THE APPLICATION OF THEOREM 1.1 AND THE CALCULATION OF K_0-GROUP

2.1. SEVERAL AUXILIARY LEMMAS AND DEFINITIONS.

VON NEUMANN-WOLD THEOREM. *Let $S \in \mathcal{L}(\mathcal{H})$ be an isometric operator and let $\mathcal{L}_\infty = \bigcap_{n=1}^{\infty} S^n\mathcal{H}$. Then $S|_{\mathcal{L}_\infty}$ is unitary and $S|_{\mathcal{L}_\infty^\perp} \simeq \bigoplus_1^l T_z$, where $l = \dim\ker S^*$.*

DEFINITION 2.5. Say S to be a *pure isometry* if $\bigcap_{n=1}^{\infty} S^n\mathcal{H} = 0$.

It is easily seen that $T_z^{(n)}$ is a pure isometry for every natural number n. The following result is well-known.

LEMMA 2.6. *Let $S \in \mathcal{L}(\mathcal{H})$ be a pure isometry. Then the following hold:*

(i) *If $l = \dim\ker S^*$, then $S \simeq \bigoplus_1^l T_z$;*

(ii) *If $l < \infty$, then $S^* \in \mathcal{B}_l(\mathbb{D})$;*

(iii) $S \in$ (SI) *if and only if* $S^* \in \mathcal{B}_1(\mathbb{D})$.

COROLLARY 2.7. *Let $S \in \mathcal{L}(\mathcal{H})$ be a pure isometry and P an idempotent of $\mathcal{A}'(S)$. Then $S|_{P\mathcal{H}} \in$ (SI) if and only if $S|_{P\mathcal{H}} \simeq T_z$.*

Proof. It is a straightforward conclusion of Lemma 2.6. ∎

LEMMA 2.8. *For each natural number n, $T_z^{(n)}$ has unique* (SI) *decomposition up to similarity.*

Proof. Since $(T_z^{(n)})^* \in \mathcal{B}_n(\mathbb{D})$, $T_z^{(n)}$ has finite (SI) decomposition. If $P \in \mathcal{A}'(T_z^{(n)})$ is idempotent and $T_n^{(n)}|_{P\mathcal{H}^{(n)}} \in$ (SI), then $T_z^{(n)}|_{P\mathcal{H}^{(n)}} \simeq T_z$ follows from Lemma 2.6. Since $(T_z^{(n)})^* \in \mathcal{B}_n(\mathbb{D})$, $m = n$. This implies that $T_z^{(n)}$ has unique (SI) decomposition up to similarity. ∎

LEMMA 2.9. *$\bigvee(H^\infty(\mathbb{D})) \cong \mathbb{N}$, $\mathrm{K}_0(H^\infty(\mathbb{D})) \cong \mathbb{Z}$.*

Proof. Note that $\mathcal{A}'(T_z) \cong H^\infty(\mathbb{D})$ and use Lemma 2.8 and Theorem 1.1, we can complete the proof of Lemma 2.9. ∎

DEFINITION 2.10. Let $T \in \mathcal{L}(\mathcal{H})$ and let $\mathcal{K}$ be a compact subset of $\mathbb{C}$. If $\sigma(T) \subset \mathcal{K}$ and for every f in $\mathrm{Rat}\,(\mathcal{K}) = \{f : f$ is rational function with poles outside $\mathcal{K}\}$, $\|f(T)\| \leqslant \|f\| \overset{\triangle}{=} \max_{z\in\mathcal{K}} \|f(z)\|$, then we call $\mathcal{K}$ *a spectral set for T*.

DEFINITION 2.11. $T \in \mathcal{L}(\mathcal{H})$ is called a *von Neumann operator*, if T has a spectral set.

The following three lemmas come from [4].

LEMMA 2.12. *Every subnormal operator T is a von Neumann operator and $\sigma(T)$ and $\sigma(T^*)$ are spectral sets for T and T^*. Furthermore,*

$$\|f(T)\| = \|f\|_{\sigma(T)}, \quad \|g(T^*)\| = \|g\|_{\sigma(T^*)},$$

where $f \in \mathrm{Rat}\,(\sigma(T))$ and $g \in \mathrm{Rat}\,(\sigma(T^*))$.

