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SOLUTION OF THE SINGULAR QUARTIC MOMENT PROBLEM

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Abstract. In this note we obtain a complete solution to the quartic problem in the case when the associated moment matrix $M(2)(\gamma)$ is singular. Each representing measure μ satisfies $\operatorname{card}\operatorname{supp}\mu \geqslant \operatorname{rank} M(2)$, and we develop concrete necessary and sufficient conditions for the existence and uniqueness of representing measures, particularly *minimal* ones. We show that $\operatorname{rank} M(2)$-atomic minimal representing measures exist in case the moment problem is subordinate to an ellipse or non-degenerate hyperbola. If the quartic moment problem is subordinate to a pair of intersecting lines, those problems subordinate to a general intersection of two conics may not have any representing measure at all. As an application, we describe the minimal quadrature rules of degree 4 for arclength measure on a parabolic arc.

Keywords: *Quartic moment problem, moment matrix extension, flat extensions, degree-one transformations, quadrature rules.*

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1. INTRODUCTION

Given complex numbers $\gamma \equiv \gamma^{(4)}$: $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{03}, \gamma_{12}, \gamma_{21}, \gamma_{30}, \gamma_{04}, \gamma_{13}, \gamma_{22}, \gamma_{31}, \gamma_{40}$, with $\gamma_{ij} = \overline{\gamma}_{ji}$, the *quartic complex moment problem* for γ entails finding conditions for the existence of a positive Borel measure μ, supported in the complex plane $\mathbb{C}$, such that

$$\gamma_{ij} = \int \overline{z}^i z^j \,\mathrm{d}\mu, \quad 0 \leqslant i + j \leqslant 4.$$

In the sequel we study the case when the *moment matrix* associated to γ, $M(2) \equiv M(2)(\gamma)$, is *singular,* where

$$M(2) = \begin{array}{c} \begin{array}{cccccc} \mathbb{1} & Z & \overline{Z} & Z^2 & Z\overline{Z} & \overline{Z}^2 \end{array} \\ \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix} \end{array}.$$

In this context, we use positivity and extension properties of $M(2)$ to develop concrete necessary and sufficient conditions for the existence and uniqueness of representing measures μ, particularly *minimal* representing measures, i.e., finitely atomic representing measures with the fewest atoms possible. The singular quartic moment problem arises quite naturally in any degree-4 quadrature problem for a measure whose support is contained in the variety of a complex polynomial $p(z, \bar{z})$ with $\deg p \leqslant 2$. Further, to find a minimal representing measure in the *quadratic* moment problem (corresponding to $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$), one necessarily solves an associated singular quartic moment problem (cf. Proposition 6.4 from [6]).

The quartic moment problem is a special case (with $n = 2$) of the following Truncated Complex Moment Problem for a prescribed moment sequence $\gamma \equiv \gamma^{(2n)}$: $\gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$:

$$\text{TCMP} \qquad \gamma_{ij} = \int \bar{z}^i z^j \, \mathrm{d}\mu, \quad 0 \leqslant i + j \leqslant 2n, \; \mu \geqslant 0, \; \operatorname{supp} \mu \subseteq \mathbb{C}.$$

TCMP is closely related to the *Full* Moment Problem ([1], [2], [3], [16], and [22]), which has attracted renewed attention in the last few years ([17], [18], [19], [20], [21], [25], [26], and [27]). Indeed, J. Stochel ([24]) has shown that TCMP is more general than the Full Moment Problem in the following sense: a full moment sequence $\gamma \equiv (\gamma_{ij})_{i,j \geqslant 0}$ admits a representing measure if and only if each truncation $\gamma^{(2n)}$ admits a representing measure.

In [6] we initiated a study of TCMP based on positivity and extension properties of the associated moment matrix $M(n) \equiv M(n)(\gamma)$. If $\gamma^{(2n)}$ admits a representing measure μ, then $M(n)$ is positive semidefinite ($M(n) \geqslant 0$), *recursively generated* (see below for terminology), and $\operatorname{card} \operatorname{supp} \mu \geqslant \operatorname{rank} M(n)$ (Corollary 3.7 from [6]). Conversely, $M(n)$ admits a $\operatorname{rank} M(n)$-atomic (minimal) representing measure if and only if $M(n) \geqslant 0$ admits a *flat extension*, i.e., an extension to a moment matrix $M(n+1)$ satisfying $\operatorname{rank} M(n+1) = \operatorname{rank} M(n)$. Let us denote the successive columns of $M(n)$ lexicographically, by $\mathbb{1}, Z, \overline{Z}, \ldots, Z^n, \overline{Z} Z^{n-1}, \ldots, \overline{Z}^n$. Results of [7] imply that for $n \geqslant 2$, if $M(n) \geqslant 0$ is recursively generated and $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is dependent in $\mathcal{C}_{M(n)}$ (the column space of $M(n)$), then $M(n)$ admits a flat extension $M(n+1)$ (and a corresponding $\operatorname{rank} M(n)$-atomic (minimal) representing measure). Other concrete sufficient conditions for flat extensions $M(n+1)$ are described below (cf. [8]), but a complete solution to the Flat Extension Problem remains unknown. For the general case, $\gamma^{(2n)}$ admits a *finitely* *atomic* representing measure if and only if, for some $k \geqslant 0$, $M(n)$ admits an extension $M(n+k) \geqslant 0$ which in turn admits a flat extension $M(n+k+1)$ ([8]).

In Theorem 6.1 from [6], for $n = 1$, we proved that if $M(1) \geqslant 0$, then $\gamma^{(2)}$ admits a rank $M(1)$-atomic (minimal) representing measure. By contrast, for $n = 3$, in Section 4 from [7] we exhibited $\gamma^{(6)}$ for which $M(3) \geqslant 0$, but $\gamma^{(6)}$ admits no representing measure (cf. [12]). For the intermediate case $n = 2$, our study of the singular quartic moment problem commenced in [8], where we established flat extensions for $M(2) \geqslant 0$ in certain cases where $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is independent and $\overline{Z}Z \in \langle \mathbb{1}, Z, \overline{Z} \rangle$. (For an elementary overview of TCMP, the reader is referred to [4].)

The aim of this note is to complete our analysis of the singular quartic moment problem, and we next outline our main results. In view of [7], we may assume that $M(2)$ is positive and that $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(2)}$. The remaining cases may then be organized as follows:

Case I. $M(2) \geqslant 0$, the set $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(2)}$, and $\overline{Z}Z \in \langle \mathbb{1}, Z, \overline{Z}, Z^2 \rangle$.

Case II. $M(2) \geqslant 0$, the set $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z\}$ is independent in $\mathcal{C}_{M(2)}$, and $\overline{Z}^2 \in \langle \mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z \rangle$.

In Section 2 we prove the following general result concerning the truncated moment problem in which $M(n) \geqslant 0$, $\{\mathbb{1}, Z, \overline{Z}\}$ is independent in $\mathcal{C}_{M(n)}$, and $\overline{Z}Z \in \langle \mathbb{1}, Z, \overline{Z} \rangle$.

THEOREM *Let $n > 1$. If $M(n) \geqslant 0$, $\{\mathbb{1}, Z, \overline{Z}\}$ is independent in $\mathcal{C}_{M(n)}$, and $\overline{Z}Z \in \langle \mathbb{1}, Z, \overline{Z} \rangle$, then $M(n)$ admits a flat extension (and $\gamma^{(2n)}$ admits a corresponding* rank $M(n)$*-atomic representing measure). Moreover,* rank $M(n) \leqslant 2n + 1$, *and if* rank $M(n) \leqslant 2n$, *then $\gamma^{(2n)}$ admits a unique representing measure. If* rank $M(n) = 2n + 1$, *then $M(n)$ admits infinitely many flat extensions, each corresponding to a distinct $(2n+1)$-atomic representing measure.*

Theorem 1.1 has the following implication for Case I of the quartic moment problem.

THEOREM *Suppose $M(2) \geqslant 0$, $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(2)}$ and $\overline{Z}Z \in \langle \mathbb{1}, Z, \overline{Z} \rangle$. Then $\overline{Z}Z = A\mathbb{1} + BZ + \overline{B}\,\overline{Z}$ in $\mathcal{C}_{M(2)}$ with $A + |B|^2 > 0$, and $\gamma^{(4)}$ admits a* rank $M(2)$*-atomic (minimal) representing measure. Moreover, each representing measure is supported in the circle $C_\gamma = \{z \in \mathbb{C} : |z - \overline{B}|^2 = A + |B|^2\}$. If $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$, then there exists a unique representing measure, which is* 4*-atomic. Otherwise, $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}^2\}$ is a basis for $\mathcal{C}_{M(2)}$, and there exist infinitely many flat extensions, each corresponding to a distinct* 5*-atomic (minimal) representing measure.*

In Section 3 we complete Case I with the following computational test.

Theorem *Suppose $M(2) \geqslant 0$, $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(2)}$, and $\overline{Z}Z = A\mathbb{1} + BZ + C\overline{Z} + DZ^2$, $D \neq 0$. The following are equivalent:*

(i) *$\gamma^{(4)}$ admits a finitely atomic representing measure;*
(ii) *$\gamma^{(4)}$ admits a 4-atomic (minimal) representing measure;*
(iii) *$M(2)$ admits a flat extension $M(3)$;*
(iv) *$M(2)$ admits a recursively generated extension $M(3) \geqslant 0$;*
(v) *there exists $\gamma_{23} \in \mathbb{C}$ such that*

$$\overline{\gamma}_{23} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23}.$$

For the case $|D| \neq 1$, we prove in Corollary 3.4 that there always exists a 4-atomic (minimal) representing measure. By contrast, for $|D| = 1$ we illustrate cases in which there exist representing measures (Example 3.6) and also a case in which $\gamma^{(4)}$ admits no representing measure (Example 3.8). The latter case is the "smallest possible" example of a positive moment matrix which admits no representing measure.

In Section 4, for Case II of $M(2)$, we study the structure of a recursively generated moment matrix extension

$$M(3) \equiv \begin{pmatrix} M(2) & B(3) \\ B(3)^* & C(3) \end{pmatrix}.$$

We show that to each $\gamma_{23} \in \mathbb{C}$ there corresponds a unique moment matrix block $B(3) \equiv B(3)[\gamma_{23}]$ satisfying $\operatorname{Ran} B(3) \subseteq \operatorname{Ran} M(2)$; thus there exists a matrix W such that $M(2)W = B(3)$. Let $C = W^*M(2)W$ $(\equiv (C_{ij})_{1 \leqslant i,j \leqslant 4})$. $M(2)$ admits a flat extension $M(3)$ if and only if C assumes the form of a moment matrix block $C(3)$, i.e., C is Toeplitz.

Theorem *Suppose $M(2) \geqslant 0$, $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z\}$ is independent in $\mathcal{C}_{M(2)}$, and $\overline{Z}^2 \in \langle \mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z\rangle$. $M(2)$ admits a flat extension $M(3)$ (and $\gamma^{(4)}$ admits a 5-atomic (minimal) representing measure) if and only if there exists $\gamma_{23} \in \mathbb{C}$ such that $C_{21} = C_{32}$.*

In Corollary 4.11 we use computer algebra to establish the existence of a flat extension in the case of Theorem 1.4 where $\overline{Z}^2 = A\mathbb{1} + BZ + C\overline{Z} + Z^2$ and the moment data are *real*. In Example 4.12 we use Theorem 1.4 to give a complete description of the minimal quadrature rules of degree 4 for arclength measure on the segment of the parabola $y = x^2$ determined by $0 \leqslant x \leqslant 1$. We note that $\Delta := C_{21} - C_{32}$ can be expressed as a *quadratic* polynomial in γ_{23} and $\overline{\gamma}_{23}$. In Example 4.3 we show that even in an apparently simple situation, when $\overline{Z}^2 = Z^2$, it is possible to have Δ nonzero for every $\gamma_{23} \in \mathbb{C}$.

We begin Section 5 by establishing that whenever $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$, then the associated $\mathcal{V}(\gamma)$ is the zero set of a real quadratic equation in $x := \operatorname{Re}[z]$ and $y := \operatorname{Im}[z]$. We then proceed to reduce Case II to subcases corresponding to the following four real conics: (i) $y = x^2$; (ii) $yx = 1$; (iii) $yx = 0$; and (iv) $x^2 + y^2 = 1$. This is done with the aid of Proposition 1.7 (invariance under degree-one transformations) and Proposition 1.12, which establishes the equivalence between TCMP and a naturally associated truncated moment problem in $\mathbb{R}^2$. Since Sub-Case (iv) is subsumed by Section 2, we devote the rest of Section 5 to establishing the existence of representing measures for Sub-Cases (i)–(iii). Our main result for Case II follows.

Theorem *Let $\gamma^{(4)}$ be given, and assume $M(2) \geqslant 0$ and $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$. Then $\gamma^{(4)}$ admits a representing measure μ. Moreover, it is possible to find μ with* $\operatorname{card}\operatorname{supp}\mu = \operatorname{rank} M(2)$, *except in some cases when $\mathcal{V}(\gamma^{(4)})$ is a pair of intersecting lines, in which cases there exist μ with* $\operatorname{card}\operatorname{supp}\mu \leqslant 6$.

When we began to study the singular quartic moment problem in [8], we believed that a positive, singular, recursively generated moment matrix $M(2)$ always corresponded to a flat extension $M(3)$ and a rank $M(2)$-atomic representing measure. The results of the present paper show, perhaps surprisingly, that the singular quartic moment problem actually displays the full range of pathology associated with multidimensional truncated moment problems. Flat extensions do exist for quartic moment problems subordinate to a circle (Section 2), or to an ellipse or non-degenerate hyperbola (Sections 4 and 5). For problems subordinate to a pair of intersecting lines, a minimal representing measure does not always correspond to a flat extension (Section 5). For the moment problems considered in Section 3, where the variety is typically the intersection of two conics, it may happen that there is no representing measure at all. For instance, Example 3.8 exhibits a quartic moment problem whose associated variety is the intersection of a parabola in a pair of intersecting lines, so the cardinality of the variety is less than the rank of the moment matrix, which results in no representing measure. (In [13], the second-named author extends the results of Section 3, together with those in Section 5 corresponding to the above mentioned "parabola case" (i), to arbitrary n; see Note added in proof). After completing the results of this paper, we received a manuscript by I.B. Jung, S.H. Lee, W.Y. Lee and C. Li ([15]), in which the authors independently establish the existence of a flat extension in the case of the singular quartic moment problem when $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is independent, and $\overline{Z}Z \in \langle \mathbb{1}, Z, \overline{Z}\rangle$ (cf. Theorem 2.1); unlike our proof, the proof in [15] is by computer algebra. [15] also contains an example of the case when $M(2) \geqslant 0$ fails to have a representing measure (cf. Examples 3.8 and 3.9 below), and it contains some positive results for the *nonsingular* quartic moment problem.

The remainder of this section is devoted to notation and technical results concerning moment matrices. Let $\mathcal{P}_n$ denote the complex polynomials $q(z,\overline{z}) = \sum a_{ij}\overline{z}^i z^j$ of total degree at most n, and for $q \in \mathcal{P}_n$, let $\widehat{q} = (a_{ij})$ denote the coefficient vector of q with respect to the basis $\{\overline{z}^i z^j\}_{0\leqslant i+j\leqslant n}$ of $\mathcal{P}_n$ (ordered lexicographically: $1, z, \overline{z}, z^2, z\overline{z}, \overline{z}^2, \ldots, z^n, \ldots, \overline{z}^n$). For $p \in \mathcal{P}_{2n}$, $p(z,\overline{z}) \equiv \sum b_{ij}\overline{z}^i z^j$, let $L_\gamma(p) := \sum b_{ij}\gamma_{ij}$; L_γ is the *Riesz functional* associated to γ. The *moment matrix* $M(n) \equiv M(n)(\gamma)$ is the unique matrix (of size $\frac{(n+1)(n+2)}{2}$) such that

$$\langle M(n)\widehat{f}, \widehat{g}\rangle = L_\gamma(f\overline{g}), \quad f, g \in \mathcal{P}_n. \tag{1.1}$$

If we label the rows and columns of $M(n)$ lexicographically as $\mathbb{1}$, Z, $\overline{Z}$, Z^2, $\overline{Z}Z$, $\overline{Z}^2$, …, Z^n, …, $\overline{Z}^n$, it follows that the row $\overline{Z}^k Z^l$, column $\overline{Z}^i Z^j$ entry of $M(n)$ is equal to

$$\langle M(n)\widehat{\overline{z}^i z^j}, \widehat{\overline{z}^k z^l}\rangle = L_\gamma(\overline{z}^{i+l} z^{j+k}) = \gamma_{i+l,j+k}.$$

For example, with $n = 1$, the Quadratic Moment Problem for $\gamma^{(2)}$: γ_{00}, γ_{01}, γ_{10}, γ_{02}, γ_{11}, γ_{20} corresponds to

$$M(1) = \begin{array}{c} \\ \mathbb{1} \\ Z \\ \overline{Z} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbb{1} & Z & \overline{Z} \end{array} \\ \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix} \end{array}.$$

If γ admits a representing measure μ, then for $f \in \mathcal{P}_n$, $\langle M(n)\widehat{f}, \widehat{f}\,\rangle = L_\gamma(|f|^2) = \int |f|^2 \,\mathrm{d}\mu \geqslant 0$, whence $M(n) \geqslant 0$.

Now let $p \in \mathcal{P}_{2n}$, $p \not\equiv 0$, and define k by $\deg p = 2k$ or $\deg p = 2k-1$. There exists a unique *localizing matrix* $M_p(n) \equiv M_p(n)(\gamma)$ (of size $\frac{(n-k+1)(n-k+2)}{2}$) such that

$$\langle M_p(n)\widehat{f}, \widehat{g}\rangle = L_\gamma(pf\overline{g}), \quad f, g \in \mathcal{P}_{n-k}.$$

Thus, if a representing measure μ for γ is supported in $K_p := \{z \in \mathbb{C} : p(z, \overline{z}) \geqslant 0\}$, then for $f \in \mathcal{P}_{n-k}$,

$$\langle M_p(n)\widehat{f}, \widehat{f}\rangle = L_\gamma(p|f|^2) = \int p|f|^2 \,\mathrm{d}\mu \geqslant 0,$$

whence $M_p(n) \geqslant 0$.

For a matrix M, $[M]_k$ denotes the compression of M to the first k rows and columns and $\langle \overline{Z}^i Z^j, \overline{Z}^k Z^l\rangle_M$ denotes the entry in row $\overline{Z}^k Z^l$ and column $\overline{Z}^i Z^j$. Similarly, for a vector $\mathbf{v}$, $[\mathbf{v}]_k$ denotes the compression of $\mathbf{v}$ to the first k entries. In the sequel, unless otherwise stated, we always assume that $\gamma^{(2n)}$ satisfies $\gamma_{00} = 1$; this amounts to rescaling the total mass, and has no effect as to existence, uniqueness or location of representing measures.

We next recall from [6] some additional necessary conditions for the existence of representing measures. Let $\mathcal{C}_{M(n)}$ denote the column space of $M(n)$, i.e., $\mathcal{C}_{M(n)} = \langle \mathbb{1}, Z, \overline{Z}, \ldots, Z^n, \ldots, \overline{Z}^n\rangle \subseteq \mathbb{C}^{m(n)}$. For $p \in \mathcal{P}_n$, $p \equiv \sum a_{ij}\overline{z}^i z^j$, we define $p(Z, \overline{Z}) \in \mathcal{C}_{M(n)}$ by $p(Z, \overline{Z}) := \sum a_{ij}\overline{Z}^i Z^j$; note that if $p(Z, \overline{Z}) = 0$, then $\overline{p}(Z, \overline{Z}) = 0$ ([6]). If μ is a representing measure for γ, then

(1.2) For $p \in \mathcal{P}_n$, $\quad p(Z, \overline{Z}) = 0 \Leftrightarrow \operatorname{supp}\mu \subseteq \mathcal{Z}(p) := \{z \in \mathbb{C} : p(z, \overline{z}) = 0\}$ (Proposition 3.1 from [6]).

It follows from (1.2) that

(1.3) If μ is a representing measure for γ, then $\operatorname{card}\operatorname{supp}\mu \geqslant \operatorname{rank} M(n)$ (Corollary 3.5 from [6]).

The main result of Theorem 5.13 from [6] shows that γ admits a $\operatorname{rank} M(n)$-atomic (minimal) representing measure if and only if $M(n) \geqslant 0$ and $M(n)$ admits an extension to a (necessarily positive) moment matrix $M(n+1)$ satisfying $\operatorname{rank} M(n+1) = \operatorname{rank} M(n)$; such an extension is called a *flat extension*.

