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# HAAGERUP APPROXIMATION PROPERTY FOR FINITE VON NEUMANN ALGEBRAS

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Abstract. We study finite von Neumann algebras $M$ that admit an approximate identity made with completely positive normal maps whose extention to $L^2(M)$ are compact operators. We prove heredity results, and we state sufficient conditions on actions of countable groups to ensure that the crossed product algebras have the same property.

Keywords: *von Neumann algebras, completely positive maps, Haagerup property, crossed products.*

MSC (2000): 46L10.

## 1. INTRODUCTION

In [14], U. Haagerup proved that the reduced $C^*$-algebra of the non Abelian free group $F_N$ has Grothendieck's metric approximation property. To do that, he proved the existence of a sequence $(\varphi_n)_{n\geqslant 1}$ of normalized, positive definite functions on $F_N$ with the following properties:

(1) for every $g \in F_N$, the sequence $\varphi_n(g)$ tends to 1 as $n \to \infty$;
(2) for every $n$, $\varphi_n$ belongs to $C_0(F_N)$, i.e. tends to 0 at infinity of $F_N$.

It turns out that many classes of locally compact, second countable groups possess such sequences (where pointwise convergence to 1 is replaced by uniform convergence on compact subsets), and it is the reason why we call it the *Haagerup property* in [7]. In [8], M. Choda observed that a countable group $\Gamma$ has the Haagerup property if and only if its associated von Neumann algebra $L(\Gamma)$ admits a sequence $(\Phi_n)$ of completely positive, normal maps such that

(1′) $\tau \circ \Phi_n \leqslant \tau$ and $\Phi_n$ extends to a compact operator on $\ell^2(\Gamma)$ ($\tau$ denotes the natural trace on $L(\Gamma)$);
(2′) for every $x \in L(\Gamma)$, $\|\Phi_n(x) - x\|_2$ tends to zero as $n \to \infty$.

(See also [1], Proposition 4.16.)

Obviously, injective finite von Neumann algebras have the Haagerup property in this sense. However, the first family of examples of finite von Neumann algebras that have this property and which are neither group algebras nor injective is given by F. Boca in [6]: he proves that the amalgamated free product factors constructed by S. Popa in [24] have the above property.

It is worth mentioning that the first use of the Haagerup property in the context of (group) von Neumann algebras is due to A. Connes and V. Jones [9]: they showed the existence, for every property T countable group, of a cocycle action on some algebra $L(F_n)$ which cannot be perturbed to a genuine action. A crucial fact is that if $N$ is a type $\mathrm{II}_1$ factor with property T, then it cannot be embedded into any factor with the Haagerup property. See also Section 6 and Appendix in [27].

Our article is organized as follows: In Section 2, we recall the precise definition of the Haagerup property for a finite von Neumann algebra and we prove first a technical result which says that one can always choose a sequence of unital completely positive maps $(\Phi_n)$ as above such that $\tau \circ \Phi_n = \tau$ for every $n$. This stronger condition is unavoidable to extend such maps to free products, for instance. Nevertheless, the weaker condition is more flexible to use while proving that any reduced algebra $eMe$ has the Haagerup property if $M$ does, for example (see Theorem 2.3). A priori, that property depends on the choice of some finite trace $\tau$, but we prove in Proposition 2.4 that if a finite von Neumann algebra $M$ has the Haagerup property with respect to some trace $\tau$, then it has the same property with respect to any other trace $\tau'$. We also prove the following heredity result:

THEOREM 1.1. *Let $1 \in N \subset M$ be finite von Neumann algebras (with separable preduals). Assume that $N$ has the Haagerup property and that the commutant $N'$ of $N$ in the standard representation $L^2(M)$ is finite. Then $M$ has also the Haagerup property.*

If $N$ and $M$ are finite factors, the finiteness hypothesis on $N'$ means that $N$ has finite Jones' index in $M$, and the result is well known (see [6], Lemma 3.10, for instance). However, a typical case where Theorem 1.1 applies is when there exists a conditional expectation $E : M \to N$ with finite probabilistic index: see Corollary 2.6 and Proposition 2.10 for an explicit example.

