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SPECTRAL PROPERTIES AND NORM ESTIMATES ASSOCIATED TO THE $C_c^{(k)}$-FUNCTIONAL CALCULUS

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Abstract. We show that the Davies functional calculus and the $AC^{(\nu)}$-calculus coincide under common hypotheses. Then we apply the calculus to operators on Banach spaces, to investigate spectral invariance and norm estimates linked to abstract Cauchy equations. This extends some previous results in the area, and unifies diverse approaches. The theory is applied to operators related to multiplier theory on Lie groups.

Keywords: *Functional calculus, holomorphic semigroups, abstract Cauchy problem, multipliers.*

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0. INTRODUCTION

In this article we are concerned with some questions related to Cauchy equations in Banach spaces, which can be treated using underlying holomorphic semigroups with moderate growth on vertical lines. The link between both subjects is given by the existence of appropriate functional calculi. It supplies a unified viewpoint on problems usually approached by diverse methods.

Let X be a Banach space and let H be a closed linear operator with domain and range in X. Let us consider the abstract Cauchy equation

$$\begin{cases} u'(t) = -Hu(t), & t \geqslant 0, \\ u(0) = x, \end{cases}$$

where $x \in X$ (higher order Cauchy equations may be reduced to that using matrices). The Hille-Yosida theorem characterises those operators H for which the above problem admits a (unique) solution, given by a strongly continuous (C_0)-semigroup of bounded operators acting on X: $u(t,x) \equiv T^t x$, where $t \geqslant 0$ and $x \in X$. In this case $-H$ is the infinitesimal generator of T^t, with $T^t = \mathrm{e}^{-tH}$. There are important classes of Cauchy equations which are "ill-posed" in the sense that the elements of the formal semigroup e^{-tH} are not bounded operators on X. Well known examples of it are the Schrödinger operators $H = \mathrm{i}(\Delta + V)$ (where Δ is the usual Laplacian on $X = L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, $p \neq 2$ and V lies in the Kato class) or cosine functions linked to wave equations. A way to overcome this difficulty consists of looking for large sets of functions f giving rise to bounded operators of the form $\mathrm{e}^{-tH} f(H)$. So it is clear there is an interest in having suitable functional calculi involving H and e^{-tH} ([36], [35], [15]).

There have been recent developments in this direction connected with the formula

$$f(H) := -\frac{1}{\pi} \int\limits_{\mathbb{C}} \frac{\partial \widetilde{f}}{\partial \overline{z}}(z)(z-H)^{-1}\, \mathrm{d}x\, \mathrm{d}y \tag{0.1}$$

(here $\widetilde{f}$ is an almost analytic extension to $\mathbb{C}$ of the function f initially defined on $\mathbb{R}$, see Section 1). The formula (0.1) was introduced by Helffer and Sjöstrand for self-adjoint operators H ([28], [11], [12], [16]), and Jensen and Nakamura realized its validity to estimate expressions like $\mathrm{e}^{-\mathrm{i}tH} f(H)$ in their study of Schrödinger operators on $L^p(\mathbb{R}^n)$ ([36], [35]). One of the goals of this paper is to show in a fairly simple way how to extend or recover such results, provided that H can be given an appropriate version of a functional calculus linked to formula (0.1). In this respect, Davies has shown that an abstract calculus can be constructed on the basis of (0.1), whenever it is assumed that $\sigma(H) \subset \mathbb{R}$ and the resolvent function of H satisfies property

$$\|(z-H)^{-1}\| \leqslant C \frac{\langle z\rangle^{\alpha}}{|\Im z|^{\alpha+1}}, \tag{R_α}$$

for every $z \notin \mathbb{R}$, where $\langle z \rangle = \sqrt{1+|z|^2}$ and $\alpha \geqslant 0$ is fixed ([11]).

As an application, it is possible to give a very general statement about L^p-spectral independence of (initially defined) self-adjoint operators on L^2. This result covers a wide variety of cases ([12]). Let us pay attention to the method of proof followed by Davies. In this, the required condition on H affects the (C_0)-semigroup ("exponential function") e^{-tH} generated by H rather than its resolvent function. Indeed, the argument begins with the proof that H on L^p, say H_p, is such that $L = \varepsilon + H_p$ satisfies property

$$\|\mathrm{e}^{-zL}\| \leqslant C_\varepsilon \left(\frac{|z|}{\Re z}\right)^{\alpha} \quad \text{where } \Re z > 0, \tag{HG_α}$$

for every $\varepsilon > 0$ and fixed $\alpha \geqslant 0$. This implies that $\sigma(H_p) \subset [0,\infty)$ and, more important, implies property (R_α). Thus the functional calculus applies and one obtains $\sigma(H_p) = \sigma(H_2)$ ([12], p. 182).

On the other hand, it is possible to establish an accurate functional calculus for operators H satisfying (HG_α). This is done using the inverse Laplace transform

$$G^{\nu}(u) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\Re z = 1} \frac{\mathrm{e}^{-zH}}{z^{\nu+1}}\, \mathrm{e}^{uz}\, \mathrm{d}z,$$

where $u \in \mathbb{R}$ and $\nu > \alpha$, and Weyl fractional derivatives $W^{\nu} f$, with $\nu > 0$. A function g operates in this way on H whenever it has absolutely continuous (fractional) derivative up to the order $\nu + \frac{1}{2}$, with $\nu > \alpha$, and $g(H)$ can be obtained by refinement of the formula

$$f(H) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} W^{\nu+1} f(u) G^{\nu}(u) \, \mathrm{d}u, \tag{0.2}$$

where $\nu > \alpha$ and $f \in C_{\mathrm{c}}^{(\infty)}([0, \infty))$ ([20], Theorem 6.3). We will refer to this calculus as the *$AC^{(\nu)}$-calculus*. More generally, $f(H)$ has a sense as defined by (0.2) if $f \in C_{\mathrm{c}}^{(\infty)}([0, \infty))$ for example, and H is subject to the condition, that we call (G_α), with $\alpha \geqslant 0$, that is,

$$\sup_{\Re z \geqslant 1} \frac{\|\mathrm{e}^{-zH}\|}{|z|^{\alpha}} < \infty. \tag{G_α}$$

The present paper has been planned in an abstract setting. Our first main result is Theorem 2.1 which asserts that, under common hypotheses, the Davies functional calculus (also called *Davies-Helffer-Sjöstrand calculus* in [18]) and the $AC^{(\nu)}$-calculus of [20] coincide on common domains. Moreover they can be expressed by the operator-valued formula which is naturally based on the Fourier transform. These facts suggest a structural approach to the subject, which is started in Section 2. We show that $\sigma(H)$ is an invariant for the homomorphisms involved in that calculus and, as a consequence, we obtain an "internal" proof of spectral independence for operators H enjoying property (G_α). We mean for this a proof which depends directly on the growth on vertical lines of the semigroup generated by H rather than on the growth of its resolvent. It extends results of [12], [13] in particular.

Whereas all of the above is valid for general Banach spaces, there is another method to deduce spectral independence when we are dealing with interpolated Banach spaces. It turns out that the operators of semigroups e^{-zH} with property (G_α) are decomposable, and in this case an earlier theorem by E. Albrecht establishes the constancy of spectra for all interpolation spaces ([1]). This provides us with a shorter (but less direct) way to prove the results of [12], [13], [54] as well as those of this paper, about p-independence; see Theorem 2.7. Note that the decomposability of an operator is closely related to the existence of a sufficiently rich functional calculus ([2], [3], [4]). So functional calculi are still relevant in this method of proof.

Application of techniques associated with the $AC^{(\nu)}$ formulation implies bounds for the operators $f(H)\mathrm{e}^{\mathrm{i}tH}$. These generalise, or extend partially, similar results obtained in $\mathbb{R}^n$ by multiplier methods, amalgams, etc., and in Banach spaces by methods of regularized or integrated semigroups and so on (a more detailed analysis of the relationships between the functional calculus and these notions will be given in a forthcoming paper). See Sections 3 and 5.

Calculations in Section 3 show in particular that if a semigroup $(\mathrm{e}^{-zH})_{\Re z > 0}$ satisfies (HG_α) then $((1 + H)^{-z})_{\Re z > 0}$ is also an analytic semigroup which satisfies $(\mathrm{HG}_{\nu + \frac{1}{2}})$ for $\nu > \alpha$. In Section 4 we give a converse of this result for property (G_α). The argument is related to Hadamard fractional integration.

Finally we do observe in Section 5 the close connection which exists between questions on spectral independence or norm estimates and multiplier theory on stratified or solvable Lie groups. Interesting examples constructed in recent papers are then considered, to illustrate and apply the results of preceding sections.

1. PRELIMINARIES

We collect the main features of the functional calculi defined in [11] and [20], and make some related observations. We use the variable constant convention, in which C denotes a constant which may not be the same in different lines.

For f in $C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$ let $\widetilde{f}$ denote the following particular almost-analytic extension of f to $\mathbb{C}$, of degree n. Take τ in $C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$, non-negative, such that $\tau(s) = 1$ if $|s| \leqslant 1$ and $\tau(s) = 0$ if $|s| \geqslant 2$, and define $\psi(x, y) := \tau(\frac{y}{\langle x\rangle})$. Set $\widetilde{f}(z) = \psi(x, y) \sum\limits_{r=0}^{n} \frac{1}{(r!)} f^{(r)}(x)(\mathrm{i}y)^r$ for every $z = x + \mathrm{i}y \in \mathbb{C}$. It is a simple fact that $\left|\frac{\partial \widetilde{f}}{\partial \overline{z}}(z)\right| = \mathrm{O}(|y|^n)$ as $|y| \to 0$, where $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)$ ([11]).

For $n \in \mathbb{N}$, let $\mathcal{A}_n$ be the completion of $C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$ with respect to the norm $\|f\|_n := \sum\limits_{r=0}^{n} \int\limits_{\mathbb{R}} |f^{(r)}(x)|\ \langle x\rangle^{r-1} \mathrm{d}x$, $f \in C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$. Then $\mathcal{A}_n$ is a Banach algebra for pointwise multiplication. By putting $f^{(r)}$ in terms of $f^{(r+1)}$ and then applying the Fubini Theorem one gets that the above norm is equivalent to $\int\limits_{\mathbb{R}} |f(x)| \langle x\rangle^{-1}\, \mathrm{d}x + \int\limits_{\mathbb{R}} |f^{(n)}(x)| \langle x\rangle^{n-1}\, \mathrm{d}x$. Another stronger norm has been considered in [18], p. 117.

Let X be a Banach space and denote by $\mathcal{L}(X)$ the algebra of bounded operators on X endowed with the operator norm. Suppose that H is a closed densely defined operator on X with real spectrum, $\sigma(H) \subset \mathbb{R}$, and assume that property (R_α) mentioned in the introduction holds for some $\alpha \geqslant 0$.

Theorem 1.1. ([11]) *Assume that* $n > \alpha$. *Then there exists a bounded algebra homomorphism* $\Theta : \mathcal{A}_{n+1} \longrightarrow \mathcal{L}(X)$ *given by*

$$\Theta(f) = -\frac{1}{\pi} \int\limits_{\mathbb{C}} \frac{\partial \widetilde{f}}{\partial \overline{z}}(z)(z - H)^{-1}\, \mathrm{d}x\, \mathrm{d}y$$

for every $f \in C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$.

The above formula for $\Theta(f)$ does not depend on the choice of either τ or n in the expression for $\widetilde{f}$. A property of the calculus Θ is that $\Theta((z-u)^{-1}) = (z-H)^{-1}$ for every $z \notin \mathbb{R}$. Also, if $\mathrm{supp} f \cap \sigma(H) = \emptyset$ then $\Theta(f) = 0$. This fact is crucial in the Davies proof of L^p spectral independence ([11], [12]).

Let us now suppose that the spectrum $\sigma(H)$ of H is contained in $[0, \infty)$. For $\varepsilon > 0$, take φ_ε in $C^{(\infty)}(\mathbb{R})$ such that $\varphi_\varepsilon(u) = 0$ in $u < -\varepsilon$ and $\varphi_\varepsilon(u) = 1$ in $u > -\frac{\varepsilon}{2}$. If $e_z(u) = \mathrm{e}^{-zu}$, where $u \in \mathbb{R}$ and $\Re z > 0$, then the function $(\varphi_\varepsilon - \varphi_{\varepsilon'})e_z$ is compactly supported on $(-\infty, 0)$, for every $\varepsilon, \varepsilon' > 0$, and therefore the family $\Theta(\varphi_\varepsilon e_z)$ is a holomorphic (C_0)-semigroup, *independent* of ε ([11], p. 174).

As a matter of fact, $\Theta(\varphi e_z) \equiv \Theta(\varphi_\varepsilon e_z)$ is generated by $-H$: let $-A$ be the infinitesimal generator of $\Theta(\varphi e_z)$. For λ such that $\Re\lambda > 0$ and $\Im\lambda \neq 0$ we have that $(\lambda + A)^{-1} = \int_0^\infty e^{-\lambda t}\Theta(\varphi e_t)\,dt = \Theta\Big(\int_0^\infty \varphi(u)e^{-(\lambda+u)t}\,dt\Big) = \Theta(\varphi(u)(\lambda + u)^{-1})$. Now, $\Theta([1-\varphi(u)](\lambda+u)^{-1}) = 0$ as above (by approximating with $C_c^{(\infty)}$ functions), and then $(\lambda + A)^{-1} = \Theta((\lambda + u)^{-1}) = (\lambda + H)^{-1}$ ([11], Theorem 5). (So, if $-H$ satisfying (R_α) is originally assumed to generate a semigroup, the calculus Θ allows us to recover it.)

The calculus that we have just described relies upon almost analytic extensions and the Helffer-Sjöstrand formula. The next one appeals to fractional derivation in the sense of Weyl, and its corresponding reproducing formula.

For $f \in C_c^{(\infty)}([0,\infty))$ the *Weyl fractional integral* of f of order $\nu > 0$ is defined by

$$W^{-\nu} f(u) = \frac{1}{\Gamma(\nu)} \int_u^\infty (t-u)^{\nu-1} f(t)\,dt, \quad u \geqslant 0.$$

The operator $W^{-\nu} : C_c^{(\infty)}([0,\infty)) \to C_c^{(\infty)}([0,\infty))$ is bijective and its inverse operator, denoted by W^ν, is called the *Weyl fractional derivative* (of f) of order ν. When $\nu \in \mathbb{N}$ then $W^\nu f = (-1)^\nu f^{(\nu)}$. Moreover $W^{\alpha+\beta} = W^\alpha W^\beta$ for any $\alpha, \beta \in \mathbb{R}$ where W^0 is to be taken as the identity operator. For $\nu > 0$ we define the Banach spaces $AC_{\exp}^{(\nu)}$, $AC_{\exp,2}^{(\nu)}$, and $AC^{(\nu)}$, $AC_{2,1}^{(\nu)}$, as the completions of $C_c^{(\infty)}([0,\infty))$ in the respective norms

$$\|f\|_{(\nu),e} := \int_0^\infty |W^\nu f(u)| e^u\,du, \quad \|f\|_{(\nu),e;2} := \Big(\int_0^\infty |W^\nu f(u)|^2 e^{2u}\,du\Big)^{\frac{1}{2}},$$

and

$$\|f\|_{(\nu)} := \int_0^\infty |W^\nu f(u)| u^{\nu-1}\,du, \quad \|f\|_{(\nu),2,1} := \int_0^\infty \Big[\int_y^{2y} |W^\nu f(u) u^\nu|^2 \frac{du}{u}\Big]^{\frac{1}{2}} \frac{dy}{y}.$$

These spaces satisfy the continuous inclusions $AC^{(\nu+\frac{1}{2})} \subset AC_{2,1}^{(\mu)} \subset AC^{(\mu)}$ and $AC_{\exp}^{(\nu+\frac{1}{2})} \subset AC_{\exp,2}^{(\mu)}$ for $\nu > \mu > 0$. Moreover, $AC_{\exp}^{(\nu)}$, $AC^{(\nu)}$ for $\nu \geqslant 1$, and $AC_{\exp,2}^{(\nu)}$, $AC_{2,1}^{(\nu)}$ for $\nu > \frac{1}{2}$, are in fact Banach algebras. All this is proved in [20] for $AC^{(\nu)}$ and $AC_{2,1}^{(\nu)}$. The same arguments work for the exponential weighted cases $AC_{\exp}^{(\nu)}$, $AC_{\exp,2}^{(\nu)}$.