Given $\gamma \equiv \gamma^{(2n)}$, for $0 \leqslant i, j \leqslant n$ we define the $(i+1) \times (j+1)$ matrix B_{ij} whose entries are the moments of order $i+j$:

$$B_{ij} := \begin{pmatrix} \gamma_{ij} & \gamma_{i+1,j-1} & \cdots & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{ij} & \gamma_{i+1,j-1} & \cdots & \vdots \\ \vdots & \gamma_{i-1,j+1} & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_{0,j+i} & \cdots & \cdots & \cdots & \gamma_{ji} \end{pmatrix}. \tag{1.4}$$

It follows from equation (1.1) that $M(n)(\gamma)$ admits a block decomposition $M(n) = (B_{ij})_{0 \leqslant i,j \leqslant n}$.

We may also define blocks $B_{0,n+1}, \ldots, B_{n-1,n+1}$ via (1.4). Given "new moments" of degree $2n+1$ for a prospective representing measure, let $B_{n,n+1}$ denote the corresponding moment matrix block given by (1.4), and let

$$B(n+1) := \begin{pmatrix} B_{0,n+1} & \cdots & B_{n-1,n+1} & B_{n,n+1} \end{pmatrix}^{\mathrm{t}}.$$

Given a moment matrix block $C(n+1)$ of the form $B_{n+1,n+1}$ (corresponding to "new moments" of degree $2n+2$), we may describe the moment matrix *extension* $M(n+1)$ via the block decomposition

$$M(n+1) = \begin{pmatrix} M(n) & B(n+1) \\ B(n+1)^* & C(n+1) \end{pmatrix}. \tag{1.5}$$

A theorem of Smul'jan ([23]) shows that a block matrix

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \tag{1.6}$$

is positive if and only if (i) $A \geqslant 0$, (ii) there exists a matrix W such that $B = AW$, and (iii) $C \geqslant W^*AW$ (since $A = A^*$, W^*AW is independent of W provided $B = AW$). Note also that if $M \geqslant 0$, then $\operatorname{rank} M = \operatorname{rank} A$ if and only if $C = W^*AW$; conversely, if $A \geqslant 0$ and there exists W such that $B = AW$ and $C = W^*AW$, then $M \geqslant 0$ and $\operatorname{rank} M = \operatorname{rank} A$. A block matrix M as in (1.6) is an *extension* of A, and is a *flat extension* if $\operatorname{rank} M = \operatorname{rank} A$. A flat extension of a positive matrix A is completely determined by a choice of block B satisfying $B = AW$ and $C = W^*AW$ for some matrix W; we denote such a flat extension by $[A\,;B]$.

For an $(n+1)\times(n+2)$ matrix $B_{n,n+1}$, representing "new moments" of degree $2n+1$ for a prospective representing measure of $\gamma^{(2n)}$, let

$$B := \begin{pmatrix} B_{0,n+1} & \cdots & B_{n-1,n+1} & B_{n,n+1} \end{pmatrix}^{\mathrm{t}}.$$

By Smul'jan's theorem, $M(n) \geqslant 0$ admits a (necessarily positive) flat extension

$$[M(n)\,;B] = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$$

in the form of a moment matrix $M(n+1)$ (cf. (1.5)) if and only if

(1.7) $B = M(n)W$ for some W (i.e., $\operatorname{ran} B \subseteq \operatorname{ran} M(n)$);
$C := W^*M(n)W$ is Toeplitz, i.e., has the form of a moment matrix block $B_{n+1,n+1}$.

Theorem (Flat Extension Theorem) (Remark 3.15, Theorem 5.4, Corollary 5.12, Theorem 5.13, and Corollary 5.15 from [6], Lemma 1.9 from [7], and [11]) *Suppose $M(n)(\gamma)$ is positive and admits a flat extension $M(n+1)$, so that $Z^{n+1} = p(Z, \overline{Z})$ in $\mathcal{C}_{M(n+1)}$ for some $p \in \mathcal{P}_n$. Then there exist unique successive flat (positive) moment matrix extensions $M(n+2), M(n+3), \ldots$, which are determined by the relations*

$$Z^{n+k} = (z^{k-1}p)(Z, \overline{Z}) \in \mathcal{C}_{M(n+k)}, \quad k \geqslant 2. \tag{1.8}$$

Let $r := \operatorname{rank} M(n)$. There exist unique scalars $a_0, \ldots, a_{r-1}$ such that in $\mathcal{C}_{M(r)}$,

$$Z^r = a_0 \mathbb{1} + \cdots + a_{r-1} Z^{r-1}.$$

The characteristic polynomial $g_\gamma(z) := z^r - (a_0 + \cdots + a_{r-1} z^{r-1})$ has r distinct roots, $z_0, \ldots, z_{r-1}$, and γ has a $\operatorname{rank} M(n)$-atomic minimal representing measure of the form

$$\nu = \nu[M(n+1)] = \sum \rho_i \delta_{z_i},$$

where the densities $\rho_i > 0$ are determined by the Vandermonde equation

$$V(z_0, \ldots, z_{r-1})(\rho_0, \ldots, \rho_{r-1})^{\mathrm{t}} = (\gamma_{00}, \ldots, \gamma_{0,r-1})^{\mathrm{t}}. \tag{1.9}$$

The measure $\nu[M(n+1)]$ is the unique representing measure for $\gamma^{(2n+2)}$, and is also the unique representing measure for $M(\infty)$.

We note in connection with (1.8) that due to the structure of moment matrix blocks B_{ij}, an extension $M(n+1)$ is completely determined from $M(n)$ once column Z^{n+1} is specified.

We also recall from [6] and [11] that $M(n) \geqslant 0$ is *recursively generated* if the following property holds:

$$p, q, pq \in \mathcal{P}_n,\ p(Z, \overline{Z}) = 0 \quad \Rightarrow \quad (pq)(Z, \overline{Z}) = 0. \tag{RG}$$

If $M(n) \geqslant 0$ admits a flat extension $M(n+1)$, then $M(n+1)$, and all of its successive flat extensions $M(n+1+d)$ (described by Theorem 1.6), are recursively generated (Remark 3.15-(ii) from [6]). More generally, if $\gamma^{(2n)}$ admits a representing measure, then $M(n)$ is recursively generated (Corollary 3.4 from [6]).

We conclude this section with some results that will allow us to convert a given moment problem into a simpler, equivalent, moment problem. For $a, b, c \in \mathbb{C}$, $|b| \neq |c|$, let $\varphi(z) := a + bz + c\overline{z}$, $z \in \mathbb{C}$. Given $\gamma^{(2n)}$, define $\widetilde{\gamma}^{(2n)}$ by $\widetilde{\gamma}_{ij} := L_\gamma(\overline{\varphi}^i \varphi^j)$, $0 \leqslant i + j \leqslant 2n$, where L_γ denotes the Riesz functional associated with γ. It is straightforward to verify that if $\Phi(z, \overline{z}) := (\varphi(z), \overline{\varphi(z)})$, then $L_{\widetilde{\gamma}}(p) = L_\gamma(p \circ \Phi)$ for every $p \in \mathcal{P}_n$. (Note that for $p(z, \overline{z}) \equiv \sum a_{ij} \overline{z}^i z^j$, $(p \circ \Phi)(z, \overline{z}) = p(\varphi(z), \overline{\varphi(z)}) \equiv \sum a_{ij} \overline{\varphi(z)}^i \varphi(z)^j$.)

Proposition (Invariance under degree-one transformations.) *Let $M(n)$ and $\widetilde{M}(n)$ be the moment matrices associated with γ and $\widetilde{\gamma}$, and let $J\widehat{p} := \widehat{p \circ \Phi}$ $(p \in \mathcal{P}_n)$.*

(i) *$\widetilde{M}(n) = J^* M(n) J$.*

(ii) *J is invertible.*

(iii) *$\widetilde{M}(n) \geqslant 0 \Leftrightarrow M(n) \geqslant 0$.*

(iv) $\operatorname{rank} \widetilde{M}(n) = \operatorname{rank} M(n)$.

(v) *The formula* $\mu = \widetilde{\mu} \circ \Phi$ *establishes a one-to-one correspondence between the sets of representing measures for* γ *and* $\widetilde{\gamma}$, *which preserves measure class and cardinality of the support; moreover,* $\varphi(\operatorname{supp} \mu) = \operatorname{supp} \widetilde{\mu}$.

(vi) $M(n)$ *admits a flat extension if and only if* $\widetilde{M}(n)$ *admits a flat extension.*

(vii) *For* $p \in \mathcal{P}_n$, $p(\widetilde{Z}, \widetilde{\overline{Z}}) = J^*((p \circ \Phi)(Z, \overline{Z}))$.

Proof. It is clear that (iii) and (iv) follow from (i) and (ii). Note that $\varphi^{-1}(w) = \frac{\overline{b}w - c\overline{w} + \overline{a}c - a\overline{b}}{|b|^2 - |c|^2}$, so φ^{-1} is also a degree-one map. To prove (v), assume $\widetilde{\mu}$ is a representing measure for $\widetilde{\gamma}$, and observe that

$$\begin{aligned}\gamma_{ij} &= L_\gamma(\overline{z}^i z^j) = L_\gamma(\overline{\varphi^{-1} \circ \varphi}^{\,i} (\varphi^{-1} \circ \varphi)^j) = L_{\tilde{\gamma}}(\overline{\varphi^{-1}}^{\,i} (\varphi^{-1})^j) \\ &= \int \overline{\varphi^{-1}(w)}^{\,i} (\varphi^{-1}(w))^j \, \mathrm{d}\widetilde{\mu}(w, \overline{w}) = \int \overline{z}^i z^j \, \mathrm{d}\widetilde{\mu}(\Phi(z, \overline{z})) = \int \overline{z}^i z^j \, \mathrm{d}\mu(z, \overline{z}),\end{aligned}$$

which shows that μ is a representing measure for γ. Conversely, it follows as above that if μ is a representing measure for γ, then $\widetilde{\mu} := \mu \circ \Omega$ is a representing measure for $\widetilde{\gamma}$ (where $\Omega(w) := (\varphi^{-1}(w), \overline{\varphi^{-1}(w)}\,))$. The rest of (v) is straightforward. We now establish (i). First, recall that $\langle M(n)\widehat{p}, \widehat{q}\,\rangle = L_\gamma(p\overline{q})$, $p, q \in \mathcal{P}_n$. Thus,

$$\begin{aligned}\langle J^* M(n) J\widehat{p}, \widehat{q}\,\rangle &= \langle M(n) J\widehat{p}, J\widehat{q}\,\rangle = \langle M(n)\widehat{p \circ \Phi}, \widehat{q \circ \Phi}\rangle \\ &= L_\gamma((p \circ \Phi)\overline{(q \circ \Phi)}\,) = L_\gamma((p\overline{q}) \circ \Phi) = L_{\tilde{\gamma}}(p\overline{q}) = \langle \widetilde{M}(n)\widehat{p}, \widehat{q}\,\rangle.\end{aligned}$$

For (ii), $J\widehat{p} = 0 \Rightarrow \widehat{p \circ \Phi} = 0 \Rightarrow p \circ \Phi = 0$. Therefore, $p(\varphi(z), \overline{\varphi(z)}\,) = 0$ for all $z \in \mathbb{C}$, whence $p(w, \overline{w}) = p(\varphi[\varphi^{-1}(w)], \overline{\varphi[\varphi^{-1}(w)]}\,) = 0$ for all $w \in \mathbb{C}$. It readily follows (e.g., using partial derivatives) that $p \equiv 0$, whence $\widehat{p} = 0$, which proves that J is invertible.

For (vi), suppose $M(n)$ admits a flat extension. Then $M(n)$ admits a $\operatorname{rank} M(n)$-atomic representing measure, so (v) implies that $\widetilde{M}(n)$ admits a $\operatorname{rank} M(n)$-atomic representing measure $\widetilde{\mu}$. Now (iv) implies that $\widetilde{\mu}$ is $\operatorname{rank} \widetilde{M}(n)$-atomic, and it follows that $\widetilde{M}(n)$ admits a flat extension. The converse is entirely similar.

Finally, to prove (vii), observe that

$$\begin{aligned}p(\widetilde{Z}, \widetilde{\overline{Z}}) &= \widetilde{M}(n)\widehat{p} = J^* M(n) J\widehat{p} \quad \text{(by (i))} \\ &= J^* M(n)\widehat{p \circ \Phi} = J^*[(p \circ \Phi)(Z, \overline{Z})]. \quad ∎\end{aligned}$$

COROLLARY *The following are equivalent:*

(i) $\widetilde{\gamma}$ *has a representing measure supported in* $\mathcal{Z}(p)$;

(ii) $\widetilde{\gamma}$ *has a representing measure and* $p(\widetilde{Z}, \widetilde{\overline{Z}}) = 0$;

(iii) γ *has a representing measure and* $p(\varphi, \overline{\varphi})(Z, \overline{Z}) = 0$;

(iv) γ *has a representing measure supported in* $\mathcal{Z}(p \circ \Phi)$.

COROLLARY *Given a family* $\gamma \equiv \gamma^{(2n)}$ *with* $\gamma_{00} = 1$ *and* $\gamma_{11} > |\gamma_{01}|^2$, *for the purposes of solving* TCMP, *we can always additionally assume that* $\gamma_{01} = 0$ *and* $\gamma_{11} = 1$.

Proof. For $a, b \in \mathbb{C}$, $b \neq 0$, let $\varphi(z) := \frac{z-a}{b}$, $z \in \mathbb{C}$. Using Proposition 1.7, we observe that

$$\begin{aligned}
\widetilde{\gamma}_{00} &= L_\gamma(1) = \gamma_{00} = 1, \\
\widetilde{\gamma}_{01} &= L_\gamma g(\varphi) = L_\gamma\Big(\frac{z-a}{b}\Big) = \frac{\gamma_{01} - a\gamma_{00}}{b} = \frac{\gamma_{01} - a}{b}, \\
\widetilde{\gamma}_{11} &= L_\gamma(\overline{\varphi}\varphi) = L_\gamma(\frac{\overline{z}z - a\overline{z} - \overline{a}z + |a|^2}{|b|^2}) \\
&= \frac{\gamma_{11} - a\gamma_{10} - \overline{a}\gamma_{01} + |a|^2\gamma_{00}}{|b|^2} = \frac{\gamma_{11} - a\gamma_{10} - \overline{a}\gamma_{01} + |a|^2}{|b|^2}.
\end{aligned}$$

It is now clear that choosing $a = \gamma_{01}$ and $b = \sqrt{\gamma_{11} - |\gamma_{01}|^2}$, we obtain $\widetilde{\gamma}_{01} = 0$ and $\widetilde{\gamma}_{11} = 1$, as desired. ∎

The following consequence of Proposition 1.7 is also a mild extension of Lemma 2.1 from [10].

Proposition *Let $M(n)$ be the moment matrix for $\gamma \equiv \gamma^{(2n)}$. For $0 \neq \lambda \in \mathbb{C}$, let $J_\lambda \in M_{m(n)}$ be the diagonal matrix whose entry in row $\overline{Z}^i Z^j$, column $\overline{Z}^i Z^j$ is $\overline{\lambda}^i \lambda^j$, $0 \leqslant i+j \leqslant n$. Then $\widetilde{M}(n) = J_\lambda^* M(n) J_\lambda$ satisfies the following properties:*

(i) *$\widetilde{M}(n)$ is the moment matrix associated with $\widetilde{\gamma}^{(2n)}$, where $\widetilde{\gamma}_{ij} = \overline{\lambda}^i \lambda^j \gamma_{ij}$ $(0 \leqslant i+j \leqslant 2n)$;*

(ii) *$\widetilde{M}(n) \geqslant 0$ if and only if $M(n) \geqslant 0$;*

(iii) *there exist scalars α_{rs} such that $\overline{Z}^i Z^j = \sum\limits_{r,s} \alpha_{rs} \overline{Z}^r Z^s$ in $\mathcal{C}_{M(n)}$ if and only if $\widetilde{\overline{Z}}^i \widetilde{Z}^j = \sum\limits_{r,s} (\overline{\lambda}^{i-r} \lambda^{j-s} \alpha_{rs}) \widetilde{\overline{Z}}^r \widetilde{Z}^s$ in $\mathcal{C}_{\widetilde{M}(n)}$;*

(iv) *$\{\overline{Z}^i Z^j\}_{(i,j)\in I}$ is a basis for $\mathcal{C}_{M(n)}$ if and only if $\{\widetilde{\overline{Z}}^i \widetilde{Z}^j\}_{(i,j)\in I}$ is a basis for $\mathcal{C}_{\widetilde{M}(n)}$;*

(v) *$\operatorname{rank} \widetilde{M}(n) = \operatorname{rank} M(n)$;*

(vi) *γ admits a finitely atomic representing measure $\mu \equiv \sum \rho_k \delta_{z_k}$ if and only if $\widetilde{\gamma}$ admits a finitely atomic representing measure $\widetilde{\mu} \equiv \sum \rho_k \delta_{\widetilde{z}_k}$, where $\widetilde{z}_k = \lambda z_k$.*

In brief, $\gamma^{(2n)}$ and $\widetilde{\gamma}^{(2n)}$ give rise to equivalent truncated moment problems, whose representing measures satisfy the relation $\operatorname{supp} \widetilde{\mu} = \lambda \operatorname{supp} \mu$.

Proof. (i), (ii), (v), and (vi) are contained in Lemma 2.1 from [10], and also follow from Proposition 1.7 using $\varphi(z) = \lambda z$ (i.e., $a = 0$, $b = \frac{1}{\lambda}$). Since (iii) $\Rightarrow$ (iv) is clear, it suffices to prove (iii). Suppose $\overline{Z}^i Z^j = \sum\limits_{r,s} \alpha_{rs} \overline{Z}^r Z^s$; then for

$0 \leqslant k+l \leqslant n$,

$$\begin{aligned}
&\langle \widetilde{\overline{Z}}^i \widetilde{Z}^j, \widetilde{\overline{Z}}^k \widetilde{Z}^l \rangle_{\widetilde{M}(n)} \\
&\quad = \widetilde{\gamma}_{i+l,j+k} = \overline{\lambda}^{i+l} \lambda^{j+k} \gamma_{i+l,j+k} = \overline{\lambda}^{i+l} \lambda^{j+k} \langle \overline{Z}^i Z^j, \overline{Z}^k Z^l \rangle_{M(n)} \\
&\quad = \overline{\lambda}^{i+l} \lambda^{j+k} \Big\langle \sum_{r,s} \alpha_{rs} \overline{Z}^r Z^s, \overline{Z}^k Z^l \Big\rangle_{M(n)} = \overline{\lambda}^{i+l} \lambda^{j+k} \sum_{r,s} \alpha_{rs} \gamma_{r+l,s+k} \\
&\quad = \sum_{r,s} (\overline{\lambda}^{i-r} \lambda^{j-s} \alpha_{rs}) \overline{\lambda}^{r+l} \lambda^{s+k} \gamma_{r+l,s+k} = \sum_{r,s} (\overline{\lambda}^{i-r} \lambda^{j-s} \alpha_{rs}) \widetilde{\gamma}_{r+l,s+k} \\
&\quad = \Big\langle \sum_{r,s} (\overline{\lambda}^{i-r} \lambda^{j-s} \alpha_{rs}) \widetilde{\overline{Z}}^r \widetilde{Z}^s, \widetilde{\overline{Z}}^k \widetilde{Z}^l \Big\rangle_{\widetilde{M}(n)}.
\end{aligned}$$

Thus $\widetilde{\overline{Z}}^i \widetilde{Z}^j = \sum_{r,s} (\overline{\lambda}^{i-r} \lambda^{j-s} \alpha_{rs}) \widetilde{\overline{Z}}^r \widetilde{Z}^s$, and the converse implication is proved similarly. ∎

COROLLARY *Let $M(n)$ be the moment matrix for $\gamma^{(2n)}$, with $\gamma_{01} \neq 0$, and define $\lambda = \frac{\gamma_{10}}{|\gamma_{01}|}$ and $\widetilde{\gamma}_{ij} = \overline{\lambda}^i \lambda^j \gamma_{ij}$, $0 \leqslant i+j \leqslant 2n$. Then the equivalent family $\widetilde{\gamma}^{(2n)}$ satisfies $\widetilde{\gamma}_{01} > 0$ and $\widetilde{\gamma}_{ii} = \gamma_{ii}$, $0 \leqslant i \leqslant n$.*

By virtue of Proposition 1.10 (iv) and Corollary 1.11, whenever we analyze a quartic moment problem for which $\mathcal{C}_{M(2)}$ has a particular basis, we may further assume that $\gamma_{01} \geqslant 0$. This extra assumption reduces the algebraic complexity of certain calculations that can sometimes be used to solve the moment problem via computer algebra. This approach was successful in our original proof of the rank-4 case of Theorem 1.2; an earlier attempt to prove this case by computer algebra, with γ_{01} complex, failed due to memory overflow.

