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SEQUENCES IN NON-COMMUTATIVE L^p-SPACES

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ABSTRACT. Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a faithful, normal, semi-finite trace τ. We introduce the notion of equi-integrability in non-commutative spaces and show that if a rearrangement invariant quasi-Banach function space E on the positive semi-axis is α-convex with constant 1 and satisfies a non-trivial lower q-estimate with constant 1, then the corresponding non-commutative space of measurable operators $E(\mathcal{M},\tau)$ has the following property: every bounded sequence in $E(\mathcal{M},\tau)$ has a subsequence that splits into an E-equi-integrable sequence and a sequence with pairwise disjoint projection supports. This result extends the well known Kadec-Pełczyński subsequence splitting lemma for Banach lattices to non-commutative spaces. As applications, we prove that for $1 \leqslant p < \infty$, every subspace of $L^p(\mathcal{M},\tau)$ either contains almost isometric copies of ℓ^p or is strongly embedded in $L^p(\mathcal{M},\tau)$.

KEYWORDS: *von Neumann algebras, function spaces, symmetric spaces of operators.*

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1. INTRODUCTION

In [9], Kadec and Pełczyński proved the fundamental result that if $1 \leqslant p < \infty$ then every bounded sequence $\{f_n\}_{n=1}^{\infty}$ in $L^p[0,1]$ has a subsequence that can be decomposed into two extreme sequences $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$, where the h_k's are pairwise disjoint and the g_k's are L_p-equi-integrable that is $\lim\limits_{m(A)\to 0} \sup\limits_{k} \|\chi_A g_k\|_p \to 0$ and $h_k \perp g_k$ for every $k \geqslant 1$. This result was used to study different structures of subspaces of $L^p[0,1]$. Later, the same decomposition property was proved for larger classes of Banach function spaces (see [7] for Orlicz spaces with Δ_2-condition and q-concave lattices, [8] for some symmetric spaces). There are however examples of Banach lattices with sequences for which the above decomposition is not possible. For instance, examples of reflexive, p-convex Banach lattices without the subsequence splitting property can be found in a paper of Figiel et al. ([7]). Subsequently, Weis ([18]) characterized, in terms of uniform order continuity conditions and ultraproducts, the class of all Banach lattices where such property is possible. For the case of rearrangement invariant function spaces, the spaces in which the subsequence splitting lemma holds are exactly those with order continuous norm and satisfying the so called Fatou property (equivalently, those that contain no subspace isomorphic to c_0). The subsequence splitting lemma has played an important role in the investigation of Banach space structures of function spaces.

It is the intention of the present paper to give an extension of the Kadec-Pełczyński decomposition stated above to the case of bounded sequences in general non-commutative symmetric spaces of measurable operators. Let $\mathcal{M}$ be a von Neumann algebra, equipped with a faithful, normal, semi-finite trace τ and E be a rearrangement invariant Banach function space on $[0,1]$ or the half line $(0,\infty)$ according to whether $\mathcal{M}$ is finite or infinite. We define equi-integrability in the non-commutative setting as generalization of Akemann's characterization of weak compactness on preduals of von Neumann algebras. Using such notion, we provide an analogue of the Kadec-Pełczyński subsequence splitting lemma for non-commutative spaces. More precisely, we proved that if E is order continuous and satisfies the Fatou property then the corresponding symmetric space of measurable operators $E(\mathcal{M},\tau)$ has the subsequence splitting property. Our approach allows one to consider more general spaces such as quasi-Banach rearrangement invariant spaces that are α-convex with constant 1 and satisfy non trivial q-lower estimate with constant 1. In particular, splitting of bounded sequences is valid in non-commutative L^p-spaces for $0 < p < \infty$. It should be noted that Sukochev ([16]) obtain a similar result for the case of finite von Neumann algebras.

As application of the main result, we study the structure of subspaces of $L^p(\mathcal{M},\tau)$ for $1 \leqslant p < \infty$.

We refer to [10] and [17] for general information concerning von Neumann algebras as well as non-commutative integration, to [12] and [15] for Banach lattice theory.

2. DEFINITIONS AND PRELIMINARY RESULTS

Throughout, H is a given Hilbert space and $\mathcal{M} \subset \mathcal{B}(H)$ denotes a semi-finite von Neumann algebra with a normal, faithful semi-finite trace τ. The identity in $\mathcal{M}$ will be denoted by $\mathbf{1}$ and $\mathcal{M}_p$ will stand for the set of all (self adjoint) projections in $\mathcal{M}$. A closed and densely defined operator a on H is said to be *affiliated* with $\mathcal{M}$ if $u^*au = a$ for all unitary operator u in the commutant $\mathcal{M}'$ of $\mathcal{M}$.

A closed and densely defined operator x, affiliated with $\mathcal{M}$, is called *τ-measurable* if for every $\varepsilon > 0$, there exists an orthogonal projection $p \in \mathcal{M}$ such that $p(H) \subseteq \mathrm{dom}(x)$, $\tau(\mathbf{1} - p) < \varepsilon$ and $xp \in \mathcal{M}$. The set of all τ-measurable operators will be denoted by $\widetilde{\mathcal{M}}$. The set $\widetilde{\mathcal{M}}$ is a $*$-algebra with respect to the strong sum, the strong product and the adjoint operation. Given a self-adjoint operator x in $\widetilde{\mathcal{M}}$, we denote by $\mathrm{e}^x(\cdot)$ its spectral measure. Recall that $\mathrm{e}^{|x|}(B) \in \mathcal{M}$ for all Borel sets $B \subseteq \mathbb{R}$ and $x \in \widetilde{\mathcal{M}}$. For fixed $x \in \widetilde{\mathcal{M}}$, the *generalized singular value function* $\mu(x)$ of x is defined by

$$\mu_t(x) = \inf\{s \geqslant 0 : \tau(\mathrm{e}^{|x|}(s,\infty)) \leqslant t\}, \quad \text{for } t \geqslant 0.$$

The function $\mu_{(\cdot)}(x) : [0,\infty) \to [0,\infty]$ is right continuous, non-decreasing. We note that $\mu_t(x) < \infty$ for every $t > 0$. For a complete study of $\mu_{(\cdot)}$, we refer to [6].

The topology defined by the metric on $\widetilde{\mathcal{M}}$ obtained by setting:

$$d(x,y) = \inf\{t \geqslant 0 : \mu_t(x-y) \leqslant t\}, \quad \text{for } x,y \in \widetilde{\mathcal{M}},$$

is called the *measure topology*. It is well-known that a net $(x_\alpha)_{\alpha\in I}$ in $\widetilde{\mathcal{M}}$ converges to $x \in \widetilde{\mathcal{M}}$ in measure topology if and only if for every $\varepsilon > 0$, $\delta > 0$, there exists $\alpha_0 \in I$ such that whenever $\alpha \geqslant \alpha_0$, there exists a projection $p \in \mathcal{M}_p$ such that

$$\|(x_\alpha - x)p\|_{\mathcal{M}} < \varepsilon \quad \text{and} \quad \tau(\mathbf{1}-p) < \delta.$$

Such criteria will be used in the sequel. It was shown in [13] that $(\widetilde{\mathcal{M}}, d)$ is a complete metric space.

Remark that if we consider $\mathcal{M} = L^\infty(\mathbb{R}^+, m)$, where m is the Lebesgue measure on $\mathbb{R}^+$ then $\mathcal{M}$ is an abelian von Neumann algebra acting on $L^2(\mathbb{R}^+, m)$ via the multiplication operators. With the trace being the usual integration with respect to m, $\widetilde{\mathcal{M}} = L^0(\mathbb{R}^+, m)$ (the usual space of all measurable functions on $\mathbb{R}^+$) and the generalized singular value $\mu(f)$ is precisely the decreasing rearrangement of the function $|f|$ (usually denoted by f^* in Banach lattice theory).

DEFINITION 2.1. A *symmetric quasi-Banach function space* on $\mathbb{R}^+$ is a quasi-Banach lattice E of measurable functions with the following properties:

(i) E is an order ideal in $L^0(\mathbb{R}^+, m)$;
(ii) E is rearrangement invariant in the sense of [12] (p. 114);
(iii) E contains all finitely supported simple functions.