Now, let X and H be as above and assume that $-H$ is the infinitesimal generator of a holomorphic semigroup $a^z := e^{-zH}$, $\Re z > 0$, in $\mathcal{L}(X)$ which satisfies, for some fixed $\alpha \geqslant 0$,

$$(G_\alpha) \qquad \sup_{\Re z \geqslant 1} \frac{\|a^z\|}{|z|^\alpha} < \infty,$$

or the stronger condition

$$(\mathrm{HG}_\alpha) \qquad \|a^z\| \leqslant C\left(\frac{|z|}{\Re z}\right)^\alpha \quad \text{for all } z \in \mathbb{C}^+,$$

where $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z > 0\}$. We also assume that a^z is a (C_0)-semigroup, i.e., $\lim\limits_{t\to 0^+} a^t x = x$ for each $x \in X$. Under hypothesis (G_α) it is the case that $\sigma(H) \subset [0,\infty)$ and a functional calculus for H can be defined via the kernel $G^\nu(u)$ given by

$$G^\nu(u)x = \frac{1}{2\pi \mathrm{i}} \int\limits_{\Re z=1} \frac{a^z x}{z^{\nu+1}} \mathrm{e}^{uz}\, \mathrm{d}z,$$

where $u \in \mathbb{R}$, $\nu > \alpha$ and $x \in X$. We have that $G^\nu(u) \in \mathcal{L}(X)$. Indeed, $G^\nu(u)$ lies in $\mathcal{A}$ for every $u \in \mathbb{R}$, where $\mathcal{A}$ is the Banach subalgebra of $\mathcal{L}(X)$ generated by the semigroup $(a^z)_{\Re z>0}$. Moreover, the mapping $u \in \mathbb{R} \to G^\nu(u) \in \mathcal{A}$ is continuous, satisfies $\|G^\nu(u)\| \leqslant C\mathrm{e}^u$, for every $u \in \mathbb{R}$, and $G^\nu(u) = 0$ if $u \leqslant 0$.

Theorem 1.2. ([20]). *Let $(a^z)_{\Re z>0}$ be a holomorphic (C_0)-semigroup in $\mathcal{L}(X)$ with infinitesimal generator $-H$.*

(i) *Suppose that $(a^z)_{\Re z>0}$ satisfies (G_α). Then the formula*

$$\Phi(f) = \int\limits_0^\infty W^{\nu+1} f(u) G^\nu(u)\, \mathrm{d}u$$

does not depend on $\nu > \alpha$ and defines a bounded algebra homomorphism $\Phi : AC^{(\nu+1)}_{\exp} \to \mathcal{L}(X)$ such that $\Phi(f)H \subset H\Phi(f) = \Phi(g)$ where $g(t) = tf(t)$ whenever $f \in C^{(\infty)}_{\mathrm{c}}([0,\infty))$. In fact, Φ can be extended to a bounded algebra homomorphism $\widetilde{\Phi} : AC^{(\nu+\frac12)}_{\exp,2} \to \mathcal{L}(X)$ given, for every $f \in C^\infty_{\mathrm{c}}([0,\infty))$, by

$$\widetilde{\Phi}(f) = \lim_{h\to 0^+} \int\limits_0^\infty W^{\nu+\frac12} f(u) \frac{G^{\nu+\frac12}(u) - G^{\nu+\frac12}(u-h)}{h}\, \mathrm{d}u.$$

(ii) *If $(a^z)_{\Re z>0}$ satisfies (HG_α) then $\|G^\nu(u)\| \leqslant u^\nu$ for $u \geqslant 0$, in which case Φ extends automatically to a bounded homomorphism $AC^{(\nu+1)} \to \mathcal{L}(X)$. Moreover, the formula for $\widetilde{\Phi}$ defines in this case a bounded algebra homomorphism $AC^{(\nu+\frac12)}_{2,1} \to \mathcal{L}(X)$.*

In the sequel we will use the letter Φ to also denote the homomorphism $\widetilde{\Phi}$.

Remarks 1.3. (i) The growth hypothesis on a^z which is actually considered in [20], p. 327 is not what we have here called (G_α), but the stronger one

$$\sup_{\Re z \geqslant \varepsilon} \frac{\|a^z\|}{|z|^\alpha} \equiv C_\varepsilon < \infty,$$

for some fixed $\alpha \geqslant 0$, and *each* $\varepsilon > 0$. Nevertheless, Theorem 1.2 (i) is still true with a similar proof. Our weaker hypothesis (G_α) has the advantage that one does not need to take care about the behaviour of a^z close to $\mathrm{i}\mathbb{R}$, at least in what concerns Section 2 and examples in accordance with it. Note also that, under condition (G_α), the map Φ enables us to recover a^z: if $\Re\lambda > 1$ then

$$\Phi(\mathrm{e}^{-\lambda u}) = \lambda^{m+1}\int_0^\infty \mathrm{e}^{-\lambda u}G^m(u)\,\mathrm{d}u = \frac{1}{2\pi\mathrm{i}}\int_{\Re z=1}\int_0^\infty \mathrm{e}^{-(\lambda-z)u}\,\mathrm{d}u\Big(\frac{\lambda}{z}\Big)^{m+1}a^z\,\mathrm{d}z$$
$$= \frac{1}{2\pi\mathrm{i}}\int_{\Re z=1}\frac{a^z(\frac{\lambda}{z})^{m+1}}{\lambda-z}\,\mathrm{d}z = a^\lambda.$$

(ii) The kernel $G^\nu(u)$ also enjoys the following properties:

(1) $\|G^\nu(u)\| \leqslant Cu^{\nu-\alpha}$ if $u \leqslant 1$. To see this notice that for $u < 1$ the map $z \to a^z z^{-(\nu+1)}\mathrm{e}^{uz}$ is holomorphic in the strip $1 \leqslant \Re z \leqslant u^{-1}$ and tends to zero at infinity within this domain. By the residue theorem we then get $G^\nu(u) = (\frac{1}{2\pi\mathrm{i}})\int_{\Re z=\frac{1}{u}} a^z z^{-(\nu+1)}\mathrm{e}^{uz}\,\mathrm{d}z$ and the inequality follows.

(2) $\Gamma(\mu+\nu+1)G^{\mu+\nu}(u) = \Gamma(\mu+1)G^\mu(u)\Gamma(\nu+1)G^\nu(u)$ for $\mu,\nu > \alpha$ and $u \in \mathbb{R}$. This is readily seen using, for $\beta > \alpha$, that $\Phi(R_u^\beta) = G^\beta(u)$ where R_u^β denotes the Bochner-Riesz function defined by $R_u^\beta(x) = \frac{1}{\Gamma(\beta+1)}(u-x)_+^\beta$, $x \in [0,\infty)$ ([20], p. 332).

Put $\omega(u) := \Gamma(\nu+1)\|G^\nu(u)\|$ for all $u \in \mathbb{R}$ and let $AC_\omega^{(\nu)}$ be defined using the norm for $AC_{\exp}^{(\nu)}$ but with the weight e^u replaced with ω. It is then obvious that every f in $AC_\omega^{(\nu)}$ *operates* on H in the sense of [20], p. 331. It would be interesting to know whether $AC_\omega^{(\nu)}$ is a Banach algebra but unfortunately, this is not clear, even though ω is submultiplicative by (ii) (2).

(iii) In practice it is sometimes necessary to consider a more general version of (G_α) where the growth of the semigroup at infinity over $(0,\infty)$ is allowed to be subexponential, i.e.,

$$\sup_{\Re z\geqslant 1}\frac{\|a_\varepsilon^z\|}{|z|^\alpha} < \infty,$$

for every $\varepsilon > 0$, where $a_\varepsilon^z = \mathrm{e}^{-\varepsilon z}a^z$, $a_0^z = a^z$.

Then, if $H_\varepsilon = \varepsilon + H$, we have that $\sigma(H_\varepsilon) \subset [0,\infty)$ for every $\varepsilon > 0$ whence $\sigma(H) \subset [0,\infty)$. Moreover, note that

$$G_\varepsilon^\nu(u) := \frac{1}{2\pi\mathrm{i}}\int_{\Re z=1}\frac{a_\varepsilon^z}{z^{\nu+1}}\mathrm{e}^{uz}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\int_{\Re z=1}\frac{a^z}{z^{\nu+1}}\mathrm{e}^{(u-\varepsilon)z}\,\mathrm{d}z =: G^\nu(u-\varepsilon)$$

for $\nu > \alpha$. Property (G_α) implies that $G_\varepsilon^\nu(u) = 0$ if $u < 0$, and so $G^\nu(r) = G_\varepsilon^\nu(r+\varepsilon) = 0$ if $r < -\varepsilon$ *for every* $\varepsilon > 0$. Hence $G^\nu(u) = 0$ whenever $u < 0$. Coming back to a fixed ε we get that $G_\varepsilon^\nu(u) = 0$ for any $u < \varepsilon$. We also have that $\|G^\nu(u)\| \leqslant C\mathrm{e}^u$ for u in $\mathbb{R}$.

We define

$$\Phi(f) = \int_0^\infty W^{\nu+1}f(u)G^\nu(u)\,\mathrm{d}u,$$

if $\nu > \alpha$ and $f \in C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$. Let us denote by Φ_ε the calculus for H_ε as in Theorem 1.2, and put $f_\varepsilon(u) = f(u-\varepsilon)$ if $f \in C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$. Then $W^{\nu+1}f_\varepsilon(u) = W^{\nu+1}f(u-\varepsilon)$ and so

$$\Phi_\varepsilon(f_\varepsilon) = \int_\varepsilon^\infty W^{\nu+1}f(u-\varepsilon)G^\nu(u-\varepsilon)\,\mathrm{d}u = \int_0^\infty W^{\nu+1}f(u)G^\nu(u)\,\mathrm{d}u = \Phi(f),$$

for every $\varepsilon > 0$. This allows us to easily check that $\Phi(fg) = \Phi(f)\Phi(g)$, and $\Phi(f)H \subset H\Phi(f) = \Phi(tf(t))$. So, Φ is also a functional calculus which extends automatically to $AC_{\exp}^{(\nu)}$.

Note that we can recover a^z from Φ for $\Re z > 1$ to get $\|a^z\| \leqslant C_\delta |z|^{\nu+1}$ if $\Re z \geqslant \delta$, for $\delta > 1$, and $\nu > \alpha$. As a consequence the following property holds: any holomorphic semigroup a^z having polynomial growth on vertical lines and which has *subexponential* growth at infinity on the real line has in fact *polynomial* growth at infinity on the real line. This coincides with what is known to happen for concrete examples in L^p spaces.

The subexponential version of (HG_α), over $(0,\infty)$, is

$$\|a^z\| \leqslant C_\varepsilon \mathrm{e}^{\varepsilon \Re z}\left(\frac{|z|}{\Re z}\right)^\alpha \quad \text{for all } z \in \mathbb{C}^+, \varepsilon > 0,$$

or, equivalently, that $(a_\varepsilon^z)_{\Re z > 0}$ satisfies (HG_α) for every $\varepsilon > 0$. Under this assumption it is readily seen that Φ extends to the completion $\mathcal{K}$ of $C_{\mathrm{c}}^{(\infty)}([0,\infty))$ with respect to the norm $f \to \int_0^\infty \left(\int_y^{2y+1} |W^{\nu+\frac{1}{2}}f(u)(u+1)^\nu|^2\,\mathrm{d}u\right)^{\frac{1}{2}} \frac{\mathrm{d}y}{y+1}$. Using Proposition 2.5 from [20] it is possible to show that $\mathcal{K}$ is a Banach algebra. Hence, assuming the subexponential version of (HG_α) provides us with a calculus Φ, for H, acting on $\mathcal{K}$.

In order to highlight the basic aspects of the matter, we only deal with properties (G_α) or (HG_α) in general statements throughout this paper. Corresponding results for the subexponential version of the (G_α)–(HG_α) hypotheses can be obtained for each $\varepsilon + H$ with $\varepsilon > 0$. For results given in terms of H itself one should take the algebra $\mathcal{K}$ as a substitute of $AC_{2,1}^{(\nu+\frac{1}{2})}$. This point is left to the reader.

There is a clear relationship between the calculi Θ and Φ from the very beginning. Assume that the operator H satisfies $\sigma(H) \subset [0,\infty)$. Then property (HG_α) for each $H_\varepsilon = \varepsilon + H$, $\varepsilon > 0$, implies (R_α) ([12], p. 181). In turn (R_α) enables us to obtain (G_{n+1}) for every H_ε, whenever $n > \alpha$, via Θ: choosing smooth functions $\varphi = \varphi_\varepsilon$ as in the discussion after Theorem 1.1, the boundedness condition

$$\sup_{\Re z \geqslant 1} \frac{\mathrm{e}^{-\varepsilon z}\|a^z\|}{|z|^{n+1}} < \infty,$$

for $a^z \equiv \mathrm{e}^{-zH} = \Theta(\varphi e_z)$, $z \in \mathbb{C}^+$, follows readily from estimates for φe_z in $\mathcal{A}_{n+1}$.

In particular, the calculus Φ can be established on $AC_{\exp}^{(\nu+1)}$, $\nu > n+1$, according to Remark 1.3 (iii). A natural conjecture here is that Θ and Φ are equal under common hypotheses. This is confirmed in the next section.

2. UNIQUENESS AND SPECTRAL INVARIANCE OF THE $C_c^{(k)}$-FUNCTIONAL CALCULUS

We begin with a uniqueness result.

THEOREM 2.1. *Let $(a^z)_{\Re z>0}$ be a holomorphic (C_0)-semigroup in $\mathcal{L}(X)$ with infinitesimal generator $-H$ satisfying $\sigma(H) \subset [0,\infty)$. Assume that H satisfies condition (R_α) for some $\alpha \geqslant 0$. Then $\Theta(f) = \Phi(f)$ for every $f \in C_c^{(\infty)}(\mathbb{R})$.*

Proof. As seen in the previous section (R_α) implies (G_{n+1}) for every $\varepsilon + H$, if $n > \alpha$ and $\varepsilon > 0$, so that Φ is defined on $C_c^{(\infty)}(\mathbb{R})$ in particular. Fix φ in $C^\infty(\mathbb{R})$ such that $\varphi(u) = 0$ in $u < -\varepsilon$ and $\varphi(u) = 1$ in $u > -\frac{\varepsilon}{2}$. Then it follows that $a^z = \Theta(\varphi e_z)$. Moreover, as also noticed before, if $\beta > n+1$ then $\Phi(R_u^\beta) = G^\beta(u)$, where R_u^β is the Bochner-Riesz function defined by

$$R_u^\beta(x) = \frac{1}{\Gamma(\beta+1)}(u-x)_+^\beta = \frac{1}{2\pi \mathrm{i}} \int\limits_{\Re z=1} \frac{\mathrm{e}^{-zx}}{z^{\beta+1}}\,\mathrm{e}^{uz}\,\mathrm{d}z,$$

for $u \geqslant 0$ and $x \in \mathbb{R}$. Since $G^\beta(u) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\Re z=1} a^z z^{-(\beta+1)} \mathrm{e}^{uz}\,\mathrm{d}z$ and $a^z = \Theta(\varphi e_z)$ one has

$$\Theta(\varphi R_u^\beta) = \Theta\bigg(\frac{1}{2\pi \mathrm{i}} \int\limits_{\Re z=1} \frac{\varphi(x)\mathrm{e}^{-zx}}{z^{\beta+1}}\,\mathrm{e}^{uz}\,\mathrm{d}z\bigg) = G^\beta(u).$$

Finally, if $f \in C_c^{(\infty)}(\mathbb{R})$ then $f = \int\limits_0^\infty W^{\beta+1} f(u)\, R_u^\beta\,\mathrm{d}u$ is in $\mathcal{A}_{n+1}$ which implies that

$$\Theta(f) = \Theta(\varphi f) = \int\limits_0^\infty W^{\beta+1} f(u)\,\Theta(\varphi R_u^\beta)\,\mathrm{d}u = \Phi(f). \qquad ∎$$

The theorem says that the maps Φ and Θ are different expressions of the same calculus for H, *provided that* $\sigma(H) \subset [0,\infty)$. For $\nu > \alpha$ the strongest form $\Phi : AC_{2,1}^{(\nu+\frac{1}{2})} \to \mathcal{L}(X)$ corresponds to the most restrictive assumption (HG_α), whereas the weakest form $\Phi : AC_{\exp}^{(\nu+1)} \to \mathcal{L}(X)$ is associated with the most general condition (G_α), or its "subexponential" version. In between we find $\Theta : \mathcal{A}_{n+1} \to \mathcal{L}(X)$, $n > \alpha$, and (R_α).

