J. OPERATOR THEORY
48(2002), 369–383

MULTIPLICATIVITY OF EXTREMAL POSITIVE MAPS ON ABELIAN PARTS OF OPERATOR ALGEBRAS

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Communicated by William B. Arveson

Abstract. It is shown that finitely many mutually orthogonal pure states on a JB algebra with σ-finite covers restrict simultaneously to pure (i.e. multiplicative) states on some maximal associative JB subalgebra. This result does not hold for any infinite system of orthogonal pure states; a counterexample is constructed on any infinite dimensional, separable, irreducible C^*-algebra with non-commutative quotient by the compact operators. Nevertheless, under some natural additional conditions the restriction property does hold for all systems of orthogonal pure states. Finally, it is shown that any C^*-extreme completely positive map on a C^*-algebra $\mathcal{A}$ with σ-finite representation and values in a finite dimensional algebra is multiplicative (even $\mathcal{B}$-morphism) on some maximal abelian subalgebra $\mathcal{B}$ of $\mathcal{A}$.

Keywords: *Pure states on JB and C^*-algebras, C^*-extreme completely positive maps, restriction property.*

MSC (2000): 46L30, 46L70, 46L60, 81P10.

1. INTRODUCTION

One of the classical results in the theory of operator algebras says that any pure completely positive map and, in particular, any pure state on an abelian C^*-algebra is multiplicative ([8], [22]). This fact illustrates a nice interplay between algebraic properties of states and geometric and order structure of the state space. For non-abelian algebras extremal positive maps are far from being multiplicative in general. Therefore a natural question arises as to whether, at least, a pure (respectively extremal) positive map restricts to a multiplicative (respectively extremal) map on some maximal abelian subalgebra. This problem, which is a central topic of this paper, has been studied widely in the realm of C^*-algebras. Besides its importance for the general theory of states on operator algebras it is relevant to the axiomatic foundations of quantum theory because it relates C^*-algebraic quantum mechanics to classical one. This line of research becomes topical in the light of the recent revival of the interest in the basic issues of the quantum theory (see e.g. [12], [23]).

The aim of this paper is to study the *restriction property* of extremal positive maps on both C^*-algebras and Jordan algebras. Our discussion falls into two parts.

In the first part we study systems of pairwise orthogonal pure states. It was proved in [1] and [3] that given a finite system of orthogonal pure states on a separable C^*-algebra we can find a maximal abelian subalgebra such that all states are pure (i.e. multiplicative) on it. We say in that case that the system of states in question enjoys the *restriction property.* On the other hand it was proved in [11] that any pure state on a C^*-algebra (not necessarily separable) whose GNS representation acts on a separable Hilbert space is pure on some maximal abelian subalgebra. We provide a simultaneous generalization of the above mentioned results for finite systems of orthogonal pure states with separable GNS representations. Moreover, as it turns out that the restriction property depends purely on the Jordan structure of operator algebras, we will formulate the results in terms of Jordan-Banach algebras. The transition from the C^*-case to the Jordan case requires some new ideas. Our main result is the following: Let $\varrho_1, \varrho_2, \ldots, \varrho_n$ be a system of mutually orthogonal pure states on a JB algebra A such that the central covers $c(\varrho_1), c(\varrho_2), \ldots, c(\varrho_n)$ are σ-finite in the double dual A^{**}. Then there is a maximal associative subalgebra B of A such that $\varrho_1, \varrho_2, \ldots, \varrho_n$ are multiplicative on B. This result cannot be generalized to infinite systems of orthogonal pure states. We construct a counterexample showing that nearly all irreducible separable C^*-algebras admit systems of orthogonal pure states which are not simultaneously multiplicative on any maximal abelian subalgebra. However, we prove that the restriction property does hold for infinite systems if one assumes some additional natural conditions such as approaching to infinity and inequivalence (compare [5]).

In the second part of the paper we deal with the restriction property of extreme completely positive maps between C^*-algebras. As the main result of this part we prove that any C^*-extreme completely positive map with separable representation which has values in a finite dimensional algebra is multiplicative, and thereby C^*-extreme by [14] Proposition 1.2, on some maximal abelian subalgebra. Since states are very special (one-dimensional) completely positive maps the results of this part strengthen hitherto known results on the restriction property of pure states obtained in [1], [3], [11], [19].

2. RESTRICTING ORTHOGONAL PURE STATES ON JB ALGEBRAS

We recall a few definitions and fix the notation. Throughout this part let A be a JB algebra, i.e. a real Banach algebra with a product $\circ$, such that the following conditions hold for all $a, b \in A$:

(i) $a \circ b = b \circ a$,
(ii) $a \circ (a^2 \circ b) = a^2 \circ (a \circ b)$,
(iii) $\|a^2\| = \|a\|^2$,
(iv) $\|a^2 + b^2\|^2 \geqslant \|a^2\|$.

For all unmentioned details on Jordan algebras we refer the reader to the monograph [18]. For $a \in A$ we shall denote by U_a the positive linear map of A into A defined by $U_a(x) = 2a \circ (a \circ x) - a^2 \circ x$. We say that two elements $a, b \in A$ operator commute if $a \circ (b \circ x) = b \circ (a \circ x)$ for each $x \in A$. Let $M \subset A$. The symbol M' will be reserved for the set of all elements of A operator commuting with all elements in M. The set M' is always a JB subalgebra. Indeed, by [18], p. 44, elements a and b in A operator commute if and only if they generate an associative subalgebra. Hence, if a and b operator commute then a and b^2 also operator commute. Therefore, M' is closed with respect to squares. Since M' is obviously a closed subspace, we see that it is a subalgebra. The algebra A will always be identified canonically with the weak*-dense subalgebra of its double dual A^{**}. In the same way we shall identify functionals in A^* with their canonical normal extension to A^{**}. A pure state of A is an extreme point of the positive part of the unit sphere of the dual A^*. For any pure state ϱ on A there is a uniquely defined minimal projection in A^{**}, denoted by $s(\varrho)$, such that $U_{s(\varrho)}(a) = \varrho(a)s(\varrho)$ for each $a \in A$. By $c(\varrho)$ we shall denote the central support of ϱ, i.e. the smallest central projection in A^{**} majorizing the projection $s(\varrho)$. Two pure states ϱ and φ are either equivalent, meaning that $c(\varrho) = c(\varphi)$, or inequivalent, meaning that $c(\varrho) \circ c(\varphi) = 0$. Recall that when A is a self-adjoint part of a C^*-algebra endowed with the standard anticommutant product then $c(\varrho)A^{**}$ $(= U_{c(\varrho)}(A^{**}))$ is isomorphic to the self-adjoint part of the algebra $B(H_\varrho)$ of all bounded operators acting on the Hilbert space H_ϱ resulting from the GNS construction corresponding to ϱ. Pure states ϱ and φ are said to be orthogonal if $\|\varrho - \varphi\| = 2$, or equivalently, if $s(\varrho) \circ s(\varphi) = 0$.