DEFINITION 2.2. A quasi-Banach function space E is said to satisfy a *lower q-estimate* if there exists a positive constant $C > 0$ such that for all finite sequences $\{x_n\}$ of mutually disjoint elements in E,

$$\Big(\sum \|x_n\|^q\Big)^{\frac{1}{q}} \leqslant C\Big\|\sum x_n\Big\|.$$

The least such constant C is called the *constant of the lower q-estimate.* Recall that if E is a quasi-Banach function space and $1 < p < \infty$,

$$E^{(p)} = \{x \in L^0(\mathbb{R}^+, m) : |x|^p \in E\} \quad \text{with} \quad \|x\|_{E^{(p)}} = \|\,|x|^p\,\|_E^{\frac{1}{p}}.$$

DEFINITION 2.3. Let E be a rearrangement invariant quasi-Banach function space on $(0, \tau(\mathbf{1}))$. We define the symmetric space of measurable operators $E(\mathcal{M}, \tau)$ by setting:

$$E(\mathcal{M},\tau) := \{x \in \widetilde{\mathcal{M}} : \mu(x) \in E\}$$

and

$$\|x\|_{E(\mathcal{M},\tau)} = \|\mu(x)\|_E \quad \text{for all } x \in E(\mathcal{M},\tau).$$

It was shown in [19], Lemma 4.1 that if E is α-convex (for some $0 < \alpha \leqslant 1$) with constant 1, then $\|\cdot\|_{E(\mathcal{M},\tau)}$ is a α-norm, that is, for every $x, y \in E(\mathcal{M},\tau)$,

$$\|x+y\|^\alpha_{E(\mathcal{M},\tau)} \leqslant \|x\|^\alpha_{E(\mathcal{M},\tau)} + \|y\|^\alpha_{E(\mathcal{M},\tau)}.$$

Equipped with $\|\cdot\|_{E(\mathcal{M},\tau)}$, the space $E(\mathcal{M},\tau)$ is a α-Banach space. The space $E(\mathcal{M},\tau)$ is often referred to as the non-commutative analogue of the function space E. We remark that if $0 < p < \infty$ and $E = L^p(\mathbb{R}^+, m)$ then $E(\mathcal{M},\tau)$ coincides with the usual non-commutative L^p-space associated to the semi-finite von Neumann algebra $\mathcal{M}$. Also if $E = L^\infty(\mathbb{R}^+, m)$, then $L^\infty(\mathcal{M},\tau)$ is the von Neumann algebra $\mathcal{M}$. We refer to [3], [4] and [19] for some background on the space $E(\mathcal{M},\tau)$.

We will need the following known result. A proof can be found in [5].

Proposition 2.4. *Assume that E is order-continuous and α-convex with constant* 1 *for some* $0 < \alpha \leqslant 1$.

(i) *If $x \in E(\mathcal{M},\tau)$ and $e \leqslant f$ are projections in $\mathcal{M}$ then $\|exe\|_{E(\mathcal{M},\tau)} \leqslant \|fxf\|_{E(\mathcal{M},\tau)}$;*

(ii) *If $x \in E(\mathcal{M},\tau)$ and $e_\beta \downarrow_\beta 0$ is a net of projections in $\mathcal{M}$ then* $\|xe_\beta\|_{E(\mathcal{M},\tau)} \downarrow_\beta 0$.

The following definition isolates the main topic of this paper.

Definition 2.5. Let E be a quasi-Banach function space on $\mathbb{R}^+$ and K be a bounded subset of $E(\mathcal{M},\tau)$. We will say that K is *E-equi-integrable* if $\lim\limits_{n\to\infty} \sup\limits_{x\in K} \|e_n x e_n\|_{E(\mathcal{M},\tau)} = 0$ for every decreasing sequence $\{e_n\}_{n=1}^\infty$ of projections with $e_n \downarrow_n 0$.

Remark 2.6. Since $\{e_n\}_{n=1}^\infty$ is decreasing, it is clear from Proposition 2.4 that the sequence $\left\{ \sup\limits_{x\in K} \|e_n x e_n\|_{E(\mathcal{M},\tau)} \right\}_{n=1}^\infty$ is decreasing and therefore the limit in the definition above always exists. This notion of equi-integrability was motivated by the commutative case on one hand and the characterization of weakly compact subsets of $L^1(\mathcal{M},\tau)$ by Akemann [1] (see also [17], p. 150) on the other. Using this terminology, Akemann's characterization can be stated as in the commutative case: relatively weakly compact subsets of $L^1(\mathcal{M},\tau)$ are exactly the equi-integrable sets.

In general, relatively weakly compact sets are not necessarily equi-integrable. For example, if $1 < p < \infty$, any normalized disjoint sequence cannot be L^p-integrable but since L^p is reflexive, such set is relatively weakly compact. Our next result shows that the converse always holds.

Proposition 2.7. *Assume that E is an order-continuous symmetric Banach function space and K is an E-equi-integrable set in $E(\mathcal{M},\tau)$. Then K is relatively weakly compact.*

The proposition will be proved in several steps. Recall that $E(\mathcal{M},\tau)$ is a subset of $L^1(\mathcal{M},\tau) + \mathcal{M}$ and therefore if p is a projection in $L^1(\mathcal{M},\tau) \cap \mathcal{M}$ and K is a subset of $E(\mathcal{M},\tau)$, then pK and Kp are subsets of $L^1(\mathcal{M},\tau)$.

LEMMA 2.8. *Let p be a projection in $L^1(\mathcal{M},\tau)\cap\mathcal{M}$ and K be an E-equi-integrable subset of $E(\mathcal{M},\tau)$. The sets pKp and $pK(1-p)$ are relatively weakly compact in $L^1(\mathcal{M},\tau)$.*

Proof. It is enough to check that these sets are L^1-equi-integrable. Let $T : E(\mathcal{M},\tau)\to L^1(\mathcal{M},\tau)$ be the linear map defined by $x\to Tx = pxp$. This map is well-defined and one can deduce from the closed graph theorem that it is bounded. Let $\{e_n\}_{n=1}^\infty$ be a sequence of projections with $e_n\downarrow_n 0$. For each $n\geqslant 1$, set f_n to be the right support projection of e_np. By the definition of support projections, $f_n\leqslant p$. So $\{f_n\}_{n=1}^\infty$ is a sequence of finite projections. We also note that (see for instance the proof of [17], Proposition 1.6, p. 292),

$$f_n = e_n\vee(\mathbf{1}-p)-(\mathbf{1}-p)$$

and by Kaplansky formula (see for instance [10], Theorem 6.1.6, p. 403),

$$f_n\sim e_n - e_n\wedge(\mathbf{1}-p).$$

Since $\tau(f_n)=\tau(e_n-e_n\wedge(\mathbf{1}-p))\leqslant\tau(p)$ and $\{e_n-e_n\wedge(\mathbf{1}-p)\}_{n=1}^\infty$ converges to zero, $\{f_n\}_{n=1}^\infty$ converges to zero. Now since the f_n's are finite projections, we conclude that if $g_n=\bigwedge_{k\geqslant n} f_k$, then $\{g_n\}_{n=1}^\infty$ converges to zero. Therefore, for every $x\in K$,

$$\|e_npxpe_n\|_1=\|e_np(g_nxg_n)pe_n\|_1\leqslant\|p(g_nxg_n)p\|_1\leqslant\|T\|\cdot\|g_nxg_n\|_{E(\mathcal{M},\tau)}.$$

Since K is E-equi-integrable, one obtains that

$$\lim_{n\to\infty}\sup_{y\in pKp}\|e_nye_n\|_1\leqslant\|T\|\cdot\lim_{n\to\infty}\sup_{x\in K}\|g_nxg_n\|_{E(\mathcal{M},\tau)}=0$$

which concludes that pKp is relatively weakly compact in $L^1(\mathcal{M},\tau)$.

For $pK(\mathbf{1}-p)$, let $S : E(\mathcal{M},\tau)\to L^1(\mathcal{M},\tau)$ be the map defined by $x\to Sx = px(\mathbf{1}-p)$. As above, S is bounded. Let $\{e_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ be sequences of projections as described above. For each $n\geqslant 1$, let s_n be the left support projection of $(\mathbf{1}-p)e_n$. Then $s_n = e_n\vee p - p$ for every $n\geqslant 1$ and the sequence $\{s_n\}_{n=1}^\infty$ is decreasing. It is claimed that $s_n\downarrow_n 0$.

For this, it is enough to check that $e_n\vee p\downarrow_n p$. In fact, $e_n\vee p-e_n\sim p-e_n\wedge p$ and the sequence defined by the right hand side of the equivalence converges to p which implies that

$$\lim_{n\to\infty}\tau(e_n\vee p-e_n)=\lim_{n\to\infty}\tau(p-e_n\wedge p)=\tau(p)$$

and therefore,

$$\lim_{n\to\infty}\tau(e_n\vee p-p-e_n)=0.$$

But since $(e_n\vee p-p-e_n)^2=(e_n\vee p-p-e_n)+e_np+pe_n$, we can conclude that

$$\lim_{n\to\infty}\|e_n\vee p-p-e_n\|_2=0.$$

From this, we get (by passing to a subsequence if necessary) that $\{e_n\vee p-p-e_n\}_{n=1}^\infty$ converges to zero in measure. Similarly, $\{e_n\vee p-p-pe_n\}_{n=1}^\infty$ converges to zero in measure so $e_n\vee p\downarrow_n p$ hence $s_n\downarrow_n 0$.

