J. OPERATOR THEORY
48(2002), 355–367

ENTROPY OF CROSSED PRODUCTS AND ENTROPY OF FREE PRODUCTS

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Communicated by William B. Arveson

Abstract. An entropical invariant is defined for automorphisms of countable discrete amenable groups, and relations are shown between two entropies for an automorphism on the C^*-crossed product algebra and for its restriction to the original algebra. As an application, given an automorphism β and an amenable group G, we have the equality for entropy that $ht(\underbrace{\beta * \cdots * \beta}_{|G|}) = ht(\beta * \mathrm{id})$.

Keywords: *C^*-algebra, entropy, crossed product, reduced free product.*

MSC (2000): 46L55, 46L40.

1. INTRODUCTION

A non-commutative version of the Kolmogorov-Sinai entropy was introduced by Connes and Størmer in [6] for a trace preserving automorphism of a finite von Neumann algebra, and by Connes, Narnhofer and Thirring in [5], the notion is extended to the CNT-entropy $h_\phi(\alpha)$ for an automorphism α of a C^*-algebra A preserving a given state ϕ of A.

Topological entropies for automorphisms of C^*-algebras were invented by Hudetz ([11]), Thomsen ([16]) and Voiculescu ([18]). Voiculescu's topological entropy $ht(\alpha)$ for an automorphism α of a nuclear C^*-algebra A was extended by Brown ([2]) to automorphisms of exact C^*-algebras. In general, $ht(\alpha) \geqslant h_\phi(\alpha)$, by Voiculescu ([18]) and Dykema ([9]).

In this paper, we show some results on relations between the topological entropy and the free products of automorphisms. We have our results by considering the free product of some automorphisms as automorphisms on the crossed product satisfying some conditions.

To compute topological entropy of such automorphisms, in Section 2, we define an invariant $h(\alpha)$ for an automorphism α of a discrete countable amenable group G, and we show that $h(\cdot)$ enjoys properties one would expect of entropy.

In Section 3, we consider an automorphism γ of the crossed product $A \rtimes_\alpha G$ of an exact C^*-algebra A by a discrete amenable group G (with respect to an action α) such that both A and the unitary group G in $A \rtimes_\alpha G$ are globally invariant under γ. Such automorphisms on $A \rtimes_\alpha G$ arise naturally when we consider free products of automorphisms (cf. Lemma 4.2). We show some relations among $ht(\gamma)$, $h(\gamma_G)$ and $ht(\gamma_A)$ for the restrictions γ_G and γ_A of γ to G and A respectively.

In Section 4, we apply our result in Section 3 to automorphisms on the reduced free product C^*-algebras. For every automorphism β of an exact C^*-algebra, the topological entropy for the free product $\mathop{*}_{g\in G}\beta_g$ of $\{\beta_g\}_{g\in G}$ equals to that for the free product $\beta * \mathrm{id}$ of β and the identity on $C^*_{\mathrm{r}}(G)$ (Theorem 4.3). Here $\beta_g = \beta$ for all g in an amenable group G. Furthermore, if θ is an automorphism of G with $h(\theta) = 0$, then $ht(\widehat{\theta} * \sigma_*) = 0$ (Corollary 4.4). Here $\widehat{\theta}$ is the automorphism of the reduced group C^*-algebra $C^*_{\mathrm{r}}(G)$ induced by θ, and σ_* is the automorphism of the Cuntz algebra $\mathcal{O}_\infty$ (respectively $C^*_{\mathrm{r}}(F_\infty)$ of the free group F_∞) which is a permutation of generators.

2. AUTOMORPHISMS OF AMENABLE GROUPS

Let G be a discrete countable group. We denote by $\mathcal{F}(G)$ the set of all finite subsets of G. Remark that a discrete countable group G is amenable ([13]) if and only if G satisfies Følner's condition, that is, for a given $K \in \mathcal{F}(G)$ and $\delta > 0$, there exists a non-empty $F \in \mathcal{F}(G)$ such that

$$\frac{|gF \triangle F|}{|F|} < \delta \qquad \text{for all } g \in K.$$

Here $|S|$ means the cardinality of $S \in \mathcal{F}(G)$.

We call such a set F a *Følner's set* for (K, δ).

2.1. DEFINITION. Let G be a discrete countable amenable group and let $\alpha \in \mathrm{Aut}(G)$ (the group of automorphisms of G). For a $K \in \mathcal{F}(G)$, we put

$$c(K,\delta) = \inf\left\{|F| : F \neq \emptyset,\ \frac{|gF \triangle F|}{|F|} < \delta \text{ for all } g \in K\right\},$$

$$h(\alpha, K, \delta) = \limsup_{n\to\infty} \frac{1}{n} \log c\biggl(\bigcup_{i=0}^{n-1} \alpha^i(K), \delta\biggr),$$

and

$$h(\alpha, K) = \sup_{\delta>0} h(\alpha, K, \delta).$$

Then we define $h(\alpha)$ for α by

$$h(\alpha) = \sup_{K\in\mathcal{F}(G)} h(\alpha, K).$$

REMARK. If G is generated by an increasing sequence of finite subgroups of G, then $h(\alpha)$ is given as the supremum of $h(\alpha, K)$ for all finite subgroups K of G.

The following proposition shows that $h(\cdot)$ satisfies the basic properties of "entropy".

2.2. PROPOSITION. *Let G be a discrete countable amenable group. Then:*
(i) *$h(\alpha^k) = |k|h(\alpha)$, for all $\alpha \in \mathrm{Aut}(G)$ and all $k \in \mathbb{Z}$;*
and
(ii) *$h(\alpha) = h(\beta)$, for $\alpha, \beta \in \mathrm{Aut}(G)$ which are conjugate in $\mathrm{Aut}(G)$.*

Proof. (i) It is clear that $h(\mathrm{id}) = 0$ for the identity automorphism id of G.