Section 3 deals with crossed products of finite von Neumann algebras $N$ with the Haagerup property by trace-preserving actions of groups $\Gamma$ with the same property, and our main result there is:

THEOREM 1.2. *If $\Gamma$ acts on a finite von Neumann algebra $N$ and if $\Gamma$ contains a normal subgroup $H$ such that:*

(i) *the von Neumann subalgebra $N \rtimes H$ of $N \rtimes \Gamma$ has the Haagerup property;*

(ii) *the quotient group $Q = \Gamma/H$ is amenable;*

*then $N \rtimes \Gamma$ has the Haagerup property.*

We also give examples of properly outer actions of non amenable groups on the hyperfinite type $\mathrm{II}_1$ factor $R$ such that the crossed products have the Haagerup property and we discuss briefly Boca's notion of the Haagerup property for an inclusion $N \subset M = N \rtimes \Gamma$: it turns out that the inclusion has the Haagerup property if and only if $\Gamma$ has the Haagerup property. As a consequence, we get the following result for countable groups:

Proposition 1.3. *Let $\Gamma_1$ and $\Gamma_2$ be two groups that are Orbit Equivalent (see Section 3). Then one of them has the Haagerup property if and only if the other one does.*

Finally, this paper ends with an appendix where we give a proof of a result of S. Popa ([26], Proposition 1.1) which is related to Corollary 2.6: if a conditional expectation $E$ from a von Neumann algebra $M$ onto a von Neumann subalgebra $N$ has finite index, then it is automatically faithful and normal.

## 2. TECHNICAL AND HEREDITY RESULTS

In this paper, $M, N, \dots$ denote von Neumann algebras with separable preduals, except in the Appendix. Let $\varphi$ be a faithful, normal state on $M$. Denote by $L^2(M,\varphi)$ the standard Hilbert space associated with $\varphi$ and denote by $\xi_\varphi \in L^2(M,\varphi)$ the unit vector which implements $\varphi$. We also denote by $\|\cdot\|_{2,\varphi}$ the associated Hilbert norm on $M$; we simply write $\|\cdot\|_2$ when $\varphi$ is fixed and when there is no danger of confusion. If $e$ is a non zero projection of $M$, we denote by $\varphi_e$ the state on $eMe$ defined by $\varphi_e(exe) = \varphi(e)^{-1}\varphi(exe)$ for all $exe \in eMe$. If $\Phi : M \to M$ is a completely positive, normal map such that $\varphi \circ \Phi \leqslant \varphi$, then $\Phi$ extends to a contraction $T_\Phi : L^2(M,\varphi) \to L^2(M,\varphi)$ via the equality

$$T_\Phi(x\xi_\varphi) = \Phi(x)\xi_\varphi$$

for $x \in M$. We say that $\Phi$ is $L^2$-*compact* if $T_\Phi$ is a compact operator. As we mainly deal with finite von Neumann algebras here, we use the symbol $\tau$ to denote a finite, faithful, normal, normalized trace on $M$, and we shortly call it a *trace*. We recall the following definition taken from [1]:

Definition 2.1. Let $M$ be a finite von Neumann algebra and let $\tau$ be a trace on $M$ as above. We say that $M$ has the *Haagerup approximation property* with respect to $\tau$ (shortly: $M$ has the *Haagerup property*) if there exists a sequence $(\Phi_n)_{n\geqslant 1}$ of completely positive, normal maps from $M$ to itself such that:
(i) $\tau \circ \Phi_n \leqslant \tau$ and $\Phi_n$ is $L^2$-compact for every $n$;
(ii) for every $x \in M$, $\|\Phi_n(x) - x\|_{2,\tau} \to 0$ as $n \to \infty$.