Consider now a given collection $\gamma \equiv \gamma^{(2n)}$ and its associated *real* collection $\beta^{(2n)}$, where $\beta_{ij}^{(2n)} := L_\gamma(y^i x^j)$, with $x := \frac{z+\overline{z}}{2}$ and $y := \frac{z-\overline{z}}{2\mathrm{i}}$. Since L_γ clearly maps real polynomials to real numbers, $\beta_{ij}^{(2n)} \in \mathbb{R}$, for all i, j. We can then build an associated moment matrix $M_{\mathbb{R}}(n)(\beta) := (M_{\mathbb{R}}[i,j](\beta))_{i,j=0}^n$, where

$$M_{\mathbb{R}}[i,j](\beta) := \begin{pmatrix} \beta_{0,i+j} & \beta_{1,i+j-1} & \cdots & \beta_{j,i} \\ \beta_{1,i+j-1} & \beta_{2,i+j-2} & \cdots & \beta_{j+1,i-1} \\ \vdots & \vdots & & \vdots \\ \beta_{i,j} & \beta_{i+1,j-1} & \cdots & \beta_{i+j,0} \end{pmatrix}.$$

We denote the successive rows and columns of $M_{\mathbb{R}}(n)(\beta)$ by $\mathbb{1}, X, Y, X^2, YX, Y^2, \ldots, X^n, \ldots, Y^n$; observe that each block $M_{\mathbb{R}}[i,j](\beta)$ is of Hankel type. Conversely, given a collection $\beta \equiv \beta^{(2n)}$ of real numbers, with $\beta_{00}^{(2n)} > 0$, we can let $\gamma_{ij}^{(2n)} := L_\beta(\overline{z}^i z^j)$, where $z := x + \mathrm{i}y$ and $\overline{z} := x - \mathrm{i}y$. Clearly $\gamma_{ji}^{(2n)} = L_\beta(\overline{z}^j z^i) = \overline{L_\beta(\overline{z}^i z^j)} = \overline{\gamma_{ij}^{(2n)}}$, and $\gamma_{00}^{(2n)} = \beta_{00}^{(2n)} > 0$. There is, therefore, a one-to-one correspondence between TCMP's and TRMP's, at least at the Riesz functional level. The two matrices $M \equiv M(n)(\gamma)$ and $M_{\mathbb{R}} \equiv M_{\mathbb{R}}(n)(\beta)$ give rise to inner products $\langle p, q \rangle_M := (M\widehat{p}, \widehat{q})$, $p, q \in \mathcal{P}_n$, and $\langle r, s \rangle_{M_{\mathbb{R}}} := (M_{\mathbb{R}}\widetilde{r}, \widetilde{s})$, $r, s \in \mathbb{R}[x,y]_n$. (Here $\widehat{\ }$ and $\widetilde{\ }$ denote coordinates relative to the canonical bases $1, z, \overline{z}, z^2, \overline{z}z, \overline{z}^2, \ldots$ and

$1, x, y, x^2, yx, y^2, \ldots$, respectively.) The matrix $M_{\mathbb{R}}$ has properties analogous to those enjoyed by M (cf. [6], Theorem 2.1), which we omit, since we will not need them here. We are more interested, instead, in the transition matrix L which intertwines M and $M_{\mathbb{R}}$, that is $M = L^* M_{\mathbb{R}} L$. To describe L, let $\psi(x,y) := z \equiv x + \mathrm{i}y$ and let $\Psi(x,y) := (z, \bar{z})$. Exactly as in the paragraph preceding Proposition 1.7, we have $L_\gamma(p) = L_\beta(p \circ \Psi)$, so that $L\widehat{p} := \widetilde{p \circ \Psi}$. The matrix L admits a direct sum decomposition $\bigoplus_{k=0}^{n} L_k$, where L_k acts on monomials of total degree k. For instance, $L_0 = (1)$, $L_1 = \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}$,

$$L_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2\mathrm{i} & 0 & -2\mathrm{i} \\ -1 & 1 & -1 \end{pmatrix}, \quad \text{and} \quad L_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3\mathrm{i} & \mathrm{i} & -\mathrm{i} & -3\mathrm{i} \\ -3 & 1 & 1 & -3 \\ -\mathrm{i} & \mathrm{i} & -\mathrm{i} & \mathrm{i} \end{pmatrix}.$$

We now have the following analogue of Proposition 1.7.

Proposition (Equivalence of TCMP with TRMP) *Let $M(n)$ and $M_{\mathbb{R}}(n)$ be the moment matrices associated with γ and β, and define L by $L\widehat{p} := \widetilde{p \circ \Psi}$, $p \in \mathcal{P}_n$.*

(i) $M(n) = L^* M_{\mathbb{R}}(n) L$.

(ii) *L is invertible.*

(iii) $M(n) \geqslant 0 \Leftrightarrow M_{\mathbb{R}}(n) \geqslant 0$.

(iv) $\operatorname{rank} M(n) = \operatorname{rank} M_{\mathbb{R}}(n)$.

(v) *The formula $\mu_{\mathbb{R}} = \mu \circ \Psi$ establishes a one-to-one correspondence between the sets of representing measures for β and γ, which preserves measure class and cardinality of the support; moreover, $\psi(\operatorname{supp} \mu_{\mathbb{R}}) = \operatorname{supp} \mu$.*

(vi) *$M(n)$ admits a flat extension if and only if $M_{\mathbb{R}}(n)$ admits a flat extension.*

(vii) *For $p \in \mathcal{P}_n$, $p(Z, \overline{Z}) = L^*((p \circ \Psi)(X, Y))$.*

Example Consider the moment matrix

$$M(2) := \begin{pmatrix} 1 & 1+\mathrm{i} & 1-\mathrm{i} & -1 & 5 & -1 \\ 1-\mathrm{i} & 5 & -1 & 4+9\mathrm{i} & 4-9\mathrm{i} & 4+9\mathrm{i} \\ 1+\mathrm{i} & -1 & 5 & 4-9\mathrm{i} & 4+9\mathrm{i} & 4-9\mathrm{i} \\ -1 & 4-9\mathrm{i} & 4+9\mathrm{i} & 37 & -19 & 37 \\ 5 & 4+9\mathrm{i} & 4-9\mathrm{i} & -19 & 37 & -19 \\ -1 & 4-9\mathrm{i} & 4+9\mathrm{i} & 37 & -19 & 37 \end{pmatrix}.$$

A straightforward application of the formula $M(2) = L^* M_{\mathbb{R}}(2) L$ shows that

$$M_{\mathbb{R}}(2) = \begin{pmatrix} 1 & 1 & 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 4 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 9 \\ 2 & 4 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 9 & 0 & 0 & 28 \end{pmatrix}.$$

Observe that $\operatorname{rank} M(2) = \operatorname{rank} M_{\mathbb{R}}(2) = 5$, and that $\overline{Z}^2 = Z^2$ in $\mathcal{C}_{M(2)}$, while $YX = 0$ in $\mathcal{C}_{M_{\mathbb{R}}(2)}$; also, $\mathcal{V}(\gamma) = \{z \in \mathbb{C} : (z + \bar{z})(z - \bar{z}) = 0\}$ and $\mathcal{V}(\beta) = \{(x, y) \in$

$R : yx = 0\}$, so as subsets of two-dimensional real space $\mathcal{V}(\gamma)(= \mathcal{V}(\beta))$ is the pair of coordinate axes. Proposition 1.12 says that $M(2)$ admits a representing measure if and only if $M_{\mathbb{R}}(2)$ does. In Example 4.3, we use Theorem 4.1 to show that $M(2)$ admits no flat extension. In Example 5.6 we provide an alternative test to show that $M_{\mathbb{R}}(2)$ does not admit a 5-atomic representing measure, while at the same time we use Proposition 5.5 to establish that $M_{\mathbb{R}}(2)$ does admit infinitely many 6-atomic representing measures. ∎

2. THE CASE $\overline{Z}Z = A\mathbb{1} + BZ + C\overline{Z}$

In this section we analyze TCMP for the case when $M(n) \geqslant 0$, $\{\mathbb{1}, Z, \overline{Z}\}$ is independent in $\mathcal{C}_{M(n)}$, and there exist scalars A, B, C such that

$$\overline{Z}Z = A\mathbb{1} + BZ + C\overline{Z}. \tag{2.1}$$

The main result of this section is Theorem 1.1, which we restate for convenience.

THEOREM *Let $n > 1$. If $M(n) \geqslant 0$, $\{\mathbb{1}, Z, \overline{Z}\}$ is independent in $\mathcal{C}_{M(n)}$, and $\overline{Z}Z \in \langle \mathbb{1}, Z, \overline{Z}\rangle$, then $M(n)$ admits a flat extension (and $\gamma^{(2n)}$ admits a corresponding* rank $M(n)$*-atomic representing measure). Moreover,* rank $M(n) \leqslant 2n+1$, *and if* rank $M(n) \leqslant 2n$, *then $\gamma^{(2n)}$ admits a unique representing measure. If* rank $M(n) = 2n+1$, *then $M(n)$ admits infinitely many flat extensions, each corresponding to a distinct $(2n+1)$-atomic representing measure.*

We first observe that conjugation of (2.1) leads to $\overline{Z}Z = \overline{A}\cdot\mathbb{1} + \overline{B}\,\overline{Z} + \overline{C}Z$, so the linear independence of $\{\mathbb{1}, Z, \overline{Z}\}$ implies that $A \in \mathbb{R}$ and $C = \overline{B}$. We now apply Proposition 1.7 with $a := -\frac{\overline{B}}{\sqrt{A+|B|^2}}$ and $b := \frac{1}{\sqrt{A+|B|^2}}$. In fact, the first row of (2.1) states that

$$\gamma_{11} = A\gamma_{00} + B\gamma_{01} + \overline{B}\gamma_{10} = A + 2\,\mathrm{Re}(B\gamma_{01}); \tag{2.2}$$

since $M(n) \geqslant 0$ and $\{\mathbb{1}, Z\}$ is independent, then

$$\begin{aligned} 0 < \det[M(n)]_2 &= \gamma_{11} - |\gamma_{01}|^2 \leqslant \gamma_{11} - |\gamma_{01}|^2 + |\overline{B} - \gamma_{01}|^2 \\ &= A + 2\,\mathrm{Re}(B\gamma_{01}) - |\gamma_{01}|^2 + |B|^2 - 2\,\mathrm{Re}(B\gamma_{01}) + |\gamma_{01}|^2 \qquad \text{(by (2.2))} \\ &= A + |B|^2, \end{aligned}$$

showing that b is a well-defined positive number. Thus the equation $\overline{z}z = A + Bz + \overline{B}\overline{z}$ defines a nondegenerate circle $C_\gamma := \{z \in \mathbb{C} : |z - \overline{B}|^2 = A + |B|^2\}$. Let $p(z, \overline{z}) = 1 - \overline{z}z$. Now

$$(p\circ\Phi) = 1 - \overline{\varphi(z)}\varphi(z) = \frac{1}{b^2}(b^2 - |a|^2 + \overline{a}z + a\overline{z} - \overline{z}z) = \frac{1}{A+|B|^2}(A + Bz + \overline{B}\overline{z} - \overline{z}z),$$

whence $(p\circ\Phi)(Z, \overline{Z}) = 0$ in $\mathcal{C}_{M(2)}$. It follows from Proposition 1.7 (vii) that $\overline{\widetilde{Z}}\widetilde{Z} = \widetilde{\mathbb{1}}$ in $\mathcal{C}_{\widetilde{M}(n)}$. In view of Proposition 1.7 (vi), $M(n)$ admits a flat extension if and only if $\widetilde{M}(n)$ admits a flat extension. It thus follows from Proposition 1.7 and Corollary 1.8 that to prove Theorem 2.1 we may assume in the sequel that

$M(n) \geqslant 0$ and $\overline{Z}Z = \mathbb{1}$ in $\mathcal{C}_{M(n)}$. To complete the proof we require two auxiliary results.

For a sequence $\beta^{(2n)}\colon \beta_{-2n}, \ldots, \beta_0, \ldots, \beta_{2n}$ with $\beta_0 > 0$, consider the *truncated trigonometric moment problem*

$$\beta_j = \int t^j \,\mathrm{d}\nu, \quad -2n \leqslant j \leqslant 2n,\ \nu \geqslant 0,\ \operatorname{supp}\nu \subset \mathbb{T}\,(\text{unit circle}).$$

Let $T(2n) \equiv T(2n)(\beta)$ denote the Toeplitz matrix $(\beta_{j-i})_{0 \leqslant i,j \leqslant 2n}$. Observe that $T(2n)$ is a $(2n+1) \times (2n+1)$ matrix. It is well-known that $\beta^{(2n)}$ admits a representing measure (supported in $\mathbb{T}$) if and only if $T(2n) \geqslant 0$ (Theorem I.I.12 from [1] and p. 211 of [14]):

PROPOSITION (cf. [5], Theorem 6.12) *The following are equivalent for $\beta \equiv \beta^{(2n)}$:*

(i) *β has a representing measure;*
(ii) *β admits a* $\operatorname{rank} T(2n)$*-atomic (minimal) representing measure;*
(iii) $T(2n)(\beta) \geqslant 0$.

Note that a representing measure ν for $\beta^{(2n)}$ is also a representing measure (for TCMP) associated with $\gamma^{(2n)}$ defined by $\gamma_{ij} := \beta_{j-i}$; indeed, since $\operatorname{supp}\nu \subseteq \mathbb{T}$,

$$\int \overline{z}^i z^j \,\mathrm{d}\nu = \int z^{j-i} \,\mathrm{d}\nu = \beta_{j-i} = \gamma_{ij}.$$

Conversely, if $\overline{Z}Z = \mathbb{1}$ in $\mathcal{C}_{M(n)}$, or equivalently, if $\gamma_{ij} = \gamma_{i+1,j+1}$, where $0 \leqslant i+j \leqslant 2n-2$, then a representing measure for $\gamma^{(2n)}$ (necessarily supported in $\mathbb{T}$) is also a representing measure for the trigonometric problem for $\beta^{(2n)}$ defined by $\beta_k := \gamma_{0,k}$, $-2n \leqslant k \leqslant 2n$. Since $\beta_{j-i} = \gamma_{0,j-i} = \gamma_{0+i,j-i+i} = \gamma_{ij}$, it follows that $\gamma^{(2n)}$ and $\beta^{(2n)}$ have the same representing measures.

PROPOSITION (cf. [11], Proposition 4.1) *Let $n > 1$ and suppose there is a sequence $\beta_{-2n}, \ldots, \beta_0, \ldots, \beta_{2n}$ such that $\gamma^{(2n)}$ satisfies $\gamma_{ij} = \beta_{j-i}$, $0 \leqslant i+j \leqslant 2n$. Then $M(n) \geqslant 0$ if and only if $T(2n) \geqslant 0$, in which case* $\operatorname{rank} M(n) = \operatorname{rank} T(2n)$.

(The equality of rank is not part of the statement of Proposition 4.1 from [11], but a careful examination of the proof of Proposition 4.1 from [11] readily yields this conclusion.)

Proof of Theorem 2.1. From the earlier discussion, we may assume $M(n) \geqslant 0$ and $\overline{Z}Z = \mathbb{1}$. It follows from Propositions 2.2 and 2.3 (and the remarks immediately preceding Proposition 2.3) that $M(n)$ admits a flat extension. Proposition 2.3 implies $\operatorname{rank} M(n) = \operatorname{rank} T(2n)$ ($\leqslant 2n+1$), so $\operatorname{rank} M(n) \leqslant 2n$ if and only if $T(2n)$ is singular. In this case, Remark 6.13 of [5] implies that $\beta^{(2n)}$ has a unique representing measure, so $\gamma^{(2n)}$ has a unique representing measure, which is $\operatorname{rank} M(n)$-atomic. Conversely, $\operatorname{rank} M(n) = 2n+1$ implies that $T(2n)$ is invertible, so $T(2n)$ admits infinitely many positive and singular Toeplitz extensions $T(2n+1)$, parameterized by a choice of β_{2n+1} in an appropriate circle. Each such (flat) extension corresponds to a $\operatorname{rank} T(2n)$-atomic representing measure for $\beta^{(2n)}$ and thus also corresponds to a $\operatorname{rank} M(n)$-atomic representing measure for $\gamma^{(2n)}$; thus $M(n)$ admits infinitely many distinct flat extensions. ∎

Proof of Theorem 1.2 As in the proof of Theorem 2.1, $\overline{Z}Z \in \langle \mathbb{1}, Z, \overline{Z}\rangle$ and $\{\mathbb{1}, Z, \overline{Z}\}$ independent imply that there exist scalars A and B such that

$$\overline{Z}Z = A\mathbb{1} + BZ + \overline{BZ} \quad \text{and} \quad A + |B|^2 > 0. \tag{2.3}$$

Theorem 2.1 implies that $M(2)$ admits a flat extension, so $\gamma^{(4)}$ admits a rank $M(2)$-atomic representing measure, and (2.3) and (1.2) imply that the support of each representing measure is contained in C_γ. If $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$, then $4 = \operatorname{rank} M(2) = 2n$, so Theorem 2.1 implies that $\gamma^{(4)}$ admits a unique representing measure, which is 4-atomic. In the remaining case, $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}^2\}$ is a basis for $\mathcal{C}_{M(2)}$, so Theorem 2.1 implies that $M(2)$ admits infinitely many flat extensions, each corresponding to a distinct 5-atomic (minimal) representing measure. ∎

To visualize how Proposition 1.7 interacts with Theorem 1.2, we present an example which illustrates the case of Theorem 1.2 when $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$. First, we pause to parameterize the matrices $M(2)$ under consideration. By virtue of Proposition 1.7 and Corollary 1.9, we may assume $\gamma_{00} = 1$, $\gamma_{01} = 0$, and $\gamma_{11} = 1$; thus $M(1)$ assumes the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c - \mathrm{i}d \\ 0 & c + \mathrm{i}d & 1 \end{pmatrix}.$$

To ensure $M(1) > 0$, we require $c^2 + d^2 < 1$. Next let $\gamma_{12} = u + \mathrm{i}v$, $\gamma_{03} = x + \mathrm{i}y$, and $\gamma_{04} = r + \mathrm{i}s$ with $u, v, x, y, r, s \in \mathbb{R}$; then $M(2)$ assumes the form

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & c+\mathrm{i}d & 1 & c-\mathrm{i}d \\ 0 & 1 & c-\mathrm{i}d & u+\mathrm{i}v & u-\mathrm{i}v & x-\mathrm{i}y \\ 0 & c+\mathrm{i}d & 1 & x+\mathrm{i}y & u+\mathrm{i}v & u-\mathrm{i}v \\ c-\mathrm{i}d & u-\mathrm{i}v & x-\mathrm{i}y & \gamma_{22} & \gamma_{31} & r-\mathrm{i}s \\ 1 & u+\mathrm{i}v & u-\mathrm{i}v & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ c+\mathrm{i}d & x+\mathrm{i}y & u+\mathrm{i}v & r+\mathrm{i}s & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

Choose u and v arbitrarily. Since $M(1) > 0$ and $\overline{Z}Z \in \langle \mathbb{1}, Z, \overline{Z}\rangle$, it follows that

$$\gamma_{22} = (\,1 \quad u+\mathrm{i}v \quad u-\mathrm{i}v\,)\, M(1)^{-1}\, (\,1 \quad u-\mathrm{i}v \quad u+\mathrm{i}v\,)^{\mathrm{t}}, \tag{2.4}$$

$$\gamma_{13} = (\,c+\mathrm{i}d \quad x+\mathrm{i}y \quad u+\mathrm{i}v\,)\, M(1)^{-1}\, (\,1 \quad u-\mathrm{i}v \quad u+\mathrm{i}v\,)^{\mathrm{t}}. \tag{2.5}$$

Thus γ_{22} is determined by c, d, u, v and γ_{13} is determined by c, d, u, v, x, y. Finally, appropriate choices of x and y guarantee that $[M(2)]_4 > 0$, and appropriate choices of r and s ensure that $M(2) \geqslant 0$ and $\operatorname{rank} M(2) = 4$.