Assume that k is a positive integer. Since for any finite subset K of G and $\delta > 0$,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log c\Big(\bigcup_{j=0}^{n-1} \alpha^{kj}(K), \delta\Big) &\leqslant \limsup_{n\to\infty} \frac{1}{n} \log c\Big(\bigcup_{j=0}^{(n-1)k} \alpha^{j}(K), \delta\Big)\\ &= k \limsup_{n\to\infty} \frac{1}{nk} \log c\Big(\bigcup_{j=0}^{(n-1)k} \alpha^{j}(K), \delta\Big)\\ &\leqslant k \limsup_{n\to\infty} \frac{1}{nk} \log c\Big(\bigcup_{j=0}^{nk-1} \alpha^{j}(K), \delta\Big),\end{aligned}$$

we have that $h(\alpha^k) \leqslant kh(\alpha)$.

Conversely, let $[\frac{n}{k}]$ be the Gauss symbol, that is, the integer m with $m \leqslant \frac{n}{k} < m+1$. For a given finite subset K of G, we denote the set $\bigcup_{i=0}^{k-1} \alpha^i(K)$ by K'. Then

$$\begin{aligned}kh(\alpha, K, \delta) = \limsup_{n\to\infty} \frac{k}{n} \log c\Big(\bigcup_{j=0}^{n-1} \alpha^{j}(K), \delta\Big) &\leqslant \limsup_{n\to\infty} \frac{1}{[\frac{n}{k}]} \log c\Big(\bigcup_{j=0}^{n-1} \alpha^{j}(K), \delta\Big)\\ \leqslant \limsup_{n\to\infty} \frac{1}{[\frac{n}{k}]} \log c\Big(\bigcup_{j=0}^{[\frac{n}{k}]} \alpha^{kj}(K'), \delta\Big) &= h(\alpha^k, K', \delta).\end{aligned}$$

This implies that $kh(\alpha) \leqslant h(\alpha^k)$ so that $kh(\alpha) = h(\alpha^k)$ for all positive integers k. It is obvious for finite subsets K and F of G that $|F \triangle sF|/|F| < \delta$ for all $s \in K$ if and only if $|\alpha F \triangle s\alpha F|/|\alpha F| < \delta$ for all $s \in \alpha(K)$. Hence

$$c\Big(\bigcup_{j=0}^{n-1} \alpha^{-j}(K), \delta\Big) = c\Big(\alpha^{-n+1}\Big(\bigcup_{j=0}^{n-1} \alpha^{j}(K)\Big), \delta\Big) = c\Big(\bigcup_{j=0}^{n-1} \alpha^{j}(K), \delta\Big)$$

which implies that

$$h(\alpha) = h(\alpha^{-1}).$$

Therefore (i) holds.

(ii) Assume that $\alpha = \gamma\beta\gamma^{-1}$ for some $\gamma \in \mathrm{Aut}(G)$. Then $|F \triangle sF|/|F| < \delta$ for all $s \in \bigcup_{j=0}^{n-1} \beta^j(K)$ if and only if $|\gamma(F) \triangle s\gamma(F)|/|\gamma(F)| < \delta$ for all $s \in \bigcup_{j=0}^{n-1} \alpha^j(\gamma(K))$, and $h(\alpha) = h(\beta)$. ∎

2.3. The restricted direct product $\coprod_{i\in I} G_i$ of discrete groups $(G_i)_{i\in I}$ is the subgroup of the cartesian product $\prod_{i\in I} G_i$ formed by the elements $(g_i)_{i\in I}$ such that $g_i \in G_i$ is the unit e_i of G_i for all but a finite number of indices. It is well known that if all G_i are amenable, then $\coprod_{i\in I} G_i$ is amenable.

PROPOSITION. *Let G_0 be a finite group, and let $G = \coprod_{i\in\mathbb{Z}} G_i$. Here G_i is a copy of G_0 for all $i \in \mathbb{Z}$. If α is the automorphism of G induced by the map $i \in \mathbb{Z} \to i+1$, then*

$$h(\alpha) \leqslant \log |G_0|.$$

Proof. Given $K \in \mathcal{F}(G)$, there exists a $k \in \mathbb{N}$ such that

$$K \subset \{(g_i)_i \in G : g_i = e_i, \text{ if } i \notin [-k, k]\}.$$

For $n \in \mathbb{N}$, let

$$F(n) = \{(g_i)_i \in G : g_i = e_i, \text{ if } i \notin [-k, k+n]\}.$$

If $g \in \bigcup_{i=0}^{n-1} \alpha^i(K)$ and $h \in F(n)$, then $gh \in F(n)$ and $g^{-1}h \in F(n)$. Hence $gF(n) \triangle F(n) = \emptyset$ for all $g \in \bigcup_{i=0}^{n-1} \alpha^i(K)$ so that for any $\delta > 0$ we have

$$c\Big(\bigcup_{i=0}^{n-1} \alpha^i(K), \delta\Big) \leqslant |F(n)| = |G_0|^{2k+n+1}.$$

This implies that $h(\alpha, K, \delta) \leqslant \log |G_0|$ for all $K \in \mathcal{F}(G)$ and $\delta > 0$ and we have $h(\alpha) \leqslant \log |G_0|$. ∎

2.4. An automorphism α of a group G induces an automorphism $\widehat{\alpha}$ of the C^*-algebra $C^*_{\mathrm{r}}(G)$ generated by the left regular representation λ:

$$\widehat{\alpha}(\lambda_g) = \lambda_{\alpha(g)}, \quad g \in G.$$

COROLLARY. *Let G and α be the same as in Proposition* 2.3. *If G is abelian (that is, G_0 is abelian), then $ht(\widehat{\alpha}) = h(\alpha) = \log |G_0|$.*

Proof. We show in Corollary 3.6 that in general $ht(\widehat{\alpha}) \leqslant h(\alpha)$. The C^*-algebra $C^*_{\mathrm{r}}(G)$ is represented as $\bigotimes_{i\in\mathbb{Z}} C^*_{\mathrm{r}}(G_i)$, and the shift automorphism $\widehat{\alpha}$ has $ht(\widehat{\alpha}) \geqslant \log(\operatorname{rank}(C^*_{\mathrm{r}}(G_0)))$ ([19]). If G_0 is abelian, then $\operatorname{rank}(C^*_{\mathrm{r}}(G_0)) = |G_0|$. Hence $ht(\widehat{\alpha}) = \log |G_0|$. ∎