Apparently, the above property depends on the trace $\tau$; however, we will prove below that it is not the case. Moreover, we prove that the sequence $(\Phi_n)$ can be chosen to be $\tau$-preserving:

Proposition 2.2. *Let $M$ be a finite von Neumann algebra which has the Haagerup property with respect to some trace $\tau$. Then there exists a sequence $(\Psi_n)_{n\geqslant 1}$ of completely positive normal maps on $M$ which satisfy:*
*(i′) $\tau \circ \Psi_n = \tau$, $\Psi_n(1) = 1$ and $\Psi_n$ is $L^2$-compact for every $n \geqslant 1$;*
*(ii′) for every $x \in M$, $\|\Psi_n(x) - x\|_{2,\tau} \to 0$ as $n \to \infty$.*

*Proof.* We prove first that we can choose a sequence $(\Phi'_n)$ that fulfills conditions (i) and (ii) of Definition 2.1 and such that $\Phi'_n(1) \leqslant 1$ for every $n$. To do that, let $(a_k)_{k\geqslant 1}$ be a $\|\cdot\|_2$-dense sequence of the unit ball of $M$ with $a_1 = 1$. For every $n \geqslant 1$, set $F_n = \{a_k : \text{where } 1 \leqslant k \leqslant n\}$. Let $(\Phi_n)$ be a sequence as in Definition 2.1. Extracting a subsequence if necessary, we assume that the following inequality holds for every positive integer $n$:

$$\sup_{x\in F_n} \|\Phi_n(x) - x\|_2 \leqslant \frac{1}{4^{n+1}}.$$

We are going to define a sequence $(\Phi'_n)$ of completely positive normal maps that satisfies condition (i), such that $\Phi'_n(1) \leqslant 1$ and morever

$$\sup_{x\in F_n} \|\Phi'_n(x) - x\|_2 \leqslant \frac{1}{2^{n-1}}$$

for every $n$.

Fix $n \geqslant 1$, set $\varepsilon_n = \frac{1}{2^{n+1}}$ and let $e_n$ denote the spectral projection of $\Phi_n(1)$ corresponding to the interval $[1-\varepsilon_n, 1+\varepsilon_n]$. Then $(\Phi_n(1)-1)^2 \geqslant \varepsilon_n^2(1-e_n)$ and we get:

$$\varepsilon_n^2\tau(1-e_n) \leqslant \tau((\Phi_n(1)-1)^2) = \|\Phi_n(1)-1\|_2^2 \leqslant \frac{1}{4^{2n+2}}.$$

This implies that

$$\tau(1-e_n) \leqslant \frac{4}{16}\left(\frac{2}{4}\right)^{2n} = \frac{1}{2^{2n+2}}.$$

Now, set for $x \in M$: $\Phi'_n(x) = \frac{1}{1+\varepsilon_n} e_n\Phi_n(x)e_n$. Then $\Phi'_n(1) \leqslant 1$, and, if $x \in M_+$,

$$\tau\circ\Phi'_n(x) \leqslant \tau(e_n\Phi_n(x)e_n) = \tau(\Phi_n(x)^{1/2}e_n\Phi_n(x)^{1/2}) \leqslant \tau\circ\Phi_n(x) \leqslant \tau(x).$$

This shows that $\Phi'_n$ satisfies condition (i) and that it is $L^2$-compact since $T_{\Phi'_n} = \frac{1}{1+\varepsilon_n}e_nJe_nJT_{\Phi_n}$. Finally, let $x \in F_n$. Then:

$$\begin{aligned}
\|\Phi'_n(x) - x\|_2 &\leqslant \|e_n\Phi_n(x)e_n - x - \varepsilon_n x\|_2 \leqslant \|e_n\Phi_n(x)e_n - x\|_2 + \frac{1}{2^{n+1}}\\
&\leqslant \|e_n(\Phi_n(x)-x)e_n\|_2 + \|e_nxe_n - e_nx\|_2 + \|e_nx - x\|_2 + \frac{1}{2^{n+1}}\\
&\leqslant \frac{1}{4^{n+1}} + \frac{1}{2^{n+1}} + 2\cdot\frac{1}{2^{n+1}} \leqslant \frac{1}{2^{n-1}}.
\end{aligned}$$

(We used the fact that $\|x\| \leqslant 1$ for every $x \in F_n$.)