To conclude the proof of Lemma 2.8, note that $g_n \perp s_n$ so $g_n \vee s_n = g_n + s_n$. In particular, $g_n \vee s_n \downarrow_n 0$ and we get that

$$\begin{aligned}\lim_{n\to\infty} \sup_{y\in pK(\mathbf{1}-p)} \|e_n y e_n\|_1 &= \lim_{n\to\infty} \sup_{x\in K} \|e_n px(\mathbf{1}-p)e_n\|_1 \\ &= \lim_{n\to\infty} \sup_{x\in K} \|e_n p(g_n \vee s_n)x(g_n \vee s_n)(\mathbf{1}-p)e_n\|_1 \\ &\leqslant \lim_{n\to\infty} \sup_{x\in K} \|p(g_n \vee s_n)x(g_n \vee s_n)(\mathbf{1}-p)\|_1 \\ &\leqslant \|S\| \cdot \lim_{n\to\infty} \sup_{x\in K} \|(g_n \vee s_n)x(g_n \vee s_n)\|_{E(\mathcal{M},\tau)} = 0\end{aligned}$$

which verifies the lemma. ∎

LEMMA 2.9. *Let p and K be as in Lemma 2.8. Then pK is relatively weakly compact in $E(\mathcal{M},\tau)$.*

Proof. Note first that pK is E-equi-integrable. This can be seen by applying the series of arguments used in Lemma 2.8, considering the operators T and S as maps from $E(\mathcal{M},\tau)$ into $E(\mathcal{M},\tau)$. Let $\{px_n\}_{n=1}^{\infty}$ be a bounded sequence in pK. From Lemma 2.8, we can assume that $\{px_n\}_{n=1}^{\infty}$ is weakly convergent in $L^1(\mathcal{M},\tau)$. Fix $\varphi \in E^*(\mathcal{M},\tau)_+$ and let $\varphi = \int_0^{\infty} t\,\mathrm{d}e_t$ be its spectral decomposition. For each $k \geqslant 1$, set $q_k := \mathrm{e}^{\varphi}((0,k))$. We remark that $\varphi q_k = q_k\varphi \in \mathcal{M}$. For $m,n \in \mathbb{N}$,

$$\begin{aligned}\langle \varphi, px_n - px_m\rangle &= \langle \varphi - \varphi q_k, px_n - px_m\rangle + \langle \varphi q_k, px_n - px_m\rangle \\ &= \langle \varphi(\mathbf{1}-q_k), px_n - px_m\rangle + \langle \varphi q_k, px_n - px_m\rangle \\ &= \langle (\mathbf{1}-q_k)\varphi(\mathbf{1}-q_k), px_n - px_m\rangle + \langle \varphi q_k, px_n - px_m\rangle.\end{aligned}$$

This implies that

$$|\langle \varphi, px_n - px_m\rangle| \leqslant |\tau(\varphi(1-q_k)(px_n - px_m)(1-q_k))| + |\langle \varphi q_k, px_n - px_m\rangle|.$$

Since φq_k belongs to $\mathcal{M}$,

$$\limsup_{n,m\to\infty} |\langle \varphi q_k, px_n - px_m\rangle| \leqslant 2\|\varphi\|_{E^*(\mathcal{M},\tau)} \cdot \sup_{a\in K} \|(\mathbf{1}-q_k)pa(\mathbf{1}-q_k)\|_{E(\mathcal{M},\tau)}.$$

Since $\mathbf{1} - q_k \downarrow_k 0$ and pK is E-equi-integrable, we conclude that

$$\lim_{n,m\to\infty} \langle \varphi q_k, px_n - px_m\rangle = 0.$$

This proves the lemma. ∎

To deduce Proposition 2.7, let $\{p_k\}_{k=1}^{\infty}$ be a sequence of projections that increases to $\mathbf{1}$ and $\tau(p_k) < \infty$ and fix $\varepsilon > 0$. Choose $k_0 \geqslant 1$ such that

$$\sup_{a\in K} \|(\mathbf{1}-p_{k_0})a(\mathbf{1}-p_{k_0})\|_{E(\mathcal{M},\tau)} \leqslant \varepsilon.$$

We have $K = p_{k_0}K + (\mathbf{1}-p_{k_0})Kp_{k_0} + (\mathbf{1}-p_{k_0})K(\mathbf{1}-p_{k_0})$ which implies that

$$K \subset p_{k_0}K + (\mathbf{1}-p_{k_0})Kp_{k_0} + \varepsilon B_{E(\mathcal{M},\tau)}$$

where $B_{E(\mathcal{M},\tau)}$ denotes the closed unit ball of $E(\mathcal{M},\tau)$. From Lemma 2.9, the sets $p_{k_0}K$ and $(\mathbf{1}-p_{k_0})Kp_{k_0}$ are relatively weakly compact which concludes that K is relatively weakly compact. The proof is complete. ∎

REMARK 2.10. If $\tau(\mathbf{1}) < \infty$, the proof above can be considerably shortened. In this case, $E(\mathcal{M},\tau) \subset L^1(\mathcal{M},\tau)$ so if K is E-equi-integrable, then it is relatively weakly compact in $L^1(\mathcal{M},\tau)$ and one can argue directly as in the last part of Lemma 2.9 to conclude that K is relatively weakly compact in $E(\mathcal{M},\tau)$.

The following proposition should be compared with Theorem 5.1 and Theorem 5.2 in [2]. It generalizes a well known property of equi-integrable sets in function spaces to the non-commutative setting.

PROPOSITION 2.11. *Let E be a symmetric quasi-Banach space function space and K be a E-equi-integrable subset of $E(\mathcal{M},\tau)$. For each sequence $\{x_n\}_{n=1}^\infty$ in K and $x \in \overline{K}$, the following are equivalent:*

(a) $\lim\limits_{n\to\infty} \|x_n - x\|_{E(\mathcal{M},\tau)} = 0$*;*

(b) $\{x_n\}_{n=1}^\infty$ *converges to x in measure (as $n \to \infty$).*

Proof. The implication $(a) \Rightarrow (b)$ is trivial. For $(b) \Rightarrow (a)$, we will assume that $x = 0$. Recall that there exists $0 < \alpha \leqslant 1$, such that $\mathcal{M} \cap L^\alpha(\mathcal{M},\tau) \subset E(\mathcal{M},\tau) \subset \mathcal{M} + L^\alpha(\mathcal{M},\tau)$ with $\|x\|_{\mathcal{M}+L^\alpha(\mathcal{M},\tau)} \leqslant \|x\|_{E(\mathcal{M},\tau)} \leqslant 2\|x\|_{\mathcal{M}\cap L^\alpha(\mathcal{M},\tau)}$ for every $x \in \mathcal{M} \cap L^\alpha(\mathcal{M},\tau)$. We will prove first the following lemma:

LEMMA 2.12. *For every $p \in \mathcal{M}_p$ with $\tau(p) < \infty$, $\lim\limits_{n\to\infty} \|x_n p\|_{E(\mathcal{M},\tau)} = 0$. Similarly, $\lim\limits_{n\to\infty} \|p x_n\|_{E(\mathcal{M},\tau)} = 0$.*

To prove this lemma, fix $\varepsilon > 0$ and let $C = \max\{1, \tau(p)\}$. Since K is equi-integrable, there exists $\delta > 0$ such that whenever $q \in \mathcal{M}_p$ satisfies $\tau(q) < \delta$, then for every $n \in \mathbb{N}$, $\|q x_n q\|_{E(\mathcal{M},\tau)} \leqslant \varepsilon/(2)^{1/\alpha}$. Since both $\{x_n\}_{n=1}^\infty$ and $\{x_n^*\}_{n=1}^\infty$ converge to zero in measure, one can choose $n_0 \geqslant 1$ such that for each $n \geqslant n_0$, there exists a projection $p_n \in \mathcal{M}_p$ with $\tau(\mathbf{1} - p_n) < \delta$,

$$\|x_n p_n\|_{\mathcal{M}} < \frac{\varepsilon}{2[4C]^{\frac{1}{\alpha}}}$$

and

$$\|x_n^* p_n\|_{\mathcal{M}} < \frac{\varepsilon}{2[4C]^{\frac{1}{\alpha}}}.$$

For every $n \geqslant n_0$,

$$\begin{aligned}
\|x_n p\|_{E(\mathcal{M},\tau)}^\alpha &\leqslant \|x_n p_n p\|_{E(\mathcal{M},\tau)}^\alpha + \|p_n x_n (\mathbf{1} - p_n) p\|_{E(\mathcal{M},\tau)}^\alpha \\
&\quad + \|(\mathbf{1} - p_n) x_n (\mathbf{1} - p_n) p\|_{E(\mathcal{M},\tau)}^\alpha \\
&\leqslant 2^\alpha \max\left\{\|x_n p_n\|_{\mathcal{M}}^\alpha, \|x_n p_n p\|_{L^\alpha(\mathcal{M},\tau)}^\alpha\right\} \\
&\quad + 2^\alpha \max\left\{\|x_n^* p_n\|_{\mathcal{M}}^\alpha, \|p(\mathbf{1} - p_n) x_n^* p_n\|_{L^\alpha(\mathcal{M},\tau)}^\alpha\right\} \\
&\quad + \|(\mathbf{1} - p_n) x_n (\mathbf{1} - p_n)\|_{E(\mathcal{M},\tau)}^\alpha \\
&\leqslant 2 \cdot 2^\alpha \max\left\{\frac{\varepsilon^\alpha}{2^\alpha 4C}, \frac{\varepsilon^\alpha}{2^\alpha 4C}\tau(p)\right\} + \frac{\varepsilon^\alpha}{2} \leqslant \varepsilon^\alpha.
\end{aligned}$$

A similar estimate works for $\{x_n^* p\}_{n=1}^\infty$. The lemma is verified.