3. ENTROPY OF CROSSED PRODUCTS

To fix our notations, we first review the definitions of $ht(\cdot)$. For a C^*-algebra A, let $\pi : A \to B(H)$ be a faithful $*$-representation of A and let $\omega \subset A$ be a finite set. For a $\delta > 0$, $rcp(\pi, \omega, \delta) = \inf\{\text{rank}\,(B) : B$ is a finite dimensional C^*-algebra which has contractive completely positive maps $\varphi : A \to B, \psi : B \to B(H)$ such that $\|\psi \cdot \varphi(a) - \pi(a)\| < \delta, (a \in \omega)\}$. Here $\text{rank}\,(B)$ means the dimension of a maximal abelian subalgebra of B. Let $\alpha \in \text{Aut}(A)$. Then

$$ht(\pi, \alpha, \omega, \delta) = \limsup_{n\to\infty} \frac{1}{n} \log \left(rcp\left(\pi, \bigcup_{i=0}^{n-1} \alpha^i(\omega), \delta\right)\right)$$

and the topological entropy is defined as $ht(\alpha) = \sup_{\omega} \sup_{\delta>0} ht(\pi, \alpha, \omega, \delta)$, which does not depend on representations π ([2]).

In this section, we study relations among entropies $ht(\gamma)$, $ht(\gamma_A)$ and $h(\gamma_G)$ for an automorphism γ on the reduced C^*-crossed product $A \rtimes_\alpha G$.

Let A be a C^*-algebra acting on a Hilbert space H, and let α be an action of a discrete countable group G on A, that is, α is a homomorphism from G to the group $\text{Aut}(A)$ of $*$-automorphisms on A. The representation π of A on $l^2(G, H)$ is given by $(\pi(a)\xi)(g) = \alpha_g^{-1}(a)\xi(g)$ for all $a \in A$, $g \in G$, $\xi \in l^2(G, H)$ and the unitary representation λ of G on $l^2(G, H)$ is given by $(\lambda_g\xi)(h) = \xi(g^{-1}h)$ for all $g, h \in G$, $\xi \in l^2(G, H)$. The reduced crossed product $A \rtimes_\alpha G$ is the C^*-algebra on $l^2(G, H)$ which is generated by $\pi(A)$ and the unitary group $\lambda_G = \{\lambda_g : g \in G\}$. Assume that a $\gamma \in \text{Aut}(A \rtimes_\alpha G)$ satisfies the following condition:

$$\gamma(\lambda_G) = \lambda_G \quad \text{and} \quad \gamma(\pi(A)) = \pi(A). \tag{3.1}$$

Then we have $\gamma_G \in \text{Aut}(G)$ and $\gamma_A \in \text{Aut}(A)$ such that

$$\lambda_{\gamma_G(g)} = \gamma(\lambda_g) \quad \text{and} \quad \pi(\gamma_A(a)) = \gamma(\pi(a)) \quad g \in G,\, a \in A.$$

3.2. Example. An automorphism γ of $A \rtimes_\alpha G$ which satisfies condition (3.1) is obtained from an automorphism of A and an automorphism of G. Let $\theta \in \text{Aut}(G)$ and let α be an action of the group G on a C^*-algebra A such that $\alpha_g = \alpha_{\theta(g)}$ for all $g \in G$. (Such a pair (α, θ) is given for an example as follows: Assume that a group G_1 acts trivially on A and let α' be an action of a group G_2 on A. Let G be the semidirect product $G_1 \rtimes G_2$. For $g = g_1g_2$, $g_i \in G_i$, we define the action α of G on A by $\alpha_g(a) = \alpha'_{g_2}(a)$. Let $\theta' \in \text{Aut}(G_1)$ be such that $\theta'(g_2g_1g_2^{-1}) = g_2\theta'(g_1)g_2^{-1}$ for $g_i \in G_i$, $i = 1, 2$. Then we have $\theta \in \text{Aut}(G)$ defined by $\theta(g_1g_2) = \theta'(g_1)g_2$ for $g_1, g_2 \in G$, and $\alpha_g = \alpha_{\theta(g)}$ for all $g \in G$.)

If $\sigma \in \text{Aut}(A)$ satisfies that $\alpha_g\sigma = \sigma\alpha_g$ for all $g \in G$, then there exists $\gamma \in \text{Aut}(A \rtimes_\alpha G)$ such that $\gamma(\pi(a)) = \pi(\sigma(a))$ for all $a \in A$ and $\gamma(\lambda_g) = \lambda_{\theta(g)}$ for all $g \in G$.

In fact, we may assume that there exists a unitary $v \in B(H)$ with $\sigma(a) = vav^*$ for all $a \in A$. Let U be the unitary defined by

$$(U\xi)(g) = v^*(\xi(\theta(g))), \quad \xi \in l^2(G, H),\, g \in G.$$

Then we have

$$U^*\pi(x)\lambda_g U = \pi(\sigma(x))\lambda_{\theta(g)}, \quad x \in A,\, g \in G$$

and the restriction γ of $\mathrm{Ad}U^*$ to $A \rtimes_\alpha G$ satisfies the condition (3.1).

We give in Section 4 other kind of examples of $\gamma \in A \rtimes_\alpha G$ with the property (3.1).

3.3. Assume that an exact C^*-algebra A is represented on a Hilbert space H. Let G be a discrete amenable countable group, and let α be an action of G on A. Remark that $A \rtimes_\alpha G$ is exact by [12]. For a finite subset K of G and a finite subset ω of A, we put

$$\omega_K = \{\pi(a)\lambda_g : a \in \omega,\ g \in K\}.$$

Under these conditions, we have the following inequality.