We define now the required sequence $(\Psi_n)$. Fix $n \geqslant 1$. As $\tau\circ\Phi'_n \leqslant \tau$, there exists $h_n \in M$ such that $0 \leqslant h_n \leqslant 1$ and that $\tau\circ\Phi'_n(x) = \tau(h_nx)$ for all $x \in M$. Notice that in particular, $\tau(\Phi'_n(1)) = \tau(h_n)$. If $\Phi'_n(1) = 1$, then set $\Psi_n = \Phi'_n$, and we are done because this implies that $h_n = 1$. Thus assume that $\Phi'_n(1) - 1 \neq 0$. Set

$$x_n = \frac{1}{\tau(1-h_n)}(1-\Phi'_n(1)) \quad \text{and} \quad y_n = 1 - h_n,$$

which are positive elements of $M$. Next define $\Psi_n : M \to M$ by

$$\Psi_n(x) = \Phi'_n(x) + x_n\tau(y_nx) = \Phi'_n(x) + x_n^{1/2}\tau(y_nx)x_n^{1/2}.$$

It is a completely positive, normal and $L^2$-compact map on $M$. Moreover, $\Psi_n(1) = 1$ and

$$\begin{aligned}
\tau\circ\Psi_n(x) &= \tau(h_nx) + \tau(x_n\tau(y_nx))\\
&= \tau(h_nx) + \frac{1}{\tau(1-h_n)}\tau(1-\Phi'_n(1))\tau((1-h_n)x) = \tau(x)
\end{aligned}$$

for all $x \in M$, since $\tau(1-h_n) = \tau(1-\Phi'_n(1))$. Finally, one has for every $x \in M_+$:

$$\Psi_n(x) - \Phi'_n(x) = x_n\tau(y_n x) \leqslant \|x\| x_n \tau(y_n) = \|x\|(1-\Phi'_n(1)),$$

which shows that $(\Psi_n)$ satisfies condition (ii′). ∎

We now gather some heredity results concerning the Haagerup property; the second one is analogous to semidiscreteness in the case of injective von Neumann algebras.

Theorem 2.3. *Let $M, M_1$ and $M_2$ be finite von Neumann algebras gifted with traces $\tau, \tau_1$ and $\tau_2$ respectively. Assume that $M_1$ and $M_2$ have the Haagerup property (with respect to their prescribed traces). Then:*

(i) *If $e$ is any non zero projection of $M$ and if $M$ has the Haagerup property with respect to $\tau$, then the reduced algebra $eMe$ has the Haagerup property with respect to the trace $\tau_e$. Moreover, if $1 \in N \subset M$ is a von Neumann subalgebra of $M$, then it has the Haagerup property with respect to $\tau|N$.*

(ii) *Assume that there exists a sequence of finite von Neumann algebras $(N_n)_{n\geqslant 1}$, each being gifted with a trace $\tau_n$ with respect to which it has the Haagerup property, and assume that for every $n$ there exist completely positive normal maps $S_n : M \to N_n$ and $T_n : N_n \to M$ such that $\tau_n \circ S_n \leqslant \tau$, $\tau \circ T_n \leqslant \tau_n$, and such that*

$$\|T_n \circ S_n(x) - x\|_{2,\tau} \to 0$$

*as $n \to \infty$. Then $M$ has the Haagerup property with respect to $\tau$. In particular, $M$ has the Haagerup property with respect to $\tau$ if it is generated by an increasing sequence of von Neumann algebras $1 \in N_1 \subset N_2 \subset \cdots$ and if there exists an increasing sequence of projections $e_n \in N_n$ with limit $1$ such that the reduced algebras $e_nN_ne_n$ all have the Haagerup property with respect to their traces $\tau_{e_n}$.*