EXAMPLE Set $c := 0$, $d := \frac{1}{2}$, $u := 1$, and $v := 0$. By (2.4),

$$\gamma_{22} := (\,1 \quad 1 \quad 1\,)\, M(1)^{-1}\, (\,1 \quad 1 \quad 1\,)^{\mathrm{t}} = \frac{11}{3}.$$

To ensure $[M(2)]_4 > 0$, we must choose x, y such that $x^2 + (y - \frac{1}{2})^2 < \frac{29}{16}$, e.g., $x = 0$, $y = \frac{1}{2}$. Thus $\gamma_{03} = \frac{\mathrm{i}}{2}$ and by (2.5),

$$\gamma_{13} := \left(\,\tfrac{\mathrm{i}}{2} \quad \tfrac{\mathrm{i}}{2} \quad 1\,\right) M(1)^{-1}\, (\,1 \quad 1 \quad 1\,)^{\mathrm{t}} = 1 + \frac{\mathrm{i}}{2}.$$

Finally, to guarantee that $M(2) \geqslant 0$ and rank $M(2) = 4$, we require $(r+\frac{1}{4})^2+(s-\frac{1}{2})^2 = (\frac{29}{12})^2$; for instance, take $r = \frac{13}{6}$, $s = \frac{1}{2}$. The associated moment matrix is

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & \frac{\mathrm{i}}{2} & 1 & -\frac{\mathrm{i}}{2} \\ 0 & 1 & -\frac{\mathrm{i}}{2} & 1 & 1 & -\frac{\mathrm{i}}{2} \\ 0 & \frac{\mathrm{i}}{2} & 1 & \frac{\mathrm{i}}{2} & 1 & 1 \\ -\frac{\mathrm{i}}{2} & 1 & -\frac{\mathrm{i}}{2} & \frac{11}{3} & 1-\frac{\mathrm{i}}{2} & \frac{13}{6}-\frac{\mathrm{i}}{2} \\ 1 & 1 & 1 & 1+\frac{\mathrm{i}}{2} & \frac{11}{3} & 1-\frac{\mathrm{i}}{2} \\ \frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{2} & 1 & \frac{13}{6}+\frac{\mathrm{i}}{2} & 1+\frac{\mathrm{i}}{2} & \frac{11}{3} \end{pmatrix},$$

$\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$, and (2.1) is satisfied with $A = 1$, $B = \frac{4}{3} + \frac{2}{3}\mathrm{i}$, and $C = \overline{B}$. Letting $a := -\frac{\overline{B}}{\sqrt{A+|B|^2}} = \frac{-4+2\mathrm{i}}{\sqrt{29}}$ and $b := \frac{1}{\sqrt{A+|B|^2}} = \frac{3}{\sqrt{29}}$, Proposition 1.7 shows that $M(2)$ is equivalent to $\widetilde{M}(2)$ with $\overline{\widetilde{Z}}\widetilde{Z} = \widetilde{\mathbb{1}}$. $\widetilde{M}(2)$ gives rise to an equivalent trigonometric moment problem with Toeplitz matrix $T(4) := (\widetilde{\beta}_{j-i})_{0\leqslant i,j\leqslant 4}$, where $\widetilde{\beta}_k := \widetilde{\gamma}_{0,k}$, $0 \leqslant k \leqslant 4$, and $\widetilde{\gamma}_{00} = 1$, $\widetilde{\gamma}_{01} = a + b\gamma_{01} = a$, $\widetilde{\gamma}_{02} = a^2 + 2ab\gamma_{01} + b^2\gamma_{02} = a^2 + b^2\gamma_{02}$, $\widetilde{\gamma}_{03} = a^3 + 3ab^2\gamma_{02} + b^3\gamma_{03}$ and $\widetilde{\gamma}_{04} = a^4 + 6a^2b^2\gamma_{02} + 4ab^3\gamma_{03} + b^4\gamma_{04}$. Concretely,

$$T(4) = \begin{pmatrix} 1 & \frac{-4+2\mathrm{i}}{\sqrt{29}} & \frac{24-23\mathrm{i}}{58} & \frac{\sqrt{29}(-86+95\mathrm{i})}{1682} & \frac{775-471\mathrm{i}}{1682} \\ \frac{-4-2\mathrm{i}}{\sqrt{29}} & 1 & \frac{-4+2\mathrm{i}}{\sqrt{29}} & \frac{24-23\mathrm{i}}{58} & \frac{\sqrt{29}(-86+95\mathrm{i})}{1682} \\ \frac{24+23\mathrm{i}}{58} & \frac{-4-2\mathrm{i}}{\sqrt{29}} & 1 & \frac{-4+2\mathrm{i}}{\sqrt{29}} & \frac{24-23\mathrm{i}}{58} \\ \frac{\sqrt{29}(-86-95\mathrm{i})}{1682} & \frac{24+23\mathrm{i}}{58} & \frac{-4-2\mathrm{i}}{\sqrt{29}} & 1 & \frac{-4+2\mathrm{i}}{\sqrt{29}} \\ \frac{775+471\mathrm{i}}{1682} & \frac{\sqrt{29}(-86-95\mathrm{i})}{1682} & \frac{24+23\mathrm{i}}{58} & \frac{-4-2\mathrm{i}}{\sqrt{29}} & 1 \end{pmatrix},$$

with columns denoted by $\mathbb{1}, Z, Z^2, Z^3, Z^4$. As expected from Theorem 2.1, $T(4) \geqslant 0$, rank $T(4) = 4$, and the last column of $T(4)$ is a linear combination of the first four columns. This readily leads to the characteristic function

$$g_{\tilde{\beta}}(z) := z^4 - \left(1 + \frac{5+4\mathrm{i}}{\sqrt{29}}z + \mathrm{i}z^2 + \frac{-5+4\mathrm{i}}{\sqrt{29}}z^3\right),$$

whose roots (all belonging to the unit circle) are

$$\widetilde{z}_0 \cong -0.527794 - 0.849373\mathrm{i}, \quad \widetilde{z}_1 \cong -0.328834 + 0.944388\mathrm{i},$$
$$\widetilde{z}_2 \cong -0.975577 + 0.219659\mathrm{i}, \quad \widetilde{z}_3 \cong 0.903728 + 0.428107\mathrm{i}.$$

Using the associated Vandermonde matrix and the moments $\widetilde{\beta}_0$, $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, $\widetilde{\beta}_3$, we obtain the densities $\rho_0 \cong 0.0370864$, $\rho_1 \cong 0.256356$, $\rho_2 \cong 0.679743$ and $\rho_3 \cong 0.0268143$. If we now use Proposition 1.7 (v) to translate and rotate/dilate the atoms $\{\widetilde{z}_i\}_{i=0}^{3}$, we see that the unique representing measure for $M(2)$ is given by $\mu := \sum_{i=0}^{3} \rho_i \delta_{z_i}$, where $z_i := b\widetilde{z}_i + a$, that is,

$$\begin{aligned} &z_0 \cong 0.385914 - 2.19134\mathrm{i}, && z_1 \cong 0.743058 + 1.02856\mathrm{i}, \\ &z_2 \cong -0.41788 - 0.272367\mathrm{i}, && z_3 \cong 2.95557 + 0.101809\mathrm{i}. \quad \blacksquare \end{aligned}$$

3. THE CASE $\overline{Z}Z = A\mathbb{1} + BZ + C\overline{Z} + DZ^2$

In this section we analyze the quartic moment problem for the case when $M(2) \geqslant 0$, $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(2)}$, and

$$\overline{Z}Z = A\mathbb{1} + BZ + C\overline{Z} + DZ^2, \quad D \neq 0. \tag{3.1}$$

In Theorem 3.1 we provide a concrete test for the existence of a representing measure, or, equivalently, for the existence of a 4-atomic (minimal) representing measure. This test is satisfied whenever $|D| \neq 1$ (Lemma 3.2). For $|D| = 1$, Example 3.6 illustrates a case in which a measure exists, while Example 3.8 illustrates a case in which no representing measure exists.

Our first goal is to study conditions for the existence of a recursively generated moment matrix extension

$$M(3) \equiv \begin{pmatrix} M(2) & B(3) \\ B(3)^* & C(3) \end{pmatrix}, \tag{3.2}$$

where

$$B(3) = \begin{array}{c} \begin{array}{cccc} Z^3 & Z^2\overline{Z} & Z\overline{Z}^2 & \overline{Z}^3 \end{array} \\ \begin{pmatrix} \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \end{pmatrix} \end{array} \quad \text{and} \quad C(3) = \begin{array}{c} \begin{array}{cccc} Z^3 & Z^2\overline{Z} & Z\overline{Z}^2 & \overline{Z}^3 \end{array} \\ \begin{pmatrix} \gamma_{33} & \gamma_{42} & \gamma_{51} & \gamma_{60} \\ \gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{51} \\ \gamma_{15} & \gamma_{24} & \gamma_{33} & \gamma_{42} \\ \gamma_{06} & \gamma_{15} & \gamma_{24} & \gamma_{33} \end{pmatrix} \end{array}.$$

To begin, we derive certain column relations that hold in $\mathcal{C}_{M(2)}$, or would necessarily hold in $\mathcal{C}_{M(3)}$.

Recall Lemma 3.10 from [6],

$$p \in \mathcal{P}_n, \quad p(Z, \overline{Z}) = 0 \text{ in } \mathcal{C}_{M(n)} \Longrightarrow \overline{p}(Z, \overline{Z}) = 0 \text{ in } \mathcal{C}_{M(n)}. \tag{3.3}$$

From (3.1) and (3.3), in $\mathcal{C}_{M(2)}$ we have

$$\overline{Z}Z = \overline{A}\mathbb{1} + \overline{B}\,\overline{Z} + \overline{C}Z + \overline{D}\,\overline{Z}^2, \tag{3.4}$$

whence (3.1) implies

$$\overline{Z}^2 = \frac{1}{\overline{D}}((A - \overline{A})\mathbb{1} + (B - \overline{C})Z + (C - \overline{B})\overline{Z} + DZ^2). \tag{3.5}$$

Thus $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$, and $\operatorname{rank} M(2) = 4$. It follows from (3.1), (3.3), and (3.4) that in any recursively generated extension $M(3)$, the following column relations must hold:

$$\overline{Z}Z^2 = AZ + BZ^2 + C\overline{Z}Z + DZ^3, \tag{3.6}$$

$$\overline{Z}^2 Z = \overline{A}\,\overline{Z} + \overline{B}\,\overline{Z}^2 + \overline{C}\,\overline{Z}Z + \overline{D}\,\overline{Z}^3, \tag{3.7}$$

$$\overline{Z}Z^2 = \overline{A}Z + \overline{B}\,\overline{Z}Z + \overline{C}Z^2 + \overline{D}\,\overline{Z}^2 Z, \tag{3.8}$$

$$\overline{Z}^2 Z = A\overline{Z} + B\overline{Z}Z + C\overline{Z}^2 + D\overline{Z}Z^2. \tag{3.9}$$

From the form of $\begin{pmatrix} M(2) & B(3) \end{pmatrix}$ in (3.2), note the following consequence of (3.6): There exists $\gamma_{23} \in \mathbb{C}$ such that

$$\overline{\gamma}_{23} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23}. \tag{3.10}$$

Our main result for this section, which follows, shows that (3.10) is actually equivalent to the existence of a representing measure.

Theorem *Suppose $M(2) \geqslant 0$, $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(2)}$, and $\overline{Z}Z = A\mathbb{1} + BZ + C\overline{Z} + DZ^2$, $D \neq 0$. The following are equivalent:*

(i) *$\gamma^{(4)}$ has a finitely atomic representing measure;*
(ii) *$\gamma^{(4)}$ admits a 4-atomic (minimal) representing measure;*
(iii) *$M(2)$ admits a flat extension;*
(iv) *$M(2)$ admits a recursively generated extension $M(3) \geqslant 0$;*
(v) *there exists $\gamma_{23} \in \mathbb{C}$ such that*

$$\gamma_{32} \equiv \overline{\gamma}_{23} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23}.$$

We defer the proof of Theorem 3.1 to consider some illustrative examples. We begin with the case $|D| \neq 1$.

Lemma *If $|D| \neq 1$, there exists a unique $\gamma_{23} \in \mathbb{C}$ satisfying*

$$A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23} = \overline{\gamma}_{23}. \tag{3.11}$$

Proof. Write $\gamma_{23} \equiv x + \mathrm{i}y$ and let $D \equiv d_1 + \mathrm{i}d_2$, with $x, y, d_1, d_2 \in \mathbb{R}$. (3.11) is equivalent to the real system

$$\begin{pmatrix} d_1 - 1 & -d_2 \\ d_2 & d_1 + 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\operatorname{Re}(A\gamma_{21} + B\gamma_{22} + C\gamma_{31}) \\ -\operatorname{Im}(A\gamma_{21} + B\gamma_{22} + C\gamma_{31}) \end{pmatrix}.$$

Since $(d_1 - 1)(d_1 + 1) + d_2^2 = |D|^2 - 1 \neq 0$, there is a unique solution, x, y, so we may define $\gamma_{23} := x + \mathrm{i}y$. ∎

Lemma *Corresponding to γ_{23} satisfying* (3.11), *there exist unique γ_{14}, $\gamma_{05} \in \mathbb{C}$ such that*

$$Z^3 \equiv \begin{pmatrix} \gamma_{03} & \gamma_{13} & \gamma_{04} & \gamma_{23} & \gamma_{14} & \gamma_{05} \end{pmatrix}^{\mathrm{t}} \tag{3.12}$$

$$\overline{Z}Z^2 \equiv \begin{pmatrix} \gamma_{12} & \gamma_{22} & \gamma_{13} & \overline{\gamma}_{23} & \gamma_{23} & \gamma_{14} \end{pmatrix}^{\mathrm{t}} \tag{3.13}$$

satisfy

$$\overline{Z}Z^2 = AZ + BZ^2 + C\overline{Z}Z + DZ^3. \tag{3.14}$$

Proof. Since $\overline{Z}Z = A\mathbb{1} + BZ + C\overline{Z} + DZ^2$ in $\mathcal{C}_{M(2)}$, it follows immediately that $[\overline{Z}Z^2]_3 = [AZ+BZ^2+C\overline{Z}Z+DZ^3]_3$, e.g., $\gamma_{13} = A\gamma_{02}+B\gamma_{03}+C\gamma_{12}+D\gamma_{04}$ is inherent in (3.1). Suppose $\gamma_{23} \in \mathbb{C}$ satisfies $\gamma_{32} \equiv \overline{\gamma}_{23} = A\gamma_{21}+B\gamma_{22}+C\gamma_{31}+D\gamma_{23}$. It follows that (3.14) holds if and only if we set

$$\gamma_{14} := \frac{1}{D}(\gamma_{23} - (A\gamma_{12} + B\gamma_{13} + C\gamma_{22})) \tag{3.15}$$

$$\gamma_{05} := \frac{1}{D}(\gamma_{14} - (A\gamma_{03} + B\gamma_{04} + C\gamma_{13})). \tag{3.16}$$ ∎

COROLLARY *Suppose $M(2) \geqslant 0$, $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is independent, and $\overline{Z}Z = A\mathbb{1} + BZ + C\overline{Z} + DZ^2$, with $D \neq 0$. If $|D| \neq 1$, then $\gamma^{(4)}$ admits a unique 4-atomic (minimal) representing measure.*

Proof. Theorem 3.1 and Lemma 3.2 imply that $M(2)$ admits a flat extension $M(3)$, and any such flat extension is recursively generated (by [6]) and is uniquely determined by $B(3)$. Lemmas 3.2 and 3.3 imply that $B(3)$ is itself uniquely determined, so it follows that $M(2)$ admits a unique flat extension and that $\gamma^{(4)}$ admits a unique 4-atomic (minimal) representing measure (Corollary 5.1 from [6]). ∎

EXAMPLE Consider the moment matrix

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\mathrm{i} & \mathrm{i} & -\frac{3\mathrm{i}}{2} \\ 0 & 0 & 1 & \frac{3\mathrm{i}}{2} & -\mathrm{i} & \mathrm{i} \\ 0 & \mathrm{i} & -\frac{3\mathrm{i}}{2} & \frac{10}{3} & -\frac{8}{3} & \frac{37}{12} \\ 1 & -\mathrm{i} & \mathrm{i} & -\frac{8}{3} & \frac{10}{3} & -\frac{8}{3} \\ 0 & \frac{3\mathrm{i}}{2} & -\mathrm{i} & \frac{37}{12} & -\frac{8}{3} & \frac{10}{3} \end{pmatrix}.$$

In $\mathcal{C}_{M(2)}$, $\overline{Z}Z = \mathbb{1} - \mathrm{i}Z + 2\mathrm{i}\overline{Z} - 2Z^2$, so the hypotheses of Corollary 3.4 are satisfied with $D = -2$. It follows that the associated $\gamma^{(4)}$ admits a unique 4-atomic (minimal) representing measure. In fact, (3.10), (3.15), and (3.16) determine a unique flat extension corresponding to the choices $\gamma_{23} = -\frac{23}{3}\mathrm{i}$, $\gamma_{14} = 8\mathrm{i}$ and $\gamma_{05} = -\frac{179}{24}\mathrm{i}$. A calculation using the Flat Extension Theorem shows that the characteristic polynomial is

$$g(z) \equiv z^4 + \frac{5\mathrm{i}}{2}z^3 - \frac{z^2}{6} - \frac{3\mathrm{i}}{2}z + \frac{2}{3},$$

with roots $z_0 = -\frac{3+\sqrt{5}}{2}\mathrm{i}$, $z_1 = -\frac{3-\sqrt{5}}{2}\mathrm{i}$, $z_2 = \frac{3\mathrm{i}-\sqrt{87}}{12}$ and $z_3 = \frac{3\mathrm{i}+\sqrt{87}}{12}$. An application of the Vandermonde equation (1.9) yields densities $\rho_0 \cong 0.063$, $\rho_1 \cong 0.109$, $\rho_2 = \rho_3 \cong 0.414$. ∎

We next illustrate cases with $|D| = 1$.

Example For $r, s \in \mathbb{R}$ with $r > s^2 = 1$, consider

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & s \\ 0 & 0 & 1 & s & 0 & 0 \\ 0 & 0 & s & r & 1-r & r-1 \\ 1 & 0 & 0 & 1-r & r & 1-r \\ 0 & s & 0 & r-1 & 1-r & r \end{pmatrix}.$$

Note that in $\mathcal{C}_{M(2)}$, $\overline{Z}Z = \mathbb{1} + s\overline{Z} - Z^2$, so the hypothesis of Theorem 3.1 is satisfied with $A = 1$, $B = 0$, $C = s$, $D = -1$. Theorem 3.1 (v) entails $\overline{\gamma}_{23} = s(1-r) - \gamma_{23}$, or $\operatorname{Re}\gamma_{23} = \frac{s(1-r)}{2}$. A calculation shows that with $\gamma_{23} = \frac{s(1-r)}{2}$, there is a unique flat extension $M(3)$, corresponding to $Z^2\overline{Z} = Z + s\overline{Z}Z - Z^3$. Indeed, $M(3)$ is determined (via Lemma 3.3) by $\gamma_{14} = \frac{s(3r-1)}{2}$, $\gamma_{05} = \frac{5s(1-r)}{2}$, and by $Z^3 = s\mathbb{1} + (1-r)Z + (r+\frac{1}{2})\overline{Z} - \frac{3}{2}sZ^2$ in $\mathcal{C}_{M(3)}$.

A calculation using the Flat Extension Theorem yields the characteristic polynomial

$$g(z) \equiv \frac{1}{2}(1-2r) - rsz - 2(1-r)z^2 + \frac{1}{2}sz^3 + z^4.$$

In the specific case of $r = 2$, $s = 1$, we obtain $z_0 = -\frac{1}{2}$, $z_1 = 1$, $z_2 = \frac{1}{2}(-1 + \sqrt{11}\,\mathrm{i})$ and $z_3 = \frac{1}{2}(-1 - \sqrt{11}\,\mathrm{i})$, with corresponding densities $\rho_0 = \frac{16}{33}$, $\rho_1 = \frac{1}{3}$, $\rho_2 = \rho_3 = \frac{1}{11}$. ∎

Example Let

$$M(2) := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 1 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & \mathrm{i} & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & -\mathrm{i} & 0 & 1 & 1 & 2 \end{pmatrix}.$$

In this case, $\overline{Z}Z = \mathbb{1} + \mathrm{i}\overline{Z} + Z^2$, so $A = 1$, $B = 0$, $C = \mathrm{i}$ and $D = 1$ in Theorem 3.1. Let $J \equiv J_{-\mathrm{i}}$ be the diagonal matrix defined in Proposition 1.10. Observe that

$$J^*M(2)J = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \\ 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \end{pmatrix},$$

which is the moment matrix of Example 3.6 with $r = 2$, $s = 1$. For the latter problem we have already obtained a representing measure $\mu \equiv \sum_{i=0}^{3} \rho_i \delta_{z_i}$. Proposition 1.10 (vi) thus implies that $M(2)$ admits a representing measure $\widetilde{\mu} \equiv \sum_{i=0}^{3} \rho_i \delta_{\widetilde{z}_i}$, where $\widetilde{z}_i := -\mathrm{i}z_i$ $(i = 0, 1, 2, 3)$; concretely, $\widetilde{z}_0 = \frac{\mathrm{i}}{2}$, $\widetilde{z}_1 = -\mathrm{i}$, $\widetilde{z}_2 = \frac{1}{2}(\sqrt{11} + \mathrm{i})$ and $\widetilde{z}_3 = \frac{1}{2}(-\sqrt{11} + \mathrm{i})$. ∎

EXAMPLE For $f > 1$, let

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & f & f-1 & f-1 \\ 1 & 0 & 0 & f-1 & f & f-1 \\ 0 & 1 & 0 & f-1 & f-1 & f \end{pmatrix}.$$

It is straightforward to verify that $M(2)$ is positive, $\{\mathbb{1}, Z, \overline{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$, and

$$\overline{Z}Z = \mathbb{1} - \overline{Z} + Z^2; \tag{3.17}$$

thus $A = 1$, $B = 0$, $C = -1$, $D = 1$. Theorem 3.1 (v) entails $\overline{\gamma}_{23} = \gamma_{21} - \gamma_{31} + \gamma_{23}$, or $\mathrm{i}\,\mathrm{Im}\,\gamma_{23} = \frac{f-1}{2}$ (> 0). It follows from Theorem 3.1 that $\gamma^{(4)}$ admits no finitely atomic representing measure. Alternately, the nonexistence of a representing measure follows from (3.17) and the fact that $\mathrm{card}\{z : \overline{z}z = 1 - \overline{z} + z^2\} = 3 < 4 = \mathrm{rank}\, M(2)$ (Corollary 3.7 from [6]). ∎

EXAMPLE For $f > 1$, consider the moment matrix

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 1 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & \mathrm{i} & f & 1-f & -1+f \\ 1 & 0 & 0 & 1-f & f & 1-f \\ 0 & -\mathrm{i} & 0 & -1+f & 1-f & f \end{pmatrix}.$$

A calculation shows that $A = 1$, $B = 0$, $C = -\mathrm{i}$ and $D = -1$ in this case. Moreover, $J_{-\mathrm{i}}^* M(2) J_{-\mathrm{i}}$ is the moment matrix in Example 3.8. Since that matrix admits no representing measure, Proposition 1.10 readily implies that the same is true for $M(2)$. Thus, $\gamma^{(4)}$ admits no representing measure. ∎

We now begin the proof of Theorem 3.1 (v) $\Rightarrow$ (iii).