A projection p in A^{**} is called open if there is a net (a_α) of positive elements in A such that $a_\alpha \nearrow p$. A projection p in A^{**} is said to be closed if $1 - p$ is open. The product $p \circ q$ of two operator commuting open projections p and q is again open (see [2], Theorem II.7 for the C^*-case, the proof for the Jordan algebras is similar). The support projection $s(\varrho)$ of a pure state ϱ is always closed. The range projection $r(a)$ of $a \in A$ is always open. (The range projection of $a \in A^{**}$ is a smallest projection $p \in A^{**}$ such that $p \circ a = a$). Finally, A is called σ-finite if any system of mutually orthogonal projections in A is at most countable. A projection $p \in A$ is called σ-finite if the algebra $U_p(A)$ is σ-finite.

In order to prove the main result we shall need the following auxiliary lemmas.

2.1. LEMMA. *Let $\varrho_1, \varrho_2, \dots, \varrho_n$ be orthogonal pure states on a* JB *algebra A such that $c(\varrho_i)A^{**}$ is σ-finite for each $i = 1, 2, \dots, n$. Suppose $x \in A$ operator commutes with all support projections $s(\varrho_1), s(\varrho_2), \dots, s(\varrho_n)$. Then there is an element $a \in A$,*

$$0 \leqslant a \leqslant r(x) \circ \left(1 - \sum_{i=1}^{n} s(\varrho_i)\right),$$

satisfying

$$e \circ r(a) = \left(e - \sum_{i=1}^{n} s(\varrho_i)\right) \circ r(x),$$

where e is the supremum projection of the set $\{c(\varrho_1), c(\varrho_2), \dots, c(\varrho_n)\}$ in the projection lattice of A^{**}.

Proof. As all minimal projections $s(\varrho_1), s(\varrho_2), \dots, s(\varrho_n)$ operator commute with $r(x)$, an element $p = \left(e - \sum_{i=1}^{n} s(\varrho_i)\right) \circ r(x)$ is a projection. If $p = 0$ we can set $a = 0$. Suppose that p is nonzero. By the assumption p is σ-finite. Any projection in eA^{**} is a union of minimal projections (see the structure theory of JW factors [18]). Therefore p can be written as

$$p = \sum_{\alpha \in I} e_\alpha,$$

where each e_α is a minimal projection in A^{**} and I is a subset of positive integers. Let us choose states ω_α's with the supports $s(\omega_\alpha) = e_\alpha$, $\alpha \in I$, and set

$$\omega = \sum_{\alpha \in I} \lambda_\alpha \omega_\alpha, \quad \text{where } 0 < \lambda_\alpha < 1, \sum_{\alpha \in I} \lambda_\alpha = 1.$$

It can be verified easily that the support of ω is exactly p. Since the projection $r(x) \circ \left(1 - \sum_{i=1}^{n} s(\varrho_i)\right)$ is open we can select a net $(a_\gamma) \subset A$ of positive elements such that $a_\gamma \nearrow r(x) \circ \left(1 - \sum_{i=1}^{n} s(\varrho_i)\right)$. As $\omega(p) = 1$ and $p \leqslant r(x) \circ \left(1 - \sum_{i=1}^{n} s(\varrho_i)\right)$, $\omega\left(r(x) \circ \left(1 - \sum_{i=1}^{n} s(\varrho_i)\right)\right) = 1$ and so $\omega(a_\gamma) \nearrow 1$. Hence, there is a subsequence (a_j) (finite or infinite) of (a_γ) such that $\omega(a_j) \nearrow 1$. The sequence $(r(a_j))$ being increasing there exists a projection $q \in A^{**}$ such that $r(a_j) \nearrow q$. Now $q \leqslant r(x) \circ \left(1 - \sum_{i=1}^{n} s(\varrho_i)\right)$ and $\omega(q) = 1$, which implies

$$r(x) \circ \left(e - \sum_{i=1}^{n} s(\varrho_i)\right) \leqslant q \leqslant r(x) \circ \left(1 - \sum_{i=1}^{n} s(\varrho_i)\right)$$

and, in turn,

$$e \circ q = \left(e - \sum_{i=1}^{n} s(\varrho_i)\right) \circ r(x).$$

Putting

$$a = \sum_j \frac{1}{2^j} a_j$$

we get an element in A satisfying $r(a) = q$ and the proof is completed by the equality above. ∎

2.2. LEMMA. *Let $\varrho_1, \varrho_2, \ldots, \varrho_n$ be orthogonal pure states on a* JB *algebra A. Then there are norm one, positive, and mutually orthogonal elements x_1, $x_2, \ldots, x_n \in A$ such that*

$$x_i \circ s(\varrho_j) = \delta_{ij} s(\varrho_j)$$

for all $i, j = 1, \ldots, n$.