(iii) *The tensor product von Neumann algebra $M_1 \overline{\otimes} M_2$ has the Haagerup property with respect to the tensor product trace $\tau_1 \otimes \tau_2$.*

(iv) *The free product von Neumann algebra $M_1 \star M_2$ has the Haagerup property with respect to the free trace $\tau_1 \star \tau_2$.*

*Proof.* Assertions (i) and (iii) are obvious. Assertion (iv) follows from Proposition 3.9 of [6]. Observe nonetheless that one has to choose sequences $(\Psi_{i,n})$ on $M_i$ satisfying conditions (i′) and (ii′) of Proposition 2.2 in order to be able to extend them to the free product $M_1 \star M_2$.

We now prove (ii). Let $F \subset M$ be a finite subset and let $\varepsilon > 0$. There exists an integer $n$ such that

$$\|T_n \circ S_n(x) - x\|_{2,\tau} \leqslant \frac{\varepsilon}{2}, \quad \forall x \in F.$$

There exists a completely positive normal map $\Phi : N_n \to N_n$ such that $\tau_n \circ \Phi \leqslant \tau_n$, $\Phi$ is $L^2$-compact and

$$\|\Phi(S_n(x)) - S_n(x)\|_{2,\tau_n} \leqslant \frac{\varepsilon}{2}, \quad \forall x \in F.$$

Then $\Phi_n = T_n \circ \Phi \circ S_n$ is a completely positive normal map on $M$, $\tau \circ \Phi_n \leqslant \tau$, and, as $T_{\Phi_n} = T_{T_n} T_\Phi T_{S_n}$, $\Phi_n$ is $L^2$-compact and satisfies for every $x \in F$:

$$\begin{aligned}\|\Phi_n(x) - x\|_{2,\tau} &\leqslant \|T_n\Phi S_n(x) - T_nS_n(x)\|_{2,\tau} + \|T_nS_n(x) - x\|_{2,\tau} \\ &\leqslant \|\Phi(S_n(x)) - S_n(x)\|_{2,\tau} + \frac{\varepsilon}{2} \leqslant \varepsilon.\end{aligned}$$

Finally, in the particular case, denoting by $E_{N_n}$ the conditional expectation from $M$ onto $N_n$ associated with $\tau$, it suffices to set $S_n(x) = \tau(e_n)e_nE_{N_n}(x)e_n$ and $T_n(e_nxe_n) = \tau(e_n)^{-1}e_nxe_n$ for $x \in M$. ∎

The following proposition shows that the Haagerup property is in fact independent of the trace:

PROPOSITION 2.4. *Let $M$ be a finite von Neumann algebra and let $\tau$ and $\tau'$ be two normal, faithful, finite, normalized traces on $M$. If $M$ has the Haagerup property with respect to $\tau$ then it also has the Haagerup property with respect to $\tau'$.*

*Proof.* Let $h$ be the positive operator affiliated with $Z(M)$, the centre of $M$, such that $\tau'(x) = \tau(hx)$ for every $x \in M$.

Let us assume first that $h$ and $h^{-1}$ are bounded operators, and let $(\Phi_n)_{n\geqslant 1}$ be a sequence of completely positive normal maps on $M$ satisfying conditions (i) and (ii) in Definition 2.1 with respect to $\tau$. Then set $\Psi_n(x) = h^{-1}\Phi_n(hx)$ for $x \in M$. Then it is easy to check that the sequence $(\Psi_n)_{n\geqslant 1}$ satisfies the same conditions with respect to $\tau'$.