Next we show that Φ and Θ also coincide with the functional calculus naturally based upon the Fourier transform.

PROPOSITION 2.2. *Let $(a^z)_{\Re z>0}$ be a holomorphic semigroup in $\mathcal{L}(X)$ which satisfies property (G_α) for some $\alpha \geqslant 0$. Then*

$$\Phi(f)a^1 = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} (\mathcal{F}^{-1} f)(t) a^{1+\mathrm{i}t}\,\mathrm{d}t$$

for every $f \in C_c^{(n+1)}(\mathbb{R})$, $n > \alpha$, where $\mathcal{F}^{-1} f$ denotes the inverse Fourier transform of f.

Proof. Let f be a function in $C_c^{(n+1)}(\mathbb{R})$. Then $\mathcal{F}^{-1}f$ is an entire function which is $\mathrm{O}(z^{-n-1})$ in every halfplane $\Im z \leqslant M$. Indeed, integrating by parts $n+1$ times the integral $\mathcal{F}^{-1}f(z) = \int_{\mathbb{R}} f(u)\mathrm{e}^{\mathrm{i}zu}\,\mathrm{d}u$ we get

$$\mathcal{F}^{-1}f(z) = \frac{1}{(\mathrm{i}z)^{n+1}} \int_{\mathbb{R}} W^{n+1}f(u)\mathrm{e}^{\mathrm{i}zu}\,\mathrm{d}u.$$

Since the semigroup $(a^z)_{\Re z>0}$ has property (G_α) the integral $\int_{\Im z=-\delta} \mathcal{F}^{-1}f(z)a^{1+\mathrm{i}z}\mathrm{d}z$ is absolutely convergent for every $\delta \geqslant 0$ and does not depend on δ. In particular

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \mathcal{F}^{-1}f(t)a^{1+\mathrm{i}t}\mathrm{d}t \\
&= \int_{\Im z=-1} \mathcal{F}^{-1}f(z)a^{1+\mathrm{i}z}\,\mathrm{d}z = \int_{\Im z=-1} \int_{\mathbb{R}} W^{n+1}f(u)\mathrm{e}^{\mathrm{i}zu}\,\mathrm{d}u \frac{a^{1+\mathrm{i}z}}{(\mathrm{i}z)^{n+1}}\,\mathrm{d}z \\
&= \int_{\mathbb{R}} W^{n+1}f(u)\,\frac{1}{\mathrm{i}} \int_{\Re z=1} \frac{a^{1+z}}{z^{n+1}}\mathrm{e}^{zu}\,\mathrm{d}z\,\mathrm{d}u = \int_{\mathbb{R}} W^{n+1}f(u)\,G^n(u)\,\mathrm{d}u = 2\pi\Phi(f)\,a^1. \qquad \blacksquare
\end{aligned}$$

The above result is of some interest, apart from applications later, because it is known that both Θ and Φ coincide with the L^∞ spectral calculus in the case that H is self-adjoint on a Hilbert space ([11], [20]) . Proposition 2.2 supplies more unity to the matter.

Our purpose now is to analyse some spectral properties of the homomorphism Φ and the semigroup $(a^z)_{\Re z>0}$. To do so we need to develop an abstract framework, which begins with the following lemma. As usual in this paper the infinitesimal generator of the semigroup $(a^z)_{\Re z>0}$ will be denoted by $-H$.

Lemma 2.3. *Suppose that $(a^z)_{\Re z>0} \subset \mathcal{L}(X)$ is a holomorphic (C_0)-semigroup which satisfies (G_α), for some $\alpha \geqslant 0$. Then for every $\omega \in \mathbb{C}^+$,*

$$\int_0^\infty \mathrm{e}^{-\omega t}a^{\tau+jt}\mathrm{d}t = j(j\omega - H)^{-1}a^\tau$$

in $\mathcal{L}(X)$ where $j \in \{+\mathrm{i}, -\mathrm{i}\}$ and $\tau > 1$.

Proof. For $s > 0$ and $x \in X$, we have $Ha^s x = -\lim_{t\to 0} t^{-1}(a^{t+s}x - a^s x)$ so that $Ha^s \in \mathcal{L}(X)$. Let now $\tau > 1$ and put $T = \int_0^\infty \mathrm{e}^{-\omega t}a^{\tau+jt}\,\mathrm{d}t$ in $\mathcal{L}(X)$. For each

$x \in X$, $Tx = a^{\tau-1} \int\limits_0^\infty \mathrm{e}^{-\omega t} a^{1+jt} x \,\mathrm{d}t$ lies in the domain of H. Then

$$\begin{aligned}(j\omega - H)Tx &= (j\omega - H)a^{\tau-1} \int\limits_0^\infty \mathrm{e}^{-\omega t} a^{1+jt} x \,\mathrm{d}t \\ &= j \int\limits_0^\infty \omega \mathrm{e}^{-\omega t} a^{\tau+jt} x \,\mathrm{d}t - \int\limits_0^\infty \mathrm{e}^{-\omega t} (Ha^{\tau-1})(a^{1+jt}x) \,\mathrm{d}t \\ &= j(-\mathrm{e}^{-\omega t} a^{\tau+jt} x)\big|_{t=0}^\infty + \int\limits_0^\infty \mathrm{e}^{-\omega t} Ha^{\tau+jt} x \,\mathrm{d}t - \int\limits_0^\infty \mathrm{e}^{-\omega t} Ha^{\tau+jt} x \,\mathrm{d}t = ja^\tau x.\end{aligned}$$

Since $\Re\omega > 0$ we get $\Im(j\omega) \neq 0$ and so $(j\omega - H)^{-1} \in \mathcal{L}(X)$. Then

$$Tx = (j\omega - H)^{-1}(j\omega - H)Tx = (j\omega - H)^{-1}(ja^\tau x). \quad ∎$$

The above lemma and Proposition 2.2 enable us to give the following spectral mapping result. The proof is standard. We include it for the sake of completeness.

Proposition 2.4. *Let $(a^z)_{\Re z>0} \subset \mathcal{L}(X)$ be a holomorphic (C_0)-semigroup satisfying (G_α), for some $\alpha \geqslant 0$. If $f \in C_{\mathrm{c}}^{(n+1)}(\mathbb{R})$, for $n > \alpha$, is such that $\operatorname{supp} f \cap \sigma(H) = \emptyset$ then $\Phi(f) = 0$.*

Proof. By Proposition 2.2, we have for $\tau > 1$ and $x \in X$

$$\begin{aligned}\Phi(f)a^\tau x &= \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} (\mathcal{F}^{-1}f)(t) a^{\tau+\mathrm{i}t} x \,\mathrm{d}t = \frac{1}{2\pi} \lim_{\varepsilon\to 0^+} \int\limits_{-\infty}^{+\infty} (\mathcal{F}^{-1}f)(t) \mathrm{e}^{-\varepsilon|t|} a^{\tau+\mathrm{i}t} x \,\mathrm{d}t \\ &= \frac{1}{2\pi} \lim_{\varepsilon\to 0^+} \int\limits_{-\infty}^{+\infty} f(s) \mathcal{F}^{-1}(\mathrm{e}^{-\varepsilon|t|} a^{\tau+\mathrm{i}t} x)(s) \,\mathrm{d}s.\end{aligned}$$

From Lemma 2.3,

$$\begin{aligned}\mathcal{F}^{-1}(\mathrm{e}^{-\varepsilon|t|} a^{\tau+\mathrm{i}t} x)(s) &= \int\limits_0^\infty \mathrm{e}^{-(\varepsilon+\mathrm{i}s)t} a^{\tau-\mathrm{i}t} x \,\mathrm{d}t + \int\limits_0^\infty \mathrm{e}^{-(\varepsilon-\mathrm{i}s)t} a^{\tau+\mathrm{i}t} x \,\mathrm{d}t \\ &= -\mathrm{i}(-\mathrm{i}(\varepsilon+\mathrm{i}s) - H)^{-1} a^\tau x + \mathrm{i}(\mathrm{i}(\varepsilon-\mathrm{i}s) - H)^{-1} a^\tau x \\ &= -\mathrm{i}(-\mathrm{i}\varepsilon + s - H)^{-1} a^\tau x + \mathrm{i}(\mathrm{i}\varepsilon + s - H)^{-1} a^\tau x.\end{aligned}$$

Since $\operatorname{supp}(f) \cap \sigma(H) = \emptyset$ and both sets are closed, we get

$$\begin{aligned}\Phi(f)a^\tau x &= \frac{\mathrm{i}}{2\pi} \lim_{\varepsilon\to 0^+} \int\limits_{\operatorname{supp} f} f(s)((\mathrm{i}\varepsilon + s - H)^{-1} a^\tau x - (-\mathrm{i}\varepsilon + s - H)^{-1} a^\tau x) \,\mathrm{d}s \\ &= \frac{\mathrm{i}}{2\pi} \int\limits_{\operatorname{supp} f} f(s) \lim_{\varepsilon\to 0^+} ((\mathrm{i}\varepsilon + s - H)^{-1} a^\tau x - (-\mathrm{i}\varepsilon + s - H)^{-1} a^\tau x) \,\mathrm{d}s = 0.\end{aligned}$$

By the uniqueness principle for holomorphic functions $\Phi(f)a^s x = 0$ for every $s > 0$ and $x \in X$, whence $\Phi(f) = 0$. ∎

Let $\mathcal{A}$ be a commutative Banach algebra with its space of (modular) maximal ideals denoted by $\mathrm{Max}(\mathcal{A})$. Let $\mathcal{B}$ be another Banach algebra. If $\Psi : \mathcal{A} \to \mathcal{B}$ is a bounded homomorphism the *spectrum* $\mathrm{Sp}(\Psi)$ of Ψ is, by definition, the zero set in $\mathrm{Max}(\mathcal{A})$ of the closed ideal $\ker(\Psi)$ of $\mathcal{A}$, i.e. $\mathrm{Sp}(\Psi) = \{\chi \in \mathrm{Max}(\mathcal{A}) : \chi(a) = 0,\ a \in \ker(\Psi)\}$. This notion plays a relevant role in the spectral study of representations.

Let $\mathcal{E}$ be a Banach algebra of functions on $\mathbb{R}$ which contains $C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$ as a dense subalgebra and such that convergence in $\mathcal{E}$ implies pointwise convergence on $\mathbb{R}$. The next result provides the invariance of the spectrum of the $C_{\mathrm{c}}^{(k)}$-calculus.

Proposition 2.5. *Let $(a^z)_{\Re z>0} \subset \mathcal{L}(X)$ be a holomorphic (C_0)-semigroup which satisfies (G_α) for some $\alpha \geqslant 0$. Let $\mathcal{E}$ be as above and suppose that there is a bounded homomorphism $\Psi : \mathcal{E} \to \mathcal{L}(X)$ such that*

$$\Psi(f) = \int_0^\infty W^{(n+1)} f(u) G^n(u)\,\mathrm{d}u$$

for each $f \in C_{\mathrm{c}}^{(n+1)}(\mathbb{R})$, whenever $n > \alpha$. Then $\mathrm{Sp}(\Psi) = \sigma(H)$.

Proof. Suppose $t \notin \sigma(H)$. Since $\sigma(H)$ is closed, a real number $\varepsilon > 0$ can be chosen such that $(t-\varepsilon, t+\varepsilon) \cap \sigma(H) = \emptyset$. Thus we can take $f \in C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$ with $f(t) = 1$ and $\mathrm{supp} f \subset (t-\varepsilon, t+\varepsilon)$. By Proposition 2.4, $\Psi(f) = 0$ and then $f \in \ker(\Psi)$. However, since evaluations of elements of $\mathcal{E}$ at points of $\mathbb{R}$ belong to $\mathrm{Max}(\mathcal{E})$ and $f(t) = 1$ we see that $t \notin \mathrm{Sp}(\Psi)$.

Conversely, let $t \in \sigma(H)$. If $\mathcal{A}$ is the closed subalgebra of $\mathcal{L}(X)$ generated by $(a^z)_{\Re z>0}$ then it is clear that $\Psi(f) \in \mathcal{A}$ for each $f \in \mathcal{E}$. By [33], p. 457, there exists a character φ of $\mathcal{A}$ such that $\varphi(a^z) = \mathrm{e}^{-tz}$, $z \in \mathbb{C}^+$. Then for $f \in C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$ and $n > \alpha$,

$$\begin{aligned}\varphi(\Psi(f)) &= (-1)^{n+1} \int_0^\infty f^{(n+1)}(u) \bigg(\frac{1}{2\pi\mathrm{i}} \int_{\Re z=1} \frac{\mathrm{e}^{(u-t)z}}{z^{n+1}}\,\mathrm{d}z\bigg)\,\mathrm{d}u \\ &= \frac{(-1)^{n+1}}{n!} \int_t^\infty f^{(n+1)}(u)(u-t)^n\,\mathrm{d}u = f(t)\end{aligned}$$

according to [20], p. 328, Remark 3. For arbitrary $f \in \ker(\Psi)$ in $\mathcal{E}$ take $(f_k) \subset C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$ with $f_k \to f$ in $\mathcal{E}$ as $k \to \infty$. By hypothesis $f_k(t) \to f(t)$. On the other hand $f_k(t) = \varphi(\Psi(f_k)) \to \varphi(\Psi(f)) = 0$ whence $f(t) = 0$ and therefore $t \in \mathrm{Sp}(\Psi)$. ∎

The above theorem applies automatically to the algebras $AC_{\exp}^{(\nu+1)}$, $AC_{2,1}^{(\nu+\frac{1}{2})}$, $\mathcal{A}_{n+1}$, and the corresponding calculi Φ, Θ under the appropriate growth assumptions and for $\nu, n > \alpha$.

REMARK 2.6. A remarkable case of the spectrum of a bounded homomorphism arises when the algebra $\mathcal{A}$ is the group algebra of an abelian locally compact group ([5]). This concept also fits well in our setting.

For $\alpha \geqslant 0$ let $L^1_\alpha(\mathbb{R})$ be the Beurling algebra formed by the functions f of $L^1(\mathbb{R})$ such that $\int_{-\infty}^{+\infty} |f(t)|(1+t^2)^{\frac{\alpha}{2}}\,\mathrm{d}t < \infty$ endowed with the norm induced by this integral. Suppose that $(a^z)_{\Re z>0}$ has property (G_α). The mapping $\Pi : L^1_\alpha(\mathbb{R}) \to \mathcal{L}(X)$ given by $\Pi(f) = \int_{-\infty}^{+\infty} f(t)a^{1+\mathrm{i}t}\,\mathrm{d}t$, for $f \in L^1_\alpha(\mathbb{R})$, is well defined, it is linear and bounded, and satisfies $\Pi(f * g)a^1 = \Pi(f)\Pi(g)$ for every f, g in $L^1_\alpha(\mathbb{R})$. Then $\ker(\Pi)$ is a closed ideal of $L^1_\alpha(\mathbb{R})$. Inspired by [6], we call the *Arveson spectrum* of the semigroup $(a^z)_{\Re z>0}$ the zero set in $\mathbb{R}$ of $\ker(\Pi)$, denoted by $\mathrm{Arv}(a^z)$, that is, $\mathrm{Arv}(a^z) := \{t \in \mathbb{R} : \mathcal{F}f(t) = 0, f \in \ker(\Pi)\}$. As we would expect, $\mathrm{Arv}(a^z) = \sigma(H)$. This follows by a simple adaptation of the proof of Proposition 2.5 to the mapping Π (note that $\Pi(f) = 2\pi\Phi(\mathcal{F}f)a^1$ via Proposition 2.2). Other standard proofs using the approximate point spectrum of H also work.

Next we establish an abstract result on spectral independence which extends that of [13], p. 147. Let X_1, X_2 be Banach spaces with $X_1 \cap X_2$ dense in each X_1 and X_2. Let $(a^z_j)_{\Re z>0}$ be two holomorphic (C_0)-semigroups in the corresponding spaces $\mathcal{L}(X_j)$, $j = 1, 2$. Then they are said to be *consistent* if $a^t_1 x = a^t_2 x$ for every $t > 0$ and $x \in X_1 \cap X_2$ ([13]). Let us write the infinitesimal generator of a^t_j as $-H_j$, $j = 1, 2$.