Proof. By [20], Proposition 2.3 there are positive, mutually orthogonal, norm one elements $x_1, x_2, \ldots, x_n \in A$ such that

$$\varrho_j(x_i) = \delta_{ij}, \quad i, j = 1, \ldots, n.$$

Fix an index i. Then

$$U_{s(\varrho_j)}(x_i) = \varrho_j(x_i) s(\varrho_j) = 0 \quad \text{whenever } i \neq j.$$

According to [18], 4.1.14, p. 98, $p \circ a = 0$ for a projection p and a positive element a in JB algebra whenever $U_p(a) = 0$. Therefore

$$x_i \circ s(\varrho_j) = 0 \quad \text{whenever } j \neq i.$$

Similarly, $\varrho_i(x_i) = 1$ implies $\varrho_i(1 - x_i) = 0$ and it follows $s(\varrho_i) = s(\varrho_i) \circ x_i$. The proof is completed. ∎

The following theorem extends Theorem 1.1 from [3] and Theorem 1 from [11], and was proved for individual state in [19].

2.3. THEOREM. *Let $\varrho_1, \varrho_2, \ldots, \varrho_n$ be orthogonal pure states on a* JB *algebra A such that $c(\varrho_i)A^{**}$ is σ-finite for all $i = 1, \ldots, n$. Then there is a maximal associative subalgebra B in A such that every state ϱ_i restricts to a pure state on B.*

Proof. Let us take positive, norm one, mutually orthogonal elements x_1, $x_2, \ldots, x_n \in A$ with the property

$$x_i \circ s(\varrho_j) = \delta_{ij} s(\varrho_j), \quad i, j = 1, \ldots, n,$$

the existence of which is guaranteed by Lemma 2.2. According to Lemma 2.1 we can find elements $a_i \in A$ satisfying for all $i = 1, \ldots, n$

$$0 \leqslant a_i \leqslant r(x_i) \circ \left(1 - \sum_{i=1}^n s(\varrho_i)\right)$$

and

$$e \circ r(a_i) = \left(e - \sum_{i=1}^n s(\varrho_i)\right) \circ r(x_i), \tag{2.1}$$

where $e = \bigvee_{i=1}^{n} c(\varrho_i)$. Let us put $x = x_1 + x_2 + \cdots + x_n$. Since $x_1, \ldots, x_n$ are mutually orthogonal $r(x) = r(x_1) + \cdots + r(x_n)$ and by (2.1)

$$e \circ \sum_{i=1}^{n} r(a_i) = \Big(e - \sum_{i=1}^{n} s(\varrho_i)\Big) \circ r(x). \tag{2.2}$$

Letting $b_i = x_i - a_i$, $i = 1, \ldots, n$, we get a system of orthogonal (and thereby operator commuting) elements in A. Moreover, b_i's are norm one since $1 \geqslant x_i \geqslant x_i - a_i \geqslant -1$ and $\varrho_i(b_i) = 1$. Let us choose a maximal associative subalgebra $\mathcal{C}$ in the algebra

$$X = U_{r(x) \circ \left(1 - \sum_{i=1}^{n} s(\varrho_i)\right)}(A^{**}) \cap A \cap \{b_1, b_2, \ldots, b_n\}'.$$

If X is zero, we set $\mathcal{C} = \{0\}$. Let $\mathcal{B}$ be the algebra generated by $b_1, b_2, \ldots, b_n$ and the set $\mathcal{C}$. We shall show that $\mathcal{B}$ is a maximal associative subalgebra of $U_{r(x)}(A^{**}) \cap A$. For this let us take an element $u \in U_{r(x)}(A^{**}) \cap A$ operator commuting with all elements in $\mathcal{B}$. For proving that $u \in \mathcal{B}$ we shall consider the following auxiliary element

$$\begin{aligned} h &= e \circ \big((1 - b_1 - b_2 - \cdots - b_n) \circ r(x)\big) \\ &= e \circ (r(x) - x_1 - x_2 - \cdots - x_n) + e \circ (a_1 + a_2 + \cdots + a_n) \\ &\geqslant e \circ (a_1 + a_2 \cdots + a_n). \end{aligned}$$

Hence,

$$e \circ r(x) \geqslant r(h) \geqslant \Big(e - \sum_{i=1}^{n} s(\varrho_i)\Big) \circ r(x) \tag{2.3}$$

by (2.2). As $a_i \circ s(\varrho_j) = 0$ for all $i, j = 1, \ldots, n$ we get $b_i \circ s(\varrho_j) = (x_i - a_i) \circ s(\varrho_j) = \delta_{ij} s(\varrho_i)$ and so

$$h \circ s(\varrho_i) = r(x) \circ s(\varrho_i) - b_i \circ s(\varrho_i) = s(\varrho_i) - s(\varrho_i) = 0 \tag{2.4}$$

for all $i = 1, \ldots, n$. (In the last equality we used the fact that $x_i \circ s(\varrho_i) = s(\varrho_i)$ and also $r(x_i) \circ s(\varrho_i) = s(\varrho_i)$). Thus, combining (2.3) and (2.4), we conclude

$$r(h) = \Big(e - \sum_{i=1}^{n} s(\varrho_i)\Big) \circ r(x).$$

As u operator commutes with h we see that

$$\Big(u \circ \sum_{i=1}^{n} s(\varrho_i)\Big) \circ h = u \circ \Big(\sum_{i=1}^{n} s(\varrho_i) \circ h\Big) = 0.$$

By this

$$\begin{aligned} 0 &= \Big(u \circ \sum_{i=1}^{n} s(\varrho_i)\Big) \circ r(h) = \Big(u \circ \sum_{i=1}^{n} s(\varrho_i)\Big) \circ \Big(\Big(e - \sum_{i=1}^{n} s(\varrho_i)\Big) \circ r(x)\Big) \\ &= u \circ \sum_{i=1}^{n} s(\varrho_i) - \Big(u \circ \sum_{i=1}^{n} s(\varrho_i)\Big) \circ \sum_{i=1}^{n} s(\varrho_i). \end{aligned}$$