To complete the proof of Proposition 2.11, choose a mutually disjoint family $\{e_i\}_{i\in I}$ of projections in $\mathcal{M}$ with $\sum\limits_{i\in I} e_i = \mathbf{1}$ for the strong operator topology and

$\tau(e_i) < \infty$ for all $i \in I$. Using a similar argument as in [19], one can get an at most countable subset $\{e_k\}_{k=1}^{\infty}$ of $\{e_i\}_{i\in I}$ such that for each e_i outside of $\{e_k\}_{k=1}^{\infty}$, $e_i x_n = x_n e_i = 0$ for every $n \in \mathbb{N}$. Let $e = \sum_{k\in\mathbb{N}} e_k$. Replacing $\mathcal{M}$ by $e\mathcal{M}e$ and τ by its restriction on $e\mathcal{M}e$, we may assume that $e = \mathbf{1}$. Let $p_n = \sum_{k\geqslant n} e_k$. It is clear that $p_n \downarrow_n 0$ and $\tau(\mathbf{1} - p_n) < \infty$ for every $n \in \mathbb{N}$. Fix $\varepsilon > 0$ and choose $n_0 \geqslant 1$ such that $\sup_{n\in\mathbb{N}} \|p_{n_0} x_n p_{n_0}\|_{E(\mathcal{M},\tau)} \leqslant \varepsilon$. We get that

$$\limsup_{n\to\infty} \|x_n\|^{\alpha}_{E(\mathcal{M},\tau)} \leqslant \lim_{n\to\infty} \|x_n(\mathbf{1}-p_{n_0})\|^{\alpha}_{E(\mathcal{M},\tau)} + \lim_{n\to\infty} \|x_n^*(\mathbf{1}-p_{n_0}\|^{\alpha}_{E(\mathcal{M},\tau)} + \varepsilon = \varepsilon$$

and since ε is arbitrary, the proof is complete. ∎

The inequality given below can be viewed as the analogue of the well-known fact on normal functionals on von Neumann algebras, $|\varphi(a)|^2 \leqslant \|\varphi\| \cdot |\varphi|(aa^*)$ whenever $a \in \mathcal{M}$ and $\varphi \in \mathcal{M}_*$ ([17], Proposition 4.6, p. 146), of the general case of symmetric spaces of measurable operators.

Proposition 2.13. *Let $x \in E(\mathcal{M},\tau)$ and $y \in \mathcal{M}$ then*

$$\|xy\|_{E(\mathcal{M},\tau)} \leqslant \|\,|x|y\|_{E(\mathcal{M},\tau)} \leqslant \|x\|^{\frac{1}{2}}_{E(\mathcal{M},\tau)} \cdot \|y^*|x|y\|^{\frac{1}{2}}_{E(\mathcal{M},\tau)}.$$

Proof. Let $x = u|x|$ be the polar decomposition of x. Then $\|xy\|_{E(\mathcal{M},\tau)} = \|u|x|y\|_{E(\mathcal{M},\tau)} \leqslant \|u\|_\infty \cdot \|\,|x|y\|_{E(\mathcal{M},\tau)}$. Also $\|\,|x|y\|_{E(\mathcal{M},\tau)} = \|\,|x|^{\frac{1}{2}}|x|^{\frac{1}{2}}y\|_{E(\mathcal{M},\tau)}$ and using Hölder's inequality,

$$\begin{aligned}\|\,|x|y\|_{E(\mathcal{M},\tau)} &\leqslant \|\,|x|^{\frac{1}{2}}\|_{E^{(2)}(\mathcal{M},\tau)} \cdot \|\,|x|^{\frac{1}{2}}y\|_{E^{(2)}(\mathcal{M},\tau)}\\ &= \|x\|^{\frac{1}{2}}_{E(\mathcal{M},\tau)} \cdot \|y^*|x|y\|^{\frac{1}{2}}_{E(\mathcal{M},\tau)}. \quad ∎\end{aligned}$$

Remark 2.14. Let K be a bounded subset of $E(\mathcal{M},\tau)$. If we set $|K| := \{|a| : a \in K\}$, then it is clear from Proposition 2.13 that if $|K|$ is E-equi-integrable then for every decreasing projections $e_n\downarrow_n 0$, $\lim_{n\to\infty}\sup_{x\in K} \|xe_n\|_{E(\mathcal{M},\tau)} = \lim_{n\to\infty}\sup_{x\in K} \|e_n x\|_{E(\mathcal{M},\tau)} = 0$. In particular, if $|K|$ is E-equi-integrable then so is K.

Proposition 2.15. *Assume that E is α-convex with constant 1 for some $0 < \alpha \leqslant 1$. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of decreasing projections in $\mathcal{M}$ and K be a bounded subset of $E(\mathcal{M},\tau)$ such that:*

(i) *$p_n \downarrow_n 0$;*

(ii) *For each $n \geqslant 1$, the sets $(\mathbf{1}-p_n)K$ and $|K(\mathbf{1}-p_n)|$ are E-equi-integrable.*

Then K is E-equi-integrable if and only if $\lim_{n\to\infty}\sup_{a\in K} \|p_n a p_n\|_{E(\mathcal{M},\tau)} = 0$.

Proof. We will show the nontrivial implication. Fix a sequence $f_k \downarrow_k 0$ in $\mathcal{M}_p$. We need to show that $\lim_{k\to\infty}\sup_{a\in K} \|f_k a f_k\|_{E(\mathcal{M},\tau)} = 0$. We will assume without loss of generality that K is a subset of the unit ball of $E(\mathcal{M},\tau)$. For every $a \in K$,

$$\begin{aligned} f_k a f_k &= f_k(\mathbf{1} - p_n)a f_k + f_k p_n a f_k\\ &= f_k(\mathbf{1} - p_n)a f_k + f_k p_n a(\mathbf{1} - p_n) f_k + f_k p_n a p_n f_k.\end{aligned}$$

Since $E(\mathcal{M},\tau)$ is α-convex, we get:

$$\begin{aligned}\|f_kaf_k\|^\alpha_{E(\mathcal{M},\tau)} &\leqslant \|f_k(\mathbf{1}-p_n)af_k\|^\alpha_{E(\mathcal{M},\tau)} + \|f_kp_na(\mathbf{1}-p_n)f_k\|^\alpha_{E(\mathcal{M},\tau)} \\ &\quad + \|f_kp_nap_nf_k\|^\alpha_{E(\mathcal{M},\tau)} \\ &\leqslant \|f_k(\mathbf{1}-p_n)af_k\|^\alpha_{E(\mathcal{M},\tau)} + \|a(\mathbf{1}-p_n)f_k\|^\alpha_{E(\mathcal{M},\tau)} + \|p_nap_n\|^\alpha_{E(\mathcal{M},\tau)}.\end{aligned}$$

Using Proposition 2.13 on the second term, we have

$$\begin{aligned}\|f_kaf_k\|^\alpha_{E(\mathcal{M},\tau)} \leqslant \|f_k(\mathbf{1}-p_n)af_k\|^\alpha_{E(\mathcal{M},\tau)} + \|a(\mathbf{1}-p_n)\|^{\frac{\alpha}{2}}_{E(\mathcal{M},\tau)} \\ \cdot \|f_k|a(\mathbf{1}-p_n)|f_k\|^{\frac{\alpha}{2}}_{E(\mathcal{M},\tau)} + \|p_nap_n\|^\alpha_{E(\mathcal{M},\tau)}.\end{aligned}$$

Let $\varepsilon > 0$, choose n_0 large enough so that $\sup\limits_{a\in K} \|p_{n_0}ap_{n_0}\|_{E(\mathcal{M},\tau)} < \varepsilon$. We conclude that

$$\begin{aligned}\lim_{k\to\infty}\sup_{a\in K} \|f_kaf_k\|^\alpha_{E(\mathcal{M},\tau)} \leqslant \lim_{k\to\infty}\sup_{a\in K} \|f_k(\mathbf{1}-p_{n_0})af_k\|^\alpha_{E(\mathcal{M},\tau)} \\ + \lim_{k\to\infty}\sup_{a\in K} \|f_k|a(\mathbf{1}-p_{n_0})|f_k\|^{\frac{\alpha}{2}}_{E(\mathcal{M},\tau)} + \varepsilon^\alpha.\end{aligned}$$

By (ii), the first two terms converge to zero so $\lim\limits_{k\to\infty}\sup\limits_{a\in K} \|f_kaf_k\|_{E(\mathcal{M},\tau)} \leqslant \varepsilon$ and since ε is arbitrary, the proof is complete. ∎

The next proposition can be found in [5], Proposition 2.5.