PROPOSITION. *Assume that an automorphism γ of the crossed product $A \rtimes_\alpha G$ satisfies the condition* (3.1). *Let K be a finite subset of G and let ω be a finite subset of the unit ball of A. Then we have*

$$ht(\mathrm{id}_{A \rtimes_\alpha G}, \gamma, \omega_K, \delta) \leqslant h(\gamma_G) + \limsup_{n\to\infty} \frac{1}{n} \log rcp\Big(\mathrm{id}_A, \bigcup_{h\in F} \alpha_{h^{-1}}\Big(\bigcup_{i=0}^{n-1} \gamma_A^i(\omega)\Big), \frac{\delta}{2}\Big).$$

Here F is a Følner's set for $\left(\bigcup_{i=0}^{n-1} \gamma_G^i(K), \delta/2\right)$ with $|F| = c\left(\bigcup_{i=0}^{n-1} \gamma_G^i(K), \delta/2\right)$.

Proof. We may assume that K contains the unit e of G. Given $\delta > 0$ and $n \in \mathbb{N}$, choose a non-empty $F \in \mathcal{F}(G)$ such that

$$|F| = c\Big(\bigcup_{i=0}^{n-1} \gamma_G^i(K), \frac{\delta}{2}\Big), \quad \frac{|gF \triangle F|}{|F|} < \frac{\delta}{2} \quad \text{for all } g \in \bigcup_{i=0}^{n-1} \gamma_G^i(K).$$

We choose a triple (ψ, φ, B) of a finite dimensional C^*-algebra B and completely positive maps $\varphi : A \to B, \psi : B \to B(H)$ such that

$$\|\psi \cdot \varphi(z) - z\| < \frac{\delta}{2}, \quad \text{for all } z \in \bigcup_{h\in F} \alpha_h^{-1}\Big(\bigcup_{i=0}^{n-1} \gamma_A^i(\omega)\Big)$$

and that

$$\mathrm{rank}\,(B) = rcp\Big(\mathrm{id}_A, \bigcup_{h\in F} \alpha_h^{-1}\Big(\bigcup_{i=0}^{n-1} \gamma_A^i(\omega)\Big), \frac{\delta}{2}\Big).$$

Let $f = |F|^{-1/2}\chi_F$, where χ_F is the characteristic function of F. Then

$$\Big|\sum_{t\in G} f(t)\overline{f(g^{-1}t)} - 1\Big| \leqslant \frac{\delta}{2}, \quad g \in \bigcup_{i=0}^{n-1} \gamma_G^i(K).$$

We denote by P_F the orthogonal projection of $l^2(G)$ onto $l^2(F)$. As in [3] following after [14], we define completely positive maps Φ and Ψ with

$$A \rtimes_\alpha G \xrightarrow{\Phi} P_F B(l^2(G)) P_F \otimes B \xrightarrow{\Psi} B(l^2(G, H))$$

by

$$\Phi(x) = (1 \otimes \varphi)((P_F \otimes 1)x(P_F \otimes 1)), \quad x \in A \rtimes_\alpha G$$

and

$$\Psi(y) = T_f((1 \otimes \psi)(y)), \quad y \in P_F B(l^2(G))P_F \otimes B,$$

where

$$T_f(x) = \sum_{t \in G} \nu_t(m_f \otimes 1)x(m_f^* \otimes 1)\nu_t^*$$

and ν is the right regular representation of G, and m_f is the multiplication operator of f. By [3], Propositions 2.5 and 2.6, we have for all $a \in \bigcup_{i=0}^{n-1} \gamma_A^i(\omega)$ and $g \in \bigcup_{i=0}^{n-1} \gamma_G^i(K)$ that

$$\|\Psi \cdot \Phi(\pi(a)\lambda_g) - \pi(a)\lambda_g\| < \Big\| \sum_{t \in F \cap gF} e_{t,g^{-1}t} \otimes (\psi \cdot \phi(\alpha_{t^{-1}}(a)) - \alpha_{t^{-1}}(a)) \Big\| + \Big| \sum_{t \in F \cap gF} f(t)\overline{f(g^{-1}t)} - 1 \Big| < \delta,$$

where $\{e_{t,s} : t, s \in G\}$ is the standard matrix units of $B(l^2(G))$. Hence

$$rcp\Big(\mathrm{id}_{A \rtimes_\alpha G}, \bigcup_{i=0}^{n-1} \gamma^i(\omega_K), \delta\Big) < |F| \cdot \mathrm{rank}\,(B).$$

This implies the inequality. ∎

3.4. Let G be a discrete amenable group and let $\theta \in \mathrm{Aut}(G)$.

CONDITION (†) FOR (G, θ): Given a finite set $K \subset G$ and $\delta > 0$, there exists a finite subgroup L such that for all $n \in \mathbb{N}$ we can choose a Følner's set $F(n)$ for $\Big(\bigcup_{i=0}^{n-1} \theta^i(K), \delta\Big)$ which satisfies that $|F(n)| = c\Big(\bigcup_{i=0}^{n-1} \theta^i(K), \delta\Big)$ and is a subset of the product set $L\theta(L) \cdots \theta^{n-1}(L)$.

COROLLARY. *Let A, G, α and γ be the same as in Proposition* 3.3.

(i) *Assume that (G, γ_G) satisfies (†). If γ_A commutes with α_g for all $g \in G$, then*

$$(\ddagger) \qquad ht(\gamma) \leqslant h(\gamma_G) + ht(\gamma_A).$$

(ii) *In particular, if (G, γ_G) is the pair in Proposition* 2.3 *and if γ_A commutes with α_g for all $g \in G$, then we have (‡).*

(iii) *Let (G, θ) be the pair in Proposition* 2.3*, and let γ be the automorphism given in* 3.2. *Then we have (‡).*

Proof. First we remark that γ_A commutes with α_g for all $g \in G$ if and only if $\alpha_g = \alpha_{\gamma_G(g)}$ for all $g \in G$. In fact, if γ_A commutes with α_g for all $g \in G$, then $\pi(\alpha_{\gamma_G(g)}\gamma_A(a)) = \lambda_{\gamma_G(g)}\gamma(\pi(a))\lambda_{\gamma_G(g)}^* = \gamma(\lambda_g\pi(a)\lambda_g^*) = \gamma(\pi(\alpha_g(a))) = \pi(\gamma_A(\alpha_g(a))) = \pi(\alpha_g(\gamma_A(a)))$ for all $g \in G$ and $a \in A$ which implies that $\alpha_g = \alpha_{\gamma_G(g)}$ for all $g \in G$. The converse relation is obtained by a similar calculation.