In other words, $u \circ \sum_{i=1}^{n} s(\varrho_i) = U_{\sum_{i=1}^{n} s(\varrho_i)}(u)$, or equivalently, u operator commutes with $\sum_{i=1}^{n} s(\varrho_i)$. Let us now put

$$v = u - \sum_{i=1}^{n} \varrho_i(u) b_i. \tag{2.5}$$

The element v operator commutes with $\sum_{i=1}^{n} s(\varrho_i)$ and so $v \circ \sum_{i=1}^{n} s(\varrho_i) = U_{\sum_{i=1}^{n} s(\varrho_i)}(v)$.
The equality

$$\varrho_j\Big(U_{\sum_{i=1}^{n} s(\varrho_i)}(v)\Big) = \varrho_j(v) = \varrho_j(u) - \varrho_j(u) = 0, \quad i = 1, \dots, n$$

implies immediately $U_{\sum_{i=1}^{n} s(\varrho_i)}(v) = 0$ and so $v \circ \sum_{i=1}^{n} s(\varrho_i) = 0$. In summary,

$$v = U_{r(x) \circ \left(1 - \sum_{i=1}^{n} s(\varrho_i)\right)}(v) \in U_{r(x) \circ \left(1 - \sum_{i=1}^{n} s(\varrho_i)\right)}(A^{**}) \cap A.$$

Therefore $v \in \mathcal{C}$ and immediately $u \in \mathcal{B}$ showing that $\mathcal{B}$ is a maximal associative subalgebra of

$$U_{r(x)}(A^{**}) \cap A.$$

Further, since $\varrho_i(b_i) = \|b_i\| = 1$ the Schwarz inequality entails that states ϱ_1, $\varrho_2, \dots, \varrho_n$ are multiplicative, and thereby pure, on $\mathcal{B}$.

Finally, let us extend $\mathcal{B}$ to a maximal associative subalgebra B of A and show that B satisfies all statements of the Theorem 2.3. Since $B \cap U_{r(x)}(A^{**}) = \mathcal{B}$, $\mathcal{B}$ is a hereditary subalgebra of B. Indeed, whenever $0 \leqslant f \leqslant g$ with $f \in B$ and $g \in \mathcal{B}$, $\|g\| = 1$, then $f \leqslant r(x)$ and so $f \in \mathcal{B}$. It follows that all states $\varrho_1, \varrho_2, \dots, \varrho_n$ are pure on B, which completes the proof. ∎

Because of the one-to-one correspondence between associative JB algebras and abelian C^*-algebras given by the complexification we get by specializing Theorem 2.3 to C^*-algebras both Theorem 1.1 from [3] and Theorem 1 from [11].

In the concluding part of this section we shall deal with possible extensions of Theorem 2.3 to systems of infinitely many orthogonal pure states. First we show that Theorem 2.3 does not hold for arbitrary system of orthogonal pure states. We exhibit a counterexample on any self-adjoint part of irreducible separable C^*-algebra which is not a commutative extension of the algebra of all compact operators.

2.4. COUNTEREXAMPLE. *Let $\mathcal{A}$ be a separable unital infinitely dimensional C^*-algebra acting irreducibly on a Hilbert space H. Let $\mathcal{K}$ be the algebra of all compact operators acting on H. Suppose that $\mathcal{A}/\mathcal{A}\cap\mathcal{K}$ is non-commutative. Then there is a sequence of pure, mutually orthogonal states on $\mathcal{A}$ which do not restrict simultaneously to pure states on any maximal abelian subalgebra of $\mathcal{A}$.*

Proof. Since $\mathcal{A}$ acts irreducibly, either $\mathcal{K}\subset\mathcal{A}$ or $\mathcal{K}\cap\mathcal{A}=\{0\}$. Therefore our discussion falls into two cases:

(a) Suppose $\mathcal{A}\cap\mathcal{K}=\{0\}$. In that case $\mathcal{A}/\mathcal{A}\cap\mathcal{K}=\mathcal{A}$ is automatically non-commutative because $\mathcal{A}$ has an infinite dimension. One can associate to any orthonormal basis (ξ_n) of H a system of vector states (ω_{ξ_n}), $\omega_{\xi_n}(a)=(a\xi_n,\xi_n)$, for $a\in\mathcal{A}$, which constitutes an orthogonal sequence of pure states on $\mathcal{A}$. We show that at least one sequence (ω_{ξ_n}) of the form stated above fulfils the statement of the counterexample. Let us suppose the contrary. By the stronger version of the Glimm's lemma [16] as stated in [5], Remark on p. 263–264, given a state f on $\mathcal{A}$, there is an orthonormal basis (ξ_n) of H such that

$$f(a)=\lim_{n\to\infty}(a\xi_n,\xi_n)\quad\text{for each } a\in\mathcal{A}.$$

Let $\mathcal{B}_f$ be a maximal abelian subalgebra such that all states (ω_{ξ_n}) are pure states on $\mathcal{B}_f$. Then, of course, f is multiplicative on $\mathcal{B}_f$. In other words, any state of $\mathcal{A}$ is multiplicative on some maximal abelian C^*-subalgebra of $\mathcal{A}$. Let us now choose a sequence (x_n) of unit vectors in H which is dense in the unit sphere of H. By the argument above, for a state $f=\sum_{n=1}^{\infty}\frac{1}{2^n}\omega_{x_n}$, there is a maximal abelian subalgebra $\mathcal{B}_f$ such that f is multiplicative on $\mathcal{B}_f$. As

$$\frac{1}{2^n}\omega_{x_n}\leqslant f\quad\text{for all } n \text{ on } \mathcal{B}_f$$

we see that all states (ω_{x_n}) coincide with f on $\mathcal{B}_f$ and so they are multiplicative on $\mathcal{B}_f$. Obviously, for any $a\in\mathcal{A}$

$$(ax_n,x_n)\geqslant 0\quad\text{for all } n \text{ if and only if } a\geqslant 0,$$

which implies that the weak*-closure of the set (ω_{x_n}) contains all pure states of $\mathcal{A}$ (see e.g. [21], Theorem 4.3.8, p. 161). It means that any pure state of $\mathcal{A}$ is multiplicative on $\mathcal{B}_f$. Take now self-adjoint elements $a\in\mathcal{B}_f$, $b\in\mathcal{A}$ and a pure state ϱ of $\mathcal{A}$. The state ϱ is definite on a. Thus,

$$\varrho(\mathrm{i}(ab-ba))=0.$$

It gives immediately $ab-ba=0$. Hence, $\mathcal{B}_f$ is contained in the center $Z(\mathcal{A})$ of $\mathcal{A}$ and so $\mathcal{B}_f=Z(\mathcal{A})$ by maximality. As the center is a maximal abelian subalgebra we infer that $\mathcal{A}$ has to be abelian — a contradiction with our assumption.