PROPOSITION 2.16. *Assume that E is α-convex with constant 1 for some $0 < \alpha \leqslant 1$ and satisfies a lower q-estimate with constant 1 for some finite $q \geqslant \alpha$. If $k = 2q/\alpha$, then for all $y \in E(\mathcal{M},\tau)$, for all projections $e, f \in \mathcal{M}$ with $e + f = 1$ and $\tau(e) < \infty$, it follows that*

$$\|eye\|^k_{E(\mathcal{M},\tau)} + \|eyf\|^k_{E(\mathcal{M},\tau)} + \|fye\|^k_{E(\mathcal{M},\tau)} + \|fyf\|^k_{E(\mathcal{M},\tau)} \leqslant \|y\|^k_{E(\mathcal{M},\tau)}.$$

3. KADEC-PEŁCZYŃSKI THEOREM FOR SYMMETRIC SPACES OF OPERATORS

The main result of the present article is the following theorem.

THEOREM 3.1. *Let E be an order continuous symmetric quasi-Banach function space in $\mathbb{R}^+$ that is α-convex with constant 1 for some $0 < \alpha \leqslant 1$ and suppose that E satisfies a lower q-estimate with constant 1 for some $q \geqslant \alpha$.*

Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in $E(\mathcal{M},\tau)$ then there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$, bounded sequences $\{y_k\}_{k=1}^\infty$ and $\{z_k\}_{k=1}^\infty$ in $E(\mathcal{M},\tau)$ and a decreasing sequence of projections $p_k \downarrow_k 0$ in $\mathcal{M}$ such that:

(i) *$x_{n_k} = y_k + z_k$ for all $k \geqslant 1$;*

(ii) *$\{y_k : k \geqslant 1\}$ is E-equi-integrable and $p_ky_kp_k = 0$ for all $k \geqslant 1$;*

(iii) *$\{z_k\}_{k=1}^\infty$ is such that $p_kz_kp_k = z_k$ for all $k \geqslant 1$.*

The proof will be divided into several steps. Without loss of generality, we will assume that the sequence $\{x_k\}_{k=1}^\infty$ is a subset of the unit ball of $E(\mathcal{M},\tau)$. Since we are dealing with sequences, we can and do assume without loss of generality that $\mathcal{M}$ is countably decomposable (see [19] and the proof of Proposition 2.11 above for the details of such reduction).

Set $\mathcal{D}_1 := \{\{e_n\}_{n=1}^{\infty} \subset \mathcal{M}_p : e_n \downarrow_n 0 \text{ and } \tau(e_1) < \infty\}$ and consider

$$\delta := \sup\Big\{ \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \|e_n|x_k|e_n\|_{E(\mathcal{M},\tau)} : \{e_n\}_{n=1}^{\infty} \in \mathcal{D}_1 \Big\}.$$

LEMMA 3.2. *There exists $\{p_n\}_{n=1}^{\infty} \in \mathcal{D}_1$ such that*

$$\delta := \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \|p_n|x_k|p_n\|_{E(\mathcal{M},\tau)}.$$

Proof. For each $m \geqslant 1$, choose a sequence $\{q_n^{(m)}\}_n$ in $\mathcal{D}_1$ such that

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \|q_n^{(m)}|x_k|q_n^{(m)}\|_{E(\mathcal{M},\tau)} \geqslant \delta(1 - 2^{-(m+1)}).$$

Since $q_n^{(m)} \downarrow_n 0$ and $\tau(q_1^{(m)}) < \infty$, $\lim_{n\to\infty} \tau(q_n^{(m)}) = 0$. For each $m \geqslant 1$, choose $n_m \in \mathbb{N}$ so that $\tau(q_{n_m}^{(m)}) \leqslant 2^{-m}$ and $\sup_{k\in\mathbb{N}} \|q_{n_m}^{(m)}|x_k|q_{n_m}^{(m)}\|_{E(\mathcal{M},\tau)} \geqslant \delta(1 - 2^{-m})$. For $j \geqslant 1$, set $p_j = \bigvee_{m\geqslant j} q_{n_m}^{(m)}$. It is clear that $\{p_j\}_j$ is a decreasing sequence of projections and $\tau(p_j) \leqslant \sum_{m=j}^{\infty} \tau(q_{n_m}^{(m)}) = \sum_{m=j}^{\infty} 2^{-m}$ so $\{p_j\}_j \in \mathcal{D}_1$. Moreover,

$$\delta(1 - 2^{-j}) \leqslant \sup_{k\in\mathbb{N}} \|q_{n_j}^{(j)}|x_k|q_{n_j}^{(j)}\|_{E(\mathcal{M},\tau)} \leqslant \sup_{k\in\mathbb{N}} \|p_j|x_k|p_j\|_{E(\mathcal{M},\tau)} \leqslant \delta$$

so $\lim_{j\to\infty} \sup_{k\in\mathbb{N}} \|p_j|x_k|p_j\|_{E(\mathcal{M},\tau)} = \delta$. ∎

LEMMA 3.3. *There exists a subsequence $\{x_n^{(1)}\}_{n=1}^{\infty}$ so that*

$$\lim_{n\to\infty} \|p_n|x_n^{(1)}|p_n\|_{E(\mathcal{M},\tau)} = \delta.$$

Proof. We will construct a sequence of integers $\{k_n\}_n$ inductively satisfying,

$$\|p_n|x_{k_n}|p_n\|_{E(\mathcal{M},\tau)} \geqslant \delta(1 - 2^{-n}).$$

Note first that $\Big\{ \sup_{k\in\mathbb{N}} \|p_n|x_k|p_n\|_{E(\mathcal{M},\tau)} \Big\}_n$ is a decreasing sequence so

$$\sup_{k\in\mathbb{N}} \|p_1|x_k|p_1\|_{E(\mathcal{M},\tau)} \geqslant \delta.$$

Choose $k_1 \geqslant 1$ so that $\|p_1|x_{k_1}|p_1\|_{E(\mathcal{M},\tau)} \geqslant \delta(1 - 2^{-1})$. Assume that the construction is done for $k_1, \ldots, k_m$. Since $\{\varphi_j : j \leqslant k_m\}$ is a finite set,

$$\lim_{n\to\infty} \sup_{k\geqslant k_m} \|p_n|x_k|p_n\|_{E(\mathcal{M},\tau)} = \delta.$$

Therefore, one can choose $k_{m+1} \geqslant k_m$ so that $\|p_{m+1}|x_{k_{m+1}}|p_{m+1}\|_{E(\mathcal{M},\tau)} \geqslant \delta(1 - 2^{-(m+1)})$. The construction is complete. ∎

Apply the argument above on $\{x_n^{(1)^*}\}_n$ to get a further subsequence $\{x_n^{(2)}\}_n$ of $\{x_n^{(1)}\}_n$ and $\{r_n\}_n \in \mathcal{D}_1$ such that

$$\begin{aligned}\delta^* &= \sup\Big\{\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|q_n|x_k^{(2)^*}|q_n\|_{E(\mathcal{M},\tau)} : (q_n)_n \in \mathcal{D}_1\Big\}\\ &= \lim_{n\to\infty}\|r_n|x_n^{(2)^*}|r_n\|_{E(\mathcal{M},\tau)}.\end{aligned}$$

We remark that since both $\tau(p_1)$ and $\tau(r_1)$ are finite numbers, $\{p_n \vee r_n\}_n \in \mathcal{D}_1$. By setting $e_n = p_n \vee r_n$, we can assume that

$$\delta = \lim_{n\to\infty}\|e_n|x_n|e_n\|_{E(\mathcal{M},\tau)} \tag{3.1}$$

and

$$\begin{aligned}\delta^* &= \lim_{n\to\infty}\|e_n|x_n^*|e_n\|_{E(\mathcal{M},\tau)}\\ &= \sup\Big\{\lim_{n\to\infty}\sup_k\|q_n|x_k^*|q_n\|_{E(\mathcal{M},\tau)} : \{q_n\}_{n=1}^\infty \in \mathcal{D}_1\Big\}.\end{aligned} \tag{3.2}$$

For each $n \geqslant 1$, set $v_n := x_n - e_nx_ne_n$ and let $V := \{v_n : n \geqslant 1\}$.

LEMMA 3.4. *There exists a sequence of projections $\{g_n\}_{n=1}^\infty$ in $\mathcal{M}$ with:*
(i) *for every $n \geqslant 1$, $g_n \leqslant \mathbf{1} - e_n$;*
(ii) *$\tau(g_n) < \infty$, in particular g_n is a finite projection;*
(iii) *$g_n \uparrow^n \mathbf{1}$.*

Proof. The lemma can be obtained inductively. Since $\mathcal{M}$ is countably decomposable, there exists φ_0 a faithful normal state in $\mathcal{M}_*$. Since $\mathbf{1} - e_n$ is a semi-finite projection, there exists a sequence of projections $\{g_j^{(n)}\}_{j=1}^\infty$ with $\tau(g_j^{(n)}) < \infty$ for every $j \geqslant 1$ and $g_j^{(n)} \uparrow^j \mathbf{1} - e_n$. One can choose $j_n \geqslant 1$ such that $\varphi_0(1 - e_n) - \varphi_0(g_{j_n}^{(n)}) < 1/n$. Set

$$\begin{cases} g_n := g_{j_1}^{(1)}, & \text{for } n = 1;\\ g_n = g_{j_n}^{(n)} \vee g_{n-1}, & \text{for } n > 1.\end{cases}$$

It is easy to verify that $\{g_n\}_{n=1}^\infty$ satisfies the requirements of the lemma. ∎

For each $n \geqslant 1$, let $p_n = \mathbf{1} - g_n$. Clearly $p_n \downarrow_n 0$, $\mathbf{1} - p_n$ is a finite projection and $p_n \geqslant e_n$ for each $n \geqslant 1$.