THEOREM 2.7. *Assume that $(a^z_j)_{\Re z>0} \subset \mathcal{L}(X_j)$ with $j = 1, 2$ are two consistent holomorphic (C_0)-semigroups satisfying the growth property (G_α) for some $\alpha \geqslant 0$. Then $\sigma(H_1) = \sigma(H_2)$.*

Proof. For $j = 1, 2$ let Φ_j be the functional calculus associated to $(a^z_j)_{\Re z>0}$, with kernel $G^\nu_j(u)$ and domain $AC^{(\nu+1)}_{\exp}$, $\nu > \alpha$. From the hypothesis $G^\nu_1(u)x = G^\nu_2(u)x$ if $u \geqslant 0$ and $x \in X_1 \cap X_2$ and therefore we get $\Phi_1(f)x = \Phi_2(f)x$ for every $x \in X_1 \cap X_2$ and $f \in AC^{(\nu+1)}_{\exp}$. By density we obtain that $\ker(\Phi_1) = \ker(\Phi_2)$. On the other hand, Proposition 2.5 implies that $\sigma(H_j) = \mathrm{Sp}(\Phi_j)$, $j = 1, 2$, so the proof is completed. ∎

Suppose now that (X_1, X_2) is a *compatible* couple of Banach spaces in the sense that they are continuously embedded in a Hausdorff topological vector space. Recall that $X_1 \cap X_2$ and $X_1 + X_2$ are Banach spaces endowed with the respective norms $\max(\|x\|_{X_1}, \|x\|_{X_2})$ if $x \in X_1 \cap X_2$, and $\inf\{\|x\|_{X_1} + \|x\|_{X_2};\, x_1 \in X_1, x_2 \in X_2, x_1 + x_2 = x\}$ if $x \in X_1 + X_2$. Theorem 2.6 can be used to show spectral invariance with respect to an interpolation space Y for (X_1, X_2). However we give here another proof of this fact, because of its significance in applications. The argument relies on the following result of E. Albrecht. For a morphism T of (X_1, X_2) we write T_Y to denote its restriction to Y.

Let T be a morphism of (X_1, X_2) such that $T_{X_1\cap X_2}$ and $T_{X_1+X_2}$ are decomposable. Then $\sigma(T_Y) = \sigma(T_j)$ for $j = 1, 2$ ([1], Corollary 2.6).

As before, assume that $(a^z_j)_{\Re z>0} \subset \mathcal{L}(X_j)$ are consistent on X_1, X_2. Then there is a unique $(a^z)_{\Re z>0}$ in $\mathcal{L}(X_1 + X_2)$ whose restriction $a^z_{X_j}$ on X_j equals a^z_j,

$j = 1, 2$, and such that $a^z(Y) \subset Y$, $\Re z > 0$, for each interpolation space Y for (X_1, X_2). Let us denote by $-H$ and $-H_Y$ the respective generators of a^z and a^z_Y, $\Re z > 0$.

THEOREM 2.8. *Let (X_1, X_2) be a compatible couple of Banach spaces such that $X_1 \cap X_2$ is dense in X_j, $j = 1, 2$. Suppose that holomorphic (C_0)-semigroups $(a^z_j)_{\Re z>0}$ in $\mathcal{L}(X_j)$, $j = 1, 2$, are consistent and satisfy (G_α) for some $\alpha \geqslant 0$. Then, if Y is an interpolation space for (X_1, X_2), we have*

$$\sigma(H_Y) = \sigma(H_1) = \sigma(H_2).$$

Proof. First, for a semigroup $(a^z)_{\Re z>0}$ enjoying property (G_α) all the elements a^z, with $\Re z > 1$, are decomposable. This is because $AC^{(\nu)}_{\exp}$ is semisimple and regular on its space of modular maximal ideals $[0, \infty)$. Therefore $a^z = \Phi(\mathrm{e}^{-zu})$ is super-decomposable (and so decomposable), for $\Re z > 1$ ([39], Theorem 2.3). Also it is easy to check that a^z satisfies property (G_α) in $\mathcal{L}(X_1 + X_2)$. Thus Albrecht's result applies to yield that $\sigma(a^z_Y) = \sigma(a^z_j)$ whenever $\Re z > 1$, $j = 1, 2$. On the other hand, in all the spaces concerned, we have that $\sigma(a^2) \setminus \{0\} = \mathrm{e}^{-2\sigma(H)}$ for every $z \in \mathbb{C}^+$, because a^z is holomorphic. Since $\sigma(H)$ is real, both equalities above yield that $\sigma(H_Y) = \sigma(H_j)$. ∎

Theorems on spectral invariance related to interpolation also hold for operators other than decomposable ones. Interesting results of this type have been obtained, for instance, using the theory of analytic multifunctions ([48], [8]). In our case, the generality of this method would allow us to establish the constancy of $\sigma(H)$ for any operator H interpolated between H_1 and H_2, but we could not ensure equalities including the "boundary points" such as $\sigma(H) = \sigma(H_1) = \sigma(H_2)$, or $\sigma(H_1) = \sigma(H_2)$ ([8]).

3. NORM ESTIMATES FOR FUNTIONS OF $\mathrm{e}^{\mathrm{i}tH}$

Let us recall the abstract Cauchy problems of first and second order associated to H;

$$\text{(ACP1)} \qquad \begin{cases} u'(t) = -\mathrm{i}Hu(t), & t \in \mathbb{R}, \\ u(0) = x_0, & x_0 \in X; \end{cases}$$

(the abstract "Schrödinger type" equation for H) and

$$\text{(ACP2)} \qquad \begin{cases} u''(t) = -Hu(t), & t \geqslant 0, \\ u'(0) = x_1, & \\ u(0) = x_0, & x_0, x_1 \in X; \end{cases}$$

(the abstract "wave" equation for H). As is well known the formal solution of (ACP1) is given by $u(t) = \mathrm{e}^{-\mathrm{i}tH}x_0$ whereas $u(t) = \cos t\sqrt{H}x_0 + (\sqrt{H})^{-1}\sin t\sqrt{H}x_1$ is the formal solution of (ACP2). In most concrete cases (like those of differential operators) $\mathrm{e}^{\mathrm{i}tH}$, $t \in \mathbb{R}$, are not bounded operators and so it is relevant to find large classes of functions f so that expresions like $f(H)\mathrm{e}^{\mathrm{i}tH}$, $f(H)(\sqrt{H})^{-1}\sin t\sqrt{H}$, and other related ones, have a sense as bounded operators on X. We approach this question from the general framework supplied by the $C_c^{(k)}$-functional calculus, provided that the operator $-H$ generates a semigroup satisfying properties (G_α) or (HG_α). As we shall see, the functions usually involved in the $\mathrm{e}^{\mathrm{i}tH}$ estimates are particularly well adapted to the $AC^{(\nu)}$ calculations.

LEMMA 3.1. *Let $(a^z)_{\Re z>0}$ be a holomorphic semigroup generated by $-H$ and let $\mathcal{A}$ denote the closed subalgebra in $\mathcal{L}(X)$ generated by this semigroup. Assume that $(a^z)_{\Re z>0}$ satisfies (G_α), for some $\alpha \geqslant 0$, and let $\mu > \alpha+\frac{1}{2}$. Then $a^{\mathrm{i}t}\,\Phi(f) \in \mathcal{A}$ and*

$$\left\|a^{\mathrm{i}t}\,\Phi(f)\right\| \leqslant C\,(1+t^2)^{\frac{\alpha}{2}}\left(\int\limits_0^\infty |W^\mu f(u)\,\mathrm{e}^{2u}|^2\,\mathrm{d}u\right)^{\frac{1}{2}}$$

for every $t \in \mathbb{R}$ and every function f on $[0,\infty)$ for which the integral is finite.

Proof. Since for $t \in \mathbb{R}$ and $u \geqslant 0$ the integral

$$a^{\mathrm{i}t}G^\mu(u) = \frac{1}{2\pi\mathrm{i}}\int\limits_{\Re z=1}\frac{a^{z+\mathrm{i}t}}{z^{\mu+1}}\,\mathrm{e}^{uz}\,\mathrm{d}z$$

is absolutely convergent, the operator $a^{\mathrm{i}t}G^\mu(u)$ belongs to $\mathcal{A}$. It follows that $a^{\mathrm{i}t}\Phi(f) = \int\limits_0^\infty W^{\mu+1}f(u)\,a^{\mathrm{i}t}G^\mu(u)\,\mathrm{d}u$ also belongs to $\mathcal{A}$ for every smooth function with compact support. Note that $a^{\mathrm{i}t}\Phi(f)$ can be written (cf. [20], Corollary 4.3) as a limit in the operator norm, when $h \to 0^+$, of the operators

$$a^{\mathrm{i}t}\Phi_h(f) = \frac{1}{h}\int\limits_0^\infty\big[W^\mu f(u) - W^\mu f(u+h)\big]\,a^{\mathrm{i}t}G^\mu(u)\,\mathrm{d}u =$$
$$= \frac{1}{2\pi}\int\limits_{-\infty}^\infty\frac{a^{1+\mathrm{i}(s+t)}}{(1+\mathrm{i}s)^\mu}\int\limits_0^1\mathrm{e}^{-lh(1+\mathrm{i}s)}\,\mathrm{d}l\,\widehat{g}(s)\,\mathrm{d}s,$$

where $\widehat{g}$ denotes the Fourier transform of the function $g(u) = W^\mu f(u)\,\mathrm{e}^u$ for $u \geqslant 0$ and $g(u) = 0$ otherwise. The (G_α) condition, the Schwarz inequality and the Plancherel formula give the norm estimate

$$\left\|a^{\mathrm{i}t}\Phi_h(f)\right\| \leqslant \frac{1}{2\pi}\left(\int\limits_{-\infty}^\infty\big[1+(s+t)^2\big]^\alpha\,(1+s^2)^{-\mu}\,\mathrm{d}s\right)^{\frac{1}{2}}\|\widehat{g}\|_2 \leqslant C\,(1+t^2)^{\frac{\alpha}{2}}\,\|g\|_2,$$

which implies the result. ∎

The proof of the above lemma is a refinement of that of Theorem 1.2 (i). This is a result on Sobolev type estimates, which automatically implies the following more familiar formulation. In it, we write e^{-zH} instead of a^z, for $\Re z \geqslant 0$, as well as $f(H)$ in place of $\Phi(f)$, for accommodation of standard notation. We also do this freely in the sequel.

Proposition 3.2. *Let* $(\mathrm{e}^{-zH})_{\Re z>0}$ *be a holomorphic semigroup in* X *satisfying* (G_α) *for some* $\alpha \geqslant 0$. *Then for* $\nu > \alpha + \frac{1}{2}$ *and every* f *in the Sobolev space* $\mathcal{H}^\nu(\mathbb{R})$, *with support in* $\left[\frac{1}{2}, 1\right]$, *the operator* $f(H)$ *is bounded on* X *with* $\|f(H)\| \leqslant C_\nu \|f\|_{\mathcal{H}^\nu(\mathbb{R})}$.

Even in the abstract setting, the previous proposition may be alternatively settled on the basis of Proposition 2.2; see [26], p. 438, [45], p. 960.

The next results concern property (HG_α). For $f \in AC_{2,1}^{(\nu+\frac{1}{2})}$ define $\|f\|_{(\nu),2,1}^{(\alpha)} := \int_0^\infty [\int_y^{2y} |W^{\nu+\frac{1}{2}} f(u) u^\nu (1+u^2)^{\frac{\alpha}{2}}|^2 \,\mathrm{d}u]^{\frac{1}{2}} \frac{\mathrm{d}y}{y}$.

Theorem 3.3. *Let* $(a^z)_{\Re z>0}$, H *and* $\mathcal{A}$ *be as in Lemma 3.1 and assume that* $(a^z)_{\Re z>0}$ *satisfies* (HG_α), *for some* $\alpha \geqslant 0$. *Then* $a^{\mathrm{i}t}\Phi(f) \in \mathcal{A}$ *and* $\|a^{\mathrm{i}t}\Phi(f)\| \leqslant C_\nu (1+t^2)^{\frac{\alpha}{2}} \|f\|_{\nu;2,1}^{(\alpha)}$ *for* $t \in \mathbb{R}$, *whenever* $\nu > \alpha$, $f \in AC_{2,1}^{(\nu+\frac{1}{2})}$, *and* $\|f\|_{(\nu),2,1}^{(\alpha)} < \infty$.

Proof. The first part of the argument is as in the proof of the lemma so that we must estimate $a^{\mathrm{i}t}\Phi_h(f)$. Now, take $r > 0$ and put this time $g(u) = W^\mu f(u) \mathrm{e}^{\frac{r}{2}u} \chi_{[0,\infty)}(u)$. Applying the Hausdorff-Young inequality in

$$a^{\mathrm{i}t}\Phi_h(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{a^{r+\mathrm{i}(rs+t)}}{r^{\mu-1}(1+\mathrm{i}s)^\mu} \int_0^1 \mathrm{e}^{-\lambda h r(1+\mathrm{i}s)} \,\mathrm{d}\lambda \hat{g}(-rs) \,\mathrm{d}s,$$

one obtains

$$\begin{aligned}
\|a^{\mathrm{i}t}\Phi_h(f)\| &\leqslant \frac{C}{r^\nu} \left[\int_{-\infty}^{\infty} \frac{(1+(s+\frac{t}{r})^2)^\alpha}{(1+s^2)^\mu} \,\mathrm{d}s \right]^{\frac{1}{2}} \|g\|_2 \\
&\leqslant \frac{C}{r^\nu} \Big(1 + \frac{t^2}{r^2}\Big)^{\frac{\alpha}{2}} \Big(\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(1+s^2)^{\mu-\alpha}} \Big)^{\frac{1}{2}} \|g\|_2 \\
&\leqslant C(1+t^2)^{\frac{\alpha}{2}} \frac{(1+r^2)^{\frac{\alpha}{2}}}{r^{\nu+\alpha}} \Big(\int_0^\infty |W^{\nu+\frac{1}{2}} f(u) \,\mathrm{e}^{ru}|^2 \,\mathrm{d}u \Big)^{\frac{1}{2}} \\
&\leqslant C(1+t^2)^{\frac{\alpha}{2}} \inf_{r>0} \frac{(1+r^2)^{\frac{\alpha}{2}}}{r^{\nu+\alpha}} \Big(\int_0^\infty |W^{\nu+\frac{1}{2}} f(u) \,\mathrm{e}^{ru}|^2 \,\mathrm{d}u \Big)^{\frac{1}{2}}.
\end{aligned}$$

Dividing $(0,\infty)$ into pieces $[2^k, 2^{k+1})$, the above infimum can be estimated as in [20], p. 341 to deduce that $\|a^{\mathrm{i}t}\Phi_h(f)\| \leqslant C(1+t^2)^{\frac{\alpha}{2}} \|f\|_{(\nu),2,1}^{(\alpha)}$. ∎

Remark 3.4. We know that $AC^{(\nu+1)} \subset AC_{2,1}^{(\nu+\frac{1}{2})}$, $\nu > 0$. It will be useful later on to have a criterion to decide if $a^{\mathrm{i}t}\Phi(f)$ defines an element of $\mathcal{A}$, in terms of $AC^{(\nu+1)}$ itself. Proceeding along the same lines as in the previous theorem, even with a simpler proof, it can be shown that $a^{\mathrm{i}t}\Phi(f) \in \mathcal{A}$ and $\|a^{\mathrm{i}t}\Phi(f)\| \leqslant C(1+$

$t^2)^{\frac{\alpha}{2}}\|f\|_{\nu+1}^{(\alpha)}$, $t \in \mathbb{R}$, whenever $\nu > \alpha$, $\|f\|_{\nu+1}^{(\alpha)} := \int_0^\infty |W^{\nu+1}f(u)|u^\nu(1+u^2)^{\frac{\alpha}{2}}\,\mathrm{d}u < \infty$ and $f \in AC^{(\nu+1)}$.

Conditions like $\|f\|_{\nu;2,1}^{(\alpha)} < \infty$ or $\|f\|_{\nu+1}^{(\alpha)} < \infty$ are not superfluous hypotheses: if $\mathcal{A} = AC^{(1)}$, which is spanned by $u \to \mathrm{e}^{-zu}$, and Φ is the inclusion $AC^{(2)} \to AC^{(1)}$, then $f(u) := (1+u)^{-1}$ is in $AC^{(2)}$ but $a^{\mathrm{i}t}\Phi(f) \equiv \mathrm{e}^{\mathrm{i}tu}(1+u)^{-1}$ does not belong to $\mathcal{A}$.