Let γ_{23} be a solution to (3.10); Lemma 3.3 implies that the B-block of a recursively generated extension $M(3)$ is uniquely determined by (3.12)–(3.16). In order that $M(3) \geqslant 0$, we require $\mathrm{Ran}\, B(3) \subseteq \mathrm{Ran}\, M(2)$. Thus it is necessary that there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that in $\mathcal{C}_{M(2)}$,

$$[\overline{Z}Z^2]_6 = \alpha\mathbb{1} + \beta Z + \gamma\overline{Z} + \delta Z^2. \tag{3.18}$$

Since $M \equiv [M(2)]_4 > 0$, $\alpha, \beta, \gamma, \delta$ are uniquely determined by

$$(\alpha, \beta, \gamma, \delta)^{\mathrm{t}} = M^{-1}(\gamma_{12}, \gamma_{22}, \gamma_{13}, \overline{\gamma}_{23})^{\mathrm{t}}. \tag{3.19}$$

To establish $\overline{Z}Z^2 \in \mathrm{Ran}\, M(2)$, it thus suffices to verify that (3.19) implies (3.18), i.e.,

$$\gamma_{23} = \alpha\gamma_{11} + \beta\gamma_{12} + \gamma\gamma_{21} + \delta\gamma_{13} \tag{3.20}$$

$$\gamma_{14} = \alpha\gamma_{02} + \beta\gamma_{03} + \gamma\gamma_{12} + \delta\gamma_{04}. \tag{3.21}$$

To establish these relations we first write out (3.19) in detail:

$$\begin{aligned} \gamma_{12} &= \alpha\gamma_{00} + \beta\gamma_{01} + \gamma\gamma_{10} + \delta\gamma_{02}, \\ \gamma_{22} &= \alpha\gamma_{10} + \beta\gamma_{11} + \gamma\gamma_{20} + \delta\gamma_{12}, \\ \gamma_{13} &= \alpha\gamma_{01} + \beta\gamma_{02} + \gamma\gamma_{11} + \delta\gamma_{03}, \\ \gamma_{32} &= \alpha\gamma_{20} + \beta\gamma_{21} + \gamma\gamma_{30} + \delta\gamma_{22}. \end{aligned} \tag{3.22}$$

LEMMA $\gamma_{23} = \alpha\gamma_{11} + \beta\gamma_{12} + \gamma\gamma_{21} + \delta\gamma_{13}$.

Proof. From (3.10) and (3.22),

$$\begin{aligned} \gamma_{23} &= \overline{A}\gamma_{12} + \overline{B}\gamma_{22} + \overline{C}\gamma_{13} + \overline{D}\gamma_{32} \\ &= \overline{A}(\alpha\gamma_{00} + \beta\gamma_{01} + \gamma\gamma_{10} + \delta\gamma_{02}) + \overline{B}(\alpha\gamma_{10} + \beta\gamma_{11} + \gamma\gamma_{20} + \delta\gamma_{12}) \\ &\quad + \overline{C}(\alpha\gamma_{01} + \beta\gamma_{02} + \gamma\gamma_{11} + \delta\gamma_{03}) + \overline{D}(\alpha\gamma_{20} + \beta\gamma_{21} + \gamma\gamma_{30} + \delta\gamma_{22}) \\ &= \alpha[A\gamma_{00} + B\gamma_{01} + C\gamma_{10} + D\gamma_{02}]^- + \beta[A\gamma_{10} + B\gamma_{11} + C\gamma_{20} + D\gamma_{12}]^- \\ &\quad + \gamma[A\gamma_{01} + B\gamma_{02} + C\gamma_{11} + D\gamma_{03}]^- + \delta[A\gamma_{20} + B\gamma_{21} + C\gamma_{30} + D\gamma_{22}]^- \\ &= \alpha\gamma_{11} + \beta\gamma_{12} + \gamma\gamma_{21} + \delta\gamma_{13} \quad \text{(from (3.1))}. \quad \blacksquare \end{aligned}$$

LEMMA $\gamma_{14} = \alpha\gamma_{02} + \beta\gamma_{03} + \gamma\gamma_{12} + \delta\gamma_{04}$.

Proof. From (3.15) and Lemma 3.10,

$$\begin{aligned} \gamma_{14} &= \frac{1}{D}(\gamma_{23} - (A\gamma_{12} + B\gamma_{13} + C\gamma_{22})) \\ &= \frac{1}{D}(\alpha\gamma_{11} + \beta\gamma_{12} + \gamma\gamma_{21} + \delta\gamma_{13} - (A\gamma_{12} + B\gamma_{13} + C\gamma_{22})) \\ &= \frac{1}{D}(\alpha\gamma_{11} + \beta\gamma_{12} + \gamma\gamma_{21} + \delta_{13} - A(\alpha\gamma_{00} + \beta\gamma_{01} + \gamma\gamma_{10} + \delta\gamma_{02}) \\ &\quad - B(\alpha\gamma_{01}+\beta\gamma_{02}+\gamma\gamma_{11}+\delta\gamma_{03}) - C(\alpha\gamma_{10}+\beta\gamma_{11}+\gamma\gamma_{20}+\delta\gamma_{12})) \quad \text{(by (3.22))} \\ &= \frac{1}{D}(\alpha(\gamma_{11} - (A\gamma_{00} + B\gamma_{01} + C\gamma_{10})) + \beta(\gamma_{12} - (A\gamma_{01} + B\gamma_{02} + C\gamma_{11})) \\ &\quad + \gamma(\gamma_{21} - (A\gamma_{10} + B\gamma_{11} + C\gamma_{20})) + \delta(\gamma_{13} - (A\gamma_{02} + B\gamma_{03} + C\gamma_{12})) \\ &= \alpha\gamma_{02} + \beta\gamma_{03} + \gamma\gamma_{12} + \delta\gamma_{04} \quad \text{(by (3.1))}. \quad \blacksquare \end{aligned}$$

We now have $\overline{Z}Z^2, Z^3 \in \operatorname{Ran} M(2)$.

We next define the remaining columns of $B(3)$ and show that they belong to $\operatorname{Ran} M(2)$.

Define

$$\overline{Z}^2 Z := (\,\gamma_{21} \quad \gamma_{31} \quad \gamma_{22} \quad \gamma_{41} \quad \gamma_{32} \quad \gamma_{23}\,)^{\mathrm{t}}, \tag{3.23}$$

where $\gamma_{32} := \overline{\gamma}_{23}$ and $\gamma_{41} := \overline{\gamma}_{14}$.

Lemma *If* $R, S, T, U \in \mathbb{C}$ *satisfy* $\overline{Z}Z^2 = R\mathbb{1} + SZ + T\overline{Z} + UZ^2$ *in* $\mathcal{C}_{M(2)}$, *then* $\overline{Z}^2 Z = \overline{R}\mathbb{1} + \overline{S}\,\overline{Z} + \overline{T}Z + \overline{U}\,\overline{Z}^2$ *in* $\mathcal{C}_{M(2)}$, *whence* $\overline{Z}^2 Z \in \operatorname{Ran} M(2)$.

Proof. The relation $\overline{Z}Z^2 = R\mathbb{1} + SZ + T\overline{Z} + UZ^2$ is equivalent to

$$\begin{aligned}
\gamma_{12} &= R\gamma_{00} + S\gamma_{01} + T\gamma_{10} + U\gamma_{02}, & \gamma_{22} &= R\gamma_{10} + S\gamma_{11} + T\gamma_{20} + U\gamma_{12},\\
\gamma_{13} &= R\gamma_{01} + S\gamma_{02} + T\gamma_{11} + U\gamma_{03}, & \gamma_{32} &= R\gamma_{20} + S\gamma_{21} + T\gamma_{30} + U\gamma_{22},\\
\gamma_{23} &= R\gamma_{11} + S\gamma_{12} + T\gamma_{21} + U\gamma_{13}, & \gamma_{14} &= R\gamma_{02} + S\gamma_{03} + T\gamma_{12} + U\gamma_{04}.
\end{aligned}$$

By conjugating these relations we immediately obtain $\overline{Z}^2 Z = \overline{R}\mathbb{1} + \overline{S}\,\overline{Z} + \overline{T}Z + \overline{U}\,\overline{Z}^2$. ∎

Now define

$$\overline{Z}^3 := (\gamma_{30} \quad \gamma_{40} \quad \gamma_{31} \quad \gamma_{50} \quad \gamma_{41} \quad \gamma_{32})^{\mathrm{t}}, \quad \text{where } \gamma_{50} := \overline{\gamma}_{05}. \tag{3.24}$$

Lemma $\overline{Z}^2 Z = \overline{A}\,\overline{Z} + \overline{B}\,\overline{Z}^2 + \overline{C}\,\overline{Z}Z + \overline{D}\,\overline{Z}^3$, *whence*

$$\overline{Z}^3 = \frac{1}{\overline{D}}(\overline{Z}^2 Z - \overline{A}Z - \overline{B}\,\overline{Z}^2 - \overline{C}\,\overline{Z}Z).$$

In particular, if $\overline{Z}^2 Z \in \operatorname{Ran} M(2)$, *then* $\overline{Z}^3 \in \operatorname{Ran} M(2)$.

Proof. From Lemma 3.3 (3.14), we have $\overline{Z}Z^2 = AZ + BZ^2 + C\overline{Z}Z + DZ^3$, which entails

$$\begin{cases}
\gamma_{12} = A\gamma_{01} + B\gamma_{02} + C\gamma_{11} + D\gamma_{03},\\
\gamma_{22} = A\gamma_{11} + B\gamma_{12} + C\gamma_{21} + D\gamma_{13},\\
\gamma_{13} = A\gamma_{02} + B\gamma_{03} + C\gamma_{12} + D\gamma_{04},\\
\gamma_{32} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23},\\
\gamma_{23} = A\gamma_{12} + B\gamma_{13} + C\gamma_{22} + D\gamma_{14},\\
\gamma_{14} = A\gamma_{03} + B\gamma_{04} + C\gamma_{13} + D\gamma_{05}.
\end{cases} \tag{3.25}$$

By conjugating these relations, we immediately obtain $\overline{Z}^2 Z = \overline{A}\,\overline{Z} + \overline{B}\,\overline{Z}^2 + \overline{C}\,\overline{Z}Z + \overline{D}\,\overline{Z}^3$. ∎

By combining Lemmas 3.3, 3.10–3.13 we see that (3.10) implies the existence of a unique moment matrix block $B(3)$ satisfying $\operatorname{Ran} B(3) \subseteq \operatorname{Ran} M(2)$. Our next goal is to show that $B(3)$ determines a unique flat extension $M(3)$. We require the following result.

Lemma *In* $\mathcal{C}_{(\,M(2) \quad B(3)\,)}$,

$$\overline{Z}^2 Z = A\overline{Z} + B\overline{Z}Z + C\overline{Z}^2 + D\overline{Z}Z^2. \tag{3.26}$$

Proof. Since $\overline{Z}Z = A\mathbb{1} + BZ + C\overline{Z} + DZ^2$ in $\mathcal{C}_{M(2)}$, relation (3.26) holds through the first three rows by virtue of moment matrix structure (cf. Proposition 2.3 from [6]). It thus remains to establish (3.26) in the last three rows, i.e.,

$$\begin{cases}
\gamma_{41} = A\gamma_{30} + B\gamma_{31} + C\gamma_{40} + D\gamma_{32},\\
\gamma_{32} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23},\\
\gamma_{23} = A\gamma_{12} + B\gamma_{13} + C\gamma_{22} + D\gamma_{14}.
\end{cases} \tag{3.27}$$

The latter two identities follow immediately from (3.14). To prove (3.27), we will establish the conjugate identity $\gamma_{14} = \overline{A}\gamma_{03} + \overline{B}\gamma_{13} + \overline{C}\gamma_{04} + \overline{D}\gamma_{23}$. From (3.25),

$$\begin{aligned}
&\overline{A}\gamma_{03} + \overline{B}\gamma_{13} + \overline{C}\gamma_{04} + \overline{D}\gamma_{23} \\
&\quad = \overline{A}\frac{1}{D}(\gamma_{12} - (A\gamma_{01} + B\gamma_{02} + C\gamma_{11})) + \overline{B}\frac{1}{D}(\gamma_{22} - (A\gamma_{11} + B\gamma_{12} + C\gamma_{21})) \\
&\qquad + \overline{C}\frac{1}{D}(\gamma_{13} - (A\gamma_{02} + B\gamma_{03} + C\gamma_{12})) + \overline{D}\frac{1}{D}(\gamma_{32} - (A\gamma_{21} + B\gamma_{22} + C\gamma_{31})) \\
&\quad = \frac{\overline{A}}{D}(\alpha\gamma_{00} + \beta\gamma_{01} + \gamma\gamma_{10} + \delta\gamma_{02} - (A\gamma_{01} + B\gamma_{02} + C\gamma_{11})) \\
&\qquad + \frac{\overline{B}}{D}(\alpha\gamma_{10} + \beta\gamma_{11} + \gamma\gamma_{20} + \delta\gamma_{12} - (A\gamma_{11} + B\gamma_{12} + C\gamma_{21})) \\
&\qquad + \frac{\overline{C}}{D}(\alpha\gamma_{01} + \beta\gamma_{02} + \gamma\gamma_{11} + \delta\gamma_{03} - (A\gamma_{02} + B\gamma_{03} + C\gamma_{12})) \\
&\qquad + \frac{\overline{D}}{D}(\alpha\gamma_{20} + \beta\gamma_{21} + \gamma\gamma_{30} + \delta\gamma_{22} - (A\gamma_{21} + B\gamma_{22} + C\gamma_{31})) \quad \text{(by (3.22))} \\
&\quad = \frac{1}{D}[\alpha(\overline{A}\gamma_{00} + \overline{B}\gamma_{10} + \overline{C}\gamma_{01} + \overline{D}\gamma_{20}) + \beta(\overline{A}\gamma_{01} + \overline{B}\gamma_{11} + \overline{C}\gamma_{02} + \overline{D}\gamma_{21}) \\
&\qquad + \gamma(\overline{A}\gamma_{10} + \overline{B}\gamma_{20} + \overline{C}\gamma_{11} + \overline{D}\gamma_{30}) + \delta(\overline{A}\gamma_{02} + \overline{B}\gamma_{12} + \overline{C}\gamma_{03} + \overline{D}\gamma_{22}) \\
&\qquad - A(\overline{A}\gamma_{01} + \overline{B}\gamma_{11} + \overline{C}\gamma_{02} + \overline{D}\gamma_{21}) - B(\overline{A}\gamma_{02} + \overline{B}\gamma_{12} + \overline{C}\gamma_{03} + \overline{D}\gamma_{22}) \\
&\qquad - C(\overline{A}\gamma_{11} + \overline{B}\gamma_{21} + \overline{C}\gamma_{12} + \overline{D}\gamma_{31})] \\
&\quad = \frac{1}{D}[\alpha\gamma_{11} + \beta\gamma_{12} + \gamma\gamma_{21} + \delta\gamma_{13}] - \frac{1}{D}[A\gamma_{12} + B\gamma_{13} + C\gamma_{22}] \quad \text{(by (3.1))} \\
&\quad = \frac{1}{D}[\gamma_{23} - (\gamma_{23} - D\gamma_{14})] \\
&\quad = \gamma_{14} \quad \text{(by (3.22) and (3.14))}. \qquad ∎
\end{aligned}$$

LEMMA *Suppose* $M \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \in \mathcal{M}_{m(n+1)}$ *is a flat extension of the positive matrix* $A \in \mathcal{M}_{m(n)}$. *If* $\sum_{0\leqslant i+j\leqslant n} a_{ij}[\overline{Z}^i Z^j]_n + \sum_{i+j=n+1} b_{ij}[\overline{Z}^i Z^j]_n = 0$ *in* $\mathcal{C}_{\begin{pmatrix} A & B \end{pmatrix}}$, *then* $\sum_{0\leqslant i+j\leqslant n} a_{ij}\overline{Z}^i Z^j + \sum_{i+j=n+1} b_{ij}\overline{Z}^i Z^j = 0$ *in* $\mathcal{C}_M$.

Proof. Since $M \geqslant 0$, $\operatorname{Ran} B \subseteq \operatorname{Ran} A$, so for $i + j = n + 1$, there exists $p_{ij} \in \mathcal{P}_n$ such that $[\overline{Z}^i Z^j]_n = p_{ij}(Z, \overline{Z})$. Since $M \geqslant 0$ and $\sum a_{ij}[\overline{Z}^i Z^j]_n + \sum b_{ij}p_{ij}(Z, \overline{Z}) = 0$ in $\mathcal{C}_A$, the Extension Principle (Proposition 2.4 from [11]) implies that $\sum a_{ij}\overline{Z}^i Z^j + \sum b_{ij}p_{ij}(Z, \overline{Z}) = 0$ in $\mathcal{C}_M$. Since M is a flat extension of A, for $i + j = n + 1$, $p_{ij}(Z, \overline{Z}) = \overline{Z}^i Z^j$ in $\mathcal{C}_M$, and the result follows. ∎

LEMMA *Assume* $M(2) \geqslant 0$ *admits a moment matrix extension block* $B(3)$ *such that*

$$B(3) = M(2)W \quad \text{for some matrix } W.$$

Assume also that there exist scalars A, B, C, D *such that in* $\mathcal{C}_{\begin{pmatrix} M(2) & B(3) \end{pmatrix}}$,

(i) $\overline{Z}Z^2 = AZ + BZ^2 + C\overline{Z}Z + DZ^3$, *and*
(ii) $\overline{Z}^2Z = A\overline{Z} + B\overline{Z}Z + C\overline{Z}^2 + D\overline{Z}Z^2$.
Then $C \equiv C(3) = W^*M(2)W$ *is Toeplitz.*

Proof. From [8], it suffices to prove that in $M \equiv \begin{pmatrix} M(2) & B(3) \\ B(3)^* & C(3) \end{pmatrix}$ we have

$$\langle \overline{Z}Z^2, \overline{Z}^2Z\rangle_M = \langle \overline{Z}^2Z, \overline{Z}^3\rangle_M \tag{3.28}$$

$$\langle Z^3, Z^3\rangle_M = \langle \overline{Z}Z^2, \overline{Z}Z^2\rangle_M. \tag{3.29}$$

We first establish (3.28). Lemma 3.15 and (i) imply that $\overline{Z}Z^2 = AZ + BZ^2 + C\overline{Z}Z + DZ^3$ in $\mathcal{C}_M$, so

$$\begin{aligned} \langle \overline{Z}Z^2, \overline{Z}^2Z\rangle_M &= \langle AZ + BZ^2 + C\overline{Z}Z + DZ^3, \overline{Z}^2Z\rangle_M \\ &= A\langle Z, \overline{Z}^2Z\rangle_M + B\langle Z^2, \overline{Z}^2Z\rangle_M + C\langle \overline{Z}Z, \overline{Z}^2Z\rangle_M + D\langle Z^3, \overline{Z}^2Z\rangle_M. \end{aligned} \tag{3.30}$$

Since $B(3)^*$ is a moment matrix block, $\langle Z, \overline{Z}^2Z\rangle_M = \langle \overline{Z}, \overline{Z}^3\rangle_M$, $\langle Z^2, \overline{Z}^2Z\rangle_M = \langle \overline{Z}Z, \overline{Z}^3\rangle_M$, and $\langle \overline{Z}Z, \overline{Z}^2Z\rangle_M = \langle \overline{Z}^2, \overline{Z}^3\rangle_M$. Further, Theorem 2.1(4) from [6] implies that $\langle Z^3, \overline{Z}^2Z\rangle_M = \langle \overline{Z}Z^2, \overline{Z}^3\rangle_M$; thus (3.30) implies $\langle \overline{Z}Z^2, \overline{Z}^2Z\rangle_M = \langle A\overline{Z} + B\overline{Z}Z + C\overline{Z}^2 + D\overline{Z}Z^2, \overline{Z}^3\rangle_M = \langle \overline{Z}^2Z, \overline{Z}^3\rangle_M$, since, by Lemma 3.5 and (ii),

$$\overline{Z}^2Z = A\overline{Z} + B\overline{Z}Z + C\overline{Z}^2 + D\overline{Z}Z^2 \quad \text{in } \mathcal{C}_M. \tag{3.31}$$

Next, to establish (3.29), Lemma 3.15 and (i) imply that

$$\langle Z^3, Z^3\rangle_M = \frac{1}{D}\langle \overline{Z}Z^2, Z^3\rangle_M - \frac{A}{D}\langle Z, Z^3\rangle_M - \frac{B}{D}\langle Z^2, Z^3\rangle_M - \frac{C}{D}\langle \overline{Z}Z, Z^3\rangle_M. \tag{3.32}$$

Since $C = C^*$, it follows from Proposition 2.3 of [6] and (3.28) that $\langle \overline{Z}Z^2, Z^3\rangle_M = \langle \overline{Z}^2Z, \overline{Z}Z^2\rangle_M$. Thus (3.32) and the moment matrix structure of $B(3)^*$ imply that

$$\begin{aligned} &\langle Z^3, Z^3\rangle_M \\ &= \frac{1}{D}\langle \overline{Z}^2Z, \overline{Z}Z^2\rangle_M - \frac{A}{D}\langle \overline{Z}, \overline{Z}Z^2\rangle_M - \frac{B}{D}\langle \overline{Z}Z, \overline{Z}Z^2\rangle_M - \frac{C}{D}\langle \overline{Z}^2, \overline{Z}Z^2\rangle_M \\ &= \Big\langle \frac{1}{D}(\overline{Z}^2Z - A\overline{Z} - B\overline{Z}Z - C\overline{Z}^2), \overline{Z}Z^2\Big\rangle_M = \langle \overline{Z}Z^2, \overline{Z}Z^2\rangle_M \quad \text{(by (3.31))}. \quad \blacksquare \end{aligned}$$

Proof of Theorem 3.1. (v) $\Rightarrow$ (iii). Given γ_{23} satisfying (3.10), Lemmas 3.3, 3.10, 3.11, 3.12 and 3.13 establish the existence of a unique moment matrix block $B(3)$ for a recursively generated extension $M(3) \geqslant 0$, and $\operatorname{Ran} B(3) \subseteq \operatorname{Ran} M(2)$. Lemma 3.16 shows that $B(3)$ corresponds to a flat extension $M(3)$, so from [6], $\gamma^{(4)}$ admits a 4-atomic representing measure, which is minimal since $\operatorname{rank} M(2) = 4$.