LEMMA 3.5. *For each $n \geqslant 1$, the sets $|V(\mathbf{1} - p_n)|$ and $|(\mathbf{1} - p_n)V|$ are E-equi-integrable.*

Proof. We will prove that for every $n \geqslant 1$, $|V(\mathbf{1}-p_n)|$ is an E-equi-integrable set. Assume that there exists $k_0 \geqslant 1$ such that $|V(\mathbf{1} - p_{k_0})|$ is not E-equi-integrable. There exists a decreasing sequence of projections $q_n \downarrow_n 0$ such that $\lim_{n\to\infty}\sup_{a\in|V(\mathbf{1}-p_{k_0})|}\|q_naq_n\|_{E(\mathcal{M},\tau)} > 0$, that is

$$\lim_{n\to\infty}\sup_{m\in\mathbb{N}}\|q_n|v_m(\mathbf{1} - p_{k_0})|q_n\|_{E(\mathcal{M},\tau)} > 0.$$

Choose a strictly increasing sequence $\{m_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ such that

$$\lim_{n\to\infty} \|q_n|v_{m_n}(1-p_{k_0})|q_n\|_{E(\mathcal{M},\tau)} > 0.$$

Let u_{n,k_0} be a bounded operator such that $|v_{m_n}(\mathbf{1}-p_{k_0})| = u_{n,k_0}v_{m_n}(\mathbf{1}-p_{k_0})$. We get that

$$\begin{aligned}\|q_n|v_{m_n}(\mathbf{1}-p_{k_0})|q_n\|_{E(\mathcal{M},\tau)} &= \|q_n u_{n,k_0}v_{m_n}(\mathbf{1}-p_{k_0})q_n\|_{E(\mathcal{M},\tau)}\\ &= \|q_n u_{n,k_0}[x_{m_n} - e_{m_n}x_{m_n}e_{m_n}](\mathbf{1}-p_{k_0})q_n\|_{E(\mathcal{M},\tau)}.\end{aligned}$$

We recall that $e_{k_0} \leqslant p_{k_0}$ and since $\{e_n\}_{n=1}^{\infty}$ is decreasing, for $m_n \geqslant k_0$, $e_{m_n} \leqslant p_{k_0}$ and therefore $e_{m_n}(\mathbf{1}-p_{k_0}) = 0$ and since $\|q_n u_{n,k_0}\|_\infty \leqslant 1$, we obtain that for n large enough,

$$\begin{aligned}\|q_n|v_{m_n}(\mathbf{1}-p_{k_0})|q_n\|_{E(\mathcal{M},\tau)} &= \|q_n u_{n,k_0}(x_{m_n})(\mathbf{1}-p_{k_0})q_n\|_{E(\mathcal{M},\tau)}\\ &\leqslant \|x_{m_n}(\mathbf{1}-p_{k_0})q_n\|_{E(\mathcal{M},\tau)}.\end{aligned}$$

Using Proposition 2.13, with $x = x_{m_n}$ and $y = (\mathbf{1}-p_{k_0})q_n$, we get

$$\begin{aligned}\|q_n|v_{m_n}(\mathbf{1}-p_{k_0})|p_n\|_{E(\mathcal{M},\tau)} &\leqslant \|x_{m_n}\|_{E(\mathcal{M},\tau)}^{\frac{1}{2}} \cdot \|q_n(1-p_{k_0})|x_{m_n}|(\mathbf{1}-p_{k_0})q_n\|_{E(\mathcal{M},\tau)}^{\frac{1}{2}}\\ &\leqslant \|q_n(\mathbf{1}-p_{k_0})|x_{m_n}|(\mathbf{1}-p_{k_0})q_n\|_{E(\mathcal{M},\tau)}^{\frac{1}{2}}.\end{aligned}$$

This implies that

$$\limsup_{n\to\infty} \|q_n(\mathbf{1}-p_{k_0})|x_{m_n}|(\mathbf{1}-p_{k_0})q_n\|_{E(\mathcal{M},\tau)} > 0.$$

Let s_n be the left support projection of $(\mathbf{1}-p_{k_0})q_n$ (this is equal to the right support projection of $q_n(\mathbf{1}-p_{k_0})$). We have

$$\begin{aligned}\|q_n(\mathbf{1}-p_{k_0})|x_{m_n}|(\mathbf{1}-p_{k_0})q_n\|_{E(\mathcal{M},\tau)} &= \|q_n(\mathbf{1}-p_{k_0})s_n|x_{m_n}|s_n(\mathbf{1}-p_{k_0})q_n\|_{E(\mathcal{M},\tau)}\\ &\leqslant \|s_n|x_{m_n}|s_n\|_{E(\mathcal{M},\tau)}.\end{aligned}$$

By the definition of support projection, $s_n \leqslant (\mathbf{1}-p_{k_0})$ for every $n \geqslant 1$, so $\{s_n\}_{n=1}^{\infty}$ is a sequence of finite projections. As in proof of Lemma 2.8, we note that $s_n = q_n \vee p_{k_0} - p_{k_0}$ and as before, $s_n \sim q_n - q_n \wedge p_{k_0}$. Now since $q_n \downarrow_n 0$, $q_n - q_n \wedge p_{k_0} \downarrow_n 0$ hence $\tau(s_n) = \tau(q_n - q_n \wedge p_{k_0})$ converges to zero which implies that $s_n \downarrow_n 0$. Therefore, $\{s_n\}_{n=1}^{\infty} \in \mathcal{D}_1$.

In summary, we get $\{s_n\}_{n=1}^{\infty} \in \mathcal{D}_1$ with $s_n \leqslant 1 - p_{k_0}$ for each $n \geqslant 1$ and for some $\gamma > 0$,

$$\limsup_{n\to\infty} \|s_n|x_{m_n}|s_n\| = \gamma. \tag{3.3}$$

Let $f_n := s_n \vee e_{m_n}$.

For each $m_n \geqslant k_0$, $s_n \leqslant \mathbf{1} - p_{k_0} \leqslant \mathbf{1} - e_{k_0}$ so $s_n \perp e_{m_n}$ hence $f_n = s_n + e_{m_n}$. In particular $\{f_n\}_{n=1}^{\infty} \in \mathcal{D}_1$.

Using Proposition 2.16 (it applies since $\tau(f_n) < \infty$),

$$\begin{aligned}\|f_n|x_{m_n}|f_n\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} &\geqslant \|s_n|x_{m_n}|s_n\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} + \|e_{m_n}|x_{m_n}|s_n\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}}\\ &\quad + \|s_n|x_{m_n}|e_{m_n}\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} + \|e_{m_n}|x_{m_n}|e_{m_n}\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}}\\ &\geqslant \|s_n|x_{m_n}|s_n\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} + \|e_{m_n}|x_{m_n}|e_{m_n}\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}}.\end{aligned}$$

Taking the limit as n tends to ∞, one gets from (3.1) and (3.3) that $\delta^{\frac{2q}{\alpha}} \geqslant \gamma^{\frac{2q}{\alpha}} + \delta^{\frac{2q}{\alpha}}$. This is a contradiction since $\gamma > 0$.

We conclude that for every $n \geqslant 1$, the set $|V(\mathbf{1}-p_n)|$ is an E-equi-integrable set.

For the case of $|(\mathbf{1}-p_n)V|$, it is enough to repeat the argument above for $V^*(\mathbf{1}-p_n)$ using the definition of δ^* (instead of δ). Details are left to the reader. This ends the proof of the lemma. ∎

We will proceed with the proof of Theorem 3.1. Consider two cases.

Case 1: *Assume that V is E-equi-integrable.*

It is enough to set $y_n = x_{m_n} - e_{m_n}x_{m_n}e_{m_n}$ and $z_n = e_{m_n}x_{m_n}e_{m_n}$.