In the rest of this section we write e^{-zH} and $f(H)$, as mentioned before. Note that all operators which are listed as bounded in subsequent statements have a previous sense as closed densely defined operators on X.

Corollary 3.5. *Let $(a^z)_{\Re z>0}$ be a holomorphic semigroup satisfying* (HG_α) *for some $\alpha \geqslant 0$. Then, if $\Re\nu > \alpha$:*

(i) $\mathrm{e}^{\mathrm{i}tH}H^{-\nu}\varphi(H) \in \mathcal{L}(X)$ *and* $\|\mathrm{e}^{\mathrm{i}tH}H^{-\nu}\varphi(H)\| \leqslant C_\nu(1+|t|^\alpha)$ *for each $t \in \mathbb{R}$ and φ in $C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$ such that $\varphi(u) = 0$ if $u < \frac{1}{2}$ and $\varphi(u) = 1$ if $u > 1$.*

(ii) $I_t^\nu(H) := \int_0^t (t-s)^{\nu-1}\mathrm{e}^{\mathrm{i}sH}\,\mathrm{d}s \in \mathcal{L}(X)$ *and* $\|I_t^\nu(H)\| \leqslant C_\nu t^{\Re\nu}$, *for $t \geqslant 0$.*

(iii) $\mathrm{e}^{\mathrm{i}tH}(1+H)^{-\nu} \in \mathcal{L}(X)$ *and* $\|\mathrm{e}^{\mathrm{i}tH}(1+H)^{-\nu}\| \leqslant C_\nu(1+t^2)^{\frac{\alpha}{2}}$, *for each $t \in \mathbb{R}$.*

(iv) $(\sin tH)H^{-1}(1+H)^{-\nu+1} \in \mathcal{L}(X)$ *and, for every $t \in \mathbb{R}$ and $0 < \varepsilon < \Re\nu$,*

$$\|(\sin tH)H^{-1}(1+H)^{-\nu+1}\| \leqslant \begin{cases} C_\nu|t|(1+|t|^{\Re\nu-1}) & \text{if } \Re\nu > 1 \text{ or } \nu = 1, \\ C_{\nu,\varepsilon}|t|^{\Re\nu}(1+|t|^{-\varepsilon}) & \text{if } 0 < \Re\nu \leqslant 1, \nu \neq 1. \end{cases}$$

Proof. (i) Take $n > \alpha$ and let $f(u) = \varphi(u)u^{-\nu}$ for $u \in \mathbb{R}$. Then $W^{n+1}f(u) = \nu\cdots(\nu+n-1)u^{-(\nu+n+1)}$ for $u > 1$, whence $\int_0^\infty |W^{n+1}f(u)|u^n(1+u^2)^{\frac{\alpha}{2}}\,\mathrm{d}u < \infty$. Hence $\|\mathrm{e}^{\mathrm{i}tH}H^{-\nu}\varphi(H)\| = \|f(H)\mathrm{e}^{\mathrm{i}tH}\| \leqslant C\|f\|_{n+1}^{(\alpha)}(1+t^2)^{\frac{\alpha}{2}}$.

(ii) We follow the idea of [30], p. 9 or [52], p. 336. Choose φ as in part (i) and set $g_\nu(u) := \int_0^1 (1-s)^{\nu-1}\mathrm{e}^{\mathrm{i}su}\,\mathrm{d}s - \mathrm{e}^{-\frac{\nu}{2}\pi\mathrm{i}}\Gamma(\nu)\varphi(u)u^{-\nu}\mathrm{e}^{\mathrm{i}u}$, for $u \geqslant 0$. For $u > 1$, we have $|g_\nu^{(j)}(u)| \leqslant Cu^{-(j+\min\{1,\Re\nu\})}$, $j = 0, 1, \ldots$. This estimate follows from the equality

$$g_\nu^{(j)}(u) = C_\nu u^{-(j+\nu)}\int_0^\infty (\mathrm{i}u+x)^{\nu-1}x^j\mathrm{e}^{-x}\,\mathrm{d}x,$$

for every $u > 1$, $\Re\nu > 0$ and $j = 0, 1, \ldots$ (for *real* $\nu > 0$ it is established in [30]; the general case follows from analytic continuation on ν). Hence $g \in AC^{(n+1)}$ for $n > \alpha$ and so we have that $I_1^\nu(H) = g_\nu(H) + \mathrm{e}^{-\frac{\nu}{2}\pi\mathrm{i}}\Gamma(\nu)\varphi(H)H^{-\nu}\mathrm{e}^{\mathrm{i}H} \in \mathcal{L}(X)$ by part (i). Moreover, if $t \neq 0$, the operator $-tH$ is the generator of a semigroup which also satisfies (HG_α) with the same constant C. Hence $\|I_1^\nu(tH)\|$ is uniformly bounded in $t \geqslant 0$. Since $I_t^\nu(H) = t^\nu I_1^\nu(tH)$ it follows that $\|I_t^\nu(H)\| \leqslant Kt^{\Re\nu}$, where K is independent of t.

(iii) The argument of part (i) also works here.

(iv) For $t > 0$, $u \geqslant 0$ and $\nu \in \mathbb{C}$, put $h_{t,\nu}(u) := (1+tu)^{\nu-1}(1+u)^{-\nu+1}$ and $g_{t,\nu} := h_{t,\nu} - t^{\nu-1}$. Then the derivatives of $g_{t,\nu}$ have the form $g_{t,\nu}^{(n)}(u) = \sum_{k=1}^{n} c_k(t-1)^k h_{t,\nu-k}(u)(1+u)^{-(n+k)}$, and so belong to $AC^{(n)}$ for all $n \in \mathbb{N}$ ([20], Corollary 3.2). In fact, if $t \geqslant 1$, then

$$\int_0^\infty |h_{t,\nu-k}(u)| \frac{u^{n-1}}{(1+u)^{n+k}}\,\mathrm{d}u = \Big(\int_0^1 + \int_1^\infty\Big) \frac{(1+tu)^{\Re\nu-1-k}u^{n-1}}{(1+u)^{n+\Re\nu-1}}\,\mathrm{d}u$$

$$\leqslant \int_0^t \frac{t^{-n}r^{n-1}\,\mathrm{d}r}{(1+r)^{k+1-\Re\nu}} + 2^{(\Re\nu-1-k)_+}\int_1^\infty \frac{(tu)^{\Re\nu-1-k}u^{n-1}}{(1+u)^{n+\Re\nu-1}}\,\mathrm{d}u \leqslant C_\nu t^{\Re\nu-k-1},$$

for $k = 1, \ldots, n$, whenever $\Re\nu \geqslant 0$. Take now $0 < t < 1$. If $\Re\nu > 1$ then

$$\int_0^\infty \frac{(1+tu)^{\Re\nu-1-k}u^{n-1}}{(1+u)^{n+\Re\nu-1}}\,\mathrm{d}u \leqslant \int_0^\infty (1+u)^{(\Re\nu-1-k)_+}\frac{u^{n-1}\,\mathrm{d}u}{(1+u)^{n+\Re\nu-1}} \equiv C_\nu.$$

If $0 < \Re\nu \leqslant 1$ and $1 - \Re\nu < \delta < 1$ then

$$\int_0^\infty \frac{(1+tu)^{\Re\nu-1-k}u^{n-1}}{(1+u)^{n+\Re\nu-1}}\,\mathrm{d}u \leqslant t^{-\delta}\int_0^\infty \frac{u^{n-1-\delta}}{(1+u)^{n+\Re\nu-1}}\,\mathrm{d}u \leqslant C_\delta t^{-\delta}.$$

In summary, if $0 < \varepsilon < \Re\nu$,

$$\|g_{t,\nu}\|_{(n)} \leqslant \begin{cases} C_\nu(1+t^{\Re\nu-1}) & \text{if } \Re\nu > 1 \text{ or } \nu = 1,\\ C_{\nu,\varepsilon}t^{\Re\nu-1}(1+t^{-\varepsilon}) & \text{if } 0 < \Re\nu \leqslant 1 \text{ and } \nu \neq 1.\end{cases}$$

Assume now that H satisfies (HG_α). If $\Re\nu > \alpha$ then $(\sin H)H^{-1}(1+H)^{-\nu+1} = \sin H(1+H)^{-\nu} + \int_0^1 \cos sH(1+H)^{-\nu}\,\mathrm{d}s \in \mathcal{L}(X)$ by (iii). Also, the operator $\sin tH(tH)^{-1}(1+tH)^{-\nu+1}$ is uniformly norm bounded in $t > 0$ as in (ii). We have

$$\frac{\sin tH}{H(1+H)^{\nu-1}} = t\sin tH(tH)^{-1}(1+tH)^{-\nu+1}\Big(\frac{1+tH}{1+H}\Big)^{\nu-1}$$

and, by the $AC^{(n)}$ functional calculus with $n > \alpha+1$ applied to $g_{t,\nu}$, we obtain (iv). ∎

Note that the constants C_ν can be written explicitly as polynomials in terms of ν, and that we can make the growth of ν in this dependency more accurate if Theorem 3.3 is employed instead of its $AC^{(n+1)}$ version. The implication (HG_α) $\Rightarrow$ (iii) may be alternatively obtained using the Laplace transform of the semigroup a^z [7]. The boundedness of $I_t^\nu(H)$ in (ii) is in fact equivalent to property (HG_ν) for every $\nu > \alpha$. This has been proved applying the integrated semigroup method in [19], Theorem 2.3.

The $AC^{(\nu)}$-calculus provides us with a very practical way to define powers H^θ for every $\theta > 0$. Moreover, it turns out that $-H^\theta$ also generates a holomorphic semigroup with the (HG_ν) property, for every $\nu > \alpha + \frac{1}{2}$ ([20], p. 343). So we get the following result.

COROLLARY 3.6. *Let β, θ be real numbers such that $\beta \geqslant 0$, $\theta > 0$. Let H be as above and assume that the semigroup generated by $-H^\theta$ satisfies* (HG_β). *Then:*

(i) $\mathrm{e}^{\mathrm{i}H^\theta} H^{-\mu}\varphi(H) \in \mathcal{L}(X)$ *for each* $\varphi \in C_{\mathrm{c}}^{(\infty)}(\mathbb{R})$ *such that* $\varphi(u) = 0$ *if* $u < \frac{1}{2}$ *and* $\varphi(u) = 1$ *if* $u > 1$, *whenever* $\Re\mu > \beta\theta$;

(ii) $\int_0^1 (1-s)^{\mu-1}\mathrm{e}^{\mathrm{i}sH^\theta}\,\mathrm{d}s \in \mathcal{L}(X)$, *if* $\Re\mu > \beta$;

(iii) $\mathrm{e}^{\mathrm{i}H^\theta}(1+H^\rho)^{-\mu} \in \mathcal{L}(X)$, *if* $\rho > 0$ *and* $\rho\Re\mu > \beta\theta$;

(iv) $(\sin H^\theta)H^{-\theta}(1+H^\rho)^{-\mu} \in \mathcal{L}(X)$, *if* $\rho > 0$ *and* $\rho\Re\mu > (\beta-1)\theta$.

Proof. (i) Put $L = H^\theta$. Then $\varphi(H)H^{-\mu}\mathrm{e}^{\mathrm{i}H^\theta} = \mathrm{e}^{\mathrm{i}L}f(L)$ where we take $f(u) = \varphi(u^{\frac{1}{\theta}})u^{-\frac{\mu}{\theta}}\mathrm{e}^{\mathrm{i}u}$. The result then follows via the same argument as in Corollary 3.4 (i).

(ii) This is clear.

(iii) Choose ν such that $(\rho\Re\mu)\theta^{-1} > \Re\nu > \beta$. Using the Φ-calculus, we have $\mathrm{e}^{\mathrm{i}H^\theta}(1+H^\rho)^{-\mu} = \mathrm{e}^{\mathrm{i}H^\theta}(1+H^\theta)^{-\nu}[h(H^\theta)+1]$, where the function $h(u) = (1+u)^{-\nu}(1+u^{\frac{\rho}{\theta}})^{-\mu} - 1$ is in $AC^{(n+1)}$ for all n. The result then follows from the boundedness of $\mathrm{e}^{\mathrm{i}H^\theta}(1+H^\theta)^{-\nu}$ and $h(H^\theta)+1$.

(iv) This is similar to (iii). ∎

Under the hypothesis (HG_α) on H the previous results hold for every $\beta > \alpha + \frac{1}{2}$. However, it seems to be more convenient to keep the statements for general β because there are cases where β for H^θ is better than α for H. This occurs when $H = -\Delta$ is the Laplacian on $\mathbb{R}^n$: it is well known that $\beta = \frac{n-1}{2}$ for $\theta = \frac{1}{2}$, whereas $\alpha = \frac{n}{2}$ ([20], p. 348). Estimates of "(HG_α) type" connecting H and $\sqrt{H}$ have on the other hand been considered in [19] in order to treat (ACP2), in the language of β-times integrated cosine functions. Note that $-H$ generating such a cosine function corresponds to the boundedness of $\int_0^1 (1-s)^{\beta-1}\mathrm{e}^{-\mathrm{i}s\sqrt{H}}\,\mathrm{d}s$.

4. (G_α)-GROWTH AND RESOLVENTS

Let $b^t = \mathrm{e}^{-tH}$ be a strongly continuous (C_0)-semigroup on a Banach space X such that $\sup_{t>0}\|b^t\| < \infty$ and $\sigma(H) \subset [0,\infty)$. Then, for fixed $\lambda > 0$ and $\Re z > 0$, the bounded operator $a^z := (\lambda + H)^{-z}$ can be expressed by subordination as $a^z = \int_0^\infty I^z(u)b^u\,\mathrm{d}u$ where $I^z(u) = \dfrac{u^{z-1}}{\Gamma(z)}\mathrm{e}^{-u}$, for $u > 0, \Re z > 0$, is the fractional semigroup in $L^1(\mathbb{R}^+)$. It is then clear that $(a^z)_{\Re z>0}$ is a holomorphic (C_0)-semigroup in $\mathcal{L}(X)$. Without more specific information on X, this formula is not suitable in order to get properties such as (G_α), or similar ones, since $\|I^z\|_{L^1(\mathbb{R}^+)} = \mathrm{O}(\mathrm{e}^{\frac{\pi}{2}|\Im z|})$, as $|\Im z| \to \infty$ ([51]). But, if we suppose that $(b^t)_{t>0}$ has a (unique) analytic extension to $\mathbb{C}^+$ with property (HG_α), then the $AC_{2,1}^{(\nu+\frac{1}{2})}$-calculus can be used to deduce that a^z satisfies (HG_β) for any $\beta > \alpha + \frac{1}{2}$ (recall

Section 3; note in this respect that property (G_α) is not enough because the function $u \to (1+u)^{-1}$ does not belong to $AC^{(\nu+\frac{1}{2})}_{\exp,2}$). We are going to prove a converse involving property (G_α). This is a fact which we find interesting. For instance, its proof is related to Hadamard fractional integro-differentiation ([50]), as the first lemma shows.

Lemma 4.1. *For $u, \Re z > 0$ and $0 < \varepsilon < 1$, put*

$$J = \int_1^\infty \frac{\mathrm{e}^{-zu} - \mathrm{e}^{-zus}}{(\log s)^{\varepsilon+1}} \frac{\mathrm{d}s}{s}.$$

Then

$$|J| \leqslant C_{\varepsilon,\theta} \max\{1, \Re zu\} \left|\frac{z}{\Re z}\right|^\theta \mathrm{e}^{-\Re zu}$$

for every θ with $\varepsilon < \theta < 1$.