(iii) $\Rightarrow$ (iv). This follows from Theorem 5.4 of [6], Theorem 1.6 of [8], and the Extension Principle (Proposition 2.4, [11]).

((iv) $\Rightarrow$ (v). This follows from (3.6) (or (3.10)).

(iii) $\Rightarrow$ (ii). This follows from [6] since $\operatorname{rank} M(2) = 4$.

(i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). (i) $\Rightarrow$ (iv) follows from [6] and the remaining implications follow as above. $\blacksquare$

4. THE CASE $\overline{Z}^2 = A\mathbb{1} + BZ + C\overline{Z} + DZ^2 + E\overline{Z}Z$

In this section we analyze the quartic moment problem for Case II, when $M(2) \geqslant 0$ and $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$. In the sequel we show that the existence of a flat extension $M(3)$ (with a corresponding 5-atomic (minimal) representing measure) is equivalent to the solubility of a single quadratic equation developed from the moment data. In the following section, we then characterize solubility of this equation and establish the existence of representing measures in Case II.

Our hypothesis implies that there exist unique scalars A, B, C, D, E such that in $\mathcal{C}_{M(2)}$ there is a column relation:

$$\overline{Z}^2 = A\mathbb{1} + BZ + C\overline{Z} + DZ^2 + E\overline{Z}Z. \tag{4.1}$$

If $D = 0$, then Lemma 3.10 from [6] implies that $Z^2 = \overline{A}\mathbb{1} + \overline{B}\,\overline{Z} + \overline{C}Z + \overline{E}\,\overline{Z}Z$, which contradicts the basis hypothesis; we may thus assume $D \neq 0$. Lemma 3.10 from [6] now implies that

$$\overline{Z}^2 = -\frac{\overline{A}}{\overline{D}}\mathbb{1} - \frac{\overline{C}}{\overline{D}}Z - \frac{\overline{B}}{\overline{D}}\overline{Z} + \frac{1}{\overline{D}}Z^2 - \frac{\overline{E}}{\overline{D}}\overline{Z}Z,$$

whence (4.1) implies

$$A = -\frac{\overline{A}}{\overline{D}}, \quad B = -\frac{\overline{C}}{\overline{D}}, \quad C = -\frac{\overline{B}}{\overline{D}}, \quad D = \frac{1}{\overline{D}}, \quad E = -\frac{\overline{E}}{\overline{D}}. \tag{4.2}$$

In particular, $|D| = 1$. Fix $\lambda \in C$ such that $\lambda^4 D = 1$, and let $W := \overline{\lambda}Z$; observe that $|\lambda| = 1$. Equation (4.1) then becomes

$$\overline{\lambda}^2\overline{W}^2 = A\mathbb{1} + \lambda BW + \overline{\lambda}C\overline{W} + \lambda^2 DW^2 + E\overline{W}W. \tag{4.3}$$

Multiplication by λ^2 in (4.3) leads to $\overline{W}^2 = \lambda^2 A\mathbb{1} + \lambda^3 BW + \lambda C\overline{W} + W^2 + \lambda^2 E\overline{W}W$. It follows that, without loss of generality, we can always assume $D = 1$ in (4.1) (cf. Proposition 1.10). Observe that (4.2) then implies that $A, E \in \mathrm{i}\mathbb{R}$, and $C = -\overline{B}$. We shall use these facts in the proof of Theorem 1.5 (see Lemma 5.1).

Consider now a recursively generated moment matrix extension

$$M(3) = \begin{pmatrix} M(2) & B(3) \\ B(3)^* & C(3) \end{pmatrix} \tag{4.4}$$

(with $B(3)$ and $C(3)$ as in (3.2)). Relation (4.1) and recursiveness imply that in $\mathcal{C}_{B(3)}$ we must have

$$\overline{Z}^2 Z = AZ + BZ^2 + C\overline{Z}Z + DZ^3 + E\overline{Z}Z^2 \tag{4.5}$$

$$\overline{Z}^3 = A\overline{Z} + B\overline{Z}Z + C\overline{Z}^2 + D\overline{Z}Z^2 + E\overline{Z}^2 Z. \tag{4.6}$$

From the form of $B(3)$ (cf. (3.2)), we see that (4.5) entails a choice of $\gamma_{23}, \gamma_{14}, \gamma_{05}$ such that

$$\gamma_{41} \equiv \overline{\gamma}_{14} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23} + E\gamma_{32}, \quad \text{where } \gamma_{32} = \overline{\gamma}_{23}, \tag{4.7}$$

$$\gamma_{32} = A\gamma_{12} + B\gamma_{13} + C\gamma_{22} + D\gamma_{14} + E\gamma_{23}, \tag{4.8}$$

$$\gamma_{23} = A\gamma_{03} + B\gamma_{04} + C\gamma_{13} + D\gamma_{05} + E\gamma_{14}. \tag{4.9}$$

We will show in the sequel that to each $\gamma_{23} \in \mathbb{C}$, there correspond unique γ_{14}, γ_{05} satisfying (4.7)–(4.9), and a unique moment matrix block $B(3) \equiv B(3)[\gamma_{23}]$ satisfying $\operatorname{Ran} B(3) \subseteq \operatorname{Ran} M(2)$.

THEOREM *Suppose $M(2) \geqslant 0$ and $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$. $\gamma^{(4)}$ admits a 5-atomic (minimal) representing measure if and only if there exists $\gamma_{23} \in \mathbb{C}$ such that the C-block of $[M(2)\ ;\ B(3)[\gamma_{23}]]$ satisfies $C_{21} = C_{32}$.*

Before proving Theorem 4.1, we illustrate it with two examples.

EXAMPLE For $d > 1$,

$$\text{let } M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & -d \\ 1 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & -d & 0 & d \end{pmatrix}.$$

It is straightforward to verify that $M(2) \geqslant 0$, $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z\}$ is independent, and $\overline{Z}^2 = -Z^2$ in $C_{M(2)}$, so (4.5)–(4.6) entail $\overline{Z}^2 Z = -Z^3, \overline{Z}^3 = -Z^2 Z$. It follows from (4.7)–(4.9) that for $x = \gamma_{23}$, there is a unique moment matrix block

$$B(3) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d & 0 & -d \\ -d & 0 & d & 0 \\ x & \overline{x} & -x & -\overline{x} \\ -\overline{x} & x & \overline{x} & -x \\ -x & -\overline{x} & x & \overline{x} \end{pmatrix}$$

satisfying (4.5)–(4.6). To see that $\operatorname{Ran} B(3) \subseteq \operatorname{Ran} M(2)$, note that in $\mathcal{C}_{(M(2)\quad B(3))}$,

$$Z^3 = \frac{\overline{x}}{d-1}\mathbb{1} - d\overline{Z} + \frac{x}{d}Z^2 - \frac{\overline{x}}{d-1}\overline{Z}Z \tag{4.10}$$

$$\overline{Z}Z^2 = -\frac{x}{d-1}\mathbb{1} + dZ + \frac{\overline{x}}{d}Z^2 + \frac{x}{d-1}\overline{Z}Z. \tag{4.11}$$

Thus $Z^3, \overline{Z}Z^2 \in \operatorname{Ran} M(2)$, and conjugation implies $\overline{Z}^3, \overline{Z}^2 Z \in \operatorname{Ran} M(2)$. A calculation of the C-block of $[M(2)\ ;\ B(3)]$ shows that

$$C_{21} = \frac{x^2}{d} - \frac{\overline{x}^2}{d-1} \quad \text{and} \quad C_{32} = \frac{-\overline{x}^2}{d} + \frac{x^2}{d-1}.$$

Thus, x corresponds to a flat extension $M(3)$ if and only if $\overline{x}^2 = -x^2$, or $\operatorname{Im}\gamma_{23} = \pm \operatorname{Re}\gamma_{23}$. For example, with $x = 0$, we compute the 5-atomic representing measure by using the Flat Extension Theorem (Theorem 1.6). In the unique flat extension $M(5)$ we have a column relation $Z^5 = -d^2 Z$. The 5-atomic representing measure corresponding to $x = 0$ has atoms $z_0 = 0$ and z_1, z_2, z_3, z_4 equal to the 4th roots of $-d^2$. The weights are $\rho_0 = \frac{d-1}{d}$ and $\rho_i = \frac{1}{4d}$, $1 \leqslant i \leqslant 4$. ∎

EXAMPLE (Example 1.13 revisited) For $M(2)$ as in Example 1.13, a straightforward verification shows that $C_{21} - C_{32} = -4\mathrm{i}$. By Theorem 4.1, $M(2)$ admits no flat extension $M(3)$, and $\gamma^{(4)}$ admits no 5-atomic representing measure. In Example 5.6 we shall find yet another way to establish this, while at the same time showing that $\gamma^{(4)}$ does admit a 6-atomic representing measure. ∎

We begin the proof of Theorem 4.1 by solving (4.7)–(4.9). For $\gamma_{23} \in \mathbb{C}$, define γ_{41} via (4.7) and let $\gamma_{14} = \overline{\gamma}_{41}$. We claim that (4.8) holds. Indeed, (4.7) implies $-D\gamma_{23} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} - \gamma_{41} + E\gamma_{32}$, whence (4.2) yields

$$\gamma_{23} = \overline{A}\gamma_{21} + \overline{C}\gamma_{22} + \overline{B}\gamma_{31} + \overline{D}\gamma_{41} + \overline{E}\gamma_{32},$$

which is equivalent to (4.8). Now, let

$$\gamma_{05} = \frac{1}{D}(\gamma_{23} - A\gamma_{03} - B\gamma_{04} - C\gamma_{13} - E\gamma_{14}),$$

so that (4.9) holds. Corresponding to γ_{23} we may thus construct a unique moment matrix block $B(3)$ satisfying (4.5). Our next goal is to prove that $\operatorname{Ran} B(3) \subseteq \operatorname{Ran} M(2)$.

Since $M \equiv [M(2)]_5$ is positive and invertible, there exist unique scalars $\alpha, \beta, \gamma, \delta, \varepsilon$ such that

$$[Z^3]_5 \equiv (\gamma_{03}, \gamma_{13}, \gamma_{04}, \gamma_{23}, \gamma_{14})^{\mathrm{t}} = M(\alpha, \beta, \gamma, \delta, \varepsilon)^{\mathrm{t}}, \tag{4.12}$$

i.e., $[Z^3]_5 = \alpha[\mathbb{1}]_5 + \beta[Z]_5 + \gamma[\overline{Z}]_5 + \delta[Z^2]_5 + \varepsilon[\overline{Z}Z]_5$. Thus, $Z^3 \in \operatorname{Ran} M(2)$ if and only if, in $\mathcal{C}_{\left(M(2) \quad B(3)\right)}$, $Z^3 = \alpha\mathbb{1} + \beta Z + \gamma\overline{Z} + \delta Z^2 + \varepsilon\overline{Z}Z$, or, equivalently, $\gamma_{05} = \alpha\gamma_{02} + \beta\gamma_{03} + \gamma\gamma_{12} + \delta\gamma_{04} + \varepsilon\gamma_{13}$.

LEMMA $\gamma_{05} = \alpha\gamma_{02} + \beta\gamma_{03} + \gamma\gamma_{12} + \delta\gamma_{04} + \varepsilon\gamma_{13}$.

Proof. Indeed,

$$\begin{aligned}
&\overline{\alpha}\gamma_{20}+\overline{\beta}\gamma_{30}+\overline{\gamma}\gamma_{21}+\overline{\delta}\gamma_{40}+\overline{\varepsilon}\gamma_{31} \\
&\quad = \overline{\alpha}(A\gamma_{00}+B\gamma_{01}+C\gamma_{10}+D\gamma_{02}+E\gamma_{11})+\overline{\beta}(A\gamma_{10}+B\gamma_{11}+C\gamma_{20}+D\gamma_{12}+E\gamma_{21}) \\
&\quad\quad +\overline{\gamma}(A\gamma_{01}+B\gamma_{02}+C\gamma_{11}+D\gamma_{03}+E\gamma_{12})+\overline{\delta}(A\gamma_{20}+B\gamma_{21}+C\gamma_{30}+D\gamma_{22}+E\gamma_{31}) \\
&\quad\quad +\overline{\varepsilon}(A\gamma_{11}+B\gamma_{12}+C\gamma_{21}+D\gamma_{13}+E\gamma_{22}) \\
&\quad = A(\alpha\gamma_{00}+\beta\gamma_{01}+\gamma\gamma_{10}+\delta\gamma_{02}+\varepsilon\gamma_{11})^{-}+B(\alpha\gamma_{10}+\beta\gamma_{11}+\gamma\gamma_{20}+\delta\gamma_{12}+\varepsilon\gamma_{21})^{-} \\
&\quad\quad +C(\alpha\gamma_{01}+\beta\gamma_{02}+\gamma\gamma_{11}+\delta\gamma_{03}+\varepsilon\gamma_{12})^{-}+D(\alpha\gamma_{20}+\beta\gamma_{21}+\gamma\gamma_{30}+\delta\gamma_{22}+\varepsilon\gamma_{31})^{-} \\
&\quad\quad +E(\alpha\gamma_{11}+\beta\gamma_{12}+\gamma\gamma_{21}+\delta\gamma_{13}+\varepsilon\gamma_{22})^{-} \\
&\quad = A\gamma_{30}+B\gamma_{31}+C\gamma_{40}+D\gamma_{32}+E\gamma_{41} \qquad \text{(from (4.12))} \\
&\quad = \gamma_{50} \qquad \text{(from (4.2) and (4.9)).} \quad \blacksquare
\end{aligned}$$

Next, to show $\overline{Z}Z^2 \in \operatorname{Ran} M(2)$, we write $[\overline{Z}Z^2]_5 \equiv (\gamma_{12}, \gamma_{22}, \gamma_{13}, \gamma_{32}, \gamma_{23})^{\mathrm{t}} = M(r, s, t, u, v)^{\mathrm{t}}$, and we must verify that $\overline{Z}Z^2 = r\mathbb{1} + sZ + t\overline{Z} + uZ^2 + v\overline{Z}Z$, i.e., $\gamma_{14} = r\gamma_{02} + s\gamma_{03} + t\gamma_{12} + u\gamma_{04} + v\gamma_{13}$.

LEMMA $\gamma_{14} = r\gamma_{02} + s\gamma_{03} + t\gamma_{12} + u\gamma_{04} + v\gamma_{13}$.

Proof. We have

$$
\begin{aligned}
&\overline{r}\gamma_{20}+\overline{s}\gamma_{30}+\overline{t}\gamma_{21}+\overline{u}\gamma_{40}+\overline{v}\gamma_{31} \\
&\quad = \overline{r}(A\gamma_{00}+B\gamma_{01}+C\gamma_{10}+D\gamma_{02}+E\gamma_{11})+\overline{s}(A\gamma_{10}+B\gamma_{11}+C\gamma_{20}+D\gamma_{12}+E\gamma_{21}) \\
&\qquad +\overline{t}(A\gamma_{01}+B\gamma_{02}+C\gamma_{11}+D\gamma_{03}+E\gamma_{12})+\overline{u}(A\gamma_{20}+B\gamma_{21}+C\gamma_{30}+D\gamma_{22}+E\gamma_{31}) \\
&\qquad +\overline{v}(A\gamma_{11}+B\gamma_{12}+D\gamma_{21}+D\gamma_{13}+E\gamma_{22}) \\
&\quad = A(r\gamma_{00}+s\gamma_{01}+t\gamma_{10}+u\gamma_{02}+v\gamma_{11})^{-}+B(r\gamma_{10}+s\gamma_{11}+t\gamma_{20}+u\gamma_{12}+v\gamma_{21})^{-} \\
&\qquad +C(r\gamma_{01}+s\gamma_{02}+t\gamma_{11}+u\gamma_{03}+v\gamma_{12})^{-}+D(r\gamma_{20}+s\gamma_{21}+t\gamma_{30}+u\gamma_{22}+v\gamma_{31})^{-} \\
&\qquad +E(r\gamma_{11}+s\gamma_{12}+t\gamma_{21}+u\gamma_{13}+v\gamma_{22})^{-} \\
&\quad = A\gamma_{21}+B\gamma_{22}+C\gamma_{31}+D\gamma_{23}+E\gamma_{32} \\
&\quad = \gamma_{41} \qquad\qquad\qquad\qquad \text{(by (4.7))}. \quad \blacksquare
\end{aligned}
$$

Since $Z^3, \overline{Z}Z^2 \in \operatorname{Ran} M(2)$ (by Lemmas 4.4–4.5), it follows from (4.5) that $\overline{Z}^2 Z \in \operatorname{Ran} M(2)$. Moreover, the relation $Z^3 = \alpha\mathbb{1}+\beta Z+\gamma\overline{Z}+\delta Z^2+\varepsilon\overline{Z}Z$ readily implies $\overline{Z}^3 = \overline{\alpha}\mathbb{1}+\overline{\beta}\,\overline{Z}+\overline{\gamma}Z+\overline{\delta}\,\overline{Z}^2+\overline{\varepsilon}\overline{Z}Z$, whence $\overline{Z}^3 \in \operatorname{Ran} M(2)$. We have thus proved the following result:

LEMMA *Corresponding to* $\gamma_{23} \in \mathbb{C}$*, there exists a unique moment matrix block* $B(3) \equiv B(3)[\gamma_{23}]$ *(determined by* (4.5) and (4.7)–(4.9)) such that $\operatorname{Ran} B(3) \subseteq \operatorname{Ran} M(2)$.

In order to analyze the C-block of $[M(2)\ ;\ B(3)[\gamma_{23}]]$, we require the following result, which establishes (4.6).