Case 2: *Assume that V is not E-equi-integrable.*

Proposition 2.15 and Lemma 3.5 imply that there exists $\nu > 0$ such that

$$\lim_{n\to\infty} \sup_{v\in V} \|p_n v p_n\|_{E(\mathcal{M},\tau)} = \nu > 0.$$

Choose a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \|p_k v_{n_k} p_k\|_{E(\mathcal{M},\tau)} = \nu > 0. \tag{3.4}$$

For each $k \geqslant 1$, let $w_k := v_{n_k} - p_k v_{n_k} p_k$ and set

$$W := \{w_k :\ k \geqslant 1\}.$$

LEMMA 3.6. *The set W is E-equi-integrable.*

Proof. We note first that if $k \geqslant n$, then $(\mathbf{1}-p_n)w_k = (\mathbf{1}-p_n)v_k$ and $w_k(1-p_n) = v_k(1-p_n)$ so for fixed $n \geqslant 1$, $(\mathbf{1}-p_n)W = \{(\mathbf{1}-p_n)w_k : k < n\} \cup \{(\mathbf{1}-p_n)v_k : k \geqslant n\}$. Similarly, $W(\mathbf{1}-p_n) = \{w_k(\mathbf{1}-p_n) : k < n\} \cup \{v_k(\mathbf{1}-p_n) : k \geqslant n\}$.

Lemma 3.5 implies that for every $n \geqslant 1$, both $|W(\mathbf{1}-p_n)|$ and $(\mathbf{1}-p_n)W$ are E-equi-integrable sets. Therefore, if W is not E-equi-integrable, there would be a subsequence $\{w_{k(j)}\}_{j=1}^{\infty}$ of $\{w_k\}_{k=1}^{\infty}$ and $\varepsilon > 0$ such that

$$\lim_{j\to\infty} \|p_j w_{k(j)} p_j\|_{E(\mathcal{M},\tau)} = \varepsilon. \tag{3.5}$$

Using Proposition 2.13 on $v_{n_{k(j)}}$ and $p_j = (p_j - p_{k(j)}) + p_{k(j)}$, we obtain:

$$\begin{aligned}\|p_j v_{n_{k(j)}} p_j\|_{E(\mathcal{M},\tau)}^{\frac{2p}{\alpha}} \geqslant &\|(p_j - p_{k(j)})v_{n_{k(j)}}(p_j - p_{k(j)})\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} \\ &+ \|p_{k(j)} v_{n_{k(j)}}(p_j - p_{k(j)})\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} \\ &+ \|(p_j - p_{k(j)})v_{n_{k(j)}} p_{k(j)}\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} \\ &+ \|(p_j - p_{k(j)})v_{n_{k(j)}}(p_j - p_{k(j)})\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}}.\end{aligned}$$

Taking into account the identities, $(p_j - p_{k(j)})v_{n_{k(j)}}(p_j - p_{k(j)}) = (p_j - p_{k(j)})w_{k(j)}(p_j - p_{k(j)})$, $(p_j - p_{k(j)})v_{n_{k(j)}}p_{k(j)} = p_j w_{k(j)} p_{k(j)}$ and $p_{k(j)}v_{n_{k(j)}}(p_j - p_{k(j)}) = p_{k(j)}w_{k(j)}p_j$, one can deduce that,

$$\begin{aligned}\|p_j v_{n_{k(j)}} p_j\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} \geqslant &\|(p_j - p_{k(j)})w_{k(j)}(p_j - p_{k(j)})\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} + \|p_j w_{k(j)} p_{k(j)}\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} \\ &+ \|p_{k(j)} w_{k(j)} p_j\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} + \|p_{k(j)} v_{n_{k(j)}} p_{k(j)}\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}}.\end{aligned}$$

Let $C(q,\alpha)$ be the norm of the identity map from $\ell_3^{\frac{2q}{\alpha}}$ onto ℓ_3^{α}, where $\ell_3^{\frac{2q}{\alpha}}$ (respectively ℓ_3^{α}) denotes the 3-dimensional $\ell^{\frac{2q}{\alpha}}$-space (respectively ℓ^{α}-space). We have

$$\begin{aligned}\|p_j v_{n_{k(j)}} p_j\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} \geqslant C(q,\alpha)^{\frac{2q}{\alpha}} \Big[\Big(&\|(p_j - p_{k(j)}) w_{k(j)} (p_j - p_{k(j)})\|_{E(\mathcal{M},\tau)}^{\alpha} \\ &+ \|p_j w_{k(j)} p_{k(j)}\|_{E(\mathcal{M},\tau)}^{\alpha} + \|p_{k(j)} w_{k(j)} p_j\|_{E(\mathcal{M},\tau)}^{\alpha} \Big)^{\frac{1}{\alpha}} \Big]^{\frac{2q}{\alpha}} \\ &+ \|p_{k(j)} v_{n_{k(j)}} p_{k(j)}\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}}.\end{aligned}$$

We remark that $p_j w_{k(j)} p_j = (p_j - p_{k(j)}) w_{k(j)} (p_j - p_{k(j)}) + p_j w_{k(j)} p_{k(j)} + p_{k(j)} w_{k(j)} p_j$ and since $E(\mathcal{M},\tau)$ is α-convex (with constant 1), the above inequality implies

$$\|p_j v_{n_{k(j)}} p_j\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} \geqslant C(q,\alpha)^{\frac{2q}{\alpha}} \|p_j w_{k(j)} p_j\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}} + \|p_{k(j)} v_{n_{k(j)}} p_{k(j)}\|_{E(\mathcal{M},\tau)}^{\frac{2q}{\alpha}},$$

and taking the limit as $j \to \infty$, we get from (3.4) and (3.5) that

$$\nu^{\frac{2q}{\alpha}} \geqslant C(q,\alpha)^{\frac{2q}{\alpha}} \varepsilon^{\frac{2q}{\alpha}} + \nu^{\frac{2q}{\alpha}}.$$

This is a contradiction since $\varepsilon > 0$, so W is a E-equi-integrable set. The lemma is proved. ∎

To end the proof of Theorem 3.1, we note that $W = \{x_{n_k} - p_k x_{n_k} p_k : \ k \geqslant 1\}$; so, if we set $y_k = x_{n_k} - p_k x_{n_k} p_k$ and $z_k = p_k x_{n_k} p_k$, the proof of is complete.

REMARK 3.7. (1) If $\mathcal{M} = \mathcal{B}(\ell^2)$ with the usual trace, then every projection of finite trace is a finite rank projection so in the proof above, $\delta = \delta^* = 0$. In the particular case of unitary matrix space C_E where E is a symmetric sequence space, one proceed directly to Case 2 by setting $W := \{x_n - p_n x_n p_n : n \geqslant 1\}$ where $\{p_n\}_{n=1}^{\infty}$ is an arbitrary sequence of projections satisfying: $p_n \downarrow_n 0$ and for every $n \geqslant 1$, $\mathbf{1} - p_n$ is a finite projection.

(2) If $\mathcal{M}$ is a finite von Neumann algebra with a normalized finite trace τ and E is a symmetric space on $[0,1]$ satisfying the assumptions of Theorem 3.1, it is enough to take $p_n = e_n$ (i.e $g_n = \mathbf{1} - e_n$ on Lemma 3.4) and conclude immediately as in Lemma 3.5 that V is E-equi-integrable.

(3) In the proof above, it is clear that the projections $\{p_k\}_{k=1}^{\infty}$ are such that either $\tau(p_1) < \infty$ or $\tau(\mathbf{1} - p_k) < \infty$ for all $k \geqslant 1$. In fact, the argument above shows that if $\{e_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{D}_1$ that attained the quantities δ and δ^*, then any sequence of projections satisfying $p_n \downarrow_n 0$, $e_n \leqslant p_n$ for each $n \geqslant 1$ and $\tau(\mathbf{1} - p_n) < \infty$ for each $n \geqslant 1$, would satisfy the conclusion of Theorem 3.1.

The following extension shows that if one considers finitely many bounded sequences in $E(\mathcal{M},\tau)$, one can choose a single sequence of projections that works for each sequence.

COROLLARY 3.8. *If $\mathcal{M}$ and E are as in Theorem 3.1 and $\{x_n^{(1)}\}_{n=1}^\infty$, $\{x_n^{(2)}\}_{n=1}^\infty, \ldots, \{x_n^{(j_0)}\}_{n=1}^\infty$ be finitely many bounded sequences in $E(\mathcal{M},\tau)$. Then there exist a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ of $\mathbb{N}$ and a sequence of decreasing projections $p_k \downarrow_k 0$ in $\mathcal{M}$ such that for each $1 \leqslant j \leqslant j_0$, the set $\{x_{n_k}^{(j)} - p_k x_{n_k}^{(j)} p_k : k \geqslant 1\}$ is E-equi-integrable.*

Proof. For $1 \leqslant j \leqslant j_0$, we set, as in the proof of Theorem 3.1,

$$\delta_j := \sup\Big\{ \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \|e_n |x_k^{(j)}| e_n\|_{E(\mathcal{M},\tau)} : \{e_n\}_{n=1}^\infty \in \mathcal{D}_1 \Big\}.$$

One can choose a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ in $\mathbb{N}$ such that for each $1 \leqslant j \leqslant j_0$, there exists a sequence $\{\mathrm{e}_k^{(j)}\}_{k=1}^\infty \in \mathcal{D}_1$ with

$$\delta_j = \lim_{k\to\infty} \|\mathrm{e}_k^{(j)} |x_{n_k}^{(j)}| \mathrm{e}_k^{(j)}\|_{E(\mathcal{M},\tau)}$$

and

$$\begin{aligned}\delta_j^* &= \lim_{k\to\infty} \|\mathrm{e}_k^{(j)} |x_{n_k}^{(j)^*}| \mathrm{e}_k^{(j)}\|_{E(\mathcal{M},\tau)} \\ &= \sup\Big\{ \lim_{n\to\infty} \sup_{k} \|q_n |x_{n_k}^{(j)^*}| q_n\|_{E(\mathcal{M},\tau)} : \{q_n\}_{n=1}^\infty \in \mathcal{D}_1 \Big\}.\end{aligned}$$

For every $k \geqslant 1$, set $e_k := \bigvee_{1\leqslant j\leqslant j_0} \mathrm{e}_k^{(j)}$. Since $\tau(e_k) \leqslant \sum_{j=1}^{j_0} \tau(\mathrm{e}_k^{(j)})$, it is clear that the sequence $\{e_k\}_{k=1}^\infty$ belongs to $\mathcal{D}_1$ and each of the δ_j's and δ_j^*'s are attained at $\{e_k\}_{k=1}^\infty$. One can complete the proof by proceeding as in the proof of Theorem 3.1, simultaneously on the finite set of sequences and the fixed $\{e_k\}_{k=1}^\infty$. ∎

Our next result shows that the decreasing projections in the decomposition can be replaced by mutually disjoint projections.