(b) Let us now consider the case of $\mathcal{K}\subset\mathcal{A}$ that requires a different technique. The quotient algebra $\mathcal{A}/\mathcal{K}$ is non-abelian by the assumption, hence we can find a self-adjoint element $\widehat{a}$ in $\mathcal{A}/\mathcal{K}$ and a pure state $\widehat{\varrho}$ of $\mathcal{A}/\mathcal{K}$ such that $\widehat{\varrho}(\widehat{a})^2\neq\widehat{\varrho}(\widehat{a}^2)$ (see e.g. the reasoning in the part (a) above). We can lift $\widehat{a}$ and $\widehat{\varrho}$ to the self-adjoint element a in $\mathcal{A}$ and the pure state ϱ on $\mathcal{A}$ with $\varrho(a)^2\neq\varrho(a^2)$. Making use of the Weyl-von Neumann theorem (see e.g. [13], p. 59) we can find for any $\varepsilon>0$ an orthonormal basis (ξ_n) of H and a compact operator $k\in\mathcal{K}$, $\|k\|\leqslant\varepsilon$, such that

$$a=a_d+k,$$

where $a_d \in \mathcal{A}$ is a diagonal self-adjoint operator with respect to the basis (ξ_n) of H. By taking k sufficiently small we can arrange for $\varrho(a_d^2) \neq \varrho(a_d)^2$. Suppose that there is a maximal abelian subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that the sequence $(\omega_{\xi_n}) \cup \{\varrho\}$ of orthogonal pure states gives the sequence of pure states on $\mathcal{B}$ and try to reach a contradiction. Take a self-adjoint element $b \in \mathcal{B}$. The definitness of ω_{ξ_n} on b implies that $b-(b\xi_n, \xi_n)1 \in \{x \in A \mid \omega_{\xi_n}(x^*x) = 0\}$ ([6]). Hence, $b\xi_n = (b\xi_n, \xi_n)\xi_n$ and we have derived that each vector ξ_n is an eigenvector of b. Alternatively, b is diagonal with respect to the basis (ξ_n) and so $\mathcal{B}$ is the algebra of all such operators lying in $\mathcal{A}$ by maximality. As $a_d \in \mathcal{B}$ and ϱ is not definite on a_d we get a contradiction. The proof is completed. ∎

As a corollary of the Counterexample 2.4 we get, among others, that the restriction property does not hold for all infinite systems of orthogonal pure states on primitive antiliminal C^*-algebras. It is also interesting to remark that the preceeding counterexample implies that there is a separable, irreducible subalgebra of the Calkin algebra which admits a countable system of orthogonal pure states without the restriction property. This is in contrast with the remarkable result of J. Anderson ([7]) to the effect that any countable system (f_n) of (even not necessarily pure and not necessarily mutually orthogonal) states on the Calkin algebra does have the restriction property.

In the light of the Counterexample 2.4 we need some additional conditions for a system of orthogonal pure states to have the restriction property. Following [5] we say that a sequence of pure states (ϱ_n) on a JB algebra A approaches to infinity if $\lim_{n\to\infty} \varrho_n(a) = 0$ for all $a \in A$ such that the spectrum of a contains zero.

2.5. Theorem. *Let (ϱ_n) be a sequence of orthogonal pure states on a* JB *algebra A approaching to infinity and such that $c(\varrho_n)A^{**}$ is σ-finite for all $n = 1, 2, \ldots$. Suppose further that $\sum_{n=m}^{\infty} s(\varrho_n)$ is a closed projection for all m. Then there a maximal associative subalgebra B of A such that all states ϱ_n's restrict to pure states on B.*

Proof. The proof of the Theorem 2.5 can be obtained by modifying the arguments in the proof of the Theorem 2.3. Indeed, there is a sequence (x_n) of orthogonal, positive, norm one elements in A with

$$x_i \circ s(\varrho_j) = \delta_{ij} s(\varrho_j), \quad i, j = 1, 2, \ldots.$$

(Using arguments in Lemma 2.2 this can be obtained by modifying the proof for the C^*-case in Theorem 2.7 from [5].) Employing the fact that $1 - \sum_{i=1}^{\infty} s(\varrho_i)$ is an open projection we can show as in the proof of Lemma 2.1 that for each i there is an element a_i with

$$0 \leqslant a_i \leqslant r(x_i) \circ \left(1 - \sum_{j=1}^{\infty} s(\varrho_j)\right)$$

and

$$e \circ r(a_i) = \left(e - \sum_{j=1}^{\infty} s(\varrho_j)\right) \circ r(x_i),$$

where $e = \bigvee_{j=1}^{\infty} c(\varrho_j)$. We set

$$x = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i \quad \text{and} \quad b_i = x_i - a_i.$$

Then $r(x) = \sum_{i=1}^{\infty} r(x_i)$. Let now $\mathcal{C}$ be a maximal associative subalgebra in the algebra

$$U_{r(x)\circ\left(1-\sum_{j=1}^{\infty} s(\varrho_j)\right)}(A^{**}) \cap A \cap \{b_1, b_2, \dots\}'$$

and $\mathcal{B}$ be a subalgebra generated by the set $\{b_i\}_{i=1}^{\infty} \cup \mathcal{C}$. We show that $\mathcal{B}$ is maximal associative. For this, let us choose an auxiliary element

$$h = e \circ \left(\left(1 - \sum_{i=1}^{\infty} b_i\right) \circ r(x)\right).$$