An operator is called *subnormal* if it is (unitarily equivalent to) the restriction of a normal operator to an invariant subspace.

LEMMA 2.13. *Let $T \in \mathcal{L}(\mathcal{H})$ be a von Neumann operator. Then the following are equivalent:*
(i) *$\|f(T)\| = \|f\|_{\sigma(T)}$ for each f in $\mathrm{Rat}\,(\sigma(T))$;*
(ii) *$\sigma(f(T)) = f(\sigma(T))$ for each f in $\mathrm{Rat}\,(\sigma(T))$.*

LEMMA 2.14. *Let $\mathcal{K}$ be a spectral set for T and let $f \in \mathrm{Rat}\,(T)$. Then $f(\mathcal{K})$ is a spectral set for $A(= f(T))$. Furthermore, if $\|g(T)\| = \|g\|_{\mathcal{K}}$ holds for every g in $\mathrm{Rat}\,(\mathcal{K})$, then $\|h(A)\| = \|h\|_{f(\mathcal{K})}$ holds for every $h \in \mathrm{Rat}\,(f(\mathcal{K}))$.*

2.2. THE PROOF OF THEOREM 2.1. Since Ω is simply connected, the Riemann Mapping Theorem says that there exists a univalent analytic function f on $\mathbb{D}$ satisfying $f(\mathbb{D}) = \Omega$ and $f(\partial\mathbb{D}) = \partial\Omega$. Set $T = f(T_z) = T_f$. By Lemma 2.13 and Lemma 2.14, $\sigma(T_f) = f(\overline{\mathbb{D}}) = \overline{\Omega}$. Since f is univalent, $\mathcal{A}'(T_f) = \mathcal{A}'(T_z) = H^\infty(\mathbb{D})$ (see [5]). Clearly, T_z and T_f are subnormal. It follows from Lemma 2.12 that $\overline{\Omega}$ and $\overline{\Omega}^*$ are spectral sets for T_f and T_f^*, respectively. As usual, $\overline{\Omega}^* = \{\lambda : \overline{\lambda} \in \Omega\}$.

A simple computation shows that $T_f^* \in \mathcal{B}_1(\mathbb{D})$. By Theorem 1.12 of [6], $\mathcal{A}'(T_f^*) \cong H^\infty(\overline{\Omega}^*)$. Note that $\mathcal{A}'(T_f^*) \cong \mathcal{A}'(T_f)$. We have

$$H^\infty(\Omega) \cong H^\infty(\mathbb{D}).$$

Thus $\bigvee(H^\infty(\Omega)) \cong \mathbb{N}$ and $\mathrm{K}_0(H^\infty(\Omega) \cong \mathbb{Z}$ follows from Lemma 2.9. ∎

2.3. The proof of Theorem 2.2. Let $h = \mathrm{inn}(\varphi - \varphi(\alpha))$ be a finite Blaschke product for some α in $\mathbb{D}$. Using Theorem 4 of [5], we can find a natural number k and a finite Blaschke product ψ with k zeroes satisfying $\mathcal{A}'(T_\varphi) = \mathcal{A}'(T_\psi)$. Since T_ψ is a pure isometry, $T_\psi \cong \bigoplus_1^k (T_z)$ follows from von Neumann-Wold Theorem. By Lemma 1.6, we have

$$\bigvee(\mathcal{A}'(T_\varphi)) \cong \bigvee\Big(\mathcal{A}'\Big(\bigoplus_1^k T_z\Big)\Big) \cong \bigvee(M_k(H^\infty(\mathbb{D}))) \cong \mathbb{N}.$$

By Theorem 1.1, T_φ has unique finite (SI) decomposition up to similarity. ∎

2.4. The proof of Theorem 2.3.

Lemma 2.15. ([13], Lemma 2) *Let $A, B \in \mathcal{L}(\mathcal{H})$. If*

$$\mathcal{H} = \bigvee\{\ker(\lambda - B)^k : \lambda \in \Gamma, k \geqslant 1\}$$

for some fixed subset Γ of $\sigma_{\mathrm{p}}(B)$ satisfying $\sigma_{\mathrm{p}}(A) \cap \Gamma = \emptyset$, then $\ker \tau_{A,B} = \{0\}$.