Proof. For θ as above and $s > 1$ let us write $f(s) = \int_1^s (t-1)^{\theta-1} \mathrm{e}^{-zut}\,\mathrm{d}t$ and $g(s) = \int_s^\infty [(t-1)^{\theta-1} - (t-s)^{\theta-1}] \mathrm{e}^{-zut}\,\mathrm{d}t$. We have that $|f(s)| \leqslant \theta^{-1}(s-1)^\theta \mathrm{e}^{-\Re zu}$, if $1 < s < 2$, and $|f(s)| \leqslant \Gamma(\theta)\mathrm{e}^{-\Re zu}(\Re zu)^{-\theta}$, if $s \geqslant 2$. Also,

$$\begin{aligned}
|g(s)| &\leqslant \int_s^\infty [(t-s)^{\theta-1} - (t-1)^{\theta-1}] \mathrm{e}^{-\Re zut}\,\mathrm{d}t \\
&\leqslant \Gamma(\theta)\mathrm{e}^{-\Re zus}(\Re zu)^{-\theta} - \mathrm{e}^{-\Re zu} \int_{s-1}^\infty t^{\theta-1}\mathrm{e}^{-\Re zut}\,\mathrm{d}t \\
&\leqslant \Gamma(\theta)(\Re zu)^{-\theta}|\mathrm{e}^{-\Re zus} - \mathrm{e}^{-\Re zu}| + \mathrm{e}^{-\Re zu} \int_0^{s-1} t^{\theta-1}\mathrm{e}^{-\Re zut}\,\mathrm{d}t,
\end{aligned}$$

for every $s > 1$. So $|g(s)| \leqslant \Gamma(\theta)(\Re zu)^{1-\theta}\mathrm{e}^{-\Re zu}(s-1) + \theta^{-1}\mathrm{e}^{-\Re zu}(s-1)^\theta$, whenever $1 < s < 2$, whereas $|g(s)| \leqslant 3\Gamma(\theta)(\Re zu)^{-\theta}\mathrm{e}^{-\Re zu}$, if $s \geqslant 2$. Thus it follows that

$$\begin{aligned}
(zu)^{-\theta}\Gamma(\theta)^{-1} J &= \int_1^\infty \left(\int_0^\infty t^{\theta-1}\mathrm{e}^{-zut}\,\mathrm{d}t \right) \frac{\mathrm{e}^{-zu} - \mathrm{e}^{-zus}}{(\log s)^{\varepsilon+1}} \frac{\mathrm{d}s}{s} \\
&= \int_1^\infty \left(\int_1^\infty (t-1)^{\theta-1}\mathrm{e}^{-zut}\,\mathrm{d}t - \int_s^\infty (t-s)^{\theta-1}\mathrm{e}^{-zut}\,\mathrm{d}t \right) \frac{\mathrm{d}s}{(\log s)^{\varepsilon+1} s} \\
&= \int_1^\infty (f(s) + g(s)) \frac{\mathrm{d}s}{(\log s)^{\varepsilon+1} s},
\end{aligned}$$

and then

$$
\begin{aligned}
|zu|^{-\theta}|J| &\leqslant C_{\varepsilon,\theta}\mathrm{e}^{-\Re zu}\int\limits_1^2 \frac{(s-1)^\theta+(\Re zu)^{1-\theta}(s-1)}{(\log s)^{\varepsilon+1}}\,\frac{\mathrm{d}s}{s}\\
&\quad + C_{\varepsilon,\theta}(\Re zu)^{-\theta}\mathrm{e}^{-\Re zu}\int\limits_2^\infty \frac{\mathrm{d}s}{(\log s)^{\varepsilon+1}s}\\
&\leqslant C_{\varepsilon,\theta}[1+(\Re zu)^{1-\theta}+(\Re zu)^{-\theta}]\mathrm{e}^{-\Re zu}.
\end{aligned}
$$

Hence

$$
|J| \leqslant C[(\Re zu)^\theta+(\Re zu)+1]\left|\frac{z}{\Re z}\right|^\theta \mathrm{e}^{-\Re zu} \leqslant C\max\{1,\Re zu\}\left|\frac{z}{\Re z}\right|^\theta \mathrm{e}^{-\Re zu}. \qquad \blacksquare
$$

In the next two results the following inequality will be used,

$$
\int\limits_\delta^\infty u^N \mathrm{e}^{-\rho u}\,\mathrm{d}u \leqslant C(N)\rho^{-(N+1)}\big(1+(\rho\delta)^N\big)\mathrm{e}^{-\rho\delta}
$$

$(N,\delta,\rho>0)$. By $[\gamma]$ we denote the integer part of any real number γ.

LEMMA 4.2. *If* $h_z(v)=\exp(-z\mathrm{e}^v)$, *for* $v,\Re z>0$, *and* $\nu>-\frac{1}{2}$, $\theta>\nu-[\nu]$, *then*

$$
|W^{\nu+1}h_z(v)| \leqslant C_\nu \max\{1,(\Re z)^{[\nu]+2}\}\left|\frac{z}{\Re z}\right|^{[\nu]+\theta+1}\mathrm{e}^{([\nu]+2)v}\exp(-\Re z\mathrm{e}^v).
$$

Proof. Write $m=[\nu]+1$, $\varepsilon=\nu-[\nu]$ and take k such that $0\leqslant k\leqslant m$. First of all we note that

$$
\begin{aligned}
\int\limits_1^\infty \frac{s^k-1}{(\log s)^{\varepsilon+1}}\mathrm{e}^{-\Re zus}\,\frac{\mathrm{d}s}{s} &\leqslant \mathrm{e}^{-\Re zu}\int\limits_1^2 \frac{s^k-1}{(\log s)^{\varepsilon+1}}\,\frac{\mathrm{d}s}{s}+C\int\limits_2^\infty s^{k-1}\mathrm{e}^{-\Re zus}\,\mathrm{d}s\\
&\leqslant C_1\mathrm{e}^{-\Re zu}+C_2(k)(\Re zu)^{-k}\big((\Re zu)^{k-1}+1\big)\mathrm{e}^{-2\Re zu}\\
&\leqslant C_1\mathrm{e}^{-\Re zu}+C(k)\big(1+(\Re zu)^{-k}\big)\mathrm{e}^{-2\Re zu}
\end{aligned}
$$

for every $u>0$. Then, for $u=\mathrm{e}^v$, $v>0$, if

$$
\begin{aligned}
J_k :&= \int\limits_0^\infty \frac{\exp(-z\mathrm{e}^v)-\mathrm{e}^{kt}\exp(-z\mathrm{e}^{v+t})}{t^{1+\varepsilon}}\,\mathrm{d}t\\
&= \int\limits_1^\infty \frac{\mathrm{e}^{-zu}-\mathrm{e}^{-zus}}{(\log s)^{\varepsilon+1}}\,\frac{\mathrm{d}s}{s}+\int\limits_1^\infty \frac{1-s^k}{(\log s)^{\varepsilon+1}}\mathrm{e}^{-zus}\,\frac{\mathrm{d}s}{s}
\end{aligned}
$$

we obtain

$$
|J_k| \leqslant C_1(1+\Re zu)\left|\frac{z}{\Re z}\right|^\theta \mathrm{e}^{-\Re zu}+C_2(k)\big(1+(\Re zu)^{-k}\big)\mathrm{e}^{-2\Re zu}
$$

by Lemma 4.1 and the initial observation in this proof. Take now $v > 0$, and $u = \mathrm{e}^v$ again. Since $\nu + 1 = m + \varepsilon$, Marchaud's formula ([20], p. 312) applied to $W^\varepsilon h_z$ gives us

$$W^{\nu+1} h_z(v) = (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}v^m} \int_0^\infty \frac{h_z(v) - h_z(v+t)}{t^{1+\varepsilon}} \,\mathrm{d}t = (-1)^m \sum_{k=0}^m c_k z^k \mathrm{e}^{kv} J_k,$$

where $c_0 = 1$ if $m = 0$, $c_0 = 0$ if $m > 0$. Therefore

$$\begin{aligned} &|W^{\nu+1} h_z(v)| \\ &\leqslant C\Big(\sum_{k=0}^m |c_k|\,|z|^k u^k\Big)(1+\Re z)\left|\frac{z}{\Re z}\right|^\theta u \mathrm{e}^{-\Re z u} + C\sum_{k=0}^m |c_k|\,|z|^k u^k \big(1+(\Re z u)^{-k}\big)\mathrm{e}^{-2\Re z u} \\ &\leqslant C(1+\Re z)(1+|z|^m)\left|\frac{z}{\Re z}\right|^\theta u^{m+1}\mathrm{e}^{-\Re z u} + C(1+\Re z^m)\left|\frac{z}{\Re z}\right|^m u^m \mathrm{e}^{-2\Re z u} \\ &\leqslant C\max\{1, \Re z^{m+1}\}\left|\frac{z}{\Re z}\right|^{m+\theta} u^{m+1}\mathrm{e}^{-\Re z u}. \quad \blacksquare \end{aligned}$$

Let consider again the fractional integral semigroup I^z in $L^1(\mathbb{R}^+)$. A standard argument shows us that the closed span of the family $\{I^n\}_{n\in\mathbb{N}}$ is the whole of $L^1(\mathbb{R}^+)$. To see this take F in the dual space $L^\infty(\mathbb{R}^+)$ of $L^1(\mathbb{R}^+)$ and suppose that $\int_0^\infty I^n(t)F(t)\,\mathrm{d}t = 0$ for every $n \in \mathbb{N}$. The expression inside the integral can be seen up (to a constant) to be the $n-1$ derivative at $z = \frac{1}{2}$ of the analytic function $\int_0^\infty \mathrm{e}^{-zt}\mathrm{e}^{-\frac{t}{2}}F(t)\,\mathrm{d}t$ on $\mathbb{C}^+$. By the identity principle, $F(t) = 0$ a.e. on $\mathbb{R}^+$ and the assertion follows from the Hahn-Banach theorem.

We are ready to give a result on (G_α) estimates coming from corresponding bounds for the resolvent function.

Proposition 4.3. *Let $(b^t)_{t>0}$ be a strongly continuous (C_0)-semigroup in $\mathcal{L}(X)$ with infinitesimal generator $-H$ such that $\sup_{t>0}\|b^t\| < \infty$ and $\sigma(H) \subset [0,\infty)$. Suppose that there exist $Q > 0$ and $\alpha \geqslant 0$ such that the resolvent semigroup $a^z := (1+H)^{-z}$, $z \in \mathbb{C}^+$, satisfies $\sup_{\Re z \geqslant Q}\|a^z\|\,|z|^{-\alpha} < \infty$. Then b^t has a (unique) analytic extension b^z to $\mathbb{C}^+$ such that*

$$\|b^z\| \leqslant C\max\big\{(\Re z)^{-N},\, (\Re z)^{[\nu]+1}\big\}\left|\frac{z}{\Re z}\right|^{\nu+\frac{1}{2}},$$

$\Re z > 0$, for every $\nu > \alpha$, where $N = [\nu] + \frac{3}{2} + Q$.

Proof. Let $-A$ be the infinitesimal generator of a^z. Since the semigroup a^z satisfies property (G_α) (on $\Re z = Q$ instead of $\Re z = 1$) the functional calculus yields bounded operators $f(A)$ on X if f is a smooth function such that $\int_0^\infty |W^{\nu+\frac{1}{2}} f(v)\,\mathrm{e}^{Qv}|^2\,\mathrm{d}v < \infty$, where $\nu > \alpha$ (we could alternatively consider $c^z = a^{Qz}$ on $\Re z = 1$). Put $f_z(v) := \mathrm{e}^z h_z(v) = \exp(-z(\mathrm{e}^v - 1))$, $v \in \mathbb{R}$, $\Re z > 0$.

Our claim is that $b^t = f_t(A)$ for every $t > 0$. The following is perhaps the shortest way to demonstrate this assertion. First let us write $AC^{(m+1)}_{\exp Q}$, $m \in \mathbb{N}$, for the Banach space on $[0, \infty)$ defined by the norm $\int_0^\infty |W^{m+1} f(v)| \mathrm{e}^{Qv} \, \mathrm{d}v$ (when it is finite). Then $f_t \in AC^{(m+1)}_{\exp Q}$ with $\|f_t\| \leqslant C t^{-Q} \mathrm{e}^t$, $t > 0$, and so the Bochner vector integral $\int_0^\infty I^n(t) t^p \mathrm{e}^{-2t} f_t \, \mathrm{d}t$, where $p = [Q] + 1$, gives us an element in $AC^{(m+1)}_{\exp Q}$. At each point v,

$$\begin{aligned} \Gamma(n) \int_0^\infty I^n(t) t^p \mathrm{e}^{-2t} f_t(v) \, \mathrm{d}t &= \Gamma(n+p) \int_0^\infty I^{n+p}(t) \exp(-(1+\mathrm{e}^v)t) \, \mathrm{d}t \\ &= \Gamma(n+p)(2+\mathrm{e}^v)^{-(n+p)} = \Gamma(n+p) \mathrm{e}^{-(n+p)v} (2\mathrm{e}^{-v} + 1)^{-(n+p)} \end{aligned}$$

if $v > 0$, $\Re z > 0$. For m big enough, the calculus acts continuously on $AC^{(m+1)}_{\exp Q}$ and we get $\Gamma(n) \int_0^\infty I^n(t) t^p \mathrm{e}^{-2t} f_t(A) \, \mathrm{d}t = \Gamma(n+p) a^{n+p} (2a^1 + 1)^{-(n+p)}$. On the other hand it is apparent that

$$\begin{aligned} \Gamma(n) \int_0^\infty I^n(t) t^p \mathrm{e}^{-2t} b^t \, \mathrm{d}t &= \Gamma(n+p) \int_0^\infty I^{n+p}(t) \mathrm{e}^{-2t} b^t \, \mathrm{d}t \\ &= \Gamma(n+p)(3+H)^{-(n+p)} \\ &= \Gamma(n+p)(1+H)^{-(n+p)} (2(1+H)^{-1} + 1)^{-(n+p)} \\ &= \Gamma(n+p) a^{n+p} (2a^1 + 1)^{-(n+p)}. \end{aligned}$$

Finally note that the (vector valued) function $t \to t^p (b^t - f_t(A)) \mathrm{e}^{-2t}$ is continuous and bounded on $(0, \infty)$. Since the family $\{I^n\}_{n \in \mathbb{N}}$ generates $L^1(\mathbb{R}^+)$, taking linear functionals and using once again the Hahn-Banach theorem, we deduce that $b^t = f_t(A)$ for every $t > 0$, as claimed.

Therefore $b^z = f_z(A)$ is an analytic extension of b^t on $\mathbb{C}^+$. For fixed $\nu > \alpha$, take $\mu \in \mathbb{R}$ such that $[\mu] = [\nu]$, $\alpha < \mu < \nu$ and choose θ so that $\mu - [\mu] < \nu - [\mu] = \theta < 1$. Then

$$\begin{aligned} \|b^z\| = \|f_z(A)\| &\leqslant \mathrm{e}^{\Re z} \Big(\int_0^\infty |W^{\mu + \frac{1}{2}} h_z(v)|^2 \mathrm{e}^{2Qv} \, \mathrm{d}v \Big)^{\frac{1}{2}} \\ &\leqslant C \max\big\{1, \Re z^{[\nu] + \frac{3}{2}}\big\} \mathrm{e}^{\Re z} \left| \frac{z}{\Re z} \right|^{[\nu] + \theta + \frac{1}{2}} \Big(\int_0^\infty \exp\Big(\Big([\nu] + \frac{3}{2}\Big) 2v - 2\Re z \mathrm{e}^v\Big) \mathrm{e}^{2Qv} \, \mathrm{d}v \Big)^{\frac{1}{2}} \end{aligned}$$

by Lemma 4.2. Now, making the substitution $u = \mathrm{e}^v$ in the integral and using the bound which immediately preceeds Lemma 4.2, it readily follows that

$$\begin{aligned}\|b^z\| &\leqslant C \max\left\{1, \Re z^{[\nu]+\frac{3}{2}}\right\}\left|\frac{z}{\Re z}\right|^{\nu+\frac{1}{2}} \mathrm{e}^{\Re z}\left(\int_1^\infty u^{2([\nu]+Q+1)}\mathrm{e}^{-2\Re z u}\,\mathrm{d}u\right)^{\frac{1}{2}} \\ &\leqslant C \max\left\{(\Re z)^{-N}, (\Re z)^{[\nu]+1}\right\}\left|\frac{z}{\Re z}\right|^{\nu+\frac{1}{2}},\end{aligned}$$

where $N = [\nu] + \frac{3}{2} + Q$, as we wanted to show. ∎

5. EXAMPLES, APPLICATIONS AND COMMENTS

For the introduction to this section, as well as for general notions, we refer the reader to [10], [12], [23], [43], [49]. The examples that we are considering concern L^p-spaces, more particularly, those defined on Lie groups. In this setting questions about estimates for Cauchy problems or about spectral independence are very closely related to multiplier theory.