Then (see the proof of Theorem 2.1)

$$r(h) = \left(e - \sum_{i=1}^{\infty} s(\varrho_i)\right) \circ r(x).$$

Take an element $u \in U_{r(x)}(A^{**}) \cap A$ operator commuting with $\mathcal{B}$. Without loss of generality we can suppose that the spectrum of u contains zero and set

$$v = u - \sum_{i=1}^{\infty} \varrho_i(u) b_i.$$

Observe that the series on the right hand side converges in A since $\lim_{n\to\infty} \varrho_n(u) = 0$ and b_i's are orthogonal. Now we can proceed as in the proof of Theorem 2.3 and obtain that $v \in \mathcal{C}$ proving that $\mathcal{B}$ is maximal. The rest of the proof consists in extending $\mathcal{B}$ to a maximal associative subalgebra B of A with the desired property and it is the same as in the concluding part of the proof of Theorem 2.3. ∎

For a separable non-unital JB algebra the condition that the sequence of infinitely many pure states (ϱ_n) tends to infinity and $\sum_{n=m}^{\infty} s(\varrho_n)$ is closed for all m is equivalent to the condition that there is a strictly positive element $a \in A$ such that each ϱ_n is definite on a and $\lim_{n\to\infty} \varrho_n(a) = 0$ (see [5], Theorem 2.7 which can be directly generalized to Jordan algebras). Also, a straightforward transcription of the C^*-arguments in [5] gives that any orthogonal system of pure states $\varrho_1, \varrho_2, \dots$ on a separable JB algebra with finite equivalent classes the size of which is uniformly bounded (such a system is called *nearly inequivalent*) obeys the assumption of Theorem 2.5 and thereby enjoys the restriction property.

3. RESTRICTING COMPLETELY POSITIVE MAPS ON C^*-ALGEBRAS

In this section we shall deal with the restriction property of extreme completely positive maps on C^*-algebras. All C^*-algebras will be assumed to be unital. Let $\mathcal{A}$ be a C^*-algebra and $B(H)$ be the algebra of all bounded operators acting on a Hilbert space H. By the symbol $M_n(\mathcal{A})$ we shall denote the C^*-algebra of all $n \times n$ matrices over $\mathcal{A}$. A linear map $\varphi : \mathcal{A} \to B(H)$ is called *completely positive* if, for each n, the map $\varphi^{(n)} : M_n(\mathcal{A}) \to M_n(B(H))$ given by

$$\varphi^{(n)}((a_{ij}))_{ij} = (\varphi(a_{ij}))_{ij}$$

is positive. According to the fundamental Stinespring's theorem ([25]) any completely positive map is similar to a representation π of $\mathcal{A}$ on a Hilbert space K in the sense of the following equality

$$\varphi(a) = V^* \pi(a) V,$$

where V is a bounded operator of H into K and $[\pi(\mathcal{A})V(H)] = K$. (From now on the symbol $[X]$ shall denote the norm closed linear span of the set X). Moreover, a completely positive map is called *unital* if it preserves the unite. In that case the Stinespring's decomposition simplifies to V being an isometric embedding of H into K. So we can (and shall) identify H with the subspace of K and write

$$\varphi(a) = P\pi(a)P|H,$$

where P is the orthogonal projection of K onto H. Then $[\pi(\mathcal{A})H] = K$. There are a few generalizations of the concept of pure state to the context of completely positive maps. At first, a completely positive map $\varphi : \mathcal{A} \to B(H)$ is called *pure* if the only completely positive map ψ for which $\psi \leqslant \varphi$ is a multiple of φ. (We use the notation $\psi \leqslant \varphi$ if $\varphi - \psi$ is completely positive.) A completely positive map φ is pure if and only if its representation π in the Stinespring's decomposition is irreducible [8], Corolarry 1.4.3. A unital completely positive map φ is called *C^*-extreme* if the following condition holds: If $\varphi = \sum_{i=1}^{n} t_i^* \varphi_i t_i$, where t_i's are invertible operators in $B(H)$, $\sum_{i=1}^{n} t_i^* t_i = 1$, and φ_i's are completely positive unital maps of $\mathcal{A}$ into $B(H)$, then all maps φ_i's are unitarily equivalent to φ. Any pure completely positive unital map is C^*-extreme. The converse does not hold. Recently, it was shown by D.R. Farenick and H. Zhou ([15]) that if $\dim H < \infty$ then every completely positive C^*-extreme map $\varphi : \mathcal{A} \to B(H)$ is a special direct sum of pure completely positive maps. More precisely, a sequence $\varphi_1^\pi, \varphi_2^\pi, \ldots, \varphi_n^\pi$ of unital completely positive maps, each mapping $\mathcal{A}$ into $B(K_i)$, is called a *nested sequence corresponding to the representation* $\pi : \mathcal{A} \to B(K_0)$ if

$$\varphi_1^\pi = w_1^* \pi w_1 \quad \text{and} \quad \varphi_i^\pi = w_i^* \varphi_{i-1}^\pi w_i \quad \text{for } i \geqslant 2$$

where each w_i is an isometric embedding of K_i into K_{i-1}. By Theorem 2.1 from [15] a completely positive map $\varphi : \mathcal{A} \to B(H)$ is a C^*-extreme map if and only if there is a sequence of irreducible inequivalent representations $\pi_1, \pi_2, \dots, \pi_k$ of $\mathcal{A}$ such that φ is unitarily equivalent to the direct sum

$$\sum_{i=1}^{k} \oplus \Big(\sum_{j=1}^{n_i} \varphi_j^{\pi_i} \Big), \tag{3.1}$$

where each sequence $\varphi_1^{\pi_i}, \dots, \varphi_{n_i}^{\pi_i}$ is a nested sequence of completely positive maps corresponding to π_i. We shall call a completely positive map φ *σ-finite* if the representation in its Stinespring's decomposition acts on a separable Hilbert space.