LEMMA *In* $\mathcal{C}_{\left(\begin{smallmatrix} M(2) & B(3) \end{smallmatrix}\right)}$,

$$\overline{Z}^3 = A\overline{Z} + B\overline{Z}Z + C\overline{Z}^2 + D\overline{Z}Z^2 + E\overline{Z}^2 Z.$$

Proof. Moment matrix structure implies that the desired relation holds in the first three rows of the indicated columns (cf. [6], Proposition 2.3). The remaining required identities are:

$$
\begin{aligned}
\gamma_{41} &= A\gamma_{21}+B\gamma_{22}+C\gamma_{31}+D\gamma_{23}+E\gamma_{32} && \text{which is (4.7),} \\
\gamma_{32} &= A\gamma_{12}+B\gamma_{13}+C\gamma_{22}+D\gamma_{14}+E\gamma_{23} && \text{which is (4.8), and} \\
\gamma_{50} &= A\gamma_{30}+B\gamma_{31}+C\gamma_{40}+D\gamma_{32}+E\gamma_{41} && \text{which follows from (4.2) and (4.9).} \quad \blacksquare
\end{aligned}
$$

Let us denote the flat extension $[M(2)\ ;\ B(3)[\gamma_{23}]]$ by

$$N = \begin{pmatrix} M(2) & B(3) \\ B(3)^* & C \end{pmatrix}.$$

N is a moment matrix $M(3)$ if and only if C is Toeplitz; in the present context, Proposition 2.3 from [6] implies that C is Toeplitz if and only if $C_{33} = C_{44}$ and $C_{32} = C_{43}$.

Proposition $C_{33} - C_{44} = E(C_{32} - C_{43})$.

Proof. From Lemma 3.15 and (4.5), we have

$$C_{33} = \langle \overline{Z}^2 Z, \overline{Z}^2 Z\rangle_N = \langle AZ + BZ^2 + C\overline{Z}Z + DZ^3 + E\overline{Z}Z^2, \overline{Z}^2 Z\rangle_N$$
$$= A\langle Z, \overline{Z}^2 Z\rangle_N + B\langle Z^2, \overline{Z}^2 Z\rangle_N + C\langle \overline{Z}Z, \overline{Z}^2 Z\rangle_N + D\langle Z^3, \overline{Z}^2 Z\rangle_N + E\langle \overline{Z}Z^2, \overline{Z}^2 Z\rangle_N.$$

Similarly, Lemma 3.15 and (4.6) imply

$$C_{44} = \langle \overline{Z}^3, \overline{Z}^3\rangle_N$$
$$= A\langle \overline{Z}, \overline{Z}^3\rangle_N + B\langle \overline{Z}Z, \overline{Z}^3\rangle_N + C\langle \overline{Z}^2, \overline{Z}^3\rangle_N + D\langle \overline{Z}Z^2, \overline{Z}^3\rangle_N + E\langle \overline{Z}^2 Z, \overline{Z}^3\rangle_N.$$

The moment matrix structure of $B(3)^*$ implies

$$\langle \overline{Z}, \overline{Z}^3\rangle_N = \langle Z, \overline{Z}^2 Z\rangle_N, \quad \langle \overline{Z}Z, \overline{Z}^3\rangle_N = \langle Z^2, \overline{Z}^2 Z\rangle_N,$$

and $\langle \overline{Z}^2, \overline{Z}^3\rangle_N = \langle \overline{Z}Z, \overline{Z}^2 Z\rangle_N$. Further, the structure of any flat extension of a moment matrix $M(2)$ implies that $\langle \overline{Z}Z^2, \overline{Z}^3\rangle_N = C_{42} = C_{31} = \langle Z^3, \overline{Z}^2 Z\rangle_N$. Thus, $C_{33} - C_{44} = E(\langle \overline{Z}Z^2, \overline{Z}^2 Z\rangle_N - \langle \overline{Z}^2 Z, \overline{Z}^3\rangle_N) = E(C_{32} - C_{43})$. ∎

Corollary $C_{11} - C_{22} = E(C_{21} - C_{32})$.

Proof. The result follows from Proposition 4.8 and the fact that in any flat extension of $M(2)$, $C_{11} = C_{44}$, $C_{22} = C_{33}$, $C_{21} = C_{43}$ (Proposition 2.3 from [6]). ∎

Proof of Theorem 4.1. Lemma 4.6 establishes the existence of a moment matrix block $B(3)$ such that $\operatorname{Ran} B(3) \subseteq \operatorname{Ran} M(2)$. The corresponding flat extension $N = [M(2)\ ;\ B(3)]$ is a moment matrix if and only if, in the C-block, we have $C_{11} = C_{22}$ and $C_{21} = C_{32}$ (Proposition 2.3 from [6]). Corollary 4.9 now implies that if $C_{21} = C_{32}$, then N is indeed a moment matrix. ∎

Corollary *If $E = 0$, then for each $\gamma_{23} \in \mathbb{C}$, in the C-block of $[M(2)\ ;\ B(3)[\gamma_{23}]]$, $C_{11} = C_{22}$.*

Corollary *If $E = 0$, $D = 1$, and the moment data are real, then $M(2)$ admits a flat extension.*

Proof. We begin by presenting a parameterization of $M(2)$ whenever $\{1\!\!1, Z, \overline{Z}, Z^2, \overline{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$, $\overline{Z}^2 = A \cdot 1\!\!1 + BZ + C\overline{Z} + Z^2$, and all entries of $M(2)$ are real. Set

$$M(2) \equiv \begin{pmatrix} 1 & a & a & c & b & c \\ a & b & c & x & x & y \\ a & c & b & y & x & x \\ c & x & y & d & r & s \\ b & x & x & r & d & r \\ c & y & x & s & r & d \end{pmatrix}.$$

Since $\{1\!\!1, Z, \overline{Z}\}$ is independent and $M(2) \geqslant 0$, then $\det\begin{pmatrix} 1 & a \\ a & b\end{pmatrix} > 0$ and $\det\begin{pmatrix} b & c \\ c & b\end{pmatrix} > 0$, so $b > a^2$ and $b > |c|$. Consider the nested determinants $d_k := \det[M(2)]_k$, $1 \leqslant k \leqslant 6$. Now, $d_3 \equiv (b + c - 2a^2)(b - c) > 0$ implies $b + c > 2a^2$. Moreover, $d_4 > 0$ implies $d > f(a, b, c, x, y)$, where f is quadratic in

x and y. Similarly, $d_5 > 0$ requires that r remain in the open interval determined by two (distinct) roots of the quadratic equation $d_5(r) \equiv \alpha r^2 + \beta r + \delta = 0$. Indeed, since $\alpha = (2a^2 - b - c)(b - c) < 0$, an interval for r will be found provided $d_5(-\frac{\beta}{2\alpha}) > 0$. A calculation using *Mathematica* reveals that this is the case if and only if

$$b^3 + b^2c + 2a^2d - bd - cd - 4abx + 2x^2 < 0.$$

This in turn entails

$$d > -\frac{b^3 + b^2c - 4abx + 2x^2}{2a^2 - b - c}.$$

Finally, d_6 factors as a product $d_6^{(1)} d_6^{(2)}$, where $d_6^{(1)}(b, c, d, s, x, y)$ is an irreducible quadratic polynomial (and linear in s) and $d_6^{(2)}(a, b, c, d, r, s, x, y)$ is an irreducible quartic polynomial (and linear in s). Since we require $d_6 = 0$, we are led naturally to discuss two cases.

Case 1. $d_6^{(1)} = 0$; here $s = s_1 := \frac{(b-c)d-(x-y)^2}{b-c}$.

Case 2. $d_6^{(2)} = 0$; here $s = s_2$ is a rational expression in a, b, c, d, x, y, and r.

If we now recall that $\overline{Z}^2 = A\mathbb{1} + BZ + C\overline{Z} + Z^2$, we see that $A = 0$ in Case 1, and that Case 2 cannot occur. Thus, we have obtained the following parameterization of $M(2)$: $b > \max\{a^2, c\}$, $b + c > 2a^2$, $d > -\frac{b^3+b^2c-4abx+2x^2}{2a^2-b-c}$, r in an open interval determined by a, b, c, d, x, y, and $s = \frac{(b-c)d-(x-y)^2}{b-c}$. Assuming $M(2)$ is properly parameterized, Lemma 4.6 says that for every choice of $\gamma_{23} \in \mathbb{C}$ there exists a unique moment matrix block B such that $\operatorname{Ran} B \subseteq \operatorname{Ran} M(2)$. Once B has been built, an easy *Mathematica* calculation shows that $C_{21} = C_{32}$. Using Theorem 4.1, it follows that $M(2)$ admits a flat extension. ∎

EXAMPLE (Minimal degree-4 quadrature rules on a parabolic arc) We conclude this section by describing the minimal quadrature rules of degree 4 for arclength measure ν on the segment of the parabola $y = x^2$ corresponding to $0 \leqslant x \leqslant 1$. Let

$$K := \{(x, y) \in \mathbb{R}^2 : y = x^2,\ 0 \leqslant x \leqslant 1\} \equiv \{z \in \mathbb{C} : \operatorname{Im} z = (\operatorname{Re} z)^2,\ 0 \leqslant \operatorname{Re} z \leqslant 1\}.$$

By a K-*quadrature rule* for ν of degree 4 we mean a finite collection of points of K, $(x_0, y_0), \ldots, (x_d, y_d)$, and corresponding positive weights, $\omega_0, \ldots, \omega_d$, such that for every real polynomial $p(x, y)$ of total degree $\leqslant 4$,

$$\int_K p(x, y)\, \mathrm{d}\nu(x, y) \left(\equiv \int_0^1 p(t, t^2) \sqrt{1 + 4t^2}\, \mathrm{d}t \right) = \sum_{i=0}^d \omega_i p(x_i, y_i);$$

a *minimal* quadrature rule is one for which d is as small as possible.

To begin, we complexify the problem; thus, we seek to parameterize the minimal representing measures, supported in K, for the quartic complex moment problem associated with

$$\gamma_{kj} = \int_0^1 (t - \mathrm{i}t)^k (t + \mathrm{i}t)^j \sqrt{1 + 4t^2}\, \mathrm{d}t, \quad 0 \leqslant k + j \leqslant 4.$$

(Note that $\gamma_{00} = \frac{1}{4}[2\sqrt{5}+\ln(2+\sqrt{5})] \neq 1$.) Since $M(2)(\gamma)$ clearly has a representing measure (namely, ν), it follows that $M(2)(\gamma) \geqslant 0$. A calculation shows that $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$; moreover, we have a column relation

$$Z^2 + 2\overline{Z}Z + \overline{Z}^2 + 2\mathrm{i}Z - 2\mathrm{i}\overline{Z} = 0 \tag{4.13}$$

(corresponding to the fact that $P := \{z \in \mathbb{C} : (z+\overline{z})^2 + 2\mathrm{i}(z-\overline{z}) = 0\}$ is the complex equivalent of the parabola $y = x^2$, which contains $\operatorname{supp}\nu$ (cf. [8])).

From Theorem 4.1 and its proof, corresponding to each $\gamma_{23} \equiv r+\mathrm{i}s$, $r, s \in \mathbb{R}$, there exists a unique moment matrix block $B(3)[\gamma_{23}]$ satisfying (4.7)–(4.9) and $\operatorname{Ran} B(3)[\gamma_{23}] \subseteq \operatorname{Ran} M(2)$; moreover, γ_{23} gives rise to a flat extension $M(3)$ if and only if the relation $C_{21} = C_{32}$ holds in the C-block of $[M(2)\ ;\ B(3)[\gamma_{23}]]$. A calculation shows that $\Delta := C_{21} - C_{32}$ is of the form $\Delta = \alpha(r)+\beta(r)s$, $\beta(r) \neq 0$, so $\Delta = 0$ if and only if $s \equiv s(r) := -\frac{\alpha(r)}{\beta(r)}$. Thus, the 5-atomic (minimal) representing measures for $\gamma^{(4)}$ correspond precisely to the flat extensions $M(3)[r]$ determined by $\gamma_{23} = r+\mathrm{i}s(r)$, $r \in \mathbb{R}$, and (4.13) implies that each such measure $\nu[r]$ is supported in P.

Is $\nu[r]$ actually supported in the parabolic arc K determined by $0 \leqslant x \leqslant 1$? To resolve this, we employ results concerning the Truncated Complex K-Moment Problem ([9]). From Proposition 3.10 of [9], $\operatorname{supp}\nu[r] \subseteq K$ if and only if the *localizing matrix* $M_x(3)$ satisfies $M_0(3) \leqslant M_x(3) \leqslant M_1(3)$, where $M_0(3) = 0_{6\times 6}$, $M_1(3) = M(2)(\gamma)$, $M_x(3) = \frac{1}{2}(M_z(3) + M_z(3)^*)$, and $M_z(3)$ is the compression of $M(3)[r]$ to rows $\mathbb{1}$, Z, $\overline{Z}$, $Z^2, \overline{Z}Z, \overline{Z}^2$ and to columns $Z, Z^2, \overline{Z}Z, Z^3, \overline{Z}Z^2, \overline{Z}^2Z$ (*all "multiples" of* Z). Calculations using nested determinants show that $M_x(3) \geqslant 0$ if and only if $r \geqslant r_0 \cong 1.04984$ and that $M_x(3) \leqslant M_1(3)$ if and only if $r \leqslant r_1 \cong 1.04986$. Thus, precisely for r satisfying $r_0 \leqslant r \leqslant r_1$, $\nu[r]$ is a 5-atomic (minimal) representing measure for $\gamma^{(4)}$ supported in K. The minimal K-quadrature rules for ν of degree 4 thus correspond to $\nu[r]$, $r_0 \leqslant r \leqslant r_1$.

For a numerical example, let $r = 1.04985$. Using the Flat Extension Theorem and a *Mathematica* calculation of the flat extension $M(5)$ of $M(3)[r]$, we compute the characteristic polynomial

$$g_\gamma(z) \equiv z^5 - (c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4),$$

where $c_0 \cong -0.00751931+0.0094188\mathrm{i}$, $c_1 \cong 0.152349-0.264943\mathrm{i}$, $c_2 \cong -0.258941+1.72023\mathrm{i}$, $c_3 \cong -1.27715 - 3.46669\mathrm{i}$, $c_4 \cong 2.61617 + 1.9274\mathrm{i}$. The atoms of $\nu[r]$ are the roots of g_γ:

$$\begin{array}{ll} z_0 \cong 0.0532319 + 0.00283364\mathrm{i}, & z_1 \cong 0.259726 + 0.0674577\mathrm{i}, \\ z_2 \cong 0.542852 + 0.294689\mathrm{i}, & z_3 \cong 0.799611 + 0.639378\mathrm{i}, \\ z_4 \cong 0.960749 + 0.923038\mathrm{i}. & \end{array}$$

The corresponding densities, determined from the Vandermonde equation (1.9), are $\rho_0 \cong 0.135271$, $\rho_1 \cong 0.296841$, $\rho_2 \cong 0.420369$, $\rho_3 \cong 0.409633$, $\rho_4 \cong 0.21683$. ∎

5. SOLUTION OF THE CASE $\overline{Z}^2 = A\mathbb{1} + BZ + C\overline{Z} + DZ^2 + E\overline{Z}Z$

We are now ready to prove Theorem 1.5; we thus assume that $M(2) \geqslant 0$ and $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$. The reductions at the beginning of Section 4 allow us to assume that in $\mathcal{C}_{M(2)}$ there is a relation of the form $\overline{Z}^2 = A\mathbb{1} + BZ - \overline{B}\,\overline{Z} + Z^2 + E\overline{Z}Z$, with $A, E \in \mathrm{i}\mathbb{R}$. We begin by examining the associated variety $\{z \in \mathbb{C} : \overline{Z}^2 = A + Bz - \overline{B}\,\overline{z} + z^2 + E\overline{z}z\}$.

LEMMA *Let* $A, E \in \mathrm{i}\mathbb{R}$. *Then*

$$\overline{Z}^2 = A + Bz - \overline{B}\overline{z} + z^2 + E\overline{z}z \tag{5.1}$$

is a real quadratic equation in $x := \mathrm{Re}\,[z]$ *and* $y := \mathrm{Im}\,[z]$.

Proof. Observe that the real part of (5.1) is $\mathrm{Re}\,[\overline{Z}^2] = \mathrm{Re}\,[Bz - \overline{B}\overline{z}] + \mathrm{Re}\,[z^2] = \mathrm{Re}\,[z^2]$, which holds for every $z \in C$. Thus, (5.1) is equivalent to $-2\mathrm{i}yx \equiv \mathrm{Im}\,[\overline{Z}^2] = A + 2\mathrm{i}(\mathrm{Im}\,[B]x + \mathrm{Re}\,[B]y) + 2\mathrm{i}yx + E(x^2+y^2)$, or

$$\mathrm{i}Ex^2 - 4yx + \mathrm{i}Ey^2 - 2\mathrm{Im}\,[B]x - 2\mathrm{Re}\,[B]y + \mathrm{i}A = 0. \quad \blacksquare \tag{5.2}$$

Recall that a real quadratic form Q in x and y represents a conic $\mathcal{C}$ in the (x, y)-plane. If Q is nondegenerate then $\mathcal{C}$ is an ellipse, an hyperbola, or a parabola; if Q is degenerate then $\mathcal{C}$ is a point, a line, a pair of intersecting lines, or the empty set. When $M(2) \geqslant 0$ and $\{\mathbb{1}, Z, \overline{Z}, Z^2, \overline{Z}Z\}$ is linearly independent, it is clear that the only options available for the associated Q are to be nondegenerate or to be a pair of intersecting lines. By a judicious application of Proposition 1.7, we can then see that the study of $\overline{Z}^2 = A\mathbb{1} + BZ + C\overline{Z} + DZ^2 + E\overline{Z}Z$ can be reduced to the following four cases:

(a) $\overline{W}^2 = -2\mathrm{i}W + 2\mathrm{i}\overline{W} - W^2 - 2\overline{W}W$	parabola: $y = x^2$;
(b) $\overline{W}^2 = -4\mathrm{i} \cdot 1 + W^2$	hyperbola: $yx = 1$;
(c) $\overline{W}^2 = W^2$	pair of intersecting lines: $yx = 0$;
(d) $\overline{W}W = 1$	unit circle: $x^2 + y^2 = 1$.

To demonstrate this, consider the case in which the quadratic form in (5.2) is an hyperbola. A translation and a rotation (both among the types of transformations considered in Proposition 1.7) allow us to assume that (5.2) is of the form $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$. Letting $u := \frac{x}{p}$ and $v := \frac{y}{q}$, it follows that $u^2 - v^2 = 1$, which, after an additional rotation, becomes $\widetilde{v}\widetilde{u} = 1$. Now observe that $w \equiv u + \mathrm{i}v$ is of the form $w = \varphi(z) \equiv A + Bz + C\overline{z}$, with $A := 0$, $B := \frac{p+q}{2pq}$, and $C := \frac{-p+q}{2pq}$. Therefore, Proposition 1.7 is applicable, reducing a nondegenerate hyperbola to case (b) above.

Similarly, a nondegenerate ellipse reduces to $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$, with the same transformation $w = \varphi(z)$ bringing this conic to the unit circle $u^2 + v^2 = 1$, or, at the column level, $\overline{W}W = 1$, a case already considered in Section 2. The cases of a parabola and a pair of intersecting lines can be handled using the same approach.

In view of Theorem 1.2, to prove Theorem 1.5 it suffices to establish the existence of representing measures in cases (a), (b), and (c) above. Our attempt to directly establish flat extensions of $M(2)$ in these cases proved unsuccessful, due to the great complexity of the algebraic expressions involved. For this reason, we use Proposition 1.12 to replace $M(2)$ by the associated moment matrix $M_{\mathbb{R}}(2)$. Because the conics in cases (a)–(c) assume an especially simple form in (x, y)-coordinates, and because $M_{\mathbb{R}}(2)$ reflects this simplicity, the computer algebra needed to establish flat extensions becomes tractable, as we next demonstrate.