THEOREM 3.9. *Let E be an order continuous quasi-Banach function space as in Theorem 3.1. Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in $E(\mathcal{M},\tau)$ then there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$, bounded sequences $\{\varphi_k\}_{k=1}^\infty$ and $\{\zeta_k\}_{k=1}^\infty$ in $E(\mathcal{M},\tau)$ and mutually disjoint sequence of projections $\{e_k\}_{k=1}^\infty$ such that:*

(i) *$x_{n_k} = \varphi_k + \zeta_k$ for all $k \geqslant 1$;*

(ii) *$\{\varphi_k : k \geqslant 1\}$ is E-equi-integrable and $e_k \varphi_k e_k = 0$ for all $k \geqslant 1$;*

(iii) *$\{\zeta_k\}_{k=1}^\infty$ is such that $e_k \zeta_k e_k = \zeta_k$ for all $k \geqslant 1$.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in $E(\mathcal{M},\tau)$ and suppose (by taking a subsequence if necessary), $x_n = y_n + z_n$ with $p_n y_n p_n = 0$, the set $\{y_n : n \geqslant 1\}$ is E-equi-integrable and $p_n z_n p_n = z_n$ for all $n \geqslant 1$, be the decomposition of $\{x_n\}_{n=1}^\infty$ as in Theorem 3.1.

Let $n_1 = 1$. Since $p_n \downarrow_n 0$ and

$$p_1 z_1 p_1 - (p_1 - p_n) z_1 (p_1 - p_n) = p_n z_1 p_1 + p_1 z_1 p_n - p_n z_1 p_n,$$

Proposition 2.4 (ii) shows that

$$\lim_{n\to\infty} \|p_1 z_1 p_1 - (p_1 - p_n) z_1 (p_1 - p_n)\|_{E(\mathcal{M},\tau)} = 0.$$

Choose $n_2 > n_1 = 1$ such that

$$\|p_1 z_1 p_1 - (p_1 - p_{n_2}) z_1 (p_1 - p_{n_2})\|_{E(\mathcal{M},\tau)} < \frac{1}{2}.$$

Inductively, one can construct $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$\|p_{n_k} z_{n_k} p_{n_k} - (p_{n_k} - p_{n_{k+1}}) z_{n_k} (p_{n_k} - p_{n_{k+1}})\|_{E(\mathcal{M},\tau)} < \frac{1}{2^k}.$$

Since $z_n = p_n z_n p_n$ for every $n \geqslant 1$, one gets

$$\|z_{n_k} - (p_{n_k} - p_{n_{k+1}}) z_{n_k} (p_{n_k} - p_{n_{k+1}})\|_{E(\mathcal{M},\tau)} < \frac{1}{2^k}.$$

For every $k \geqslant 1$, set

$$\begin{aligned} e_k &:= p_{n_k} - p_{n_{k+1}} \\ \zeta_k &:= (p_{n_k} - p_{n_{k+1}}) z_{n_k} (p_{n_k} - p_{n_{k+1}}) \\ \varphi_k &:= y_{n_k} + [z_{n_k} - e_k z_{n_k} e_k]. \end{aligned}$$

Since $\{y_{n_k} : k \geqslant 1\}$ is a E-equi-integrable set and $\lim_{k\to\infty} \|z_{n_k} - e_k z_{n_k} e_k\|_{E(\mathcal{M},\tau)} = 0$, it is clear that $\{\varphi_k : k \geqslant 1\}$ is E-equi-integrable. Also $\{e_k\}_{k=1}^\infty$ is mutually disjoint. The proof is complete. ∎

COROLLARY 3.10. *Let E be an order-continuous symmetric Banach function space on $\mathbb{R}^+$ with the Fatou property. Let $\{x_n\}_{n=1}$ be a bounded sequence in $E(\mathcal{M},\tau)$ then there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$, bounded sequences $\{\varphi_k\}_{k=1}^\infty$ and $\{\zeta_k\}_{k=1}^\infty$ in $E(\mathcal{M},\tau)$ and mutually disjoint sequence of projections $\{e_k\}_{k=1}^\infty$ such that:*

(i) $x_{n_k} = \varphi_k + \zeta_k$ *for all* $k \geqslant 1$;

(ii) $\{\varphi_k : k \geqslant 1\}$ *is E-equi-integrable and* $e_k \varphi_k e_k = 0$ *for all* $k \geqslant 1$;

(iii) $\{\zeta_k\}_{k=1}^\infty$ *is such that* $e_k \zeta_k e_k = \zeta_k$ *for all* $k \geqslant 1$.

Proof. Assume that E has the Fatou property (equivalently E does not contain c_0). Since E is symmetric, $E \not\supset c_0$ is equivalent to E not containing ℓ_∞^n uniformly, and therefore E satisfies the q-lower estimate for some q and one can equip E with an equivalent norm so that it satisfies the lower q-estimate of constant 1. All hese facts can be found in [17]. ∎

The proof of Theorem 3.1 can be adjusted to obtain decompositions where the projections are taken only on one side, that is, the following result follows:

COROLLARY 3.11. *Let E be an order continuous quasi-Banach function space in $\mathbb{R}^+$ that is α-convex with constant 1 for some $0 < \alpha \leqslant 1$ and suppose that E satisfies a lower q-estimate with constant 1 for some $q \geqslant \alpha$. Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in $E(\mathcal{M},\tau)$ then there exist a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$, bounded sequences $\{y_k\}_{k=1}^\infty$ and $\{z_k\}_{k=1}^\infty$ in $E(\mathcal{M},\tau)$ and decreasing projections $e_k \downarrow_n 0$ in $\mathcal{M}$ such that:*

(i) $x_{n_k} = y_k + z_k$ *for all* $k \geqslant 1$;

(ii) $e_k y_k = 0$ *for all* $k \geqslant 1$ *and* $\lim_{n\to\infty} \sup_{k\geqslant 1} \|f_n y_k\|_{E(\mathcal{M},\tau)} = 0$ *for every* $f_n \downarrow_n 0$.

(iii) $\{z_k\}_{k=1}^\infty$ *is such that* $e_k z_k = z_k$ *for all* $k \geqslant 1$.

DEFINITION 3.12. A subspace X of $L^p(\mathcal{M},\tau)$ is called *strongly embedded* into $L^p(\mathcal{M},\tau)$ if the L^p and the measure topologies on X coincide.

The following result is a direct application of Proposition 2.11 and Theorem 3.9.

THEOREM 3.13. *Let $1 \leqslant p < \infty$. Every subspace of $L^p(\mathcal{M},\tau)$ either contains almost isometric copies of ℓ^p or is strongly embedded in $L^p(\mathcal{M},\tau)$.*

The next corollary should be compared with [16], Theorem 2.4.

COROLLARY 3.14. *Assume that $\mathcal{M}$ is finite and $p > 2$. Every subspace of $L^p(\mathcal{M},\tau)$ either contains almost isometric copies of ℓ^p or is isomorphic to a Hilbert space.*

For the commutative case, the space ℓ^p can not be strongly embedded in $L^p[0,1]$ for $0 < p < 2$. This is due to Kalton ([11]) for $0 < p < 1$ and Rosenthal ([14]) for the case $1 \leqslant p < 2$. A non-commutative analogue should be of interest.

PROBLEM. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra and $0 < p < 2$. Does ℓ^p strongly embed into $L^p(\mathcal{M},\tau)$?*

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Note added in proof. Since this paper was submitted, there have been some new developments: (1) Theorem 3.1 is also valid for Haagerup L^p-spaces when $1 \leqslant p < \infty$; (2) Corollary 3.14 was extended to the Haagerup L^p-spaces by Raynaud and Xu; (3) the problem stated above was solved positively by Haagerup, Rosenthal and Sukochev for the finite case and $1 \leqslant p < 2$ and the author for the general semi-finite case and $0 < p < \infty$.

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