As opposed to states, pure completely positive map on a separable algebra need not restrict to pure completely positive map on any maximal abelian subalgebra. Indeed, let us consider an irreducible C^*-algebra $\mathcal{A}$ acting on a Hilbert space H with dim $H > 1$. Since the identity map π is an irreducible representation of $\mathcal{A}$, π is a pure map. Any maximal abelian subalgebra $\mathcal{B}$ of $\mathcal{A}$ is of dimension at least two. Thus the restriction of π to $\mathcal{B}$ cannot be a pure map because any pure map on an abelian C^*-algebra has to be a complex-valued homomorphism. Nevertheless, π is obviously multiplicative on $\mathcal{B}$. So an appropriate formulation of the restriction property for completely positive maps is the property of being multiplicative on some maximal abelian subalgebra. The following theorem (generalization of Theorem 1 from [11]) shows that any C^*-extreme, finite dimensional, completely positive map enjoys this property.

3.1. THEOREM. *Let $\varphi : \mathcal{A} \to B(H)$ be C^*-extreme, completely positive, σ-finite map, where $\mathcal{A}$ is a C^*-algebra and $\dim H < \infty$. Then there is a maximal abelian subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $\varphi|\mathcal{B}$ is multiplicative.*

Proof. Any C^*-extreme completely positive map with values in finite dimensional Hilbert space is unitarily equivalent to a sequence of compressions of irreducible representations. So we can suppose that that φ is of the form (3.1). Assume each irreducible representations π_i acts on a Hilbert space H_i. Under obvious identification we can find, for each i, a decreasing sequence of non-zero finite dimensional orthogonal projections $P_1^i \supseteq \dots \supseteq P_{n_i}^i$ in $B(H_i)$ such that

$$\varphi_j^{\pi_i}(a) = P_j^i \pi_i(a) P_j^i | P_j(H_i) \quad i = 1, \dots, k; \ j = 1, \dots n_i. \tag{3.2}$$

Now, let us pick an orthonormal basis $(x_n)_{n=1}^l$ of finite dimensional Hilbert space

$$K = P_1^1(H_1) \oplus P_1^2(H_2) \oplus \dots \oplus P_1^k(H_k)$$

such that each Hilbert space $P_j^i(H_i)$ (viewed as a subspace of K) has an orthonormal basis which is a subsequence of $(x_n)_{n=1}^l$. Let us define states $\varrho_1, \dots, \varrho_l$ by

$$\varrho_m(a) = (\pi_s(a) x_m, x_m), \quad a \in A,$$

where s is such that $x_m \in P_1^s(H_s)$. The system $\varrho_1, \dots, \varrho_l$ is a finite family of mutually orthogonal pure states on $\mathcal{A}$. Employing Theorem 2.3 we can find a maximal abelian subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that the states $\varrho_1, \varrho_2, \dots, \varrho_l$ restrict to pure states on $\mathcal{B}$. We show that each direct summand in (3.1) is multiplicative on $\mathcal{B}$. For this it suffices to consider the case $i = 1$. Fix $1 \leqslant j \leqslant n_1$. By construction, there is an orthonormal basis $\xi_1, \dots, \xi_r$ of $P_j^1(H_1)$ such that the corresponding sequence of vector states $\psi_1, \dots, \psi_r$ is a subsequence of $\varrho_1, \dots, \varrho_l$. By the uniqueness of the GNS construction the spaces $[\pi_1(\mathcal{B})\xi_h]$, where $h = 1, 2, \dots, r$, have to be one-dimensional. Taking into account that $\pi_1(1)\xi_h = \xi_h$ we see that

$$[\pi_1(\mathcal{B})\xi_h] = [\xi_h].$$

Thus, $\pi_1(b)\xi_h = \psi_h(b)\xi_h$ for all $b \in \mathcal{B}$. Fix now $b \in \mathcal{B}$ and $z \in P_j^1(H_1)$. Then

$$\begin{aligned}\varphi_j^{\pi_1}(b)z = P_j^1\pi_1(b)P_j^1 z = P_j^1\pi_1(b)z = \sum_{h=1}^{r}(\pi_1(b)z, \xi_h)\xi_h \\ = \sum_{h=1}^{r}(z, \pi_1(b^*)\xi_h)\xi_h = \sum_{h=1}^{r}(z, \psi_h(b^*)\xi_h)\xi_h = \sum_{h=1}^{r}\psi_h(b)(z, \xi_h)\xi_h.\end{aligned}$$

In other words,

$$\varphi_j^{\pi_1}(b) = \sum_{h=1}^{r}\psi_h(b)P_{\xi_h}, \tag{3.3}$$

where P_{ξ_h} is the orthogonal projection on $[\xi_h]$. Therefore $\varphi_j^{\pi_1}$ is multiplicative on $\mathcal{B}$. The proof is complete. ∎

We conclude the paper with some comments on Theorem 3.1. First of all, when $\mathcal{A}$ is separable then any C^*-extreme completely positive map $\varphi : \mathcal{A} \to B(H)$, $\dim H < \infty$, restricts automatically to multiplicative map on some maximal abelian subalgebra $\mathcal{B}$ of $\mathcal{A}$. In the light of Theorem 2.5 the Theorem 3.1 holds for $\dim H = \infty$ on condition that we can find a sequence of vector states corresponding to φ which are separated in some way. Also assumption that φ is σ-finite can be removed if $\mathcal{A}$ is postliminal and φ is pure. (This will be done in a subsequent paper.)