(i) Given $K \in \mathcal{F}(G)$ and $\delta > 0$, we choose a finite subgroup L of G as in (†). If $h \in F(n)$, then $h = h_1\gamma_G(h_2)\cdots\gamma_G^{n-1}(h_n)$ for some $h_i \in L$, $i = 1, 2, \ldots, h_n$. Let $h' = h_1h_2\cdots h_n$, then $h' \in L$ and $\alpha_h^{-1} = \alpha_{h'}^{-1}$. Hence we have that

$$\bigcup_{h\in F(n)} \alpha_{h^{-1}}(\omega) \subset \bigcup_{h\in L} \alpha_{h^{-1}}(\omega),$$

so that

$$\limsup_{n\to\infty} \frac{1}{n} \log\ rcp\Big(\mathrm{id}_A, \bigcup_{h\in F(n)} \alpha_{h^{-1}}\Big(\bigcup_{i=0}^{n-1} \gamma_A^i(\omega)\Big), \frac{\delta}{2}\Big)$$

$$\leqslant \limsup_{n\to\infty} \frac{1}{n} \log\ rcp\Big(\mathrm{id}_A, \bigcup_{i=0}^{n-1} \gamma_A^i\Big(\bigcup_{h\in L} \alpha_{h^{-1}}(\omega)\Big), \frac{\delta}{2}\Big) \leqslant ht(\gamma_A).$$

Since $ht(\mathrm{id}_{A\rtimes_\alpha G}, \gamma, \omega_K, \delta) \leqslant h(\gamma_G) + ht(\gamma_A)$ for all K and δ by Proposition 3.3, we have that (‡).

(ii) Let (G, θ) be the pair in Proposition 2.3. For a finite set $K \subset G$, let L be the smallest subgroup of G which contains K. Then L satisfies the condition (†), and we have (‡) by (i).

(iii) The automorphism γ in 3.2 arises from $\theta \in \mathrm{Aut}(G)$ and an action α of G on A such that $\alpha_g = \alpha_{\theta(g)}$ for all $g \in G$. This condition implies that γ_A commutes with α_g for all $g \in G$. Hence we have (‡) by (ii). ∎

3.5. Corollary. *Let A, G, α be the same as in Proposition 3.3. If $\gamma \in \mathrm{Aut}(A \rtimes_\alpha G)$ satisfies*

$$\gamma(\pi(A)) = \pi(A) \quad \textit{and} \quad \gamma(\lambda_g) = \lambda_g, \quad \textit{for all } g \in G,$$

then

$$ht(\gamma) = ht(\gamma_A).$$

Proof. By the monotonicity of ht ([2], Proposition 2.1), $ht(\gamma) \geqslant ht(\gamma_A)$. Let K be a finite subset of G. If $\gamma(\lambda_g) = \lambda_g$ for all $g \in G$, then we can choose the same Følner's set for $\Big(\bigcup_{i=0}^{n-1} \gamma_G^i(K), \delta/2\Big)$ as for $(K, \delta/2)$. Let ω be a finite subset of A. If $\gamma(\lambda_g) = \lambda_g$ for all $g \in G$, then γ_A commutes with α_g for all $g \in G$. Hence by Proposition 3.3 we have

$$ht(\mathrm{id}_{A\rtimes_\alpha G}, \gamma, \omega_K, \delta) \leqslant ht\Big(\mathrm{id}_A, \gamma, \bigcup_{h\in F} \alpha_h^{-1}(\omega), \frac{\delta}{2}\Big)$$

so that $ht(\gamma) \leqslant ht(\gamma_A)$. ∎

Remark. Corollary 3.5 was shown independently by Dykema and Shlyakhtenko ([11], Proposition 1.2) for the automorphism γ on the crossed product $A \rtimes_\alpha G$ which arises from a $\sigma \in \mathrm{Aut}(A)$ and the identity on G.

3.6. Corollary. *Let α be an automorphism of a discrete amenable group G and let $\widehat{\alpha}$ be the automorphism of $C_{\mathrm{r}}^*(G)$ induced by α as in 2.4. Then*

$$ht(\widehat{\alpha}) \leqslant h(\alpha).$$

Proof. In Proposition 3.3, let A be the trivial algebra $\mathbb{C}$. Then $A \rtimes_\alpha G$ is nothing but $C_{\mathrm{r}}^*(G)$. Applying Proposition 3.3 to $\gamma = \widehat{\alpha}$, we have $ht(\widehat{\alpha}) \leqslant h(\alpha)$. ∎

4. ENTROPY OF FREE PRODUCTS

For a set I, let $A_i, i \in I$ be a unital C^*-algebra with a state ϕ_i whose GNS representation is faithful. The reduced free product $(A, \phi) = \underset{i \in I}{*}(A_i, \phi_i)$ defined by Voiculescu ([17]; see also [19]) is the pair of a unital C^*-algebra A with unital embeddings $A_i \hookrightarrow A$ for all $i \in I$ and a state ϕ such that

(i) $\phi|A_i = \phi_i$, for all $i \in I$,

(ii) the family $(A_i)_{i \in I}$ is free in (A, ϕ),

(iii) A is generated by the family $(A_i)_{i \in I}$,

(iv) the GNS reprentation of ϕ is faithful on A.