PROPOSITION *If $M_{\mathbb{R}}(2) \geqslant 0$, rank $M_{\mathbb{R}}(2) = 5$, and $Y = X^2$ in $C_{M_{\mathbb{R}}(2)}$, then $M_{\mathbb{R}}(2)$ has a flat extension $M_{\mathbb{R}}(3)$ and $\beta^{(4)}$ admits a 5-atomic representing measure.*

Proof. Since $Y = X^2$, $M_{\mathbb{R}}(2)$ is of the form

$$\begin{pmatrix} 1 & a & b & b & d & e \\ a & b & d & d & e & f \\ b & d & e & e & f & g \\ b & d & e & e & f & g \\ d & e & f & f & g & h \\ e & f & g & g & h & k \end{pmatrix},$$

where $N := M_{\mathbb{R}}(2)_{[1,X,Y,YX,Y^2]} > 0$. The B-block of a positive, recursively generated extension $M_{\mathbb{R}}(3)$ satisfies $X^3 = YX$ and $YX^2 = Y^2$, and thus assumes the form

$$B(3; p, q) := \begin{pmatrix} d & e & f & g \\ e & f & g & h \\ f & g & h & k \\ f & g & h & k \\ g & h & k & p \\ h & k & p & q \end{pmatrix},$$

where p and q are new moments, corresponding to the monomials y^4x and y^5, respectively. Since N is invertible, it follows that there exists a matrix W such that $M_{\mathbb{R}}(2)W = B(3; p, q)$, and a calculation of the C-block of the flat extension $[M_{\mathbb{R}}(2); B(3; p, q)]$ reveals that it is of the form

$$C(3; p, q) := B(3; p, q)^{\mathrm{t}} W \equiv \begin{pmatrix} g & h & k & p \\ h & k & p & q \\ k & p & C_{33} & u \\ p & q & u & v \end{pmatrix},$$

for some $u, v \in \mathbb{R}$. A further calculation shows that $C_{33} \equiv C_{33}(p)$ is independent of q. Thus, given a choice of p, we can let $q := C_{33}(p)$, and $C(3; p, q)$ then becomes a Hankel matrix, which implies that $[M_{\mathbb{R}}(2); B(3; p, q)]$ is of the form $M_{\mathbb{R}}(3)$. ∎

PROPOSITION *If $M_{\mathbb{R}}(2) \geqslant 0$, rank $M_{\mathbb{R}}(2) = 5$, and $YX = 1$ in $C_{M_{\mathbb{R}}(2)}$, then $M_{\mathbb{R}}(2)$ has a flat extension $M_{\mathbb{R}}(3)$ and $\beta^{(4)}$ admits a 5-atomic representing measure.*

Proof. Since $YX = 1$, $M_{\mathbb{R}}(2)$ can be expressed as

$$\begin{pmatrix} 1 & a & b & c & 1 & d \\ a & c & 1 & e & a & b \\ b & 1 & d & a & b & f \\ c & e & a & g & c & 1 \\ 1 & a & b & c & 1 & d \\ d & b & f & 1 & d & h \end{pmatrix},$$

where $N := M_{\mathbb{R}}(2)_{[1,X,Y,X^2,Y^2]} > 0$. The B-block of a positive, recursively generated extension $M_{\mathbb{R}}(3)$ satisfies $YX^2 = X$ and $Y^2X = Y$, and may thus be represented as

$$B(3;p,q) := \begin{pmatrix} e & a & b & f \\ g & c & 1 & d \\ c & 1 & d & h \\ p & e & a & b \\ e & a & b & f \\ a & b & f & q \end{pmatrix},$$

where p and q are new moments, corresponding to the monomials x^5 and y^5, respectively. Since N is invertible, there exists a matrix W such that $M_{\mathbb{R}}(2)W = B(3;p,q)$, and a calculation of the C-block of the flat extension $[M_{\mathbb{R}}(2); B(3;p,q)]$ reveals that it has the form

$$C(3;p,q) \equiv \begin{pmatrix} u & g & c & C_{14} \\ g & c & 1 & d \\ c & 1 & d & h \\ C_{41} & d & h & v \end{pmatrix},$$

for some $u, v \in \mathbb{R}$, where $C_{41} = C_{14}$. Thus $M_{\mathbb{R}}(2)$ admits a flat extension $M_{\mathbb{R}}(3)$ if and only if $C_{14} = 1$ for some real numbers p and q. A *Mathematica* calculation now shows that $C_{14} = \mathrm{Num}/\mathrm{Den}$, where Num and Den are polynomials in the moments (including p and q). Further, $\Delta := \mathrm{Num} - \mathrm{Den}$ can be expressed as

$$\Delta \equiv \Delta(p,q) \equiv \delta_0 + \delta_1 p + \delta_2 q + \delta_{12} pq,$$

where $\delta_0, \delta_1, \delta_2, \delta_{12}$ are independent of p and q. Observe that $\Delta = \delta_0 + \delta_1 p + (\delta_2 + \delta_{12} p)q$, so if, for some value of p, $\delta_2 + \delta_{12} p \neq 0$ (equivalently, if $\delta_2 \neq 0$ or $\delta_{12} \neq 0$), then $q := -\frac{\delta_0 + \delta_1 p}{\delta_2 + \delta_{12} p}$ satisfies $\Delta(p,q) = 0$. Similarly, $\Delta = \delta_0 + \delta_2 q + (\delta_1 + \delta_{12} q)p$, so if, for some value of q, $\delta_1 + \delta_{12} q \neq 0$ (equivalently, if $\delta_1 \neq 0$ or $\delta_{12} \neq 0$), then $p := -\frac{\delta_0 + \delta_2 q}{\delta_1 + \delta_{12} q}$ satisfies $\Delta(p,q) = 0$. Thus, if δ_1, δ_2 or δ_{12} is nonzero, then $M_{\mathbb{R}}(2)$ admits a flat extension.

Let us assume therefore that $M_{\mathbb{R}}(2)$ admits no flat extension and derive a contradiction; that is, we shall assume that $\delta_1 = \delta_2 = \delta_{12} = 0$. A calculation using *Mathematica* shows that $\delta_{12} \equiv \eta f + F$, where $\eta := -a^3 + 2ac - bc^2 - e + abe$ and $F := 1 - 3ab + a^2b^2 + 2b^2c + 2a^2d - 2cd - 2abcd + c^2d^2 - b^3e + 2bde - ad^2e$, so each of η and F is independent of f. We claim that $\eta = 0$. Indeed, if $\eta \neq 0$, then $\delta_{12} = 0$ implies $f = f_0 := -\frac{F}{\eta}$. A *Mathematica* calculation using $f = f_0$ now reveals that in this case $\det N$ admits a factorization $\det N \equiv \frac{1}{\eta^2} \det M_{\mathbb{R}}(2)_{[1,X,Y,X^2]} G$, where G is a polynomial in a, b, c, d, e, and h of degree 7, and that δ_1 admits a factorization of the form $\delta_1 \equiv -\frac{1}{\eta^3} G^2$. Since $\delta_1 = 0$, it follows that $G = 0$, whence $\det N = 0$, a contradiction. Thus $\eta = 0$. Now let

$$\varepsilon := -b^3 + 2bd - ad^2 - f + abf$$

(for general f). If $M_{\mathbb{R}}(2)$ admits no flat extension, then $\delta_{12} = \eta = 0$ contradicts the condition $\det M_{\mathbb{R}}(2)_{[1,X,Y]} > 0$, via the formula

$$(\det M_{\mathbb{R}}(2)_{[1,X,Y]})^2 = (1 - ab)\delta_{12} + \eta\varepsilon. \quad \blacksquare$$

Example Let $M(2)$ be the moment matrix associated to $\gamma^{(4)}:\gamma_{00},\ldots,\gamma_{22}$, where $\gamma_{00}=1$, $\gamma_{01}=\frac{14+(6+\sqrt{3})\mathrm{i}}{6}$, $\gamma_{02}=\frac{16-\sqrt{3}+14\mathrm{i}}{3}$, $\gamma_{11}=\frac{31+\sqrt{3}}{3}$, $\gamma_{03}=\frac{28+(54-3\sqrt{3})\mathrm{i}}{3}$, $\gamma_{12}=\frac{104+(40+3\sqrt{3})\mathrm{i}}{3}$, $\gamma_{04}=\frac{4(-17+2\sqrt{3}+42\mathrm{i})}{3}$, $\gamma_{13}=\frac{2(137-4\sqrt{3}+128\mathrm{i})}{3}$, $\gamma_{22}=\frac{8(63+\sqrt{3})}{3}$. It is straightforward to verify that rank $M(2)=5$, and that $\overline{Z}^2=-41+(4+2\mathrm{i})Z+(4-2\mathrm{i})\overline{Z}-Z^2$. An application of Proposition 1.7, using $w\equiv\varphi(z):=-(3+\mathrm{i})+(1+\mathrm{i})\bar{z}$, leads to a transition matrix

$$J:=\begin{pmatrix} 1 & -3-\mathrm{i} & -3+\mathrm{i} & 8+6\mathrm{i} & 10 & 8-6\mathrm{i} \\ 0 & 0 & 1-\mathrm{i} & 0 & -4+2\mathrm{i} & -4+8\mathrm{i} \\ 0 & 1+\mathrm{i} & 0 & -4-8\mathrm{i} & -4-2\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\mathrm{i} \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2\mathrm{i} & 0 & 0 \end{pmatrix},$$

and subsequently to a modified moment matrix $\widetilde{M(2)}$ whose columns satisfy $\widetilde{\overline{Z}}^2=-4\mathrm{i}\cdot\widetilde{1}+\widetilde{Z}^2$. By Proposition 1.12, $L(2):=L_0\bigoplus L_1\bigoplus L_2$ allows us to convert $\widetilde{M(2)}$ into an equivalent matrix $\widetilde{M_{\mathbb{R}}(2)}$, with column relation $\widetilde{Y}\widetilde{X}=\widetilde{1}$. Indeed,

$$\widetilde{M_{\mathbb{R}}(2)}=\begin{pmatrix} 1 & \frac13+\frac{1}{2\sqrt3} & \frac13-\frac{1}{2\sqrt3} & 4-\frac{2}{\sqrt3} & 1 & 4+\frac{2}{\sqrt3} \\ \frac13+\frac{1}{2\sqrt3} & 4-\frac{2}{\sqrt3} & 1 & \frac{13}{3}+\frac{5\sqrt3}{2} & \frac13+\frac{1}{2\sqrt3} & \frac13-\frac{1}{2\sqrt3} \\ \frac13-\frac{1}{2\sqrt3} & 1 & 4+\frac{2}{\sqrt3} & \frac13+\frac{1}{2\sqrt3} & \frac13-\frac{1}{2\sqrt3} & \frac{13}{3}-\frac{5\sqrt3}{2} \\ 4-\frac{2}{\sqrt3} & \frac{13}{3}+\frac{5\sqrt3}{2} & \frac13+\frac{1}{2\sqrt3} & 49-\frac{28}{\sqrt3} & 4-\frac{2}{\sqrt3} & 1 \\ 1 & \frac13+\frac{1}{2\sqrt3} & \frac13-\frac{1}{2\sqrt3} & 4-\frac{2}{\sqrt3} & 1 & 4+\frac{2}{\sqrt3} \\ 4+\frac{2}{\sqrt3} & \frac13-\frac{1}{2\sqrt3} & \frac{13}{3}-\frac{5\sqrt3}{2} & 1 & 4+\frac{2}{\sqrt3} & 49+\frac{28}{\sqrt3} \end{pmatrix},$$

and $\widetilde{M_{\mathbb{R}}(2)}_{\{1,2,3,4,6\}}$ is positive and invertible. (In the notation of Proposition 5.3, $a=\frac13+\frac{1}{2\sqrt3}$, $b=\frac13-\frac{1}{2\sqrt3}$, $c=4-\frac{2}{\sqrt3}$, $d=4+\frac{2}{\sqrt3}$, $e=\frac{13}{3}+\frac{5\sqrt3}{2}$, $f=\frac{13}{3}-\frac{5\sqrt3}{2}$, $g=49-\frac{28}{\sqrt3}$, and $h=49+\frac{28}{\sqrt3}$.) A calculation using Mathematica reveals that $\widetilde{C(3;p,q)}$ is Hankel if and only if $684pq+(204+44562\sqrt3)p+(204-44562\sqrt3)q-9338093=0$. In particular, for $q=\frac{1}{36p}$, we have a "conjugate pair" of solutions $p=\frac{362+209\sqrt3}{6}$, $q=\frac{362-209\sqrt3}{6}$. Then $\widetilde{C(3;p,q)}$ is Hankel, with $u=-26(-26+5\sqrt3)$, $v=\frac{26(9084002+5244651\sqrt3}{(362+209\sqrt3)^2}$, and the associated $\widetilde{M_{\mathbb{R}}(3)}$ is a flat extension of $\widetilde{M_{\mathbb{R}}(2)}$. Therefore, $\widetilde{M(3)}:=L(3)^*\widetilde{M_{\mathbb{R}}(3)}L(3)$ is also a flat extension of $\widetilde{M(2)}$ (Proposition 1.12; here $L(3):=L(2)\bigoplus L_3$). It follows that $\widetilde{M(2)}$ admits a 5-atomic representing measure $\widetilde{\mu}$, whose support can be obtained from the Flat Extension Theorem (Theorem 1.6). One computes the characteristic polynomial to be

$$\begin{aligned} g_{\tilde\gamma}(\tilde z)=\tilde z^5&+(-2(1+\mathrm{i})-(1-\mathrm{i})\sqrt3)\tilde z^4+4(-\mathrm{i}+2\sqrt3)\tilde z^3+(-32(1-\mathrm{i})\\ &-12(1+\mathrm{i})\sqrt3)\tilde z^2-4(1+4\sqrt3\mathrm{i})\tilde z-28(1-\mathrm{i})\sqrt3+56(1+\mathrm{i}), \end{aligned}$$

with roots $\tilde z_0=1+\mathrm{i}$, $\tilde z_1-1-\mathrm{i}$, $\tilde z_2=2+\sqrt3+(2-\sqrt3)\mathrm{i}$, $\tilde z_3=2-\sqrt3+(2+\sqrt3)\mathrm{i}$, and $\tilde z_4=\sqrt3-2-(2+\sqrt3)\mathrm{i}$. The corresponding densities are $\rho_0=\rho_1=\frac14$, and

$\rho_2 = \rho_3 = \rho_4 = \frac{1}{6}$. A final application of Proposition 1.7 reveals that $M(2)$ admits a representing measure $\mu \equiv \sum_{k=0}^{4} \rho_k \delta_{z_k}$, where $z_0 = 3+\mathrm{i}$, $z_1 = 1+\mathrm{i}$, $z_2 = 4+(1+\sqrt{3})\mathrm{i}$, $z_3 = 4+(1-\sqrt{3})\mathrm{i}$, and $z_4 = (1+\sqrt{3})\mathrm{i}$. ∎

PROPOSITION *If $M_{\mathbb{R}}(2) \geqslant 0$, rank $M_{\mathbb{R}}(2) = 5$, and $YX = 0$ in $C_{M_{\mathbb{R}}(2)}$, then $\beta^{(4)}$ admits a representing measure μ with card supp $\mu \leqslant 6$.*

Proof. In view of the hypothesis $YX = 0$, $M_{\mathbb{R}}(2)$ can be expressed as

$$\begin{pmatrix} 1 & a & b & c & 0 & d \\ a & c & 0 & e & 0 & 0 \\ b & 0 & d & 0 & 0 & f \\ c & e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & f & 0 & 0 & h \end{pmatrix},$$

where $N := M_{\mathbb{R}}(2)_{[1,X,Y,X^2,Y^2]} > 0$. The B-block of a positive, recursively generated extension $M_{\mathbb{R}}(3)$ satisfies $YX^2 = Y^2X = 0$, and thus assumes the form

$$B(3;p,q) := \begin{pmatrix} e & 0 & 0 & f \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & h \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

where p and q are new moments, corresponding to the monomials x^5 and y^5, respectively. Since N is invertible, there exists a matrix W such that $M_{\mathbb{R}}(2)W = B(3;p,q)$. A calculation of the C-block of the flat extension $[M_{\mathbb{R}}(2); B(3;p,q)]$ reveals that it has the form

$$C(3;p,q) \equiv \begin{pmatrix} C_{11} & 0 & 0 & C_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_{41} & 0 & 0 & C_{44} \end{pmatrix},$$

where $C_{41} = C_{14} \equiv F_1F_2$, with $F_1 \equiv F_1(p) := H + (c^2 - ae)p$ and $H := e^3 - 2ceg + ag^2$, and $F_2 \equiv F_2(q) := L + (d^2 - bf)q$ with $L := f^3 - 2dfh + bh^2$. Thus $M_{\mathbb{R}}(2)$ admits a flat extension $M_{\mathbb{R}}(3)$ if and only if $C_{14} = 0$, i.e., if and only if for some value of p we have $F_1(p) = 0$, or for some value of q we have $F_2(q) = 0$. Equivalently, $M_{\mathbb{R}}(2)$ admits a flat extension $M_{\mathbb{R}}(3)$ if and only if $H = 0$, or $c^2 - ae \neq 0$, or $L = 0$, or $d^2 - bf \neq 0$. (Example 5.6 below illustrates a case in which there is no flat extension $M_{\mathbb{R}}(3)$.)

We may thus assume that $c^2 = ae$, $H \neq 0$, $d^2 = bf$, and $L \neq 0$. Choose any real numbers p and q. Clearly, there exist $u, v > 0$ such that

$$\widetilde{C}(u,v) := \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v \end{pmatrix}$$

satisfies $u > C_{11}$, $\widetilde{C}(u,v) - C(3;p,q) \geqslant 0$ and $\det[\widetilde{C}(u,v) - C(3;p,q)] = 0$, i.e.,

$$(u - C_{11})(v - C_{44}) = C_{14}^2. \tag{5.3}$$

This uniquely determines v in terms of u and previous moments (including possibly p and q, although the choice of u is independent of q), so that

$$M_{\mathbb{R}}(3) := \begin{pmatrix} M_{\mathbb{R}}(2) & B(3;p,q) \\ B(3;p,q)^{\mathrm{t}} & \widetilde{C}(u,v) \end{pmatrix}$$

is a recursively generated positive moment matrix extension of $M_{\mathbb{R}}(2)$ having rank 6, with column basis $\{1\!\!1, X, Y, X^2, Y^2, X^3\}$. It then turns out that there are unique values of r and s so that

$$B(4;r,s) \equiv \begin{pmatrix} g & 0 & 0 & 0 & h \\ p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q \\ u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v \\ r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s \end{pmatrix}$$

satisfies $\operatorname{Ran} B(4;r,s) \subseteq \operatorname{Ran} M_{\mathbb{R}}(3)$, i.e., $M_{\mathbb{R}}(3)W' = B(4;r,s)$ for some matrix W'. (The value of r is of the form Numerator$/F_1(p)$, which requires $F_1(p) \neq 0$ for all values of p.) With this value of r, a calculation shows that the C-block of the flat extension $[M_{\mathbb{R}}(3); B(4;r,s)]$ is of the form

$$C(4;r,s) := B(4;r,s)^{\mathrm{t}} Z \equiv \begin{pmatrix} D_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} \end{pmatrix},$$

that is, $C(4;r,s)$ is actually Hankel. Thus, for each value of p and q, and for u sufficiently large, we get a uniquely determined flat extension and a corresponding 6-atomic representing measure. ∎

Example (cf. Example 1.13) Let $M_{\mathbb{R}}(2)$ be as in Example 1.13. Using the notation in Proposition 5.5, we see that $a = b = 1$, $c = 2$, $d = 3$, $e = 4$, $f = g = 9$, and $h = 28$. Then $H = e^3 - 2ceg + ag^2 = 1$, $F_1(p) = H + (c^2 - ae)p = H = 1$, $L = f^3 - 2dfh + bh^2 = 1$, and $F_2(q) = L + (d^2 - bf)q = 1$, showing that $M_{\mathbb{R}}(2)$ admits no flat extension $M_{\mathbb{R}}(3)$, by Proposition 5.5. We shall now use the last part of the proof of Proposition 5.5 to exhibit infinitely many 6-atomic representing measures for $M_{\mathbb{R}}(2)$. Observe that $C_{11} = 366 - 36p + p^2$, $C_{14} = C_{41} = F_1 F_2 = 1$, and $C_{44} = 7318 - 168q + q^2$. By taking $p = 18$, $q = 84$, $u = 43$, it is easy to see that $v = 263$. Now, the unique values of r and s predicted by Proposition 5.5 are $r = 81$ and $s = 784$, which give rise to $D_{11} = 211$ and $D_{55} = 2473$. This information in turn leads, via Proposition 1.12, to flat extensions $M(4), M(5), \ldots$ We are particularly interested in $M(6)$, since it allows us to capture an analytic dependence among its columns, namely $Z^6 = 25\mathrm{i}1\!\!1 - 5(11 + 7\mathrm{i})Z + 77Z^2 - 5(1 -$

i)$Z^3 - 4Z^4$. The associated characteristic polynomial (cf. Theorem 1.6) factors as $g(z) \equiv (z^3 - 7z + 5)(z^3 + 11z - 5\mathrm{i})$. Using again Theorem 1.6, we obtain the 6-atomic representing measure $\nu[M[4]]$ with atoms and densities as follows:

$$\begin{array}{ll} z_0 \cong 2.16601 & \rho_0 \cong 0.393081 \\ z_1 \cong 0.782816 & \rho_1 \cong 0.203329 \\ z_2 \cong -2.94883 & \rho_2 \cong 0.00359018 \\ z_3 \cong 0.463604\mathrm{i} & \rho_3 \cong 0.0821253 \\ z_4 \cong 3.06043\mathrm{i} & \rho_4 \cong 0.316218 \\ z_5 \cong -3.52404\mathrm{i} & \rho_5 \cong 0.00165656. \end{array}$$

As expected, all six atoms belong to the pair of coordinate axes. ∎

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Many of the examples, and portions of the proofs of some results in this paper were obtained using calculations with the software tool *Mathematica* ([28]).

Note added in proof. In a forthcaming article (R. Curto, L. Fialkow, Solution of the truncated parabolic moment problem, preprint, 2002), we treat the parabola case in $M(n)$.

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