Let M be a metric measure space and let H be a positive definite operator on $L^2(M)$. By the spectral theorem we get bounded operators $m(H)$ on $L^2(M)$, for every bounded Borel function $m : [0,\infty) \to \mathbb{C}$. In particular the *heat kernel* $(p_z)_{\Re z>0}$ of H is defined as $p_z = \mathrm{e}^{-z(\cdot)}(H) \equiv \mathrm{e}^{-z(H)}$. We are mostly dealing with Lie groups $M = G$ and $H = \Delta$, the sub-Laplacian on G. Let G be a connected Lie group with left (right) invariant measure $\mathrm{d}\xi$ and Lie algebra $\mathfrak{g}$. Suppose that $X_1, \ldots, X_k$ is a collection of right (left) invariant vector fields on G which generate $\mathfrak{g}$ algebraically. Then the expression $\Delta = -\sum\limits_{j=1}^{k} X_j^2$ is called the *sub–Laplacian* associated to $X_1, \ldots, X_k$. If, moreover, the family $\{X_1, \ldots, X_k\}$ is a linear basis of $\mathfrak{g}$ then Δ is called the *Laplacian*. Regarded as an (unbounded) operator on $L^2(G) = L^2(G, \mathrm{d}\xi)$, a sub-Laplacian Δ is formally self-adjoint and non negative and so the spectral theorem holds in the same sense as above ([49]).

Multipliers. The multiplier problem for H, on M, consists of finding appropriate conditions on m in order that $m(H) : L^p(M) \cap L^2(M) \to L^2(M)$ can be extended as a bounded operator $m(H) : L^p(M) \to L^p(M)$, for $1 \leqslant p \leqslant \infty$, $p \neq 2$. If the holomorphy of m is a necessary condition, then H is said to have a *holomorphic L^p functional calculus*. If, on the contrary, real differentiability of m, of compact support, up to a certain order is enough, then we say that H admits a *differentiable L^p functional calculus* ([43], pp. 685–686). Motivated by earlier classical results on $\mathbb{R}^n$, a natural goal for the differentiable calculus is to obtain theorems of so-called Marcinkiewicz-Mikhlin-Hörmander type, for which it is also relevant to seek the lowest degree of differentiability required of m. In this context, Müller and Stein have proved a remarkable theorem [44] and [43] (see also references therein):

Let $\mathbb{H}_n$ be the Heisenberg group $\mathbb{C}^n \times \mathbb{R}$ endowed with group law $(z,t)\cdot(z',t') := (z + z', t + t' - \frac{1}{2}\Im z \cdot z')$. The left invariant vector fields $U := \frac{\partial}{\partial u}, X_j := \frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial u}, Y_j := \frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial u}$, $j = 1, \ldots, n$, form a basis of the Lie algebra of

$\mathbb{H}_n$, and the sub-Laplacian $\mathcal{L} := \sum_{j=1}^{n}(X_j^2 + Y_j^2)$ is homogeneous of degree 2 with respect to the dilations $(x, u) \to (rx, r^2u)$, $r > 0$. Note that $\mathbb{H}_n$ has homogeneous dimension $2n + 2$ whereas its Euclidean dimension is $d = 2n + 1$. Considering the wave equation (ACP2) when $H = \mathcal{L}$ and $X = L^p(\mathbb{H}_n)$, Müller and Stein have proved the following fact.

The closed operator $\mathrm{e}^{\mathrm{i}\sqrt{\mathcal{L}}}(1+\mathcal{L})^{\frac{-\alpha}{2}}$ *extends to a bounded operator on* $L^1(\mathbb{H}_n)$ *when* $\alpha > \frac{d-1}{2}$ (L^p estimates follow from interpolation as usually).

Via the factorization $\mathrm{e}^{-(1+\mathrm{i})\sqrt{\mathcal{L}}} = [\mathrm{e}^{-\mathrm{i}\sqrt{\mathcal{L}}}(1+\sqrt{\mathcal{L}})^{-\nu}][(1+\sqrt{\mathcal{L}})^{\nu}\mathrm{e}^{-\sqrt{\mathcal{L}}}]$, the homogeneity of $\mathcal{L}$ entails that this result is equivalent to saying that $\|\mathrm{e}^{-z\sqrt{\mathcal{L}}}\| \leqslant C_\alpha\Big(\frac{|z|}{\Re z}\Big)^{\alpha}$, for any $\alpha > \frac{d-1}{2}$. Sharp theorems on multipliers of $\mathbb{H}_n$ follow from the above, on the basis of Sobolev estimates [43].

In fact, the general method of proving a multiplier theorem usually involves appropriate estimates of the heat kernel, though the explicit use, in this respect, of bounds for $\|p_{1+\mathrm{i}y}\|$ is quite recent (see [24], [25], [26], [44], [45], [47] and, by contrast, [10]). A direct relationship between property (HG_α) and theorems of Marcinkiewicz-Mikhlin-Hörmander type has been given in [17] and [18].

Solvable Lie groups. In practice the multiplier problem is also linked to growth properties of M. From now on we will restrict our discussion to Lie groups. An arbitrary connected Lie group G has always either *polynomial* or *exponential* volume growth and, for some time, certain results supported the idea that polynomial volume growth would correspond to differentiable functional calculus whereas the exponential growth would be associated with necessary holomorphy of a multiplier m (see the above subsection). Nevertheless, Hebisch eventually gave an example of a solvable Lie group of exponential volume growth which *admits* a differentiable L^1 calculus ([23]). More results of this nature were established subsequently, for different solvable Lie groups and sub-Laplacians ([10], [24], [25], [26], [27], [45]). It is to be noticed that an example of a solvable group has recently been given, for which a certain Laplacian does not have a differentiable L^p calculus ([9]). See also [21].

Let us describe the groups of [23] and [10] in some detail. For a noncompact and semisimple Lie group G, denote by $G = ANK$ an Iwasawa decomposition of G in standard notation, and put $S = AN$. Set $B_r = \{x \in S : |x| < r\}$ where $|x|$ is the distance to the identity of S. Then $\xi(B_r) \leqslant Cr^n\mathrm{e}^{\kappa r}$ for every $r > 0$ where n is the dimension of S and κ is a certain constant. Moreover, if φ_0 denotes the basic spherical function on S then $\|\chi_{B_r}\varphi_0\|_2^2 := \int_{B_r} |\varphi_0(x)|^2\,\mathrm{d}\xi(x) \leqslant C\max\{r^n, r^\delta\}$, where δ is the pseudo-dimension of S, $\delta \geqslant n$. Fix now a distinguished Laplacian Δ on S as in [10]. Its heat kernel p_t lies in $L^1(S)$ and

$$\|\chi_{B_r}p_t\|_1 \leqslant \|\chi_{B_r}\varphi_0\|_2\,\|\chi_{B_r}p_t\|_2 \leqslant C\max\big\{r^{\frac{n}{2}}, r^{\frac{\delta}{2}}\big\}\|p_t\|_2,$$

for $t > 0$ ([10], p. 107). When G is complex then $n = \delta$ and the Banach algebra $\mathcal{A}$ generated by p_t is isomorphic to the algebra $L^1_{\mathrm{rad}}(\mathbb{R}^n)$ of integrable radial functions on $\mathbb{R}^n$. The isomorphism is provided by the identification of p_t with the Gaussian semigroup in $\mathbb{R}^n$, and therefore Δ has a differentiable functional calculus. This is the Hebisch result mentioned above ([23]). As a consequence p_z satisfies property (HG_α) for $\alpha = \frac{n}{2}$. We observe that the Hebisch theorem also solves in the negative Problem 5.10 of [51].

For real G, p_z also enjoys a type of (HG_α) property. We are going to prove this as an application of our results of Section 4. The assertion is not surprising but we have not found explicit mention to it in references. As Δ is strongly elliptic, pointwise Gaussian estimates for $(p_z)_{\Re z>0}$ in complex time can be obtained by applying a theorem of Davies ([12], p. 180 and [49], p. 183). Thus we might try to control $\|p_z\|$ directly as in [17], p. 422 using polar coordinates (instead of the argument of [10], p. 110, which inspired the proof given here). This would give a better order of derivation W^ν for the calculus associated to p_z. But, it is not clear to us how to get in this way the polynomial growth of $\|p_{1+\mathrm{i}y}\|$, $|y| \to \infty$, in all cases.

Weighted L^p-spaces are included in the discussion. For $\omega_\beta(u) = (1+u^2)^{\frac{\beta}{2}}$, with $u \geqslant 0$, $\beta \in \mathbb{R}$, let $L^1_\beta(S)$ denote the convolution Banach algebra of measurable functions f on S such that $f\omega_\beta(|\cdot|) \in L^1(S)$, endowed with the norm $\|f\|_{1,\beta} = \int_S |f|\omega_\beta(|\cdot|)\,\mathrm{d}\xi$. It is known that there exists $\lambda \geqslant 1$ such that $\int_S |p_t(x)|\mathrm{e}^{|x|}\,\mathrm{d}\xi(x) \leqslant \lambda^t$ for all $t > 0$ ([34]). Put $q_t = \lambda^{-t}p_t$. Recall that $I^z(u) = \dfrac{u^{z-1}}{\Gamma(z)}\,\mathrm{e}^{-u}$, for $u > 0, \Re z > 0$, is the fractional semigroup in $L^1(\mathbb{R}^+)$. Put $\beta_+ = \max\{\beta, 0\}$.

Proposition 5.1. *For $(q_t)_{t>0}$ as above, let $(a^z)_{\Re z>0}$ be the resolvent semigroup defined by*

$$a^z = \int_0^\infty I^z(u) q_u\,\mathrm{d}u.$$

Then $a^z \in L^1_\beta(S)$ and $\|a^{\frac{n}{2}+\mathrm{i}y}\|_{1,\beta} = \mathrm{O}(|y|^{\frac{\delta}{2}+\beta_+})$ as $|y| \to \infty$.

Proof. Since $L^1(\mathbb{R}^+) \ni h \to \int_0^\infty h(u)q_u\,\mathrm{d}u \in L^1_\beta(S)$ is a bounded algebra homomorphism one has that $(a^z)_{\Re z>0}$ is an analytic semigroup in $L^1_\beta(S)$. Moreover, the map $(0,\infty) \ni u \to q_u \in L^2(S)$ is continuous, with $\|q_u\|_2^2 \leqslant \|q_u\|_\infty \|q_u\|_1 \leqslant Cu^{\frac{-n}{2}}\|q_u\|_1$ ([49], p. 183).

Then, for $Q = \frac{n}{4}$,

$$a^{z+Q} = \int_0^\infty I^{z+Q}(u)q_u\,\mathrm{d}u = \frac{\Gamma(z)}{\Gamma(z+Q)}\int_0^\infty I^z(u)u^Q q_u\,\mathrm{d}u \in L^2(S)$$

because

$$\int_0^\infty |I^z(u)|u^Q\|q_u\|_2\,\mathrm{d}u \leqslant \int_0^\infty |I^z(u)|u^Q u^{-Q}\|q_u\|_1^{\frac{1}{2}}\,\mathrm{d}u \leqslant \|I^z\|_{L^1(\mathbb{R}^+)} < \infty.$$

Thus

$$\begin{aligned}\|a^{2Q+\mathrm{i}y}\|_2 &\leqslant \|a^{Q+\mathrm{i}y}\|_{2\to 2}\|a^Q\|_2 \\ &\leqslant C\sup_{u>0}|(1+\log\lambda+u)^{-(Q+\mathrm{i}y)}| \leqslant C(1+\log\lambda)^{-Q}\end{aligned}$$

by the spectral theorem. Finally, for $r \geqslant 1$ we have

$$\begin{aligned}
\|a^{2Q+\mathrm{i}y}\|_{1,\beta} &= \int\limits_{B_r} |a^{2Q+\mathrm{i}y}(x)|\omega_\beta(|x|)\,\mathrm{d}\xi(x) + \int\limits_{B_r^c} |a^{2Q+\mathrm{i}y}(x)|\omega_\beta(|x|)\,\mathrm{d}\xi(x) \\
&\leqslant (1+r^2)^{\frac{\beta_+}{2}} \|\chi_{B_r}\varphi_0\|_2 \|a^{2Q+\mathrm{i}y}\|_2 + \int\limits_{B_r^c} |a^{2Q+\mathrm{i}y}(x)|\omega_\beta(|x|)\mathrm{e}^{|x|}\mathrm{e}^{-|x|}\,\mathrm{d}\xi(x) \\
&\leqslant Cr^{\frac{\delta}{2}+\beta_+} + \big(\sup_{|x|\geqslant r} \mathrm{e}^{-|x|}\big(1+|x|^2\big)^{\frac{\beta}{2}}\big) \int\limits_{B_r^c} |a^{2Q+\mathrm{i}y}(x)|\mathrm{e}^{|x|}\,\mathrm{d}\xi(x) \\
&\leqslant Cr^{\frac{\delta}{2}+\beta_+} + C\,\mathrm{e}^{-\frac{r}{2}} \int\limits_0^\infty |I^{2Q+\mathrm{i}y}(u)| \int\limits_{B_r^c} q_u(x)\mathrm{e}^{|x|}\,\mathrm{d}\xi(x)\,\mathrm{d}u \\
&\leqslant C(r^{\frac{\delta}{2}+\beta_+} + \mathrm{e}^{-\frac{r}{2}}\mathrm{e}^{\frac{\pi}{2}|y|}) = \mathrm{O}(|y|^{\frac{\delta}{2}+\beta_+})
\end{aligned}$$

by taking $r = \pi|y|$. ∎

Corollary 5.2. *Let $(p_z)_{\Re z>0}$ be the heat kernel of the Laplacian Δ on S defined as above. Put $q_z = \lambda^{-z}p_z$ if $z \in \mathbb{C}^+$. Then*

$$\|q_z\|_{1,\beta} \leqslant C_\nu \max\{(\Re z)^{-N},\ (\Re z)^{[\nu]+1}\}\Big|\frac{z}{\Re z}\Big|^{\nu+\frac{1}{2}},$$

for every $z \in \mathbb{C}^+$, $\nu > \frac{\delta}{2} + \beta_+$, where $N = [\nu] + \frac{3}{2} + \frac{n}{2}$.

Proof. Apply Propositions 4.3 and 5.1 to $q_z = b_z$. ∎

Paying attention on the proof, one observes that Proposition 5.1 actually holds for any (C_0)-semigroup $(b^t)_{t>0} \subset L^1_\beta(S)$ such that $\rho^{\frac{1}{2}}b^t$ is K-invariant (see [10], p. 108) and $\int_S |b^t(x)|^2\,\mathrm{d}\mu(x) \sim_{t\to 0^+} t^{-m}$, $\int_S |b^t(x)|\mathrm{e}^{|x|}\,\mathrm{d}\mu(x) < k^t$, $t > 0$, for some constants m, k. It would be interesting to know if there is such a semigroup, generated by a strict sub-Laplacian.

Denote by $\mathcal{G}$ any of the solvable Lie groups appearing in [24], [25], [26], [27] or [45]. What is precisely proven in these papers is that the sub-Laplacian Δ constructed on $\mathcal{G}$, in each case generates a holomorphic semigroup with property (G_α). In [27] a result about multipliers on certain solvable $\mathcal{G}$ is given which involves conditions of the type $f \in AC^{(\nu)}_{2,1}$. Then $\mathrm{e}^{-z\Delta}$ satisfies (HG_α) in this example.

Representations. The preceding results can be applied to representations of Lie groups. Let $U : G \to \mathcal{L}(X)$ be a strongly continuous representation of a Lie group G into a Banach space X. Choose a basis $\zeta_1, \dots, \zeta_n$ in the Lie algebra of G and define $A_i \equiv \mathrm{d}U(\zeta_i)$ as the infinitesimal generator of the strongly continuous one-parameter group $\mathbb{R} \ni t \to U(\mathrm{e}^{-t\zeta_i})$, $i = 1, \dots, n$. For a complex polynomial $P(u_1, \dots, u_n)$ over $\mathbb{R}^n$ the differential operator $D := P(A_1, \dots, A_n)$ is densely defined and closable on X. It is said to be *affiliated with* (X, G, U). Note that $P(\zeta) \equiv P(\zeta_1, \dots, \zeta_n)$ is affiliated with $U = L$, the left regular representation of G in $L^p(G)$. Operator D can be also written as $D = \mathrm{d}U(P(\zeta))$ by identification with its closure ([49], pp. 11–12).