Here, the statement (ii) means that $\phi(a_1 a_2 \cdots a_n) = 0$ whenever $a_j \in A_{\iota_j}, \phi(a_j) = 0$ and $\iota_j \neq \iota_{j+1}$ for $j \in \{1, 2, \ldots, n-1\}$.

The state ϕ is denoted by $\underset{i \in I}{*}\phi_i$. A reduced word a in $(A_i)_{i \in I}$ is an element in A given by an expression of the form $a = a_1 a_2 \cdots a_n$, where $n \geqslant 1, a_i \in A_{\iota_i}, \phi_{\iota_i}(a_i) = 0$ and $\iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n$. The number n is called the *length of the reduced word*. Following Dykema ([9]), we call the *set* $\{\iota_1, \ldots, \iota_n\} \subset I$ *the alphabet for the word* a. The linear span of all reduced words in $(A_i)_{i \in I}$ is dense in A. Let α_i be a $*$-automorphism of A_i, and let ϕ_i be an α_i-invariant state of A_i. Then there exists a ϕ-preserving automorphism α of the algebra A such that $\alpha(a_1 a_2 \cdots a_n) = \alpha_{\iota_1}(a_1)\alpha_{\iota_2}(a_2) \cdots \alpha_{\iota_n}(a_n)$ whenever $a_j \in A_{\iota_j}$, $\phi(a_j) = 0$ and $\iota_j \neq \iota_{j+1}$ for $j \in \{1, 2, \ldots, n-1\}$. The automorphism α is denoted by $\underset{i \in I}{*}\alpha_i$.

4.1. Let B be an exact C^*-algebra, and let ψ be a state of B with faithful GNS-representation. Let G be an amenable discrete group, and let λ be the left regular representation of G. Let $\mathcal{A}$ be the algebra given by the reduced free product construction:

$$(\mathcal{A}, \phi) = (C^*_{\mathrm{r}}(G), \tau_G) * (B, \psi),$$

where τ_G is the trace of $C^*_{\mathrm{r}}(G)$ such that $\tau_G(\lambda_g) = 0$ for all $g \in G$ except the unit. We use the method in [4] that $\mathcal{A}$ is decomposed into the crossed product. We put

$$A_g = \lambda_g B \lambda_g^* \quad \text{and} \quad \phi_g = \phi|A_g \quad \text{for all } g \in G.$$

Let A be the C^*-subalgebra of $\mathcal{A}$ generated by $\{\lambda_g B \lambda_g^* : g \in G\}$. Since $\{\lambda_g B \lambda_g^* : g \in G\}$ is a free family with respect to ϕ, we have that

$$(A, \phi|A) \cong \underset{g \in G}{*}(A_g, \phi_g).$$

We give the action α of G on A by $\alpha_g(a) = \lambda_g a \lambda_g^*$ for all $g \in G$ and $a \in A$. As we showed in [4], Claim 4, $\mathcal{A}$ is decomposed into the crossed product $A \rtimes_\alpha G$. In this setting, it is obvious (so we omit the proof) that automorphisms of $A \rtimes_\alpha G$ with the property (3.1) arise naturally as in the following:

4.2. LEMMA. *Under the same notations as in* 4.1*, let* $\beta \in \mathrm{Aut}(B)$ *with* $\psi \circ \beta = \psi$*, and let* $\sigma \in \mathrm{Aut}(G)$*. Then* $\gamma = \sigma * \beta \in \mathrm{Aut}(\mathcal{A})$ *is an automorphism of* $A \rtimes_\alpha G$ *which satisfies the condition* (3.1)*. In particular, if* σ *is the identity automorphism of* G*, then the restriction* γ_A *of* γ *to* A *commutes with the action* α*.*

THEOREM 4.3. *Let* B, ψ *and* G *be the same as in* 4.1*. If* β *is an automorphism of* B *preserving* ψ*, then*

$$ht\Big(\mathop{*}_{g \in G} \beta_g\Big) = ht(\mathrm{id}_G * \beta).$$

Here, β_g *is a copy of* β *for all* $g \in G$ *and* id_G *is the identity automorphism of* $C^*_{\mathrm{r}}(G)$*.*

Proof. We use the same notations as in 4.1. Remark that $\mathcal{A}$ is exact by [8]. We denote by γ the automorphism $\mathrm{id}_G * \beta$ of $A \rtimes_\alpha G$. Then γ satisfies all conditions in Corollary 3.5. Hence we have that $ht(\gamma) = ht(\gamma_A)$.

On the other hand, the automorphism γ_A is conjugate to $\mathop{*}_{g \in G}(\alpha_g\ \beta\ \alpha_g^{-1})$. We denote by γ_g the restriction of γ to the embedded copy of A_g into $\mathcal{A}$. The automorphism β_g on the embedded copy of B in $\mathcal{A}$ is given by $\beta_g = \alpha_g \circ \gamma_g$. Then $\mathop{*}_{g \in G}(\alpha_g\ \beta\ \alpha_g^{-1})$ is conjugate to $\Big(\mathop{*}_{g \in G} \alpha_g\Big) \mathop{*}_{g \in G} \beta_g \Big(\mathop{*}_{g \in G} \alpha_g^{-1}\Big)$. Hence, we have that $ht(\gamma_A) = ht\Big(\mathop{*}_{g \in G} \beta_g\Big)$ so that $ht\Big(\mathop{*}_{g \in G} \beta_g\Big) = ht(\mathrm{id}_G * \beta)$. ∎

In Theorem 4.3, we do not know the relation among the values $\Big\{ht\Big(\mathop{*}_{1 \leqslant i \leqslant n} \beta_i\Big)\Big\}_{n \in \mathbb{N}}$, where each β_i is a copy of an automorphism β. If we let $G = \coprod_{n \in \mathbb{N}} G_n$, ($G_n$ is a group with $|G_n| = n$) and if we let $\gamma = \mathrm{id}_{C^*(G)} * \beta$ for an automorphism β of a unital C^*-algebra B preserving the given state of B, then

$$ht(\mathrm{id}_{C^*_{\mathrm{r}}(\mathbb{Z}_2)} * \gamma) = ht(\underbrace{\gamma * \gamma * \cdots * \gamma}_{n \text{ times}}), \quad \text{for all } n \in \mathbb{N}.$$