Lemma 2.16. *Let φ_1 and φ_2 be two univalent analytic functions on $\mathbb{D}$. Then one of the following holds:*

(i) $T_{\varphi_1} \simeq T_{\varphi_2}$*;*

(ii) *either* $\ker \tau_{T_{\varphi_1},T_{\varphi_2}} = \{0\}$ *or* $\ker \tau_{T_{\varphi_2},T_{\varphi_1}} = \{0\}$.

Proof. Set $\varphi_i = \sum_{j=1}^{\infty} \lambda_i^j z_j$, where $i = 1, 2$. Since φ_i is univalent, $\lambda_1^i \neq 0$, $i = 1, 2$. If $\ker \tau_{T_{\varphi_1},T_{\varphi_2}} \neq \{0\}$ and $\ker \tau_{T_{\varphi_2},T_{\varphi_1}} \neq \{0\}$. We can find $X, Y \in \mathcal{L}(H^2(\mathbb{D}))$ satisfying

$$T_{\varphi_2} Y = Y T_{\varphi_1}, \quad T_{\varphi_1} X = X T_{\varphi_2}.$$

Set $\Omega_1 = \varphi_1(\mathbb{D})$, $\Omega_2 = \varphi_2(\mathbb{D})$. Then Ω_1 and Ω_2 are simply connected and

$$T_{\varphi_1}^* \in \mathcal{B}_1(\Omega_1^*), \quad T_{\varphi_2}^* \in \mathcal{B}_1(\Omega_2^*).$$

If $\Omega_1 \neq \Omega_2$, then by using Lemma 2.15, we can deduce that $\ker \tau_{T_{\varphi_1},T_{\varphi_2}} = \{0\}$ or $\ker \tau_{T_{\varphi_2},T_{\varphi_1}} = \{0\}$. This contradicts our assumption. Thus we may assume that $\Omega = \Omega_1 = \Omega_2$ and $\sigma(T_{\varphi_1}) = \sigma(T_{\varphi_2}) = \overline{\Omega}$. Without loss of generality, we can assume that $0 \in \Omega$ and $\varphi_1(0) = 0$. Let $z_0 \in \mathbb{D}$ and $\varphi_2(z_0) = 0$. Then there exists a Möbius transformation $\chi : \mathbb{D} \to \mathbb{D}$ satisfying $\chi(0) = z_0$. Therefore $\varphi_2(\chi(0)) = 0$. This shows that $T_{\varphi_2(z)}$ is unitarily equivalent to $T_{\varphi_2(\chi(z))}$. Thus we may assume that

$\varphi_2(0)=0$. Note that $T^*_{\varphi_1}$ and $T^*_{\varphi_2}$ have the following matrix representations with respect to the usual orthogonal basis, $\{1,z,z^2,\ldots\}$ of $H^2(\mathbb{D})$.

$$T^*_{\varphi_1}=\begin{bmatrix}0&\lambda^1_1&\lambda^1_2&\ldots&\\0&0&\lambda^1_1&\lambda^1_2&\ldots\\&&\ddots&&\\\vdots&\vdots&&&\end{bmatrix},\quad T^*_{\varphi_2}=\begin{bmatrix}0&\lambda^2_1&\lambda^2_2&\ldots&\\0&0&\lambda^2_1&\lambda^2_2&\ldots\\&&\ddots&&\\\vdots&\vdots&&&\end{bmatrix}.$$

Note that $T^*_{\varphi_1}Y^*=Y^*T^*_{\varphi_2}$. Then

$$Y^*=\begin{bmatrix}Y_{11}&Y_{12}&\ldots&\\0&Y_{22}&Y_{23}&\ldots\\&\ddots&\ddots&\end{bmatrix}\quad\text{and}\quad Y_{nn}=\left(\frac{\lambda^2_1}{\lambda^1_1}\right)^{n-1}Y_{11}.$$

We claim that $|\lambda^2_1/\lambda^1_1|\leqslant 1$, as otherwise since Y^* is bounded, $Y_{11}=0$ and $Y_{nn}=0$, $n=1,2,\ldots$. Similarly, we can deduce that $Y_{ij}=0$, $i,j\geqslant 1$. This contradicts $Y^*\neq 0$.