On the other hand U is always of exponential growth, i.e., $\|U(x)\| \leqslant C\mathrm{e}^{k|x|}$, $x \in G$, for some constants $C, k \geqslant 0$ and therefore it gives rise to a bounded homomorphism (still denoted by U) of "regularisations" $U : L^1(G, \mathrm{e}^{k|\cdot|}) \to \mathcal{L}(X)$ given by $U(\varphi) = \int_G U(x)\varphi(x)\,\mathrm{d}\xi(x)$ for all $\varphi \in L^1(G, \mathrm{e}^{k|\cdot|})$ ([49], p. 12). The heat kernel $(p_z)_{\Re z>0}$ of a *subelliptic* operator $P(\zeta)$ is included in $L^1(G, \mathrm{e}^{k|\cdot|})$ and the closure of $D = P(A_1, \ldots, A_n)$ is the infinitesimal generator of the semigroup $U(p_z)$ in $\mathcal{L}(X)$ ([49], pp. 301–302, 323). This observation may be useful in a number of situations concerning the growth of holomorphic semigroups which are introduced via representations. Here we apply it to the groups $S = AN$, $\mathcal{G}$ and sub-laplacians Δ defined on them.

So let $U : S \to \mathcal{L}(X)$ be a strongly continuous representation and assume that U is polynomially bounded, that is, $\|U(x)\| = \mathrm{O}(|x|^\beta)$ as $|x| \to \infty$, for some fixed $\beta \geqslant 0$. Let P be the polynomial over $\mathbb{R}^n$ which yields the distinguished Laplacian Δ on S and write $p_z = \mathrm{e}^{-z\Delta}$ as above. Denote by H the strongly elliptic operator affiliated with (X, S, U) and which corresponds to P. Take $\lambda \geqslant 1$ as used in Proposition 5.1 and set $\sigma = \log \lambda$.

Corollary 5.3. *The operator $-(\sigma + H)$ is the infinitesimal generator of a holomorphic semigroup a^z in $\mathcal{L}(X)$ such that*

$$\|a^z\| \leqslant C_\nu \max\{(\Re z)^{-N}, (\Re z)^{[\nu]+1}\} \left|\frac{z}{\Re z}\right|^{\nu+\frac{1}{2}},$$

for every $z \in \mathbb{C}^+$, $\nu > \frac{\delta}{2} + \beta$, where $N = [\nu] + \frac{3}{2} + \frac{n}{2}$.

Proof. From the assumptions the homomorphism $U : L^1(S, \mathrm{e}^{k|\cdot|}) \to \mathcal{L}(X)$, defined by $U(\varphi) = \int_S U(x)\varphi(x)\,\mathrm{d}\xi(x)$ for every $\varphi \in L^1(S, \mathrm{e}^{k|\cdot|})$, automatically admits a bounded extension $L^1_\beta(S) \to L(X)$. If $a^z := U(p_z)$ we have that $a^z = \mathrm{e}^{-zH}$, $z \in \mathbb{C}^+$. Then it suffices to apply Corollary 5.2. ∎

In the same way we get the following.

Corollary 5.4. *Let $U : \mathcal{G} \to \mathcal{L}(X)$ be a uniformly bounded representation and set $H = \mathrm{d}U(\Delta)$. Then $(\mathrm{e}^{-zH})_{\Re z>0}$ satisfies (G_α) for some $\alpha > 0$.*

Norm estimates. The approach generally followed to analyse solutions to the Cauchy problems (ACP1), (ACP2) has consisted of proving the boundedness of auxiliary operators such as those of Section 3. This has been done for Laplacians or more general elliptic operators on $\mathbb{R}^n$ in [42], [46], [38], [52], [30], [44] for instance. The main tools used in these papers come from the theory of multipliers in $\mathbb{R}^n$, which allow one to obtain sharp estimates. Our Corollary 3.4 is a partial, though quite general, extension of these results. Estimates for more general expressions of the type $f(H)\mathrm{e}^{\mathrm{i}tH}$ have been established for Schrödinger operators H on $\mathbb{R}^n$ in [36], [35] by the methods of amalgams, multiplier theorems again, or using the Sjöstrand-Helffer formula (see also [15], [19] for methods based on regularized, integrated or smooth distribution semigroups).

There are many concrete semigroups on L^p spaces, mainly defined on Lie groups, with the property (HG_α) or similar ones; [12], [13], [14], [17], [18], [41],

[20], [47], [23], [27], [44], etc. Hence, Theorem 3.3 holds in all these cases, and so it extends and generalises to this context results of [36], [35], [15]. Likewise, getting bounds for oscillatory expressions $\varphi(H)H^{-\mu}\mathrm{e}^{-\mathrm{i}H^{\theta}}$ is of interest in multiplier theory ([41]). Among those semigroups with property (HG_α) we have $\mathrm{e}^{-z\mathcal{L}}$ on $\mathbb{H}_n$. In [44] bounds for $\mathrm{e}^{\mathrm{i}t\mathcal{L}}$ are obtained via the study of the wave equation for $\mathcal{L}$ in $\mathbb{H}_n$. Thus, it seems worthwhile to write Theorem 3.3 in this specific case.

Corollary 5.5. *Let $\mathbb{H}_n, \mathcal{L}$ be as before. Take $t \in \mathbb{R}$. Then*

(i) *$f(\sqrt{\mathcal{L}})\mathrm{e}^{\mathrm{i}t\sqrt{\mathcal{L}}} \in L^1(\mathbb{H}_n)$ with $\|f(\sqrt{\mathcal{L}})\mathrm{e}^{\mathrm{i}t\sqrt{\mathcal{L}}}\| \leqslant C(1+t^2)^{\frac{\alpha}{2}}\|f\|^{(\alpha)}_{(\nu),2,1}$ whenever $\|f\|^{(\alpha)}_{(\nu),2,1} < \infty$, $\alpha > \frac{d-1}{2}$, $\nu > \alpha$.*

(ii) *$f(\mathcal{L})\mathrm{e}^{\mathrm{i}t\mathcal{L}} \in L^1(\mathbb{H}_n)$ with $\|f(\mathcal{L})\mathrm{e}^{\mathrm{i}t\mathcal{L}}\| \leqslant C(1+t^2)^{\frac{\alpha}{2}}\|f\|^{(\alpha)}_{(\nu),2,1}$ provided that $\|f\|^{(\alpha)}_{(\nu),2,1} < \infty$, $\alpha > \frac{d}{2}$, $\nu > \alpha$.*

Proof. Apply Theorem 3.3 and the remark above to Corollary 3.5. ∎

Part (ii) of the above result can be improved for $t = 0$ thanks to the interplay between $\mathcal{L}$ and $\sqrt{\mathcal{L}}$, and the bounds obtained in [44].

Corollary 5.6. *Let $\nu > \frac{n}{2}$, $n > 1$. Then $f(\mathcal{L}) \in L^1(\mathbb{H}_n)$ for every $f \in AC^{(\nu)}_{2,1}$.*

Proof. Put $a^z = \mathrm{e}^{-z\sqrt{\mathcal{L}}}$, $z \in \mathbb{C}^+$. Therefore $\|a^z\| \leqslant C\left(\frac{|z|}{\Re z}\right)^{\frac{n}{2}}$ and it follows that $g(\sqrt{\mathcal{L}}) \in L^1(\mathbb{H}_n)$ for every $g \in AC^{(\nu)}_{2,1}$, with $\nu > \frac{n}{2}$, by Theorem 1.2. Moreover, $AC^{(\nu)}_{2,1}$ is invariant under changes of variable of the type $u \to u^\gamma$; $u > 0$, $\gamma > 0$ ([20]). Then, for $f \in AC^{(\nu)}_{2,1}$, we have $f(\mathcal{L}) = g(\sqrt{\mathcal{L}}) \in L^1(\mathbb{H}_n)$ for $g(u) = f(u^2)$, $u > 0$. ∎

There are analogues of the last two corollaries for the semigroup $\mathrm{e}^{-z\Delta}$ in $L^1(S)$ for complex S (we only have to notice that $\mathrm{e}^{-z\sqrt{\Delta}}$ is the image in $\mathcal{A}$ of the Poisson semigroup in $L^1_{\mathrm{rad}}(\mathbb{R}^n)$). In particular, it extends Corollary 4 of [23].

The situation is more involved for real S. Boundedness conditions of the type $\|\mathrm{e}^{-zH}\| \leqslant C\max\{(\Re z)^{-N}, (\Re z)^m\}\left|\frac{z}{\Re z}\right|^\alpha$, where $N, m, \alpha \geqslant 0$, do not provide us with a $AC^{(\nu)}$-calculus for H although they give more precise information than property (G_α): it is readily seen, using an argument similar to that of Theorem 3.3, that f operates on H if $\int_0^\infty \big[\int_y^{2y} |W^{\nu+\frac12}f(u)(u^{\nu+N}+u^{m-\nu})|^2\,\mathrm{d}u\big]^{\frac12}\frac{\mathrm{d}y}{y} < \infty$ for $\nu > \alpha$. As a consequence we get the following.

Corollary 5.7. *Let Δ, S be as before. Then $f(\sigma+\Delta) \in L^1_\beta(S)$ for every f such that*

$$\int_0^\infty \Big[\int_y^{2y} |W^{\nu+\frac12}f(u)(u^r+u^s)|^2\,\mathrm{d}u\Big]^{\frac12}\frac{\mathrm{d}y}{y} < \infty$$

and $\mathrm{e}^{\mathrm{i}t(\sigma+\Delta)} f(\sigma+\Delta) \in L^1_\beta(S)$ *for all* $t \in \mathbb{R}$ *and* f *such that*

$$\int_0^\infty \left[\int_y^{2y} |W^{\nu+\frac{1}{2}} f(u)(u^r + u^s)|^2 (1+u^2)^{\alpha+\frac{1}{2}} \,\mathrm{d}u \right]^{\frac{1}{2}} \frac{\mathrm{d}y}{y} < \infty$$

where $\alpha > \frac{\delta}{2} + \beta$, $\nu > \alpha + \frac{1}{2}$, $r = \nu + [\alpha] + \frac{3+n}{2}$ *and* $s = [\alpha] + 1 - \nu$.

Details, norm bounds and analogous versions for representations of S are left to the reader.

Is it possible, in Corollary 5.7, to find better ν for $f(\sqrt{\Delta})$? We finish this subsection with a question concerning this point (which is more or less implicit in the literature).

The appeal to $\sqrt{H}$ is not infrequent. In certain cases estimates for $\mathrm{e}^{-z\sqrt{H}}$ are better than corresponding estimates for e^{-zH} which then allows us to improve the functional calculus for H. Examples of such operators H are the usual Laplacian on $\mathbb{R}^n$ ([20], p. 348), or the operators Δ on S ([23]), and $\mathcal{L}$ on $\mathbb{H}_n$ ([44]), as seen before. Furthermore, this fact seems to be behind obtaining the proper degree of derivation in L^p multiplier theorems ([43], p. 686). In this respect one should recall the use of $\cos t\sqrt{H}$ in arguments linked to the finite propagation speed property ([43]), or in the study of the wave equation ([46] and [19]).

We wonder if this is a general fact. In other words, let Δ be a sub-Laplacian on a Lie group G such that $(\mathrm{e}^{-z\Delta})_{\Re z>0}$, $(\mathrm{e}^{-z\sqrt{\Delta}})_{\Re z>0}$, are holomorphic semigroups in $L^1(G)$. Assume that $\|\mathrm{e}^{-(1+\mathrm{i}y)\Delta}\| = \mathrm{O}(|y|^{\frac{d}{2}})$, as $|y| \to \infty$. Does this imply that $\|\mathrm{e}^{-(1+\mathrm{i}y)\sqrt{\Delta}}\| = \mathrm{O}(|y|^{\frac{d-1}{2}})$, as $|y| \to \infty$?

The question can also be posed in other situations, for Laplace-Beltrami operators on Riemannian manifolds, for instance.

Spectral independence. We now discuss the question of spectral independence. Let M be a metric measure space and suppose that H is the infinitesimal generator of a semigroup e^{-tH} in $\mathcal{L}(L^2(M))$ which can be extended to a semigroup $(T_p^t)_{t>0}$ of bounded operators on $L^p(M)$, for $1 \leqslant p < \infty$. By H_p we denote the infinitesimal generator of T_p^t, i.e., $T_p^t = \mathrm{e}^{-tH_p}$.

Problem. Does the spectrum $\sigma(H_p)$ depend on p?

In general the answer to this question is yes ([9], [14], p. 536), but there are also many important cases where there is independence of $\sigma(H_p)$ with respect to p. This was proved by Hempel and Voigt for a variety of Schödinger operators on $L^p(\mathbb{R}^n)$, thereby solving a question posed by B. Simon. Their result has been extended in a number of works, mostly for second order differential operators ([14], p. 538, [32], [40], [37]; see also [2], [3], [4], [31] for other directions). Recently, as pointed out in the introduction to this paper, E.B. Davies has suggested a new approach to the subject which includes a quite general class of spaces $L^p(M)$ ([12], [14], [54]). The examples which are covered by Davies' method are required to satisfy a couple of hypotheses, namely that e^{-tH} is given by convolution with an integral kernel of *Gaussian* type, and that M has polynomial growth (these assumptions can be refined ([18])). Both requirements imply (HG_α) for $\varepsilon + H$ and all $\varepsilon > 0$, and then the functional calculus applies.

There are also results on spectral independence where the underlying space M is not of polynomial but of *subexponential* growth ([53] and [37]). In such cases the arguments are different from those of [12]. So, it seems reasonable to try to complete the picture by obtaining results on independence of the spectrum, for M of *exponential* growth, by just using the calculus $\Theta \equiv \Phi$.

Let G be any of the solvable Lie groups S or $\mathcal{G}$ considered earlier in this paper. Set $p_z = \mathrm{e}^{-z\Delta}$ where Δ is the associated sub-Laplacian on G. Suppose that X_1, X_2 are two Banach spaces such that $X_1 \cap X_2$ is dense in each one.

COROLLARY 5.8. *Let $U_j : G \to \mathcal{L}(X_j)$ be polynomially bounded representations (or uniformly bounded if $G = \mathcal{G}$) and put $H_j = \mathrm{d}U_j(\Delta)$ for $j = 1, 2$. If $U_1(p_z)x = U_2(p_z)x$ for all $x \in X_1 \cap X_2$, then $\sigma(H_1) = \sigma(H_2)$.*

Proof. For $G = S$ the action of U_j on p_z comes from the extension $U_j : L^1_\beta(S) \to \mathcal{L}(X_j)$ for some $\beta \geqslant 0$. Thus it suffices to apply Corollary 5.3 and Theorem 2.6 to the semigroups $\mathrm{e}^{-z(\sigma+H_j)} = U_j(\mathrm{e}^{-\sigma z}p_z)$, $j = 1, 2$. If $G = \mathcal{G}$ we use the extension $U_j : L^1(\mathcal{G}) \to \mathcal{L}(X_j)$, and the same argument works, as $U_j(p_z)$ again verifies the hypotheses of Theorem 2.6 ([24], [25], [26], [45] and [49], p. 301). ∎

The above proof relies upon property (G_α). In this respect, note that all we can deduce from Corollary 5.3 and the subordination formula for the resolvent function of Δ_p is that $\|(w-H)^{-\gamma}\| \leqslant C \max\{\langle w\rangle^{\nu+\frac{1}{2}+N}, \langle w\rangle^{\nu-[\nu]-\frac{1}{2}}\}|\Im w|^{\nu+\frac{1}{2}+\gamma}$ in place of property (R_α).

Taking X_j as the proper L^p spaces and U as the left representation in the corollary, we obtain invariance of the spectrum of Δ_p on G with respect to p, $1 \leqslant p < \infty$, and also with respect to $\beta \in \mathbb{R}$ if $G = S$. For $G = S$ and $\beta = 0$, or for $G = \mathcal{G}$ in [25], this is due to Hulanicki ([34], Proposition 5.3 and [22], Corollary 1.4).

More generally, Δ_p is decomposable (as it satisfies (G_α)) and L^p spaces can be obtained by interpolation. Thus, the constancy of $\sigma(\Delta_p)$ follows from the result of Albrecht ([1], Corollary 2.6); see Theorem 2.7. See also the comments in the introduction.

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