In fact, by Theorem 4.3, $ht(\mathrm{id}_{C^*_{\mathrm{r}}(\mathbb{Z}_3)} * \gamma) = ht(\gamma * \gamma * \gamma) \geqslant ht(\gamma * \gamma) \geqslant ht(\mathrm{id}_{C^*_{\mathrm{r}}(\mathbb{Z}_3)} * \gamma)$ because $C^*_{\mathrm{r}}(G_n) \subset C^*_{\mathrm{r}}(G) * B$ and the restriction of γ to $C^*_{\mathrm{r}}(G_n)$ is the identity. So, $ht(\gamma * \gamma * \gamma) = ht(\gamma * \gamma) = ht(\mathrm{id}_{C^*_{\mathrm{r}}(\mathbb{Z}_2)} * \gamma)$. Similarly, for all $n \in \mathbb{N}$, we have the equality. However, we don't know the relation between $ht(\beta * \mathrm{id})$ and $ht(\beta)$.

4.4. Next we show some examples of non-trivial automorphisms $\beta \in \mathrm{Aut}(B)$ that $ht(\mathrm{id}_{C^*_{\mathrm{r}}(G)} * \beta) = ht(\beta)$. They are given as free permutations of the reduced free products of C^*-algebras, and have 0 entropy. In special cases, β is the free permutation of the generators of Cuntz algebra $\mathcal{O}_\infty$ in [7] or $C^*_{\mathrm{r}}(F_\infty)$ of the free group with infinite generators and $ht(\beta) = 0$ ([3], [9]).

Let I be a finite set, and for every $\iota \in I$ let C_ι be a finite dimensional C^*-algebra with a state μ_ι whose GNS representation is faithful. Let

$$(C, \mu) = \mathop{*}_{\iota \in I}(C_\iota, \mu_\iota).$$

Let J be a set, and for every $\zeta \in J$ let (B_ζ, ψ_ζ) be a copy of (C, μ). Put

$$(B, \psi) = \mathop{*}_{\zeta \in J}(B_\zeta, \psi_\zeta).$$

Let σ be a permutation of J. Then there exists the automorphism σ_* of B sending the embedded copy of B_ζ in B identically to the embedded copy of $B_{\sigma(\zeta)}$ in B for every $\zeta \in J$.

THEOREM. *Under the same notations as in* 4.1, *assume that* (B,ψ) *is the pair arising from the above reduced free product construction. If* $\theta \in \mathrm{Aut}(G)$ *has* $h(\theta)=0$, *then*

$$ht(\widehat{\theta} * \sigma_*) = 0.$$

Proof. We denote $\widehat{\theta} * \sigma_*$ by γ. As in 4.1, $\mathcal{A}$ is decomposed into $A \rtimes_\alpha G$ and γ is the automorphism of $A \rtimes_\alpha G$ such that $\gamma(\lambda_g) = \lambda_{\theta(g)}$ and $\gamma(A_g) = A_{\theta(g)}$. Let $H_g = L^2(A_g, \phi_g)$ on which A_g acts via the GNS representation, and let ξ_g be the image of the identity of A_g in H_g. Then we may consider A as the C^*-algebra which acts on the Hilbert space H arising from the reduced free product $(H,\xi) = \mathop{*}_{g\in G}(H_g,\xi_g)$. Let $W(A)$ be the set of reduced words in $(A_g)_{g\in G}$. To compute $ht(\mathrm{id}_{A\rtimes_\alpha G}, \gamma, \omega_K, \delta)$ for finite sets $\omega \subset A$ and $K \subset G$, it is sufficient to take a finite set $\omega \subset W(A)$. We may assume that ω is contained in the unit ball of A and that each reduced word $a \in \omega$ has a form that

$$a = \lambda_{g_1} b_1 \lambda_{g_1}^* \cdots \lambda_{g_n} b_n \lambda_{g_n}^*, \quad g_1 \neq g_2, \ldots, g_{n-1} \neq g_n,$$

where each b_k is contained in the set of the reduced words in $(B_\zeta)_{\zeta\in J}$ so that b_k has a form that $b_k = b(k,1)\cdots b(k,n_k)$, where $b(k,l) \in B_{\zeta(k,l)} \cap \ker(\psi_{\zeta(k,l)})$ and $\zeta(k,l) \neq \zeta(k,l+1)$ for all l, $1 \leqslant l \leqslant n_k - 1$.

We denote by $C_{\zeta,\iota}$ the embedded copy of C_ι into B_ζ which is obtained by the natural embedding in the reduced free product construction.

Again, we may assume that each $b(k,l)$ is contained in the set of the reduced words in $(C_{\zeta,\iota})_{\zeta\in J,\iota\in I}$ so that $b(k,l) = c(k,l;1)\cdot\cdots\cdot c(k,l;m(k,l))$, where $c(k,l;t) \in C_{\iota(k,l;t)} \cap \ker(\mu_{\iota(k,l;t)})$, and $\iota(k,l;t) \neq \iota(k,l;t+1)$ for all t, $1 \leqslant t \leqslant m(k,l)-1$.

Thus we may assume that ω is a finite subset of the reduced words in $\{\lambda_g C_{\zeta,\iota}\lambda_g^* : g\in G, \iota\in I, \zeta\in J\}$ of finite dimensional C^*-algebras.