Similarly, $|\lambda^1_1/\lambda^2_1|\leqslant 1$. Therefore $|\lambda^1_1|=|\lambda^2_1|=\lambda$.

Set $\theta_j=\arg\frac{\lambda^j_1}{\lambda}$ and $U_j=\operatorname{diag}(1,\mathrm{e}^{\mathrm{i}\theta_j},\mathrm{e}^{2\mathrm{i}\theta_j},\ldots)$, $j=1,2$. Clearly, U_j is unitary and for each j,

$$R_j=U_jT^*_{\varphi_j}U^*_j=\begin{bmatrix}0&\lambda&&&\\0&0&\lambda&&*\\0&0&0&\lambda&\\&&&&\ddots\end{bmatrix}.$$

Since $U_jT^*_zU^*_j=\mathrm{e}^{-\mathrm{i}\theta_j}T^*_z$, $R_j\in\mathcal{A}'(T^*_z)$, there exists a g_j in $H^\infty(\mathbb{D})$ satisfying $R_j=T^*_{g_j}$. Since $T^*_{g_j}$ is unitarily equivalent to $T^*_{\varphi_j}$, $g_j(\mathbb{D})=\varphi_j(\mathbb{D})=\Omega$, and $T^*_{g_j}\in\mathcal{B}_1(\Omega^*)$. Clearly, each g_j is a univalent analytic function on $\mathbb{D}$ and

$$g_j(0)=0,\quad g'_j(0)=\lambda>0.$$

By the Riemann Mapping Theorem $g_1=g_2$. This shows that $T_{\varphi_1}\simeq T_{\varphi_2}$. ∎

Using Lemma 2.16, we immediately obtain

LEMMA 2.17. *Let* $T=\bigoplus\limits_{i=1}^{n}T_{\varphi_i}$ *be given by Theorem* 2.3. *Then there exists a unitary operator* U *such that the followings hold:*

(i) $UTU^*=\bigoplus\limits_{p=1}^{k}T^{(n_p)}_{\varphi_{i_p}}$ *and* $T_{\varphi_{i_{p_1}}}\not\simeq T_{\varphi_{i_{p_2}}}$ *for* $i_{p_1}\neq i_{p_2}$*;*

(ii) $\ker\tau_{T_{\varphi_{i_{p_2}}},T_{\varphi_{i_{p_2}}}}=\{0\}$ *for* $i_{p_1}<i_{p_2}$.

Now we are in position to prove Theorem 2.3. We know that if φ_i is a univalent analytic function on $\mathbb{D}$, then $\mathcal{A}'(T_{\varphi_i})\cong H^\infty(\mathbb{D})$. By Lemma 2.16, it is easy to see that $\mathcal{A}'(T)/\operatorname{Rad}\mathcal{A}'(T)\cong\sum\limits_{i=1}^{k}M_{n_i}(H^\infty(\mathbb{D}))$. By Lemma 2.9,

$$\bigvee(\mathcal{A}'(T))\cong\bigvee\left(\mathcal{A}'(T)/\operatorname{Rad}\mathcal{A}'(T)\right)\cong\bigvee\left(\sum_{i=1}^{k}M_{n_i}(H^\infty(\mathbb{D}))\right)\cong\mathbb{N}^k.$$

By Theorem 1.1, T has unique finite (SI) decomposition up to similarity. ∎

Similarly to the argument of Section 1, we can prove Corollary 2.4.

3. THE PROOF OF THEOREM 3.1

LEMMA 3.2. *Let* $A_1, A_2 \in (\mathrm{SI}) \cap \mathcal{L}(\mathcal{H})$ *satisfying*

$$\mathcal{A}'(A_i)/\mathrm{Rad}\,\mathcal{A}'(A_i) \cong \mathbb{C}, \quad i = 1, 2.$$

Then at least one of the following hold:

(i) $A_1 \sim A_2$;

(ii) *for* X *and* Y *in* $\mathcal{L}(\mathcal{H})$, *if* $AX = XB$, $YA = BY$ *then* $XY \in \mathrm{Rad}\,\mathcal{A}'(A_1)$ *and* $YX \in \mathrm{Rad}\,\mathcal{A}'(A_2)$.