Finally, Theorem 3.1 can be strengthen by showing that φ is even $\mathcal{B}$-morphism. Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. A unital completely positive map $\varphi : \mathcal{A} \to B(H)$ is called *$\mathcal{B}$-morphism* if

$$\varphi(ba) = \varphi(b)\varphi(a) \quad \text{for all } b \in \mathcal{B},\ a \in \mathcal{A}.$$

$\mathcal{B}$-morphisms play an important role in dilatation theory of completely positive maps ([9] and [10]). The maximal abelian subalgebra $\mathcal{B}$ of $\mathcal{A}$ in Theorem 3.1 as constructed in its proof is such that φ is even $\mathcal{B}$-morphism. Indeed, reviewing the proof of Theorem 3.1, and using its notation, it is enough to verify that any direct summand, let us say $\varphi_j^{\pi_1}$, in (3.1) is a $\mathcal{B}$-morphism. Take $b \in \mathcal{B}$, $a \in \mathcal{A}$, and $x \in H_1$. Then

$$P_j^1\pi_1(b)x = \sum_{h=1}^{r}(\pi_1(b)x, \xi_h)\xi_h = \sum_{h=1}^{r}(x, \pi_1(b^*)\xi_h)\xi_h = \sum_{h=1}^{r}\psi_h(b)(x, \xi_h)\xi_h.$$

Alternatively,

$$P_j^1\pi_1(b) = \sum_{h=1}^{r}\psi_h(b)P_{\xi_h}.$$

Hence,

$$\begin{aligned}\varphi_j^{\pi_1}(ba) &= P_j^1\pi_1(ba)P_j^1|P_j^1(H_1) = P_j^1\pi_1(b)\pi_1(a)P_j^1|P_j^1(H_1)\\ &= P_j^1\sum_{h=1}^{r}\psi_h(b)P_{\xi_h}\pi_1(a)P_j^1|P_j^1(H_1)\\ &= P_j^1\sum_{h=1}^{r}\psi_h(b)P_{\xi_h}P_j^1\pi_1(a)P_j^1|P_j^1(H_1) = \varphi(b)\varphi(a).\end{aligned}$$

Acknowledgements. The author would like to express his gratitude to the Alexander von Humboldt Foundation, Bonn, for supporting his research. He also acknowledges the support of the Grant Agency of the Czech Republic, Grant no. 201/00/0331, "Operator Algebras, Orthocomplemented Structures, and Non-Commutative Measure Theory", and the support of the Czech Technical University, Grant. No. MSM 210000010, "Applied Mathematics in Technical Sciences".

REFERENCES

1. J.F. Aarnes, R.V. Kadison, Pure states and approximate identities, *Proc. Amer. Math. Soc.* **21**(1969), 749–752.
2. C.A. Akemann, The general Stone-Weierstrass problem, *J. Funct. Anal.* **4**(1969), 277–294.
3. C.A. Akemann, Approximate units and maximal abelian C^*-subalgebras, *Pacific J. Math.* **33**(1970), 543–550.
4. C.A. Akemann, J. Anderson, G.K. Pedersen, Excising states on C^*-algebras, *Canad. J. Math.* **38**(1986), 1239–1260.
5. C.A. Akemann, J. Anderson, G.K. Pedersen, Approaching infinity in C^*-algebras, *J. Operator Theory* **21**(1989), 255–271.
6. J. Anderson, Extensions, restrictions, and representations of states on C^*-algebras, *Trans. Amer. Math. Soc.* **249**(1979), 383–400.
7. J. Anderson, Pathalogy in the Calkin algebra, *J. Operator Theory* **2**(1979), 159–167.
8. W. Arveson, Subalgebras of C^*-algebras, *Acta Math.* **123**(1969), 141–224.
9. W. Arveson, Subalgebras of C^*-algebras. III: Multivariable operator theory, *Acta Math.* **181**(1998), 159–228.
10. W. Arveson, Minimal E_0-semigroups, in *Operator Algebras and their Applications* (*Waterloo, ON* 1994 (95)), Fields Inst. Commun., vol. 13, Amer. Math. Soc., Providence, RI, 1997, pp. 1–12.
11. B.A. Barnes, Pure states with the restriction property, *Proc. Amer. Math. Soc.* **33**(1972), 277–294.
12. P. Bush, P. Lahti, P. Mittelstaed, *The Quantum Theory of Measurement*, 2nd Edition, Springer Verlag, Berlin–Heidelberg, 1996.
13. K.R. Davidson, *C^*-Algebras by Example*, Fields Inst. Monogr., vol. ??, Amer. Math. Soc., Providence, RI, 1996.
14. D.R. Farenick, P.B. Morenz, C^*-extreme points in the generalized state spaces of a C^*-algebra, *Trans. Amer. Math. Soc.* **37**(1997), 1725–1748.
15. D.R. Farenick, H. Zhou, The structure of C^*-extreme points in spaces of completely positive linear maps on C^*-algebras, *Proc. Amer. Math. Soc.* **126** (1998), 1467–1477.

16. J. GLIMM, A Stone-Weierstrass theorem for C^*-algebras, *Ann. Math.* **72**(1960), 216–244.
17. S.P. GUDDER, *Stochastic Methods in Quantum Mechanics*, North-Holland, Amsterdam 1979.
18. H. HANCHE-OLSEN, E. STØRMER, *Jordan Operator Algebras*, Pitman Publishing, Boston–London–Melbourne 1984.
19. J. HAMHALTER, Restricting pure states on JB algebras to maximal associative subalgebras, *Tatra Mt. Math. Publ.* **15**(1998), 599–607.
20. J. HAMHALTER, Supporting sequences of pure states on JB algebras, *Studia Math.* **136**(1999), 37–47.
21. R.V. KADISON, J.R. RINGROSE, *Fundamentals of the Theory of Operator Algebras*, vol. I, Academic Press, 1983.
22. G.K. PEDERSEN, *C^*-algebras and their Automorphism Groups*, Academic Press, 1979.
23. M. REDEI, *Quantum Logic in Algebraic Approach*, Kluwer Academic Publishers, 1998.
24. I.E. SEGAL, Postulates for general quantum mechanics, *Ann. Math.* **48**(1947), 930–938.
25. W. STINESPRING, Positive functions on C^*-algebras, *Proc. Amer. Math. Soc.* **6**(1955), 211–216.
26. E. STØRMER, Irreducible Jordan algebras of self-adjoint operators, *Trans. Amer. Math. Soc.* **130**(1968), 153–166.

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Received February 21, 2000; revised February 7, 2001.