Given a finite subset K of G and given $\delta > 0$, let $F(n)$ be a Følner's set for $\left(\bigcup_{i=0}^{n-1}\theta^i(K), \delta/2\right)$ such that $|F(n)| = c\left(\bigcup_{i=0}^{n-1}\theta^i(K), \delta/2\right)$. Since $ht(\theta)=0$, we have by Proposition 3.3 and Lemma 4.2, that

$$ht(\mathrm{id}_{A\rtimes_\alpha G}, \gamma, \omega_K, \delta) \leqslant \limsup_{n\to\infty} \frac{1}{n} \log rcp\biggl(\mathrm{id}_A, \bigcup_{h\in F(n)} \alpha_h^{-1}\biggl(\bigcup_{i=0}^{n-1}\gamma^i(\omega)\biggr), \frac{\delta}{2}\biggr).$$

We denote the set $\bigcup_{h\in F(n)} \alpha_{h^{-1}}\left(\bigcup_{i=0}^{n-1}\gamma^i(\omega)\right)$ by $\omega(n,\gamma)$.

Let q be the maximum of the lengths of the words belonging to ω. Then q is also the maximum of the lengths of the words belonging to $\omega(n,\gamma)$. Let $\mathcal{J}$ be the set of the alphabets for the elements of ω. We denote by $\mathcal{J}(n,\gamma)$ the alphabets for $\omega(n,\gamma)$, and by $d(\mathcal{J}(n,\gamma))$ the maximum over $(g,\zeta,\iota) \in \mathcal{J}(n,\gamma)$ of the dimension of $L^2(\lambda_g C_{\zeta,\iota}\lambda_g^*, \phi)$ as a Banach space. Then $d(\mathcal{J}(n,\gamma))$ is the maximum d of the dimensions of $(C_\iota)_{\iota\in I}$. Since A is represented as the C^*-algebra acting on H

given as the reduced free product Hilbert space of $(H_g)_{g\in G}$, we have by Dykema's estimate in [9], Proof of Theorem 1:

$$rcp\left(\mathrm{id}_A, \omega(n,\gamma), \frac{\delta}{2}\right) \leqslant (1 + k|\mathcal{J}(n,\gamma)|^k d^k).$$

Here k is an integer which depends only on $\delta/2$ and q. We put $\mathcal{J}_G = \{g \in G : (g,\zeta,\iota) \in \mathcal{J}(n,\gamma)$ for some $\zeta \in J, \iota \in I\}$, $\mathcal{J}_J = \{\zeta \in J : (g,\zeta,\iota) \in \mathcal{J}(n,\gamma)$ for some $g \in G, \iota \in I\}$, and $\mathcal{J}_I = \{\iota \in I : (g,\zeta,\iota) \in \mathcal{J}(n,\gamma)$ for some $g \in G, \zeta \in J\}$. Then

$$\mathcal{J}(n,\gamma) \subset \{h^{-1}\theta^i(g)\sigma^i(\zeta)\iota : h \in F(n), g \in \mathcal{J}_G, \zeta \in \mathcal{J}_J, \iota \in \mathcal{J}_I\}.$$

This implies that $|\mathcal{J}(n,\gamma)| \leqslant n^2|F(n)|\,|\mathcal{J}_G|\,|\mathcal{J}_J|\,|\mathcal{J}_I|$ so that

$$ht(\mathrm{id}_{A \rtimes_\alpha G}, \gamma, \omega_K, \delta) \leqslant kh(\theta, K, \delta) = 0.$$

Hence we have that $h(\widehat{\theta} * \sigma_*) = ht(\gamma) = 0.$ ∎

Remark. (1) The proof of Theorem 4.4 holds in the case where I is a one point set and $\sigma_* = \mathop{*}_{\iota\in I} \alpha_\zeta$, where α_ζ is a ψ_ζ-preserving automorphisms of B_ζ.

(2) The restriction γ_A of γ to A in the proof of Theorem 4.4 is the same kind of automorphism as in Theorem from [9], and $ht(\gamma_A) = 0$. Hence Theorem 4.4 gives an example for $\gamma \in \mathrm{Aut}(A \rtimes_\alpha G)$ such that $ht(\gamma) = ht(\gamma_A) + ht(\gamma_G)$.

Corollary. *Assume that $\theta \in \mathrm{Aut}(G)$ has $h(\theta) = 0$.*

(i) *If $\sigma_* \in \mathrm{Aut}(\mathcal{O}_\infty)$ is a permutation of the generators of the Cuntz algebra $\mathcal{O}_\infty$, then $ht(\widehat{\theta} * \sigma_*) = 0$.*

(ii) *If σ_* is the automorphism of the type II_1-factor $L(F_\infty)$ induced by a permutation of the generators of the free group F_∞, then the Connes-Stϕrmer entropy $H(\bar{\theta} * \sigma_*) = 0$. Here $\bar{\theta}$ is the automorphism of the finite group von Neumann algebra $L(G)$ induced by θ.*

Proof. Let $(\mathcal{T}, \mu)$ be the pair of the Toeplitz algebra $\mathcal{T}$ and the state μ with $\mu(vv^*) = 0$ for the generator v of $\mathcal{T}$. Then $(\mathcal{T}, \mu)$ is embedded into the free product (C, μ) for a suitable $(C_\iota, \mu_\iota)_{\iota\in I}$, and the pair $(C^*_{\mathrm{r}}(\mathbb{Z}), \tau_{\mathbb{Z}})$ is also embedded into the free product (C, μ) for a suitable $(C_\iota, \mu_\iota)_{\iota\in I}$ ([9], Examples 7).

By the monotonicity of $ht(\cdot)$ and by Theorem 4.4 we have $ht(\widehat{\theta} * \sigma_*) = 0$ for the σ_* of $\mathcal{O}_\infty$ or of $C^*_{\mathrm{r}}(F_\infty)$.

In general, the topological entropy dominates the CNT-entropy. Hence we have that $h_{\tau_G * \tau_{F_\infty}}(\widehat{\theta} * \sigma_*) = 0$. This implies that the Connes-Størmer entropy $H(\bar{\theta} * \sigma_*) = 0$. ∎

Acknowledgements. The author thanks Nathaniel Brown for pointing out a gap in the preliminary version of this paper and kind communications. She also thanks the referee for many valuable comments.

Note added in proof. After this paper was accepted, more general results on free products were obtained by Brown-Dykema-Shlyakhtenko.

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Received February 20, 2000.