Proof. If $A_1 \not\sim A_2$ and there exist X and $Y \in \mathcal{L}(\mathcal{H})$ such that $A_1X = XA_2$ and $YA_1 = A_2Y$, then $A_1XY = XA_2Y = XYA_1$. Hence $XY \in \mathcal{A}'(A_1)$. If $XY \notin \mathrm{Rad}\,\mathcal{A}'(A_1)$, then $XY = \lambda + R$, where $0 \neq \lambda \in \mathbb{C}$ and $R \in \mathrm{Rad}\,\mathcal{A}'(A_1)$ by $\mathcal{A}'(A_1)/\mathrm{Rad}\,\mathcal{A}'(A_1) \cong \mathbb{C}$. So XY is invertible. Since we have $YX \in \mathcal{A}'(A_2)$ and $\sigma(YX) \cup \{0\} = \sigma(XY) \cup \{0\}$, YX is also invertible by $\mathcal{A}'(A_2)/\mathrm{Rad}\,\mathcal{A}'(A_2) \cong \mathbb{C}$. This shows that X is invertible and $A_1 \sim A_2$. This contradicts $A_1 \not\sim A_2$.

Similarly we have $YX \in \mathrm{Rad}\,\mathcal{A}'(A_2)$. ∎

LEMMA 3.3. $\bigvee(M_n(\mathbb{C})) \cong \mathbb{N}$, $n \geqslant 1$ (see 5.1.3 in [3]).

Proof of Theorem 3.1. By Lemma 3.2, $\mathcal{A}'(T)/\mathrm{Rad}\,\mathcal{A}'(T) \cong \sum_{i=1}^{k} M_{n_i}(\mathbb{C})$. By Lemma 3.3

$$\bigvee(\mathcal{A}'(T)) \cong \bigvee(\mathcal{A}'(T)/\mathrm{Rad}\,\mathcal{A}'(T)) \cong \bigvee\Big(\sum_{i=1}^{k} M_{n_i}(\mathbb{C})\Big) \cong \mathbb{N}^k.$$

So T has unique decomposition up to similarity.

EXAMPLE 3.4. Suppose that

$$T = \begin{bmatrix} 0 & w_1 & & & & \\ & 0 & w_2 & & & \\ & & \ddots & \ddots & & \\ & & & & w_n & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{bmatrix},$$

where $\sum_{n=1}^{\infty} |w_n|^2 < \infty$, $|w_n| \leqslant |w_{n-1}|$, and $w_n \to 0$. A simple computation shows that

$$\mathcal{A}'(T)/\mathrm{Rad}\,\mathcal{A}'(T) \cong \mathbb{C}.$$

Example 3.4 shows that the collection of operators considered in Theorem 3.1 is not empty.

It is well known that a Jordan block is an (SI) operator on a finite-dimensional space, and if A is an (SI) operator on a finite-dimensional space, then A is similar to some Jordan block and $\mathcal{A}'(A)/\mathrm{Rad}\,\mathcal{A}'(A) \cong \mathbb{C}$. So, in the same way, we can prove the Jordan Standard Theorem, and in our proof we do not use determinants.

JORDAN STANDARD THEOREM. *Every operator A in $\mathcal{L}(\mathbb{C}^n)$ is similar to* $J = \bigoplus_{i=1}^{n}\bigoplus_{1}^{n_i}(\lambda_i I_{m_i} + J_{m_i})$, *where* $\lambda_i I_{m_i} + J_{m_i}$ *a summand of J, is uniquely determined by A and* $\|m_i - m_j\| + \|\lambda_i - \lambda_j\| > 0$ *for* $i \neq j$.

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YANG CAO
Department of Mathematics
Jilin University
Chang chun 130023
P.R. CHINA

JUNSHENG FANG
Academia Sinica
Institute of Mathematics
Beijing 100080
P.R. CHINA

CHUNLAN JIANG
Department of Applied Mathematics
Hebei University of Technology
Tianjin 300130
P.R. CHINA

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