If $h$ or $h^{-1}$ is unbounded, for every $n \geqslant 2$ let $e_n \in Z(M)$ be the spectral projection of $h$ corresponding to the interval $\left[\frac{1}{n}, n\right]$. Observe that $e_n$ increases to 1. Set

$$h_n = \frac{\tau(e_n)}{\tau(he_n)}he_n = \frac{\tau(e_n)}{\tau'(e_n)}he_n,$$

which is a positive, invertible element of $Z(e_nM)$. Then we have for every $e_nx \in e_nM$:

$$\tau_{e_n}(h_ne_nx) = \frac{\tau(e_n)}{\tau'(e_n)}\frac{1}{\tau(e_n)}\tau(he_nx) = \frac{1}{\tau'(e_n)}\tau'(e_nx) = \tau'_{e_n}(e_nx).$$

As $h_n^{\pm 1}$ are bounded operators and as $e_nMe_n$ has the Haagerup property with respect to $\tau_{e_n}$ by Theorem 2.3 (ii), the first part of the proof shows that $e_nMe_n$ has the Haagerup property with respect to $\tau'_{e_n}$ as well. Again, Theorem 2.3 (ii) shows that $M$ has the Haagerup property with respect to $\tau'$. ∎

THEOREM 2.5. *Let $1 \in N \subset M$ be a pair of finite von Neumann algebras. Assume that $N$ has the Haagerup property and that there exists a Hilbert space $\mathcal{H}$ such that $M \subset B(\mathcal{H})$ and that the commutant $N'_{\mathcal{H}}$ of $N$ in $B(\mathcal{H})$ is a finite von Neumann algebra. Then $M$ and $N'_{\mathcal{H}}$ have the Haagerup property.*

*Proof.* Denote by $M'_{\mathcal{H}}$ the commutant of $M$ in $B(\mathcal{H})$. If $A$ denotes any von Neumann algebra among $M, N, M'_{\mathcal{H}}, N'_{\mathcal{H}}$, let $\mathrm{Ctr}_A$ be the canonical $Z(A)$-valued trace on $A$. Recall that the coupling operators $c_M(\mathcal{H})$ and $c_N(\mathcal{H})$ are positive, invertible operators affiliated with $Z(M)$ and $Z(N)$ respectively and are characterized by:

$$\mathrm{Ctr}_M(e_\xi^{M'}) = c_M(\mathcal{H})\mathrm{Ctr}_{M'}(e_\xi^M), \quad \forall \xi \in \mathcal{H},$$

and similarly for $c_N(\mathcal{H})$.

Let us assume first that there exists a positive integer $m$ such that

$$\frac{1}{m} \leqslant c_N(\mathcal{H}) \leqslant m.$$

We claim that the same inequalities hold true for $c_M(\mathcal{H})$. Indeed, by Propositions 3 and 6, pp. 300 and 302 of [12], and by Lemma 1.1 of [17], there exist vectors $\xi_1, \ldots, \xi_m \in \mathcal{H}$ such that the cyclic projections $e^N_{\xi_1}, \ldots, e^N_{\xi_m}$ are pairwise orthogonal with sum 1. As $e^M_{\xi_i} \geqslant e^N_{\xi_i}$ for all $i$, we get

$$c_M(\mathcal{H}) \leqslant \sum_{i=1}^{m} c_M(\mathcal{H})\mathrm{Ctr}_{M'}(e^M_{\xi_i}) = \sum_{i=1}^{m} \mathrm{Ctr}_M(e^{M'}_{\xi_i}) \leqslant m.$$

Similarly, since $c_{M'}(\mathcal{H}) = c_M(\mathcal{H})^{-1}$, we prove that $c_M(\mathcal{H}) \geqslant \frac{1}{m}$.

Now set $n = m^2$, and denote by $M'$ (respectively $N'$) the commutant of $M$ (respectively $N$) in its standard representation $L^2(M)$ (respectively $L^2(N)$). By Lemma 2.2 of [17], there exist non zero projections $e' \in N' \otimes M_n(\mathbb{C})$ and $f' \in M' \otimes M_n(\mathbb{C})$, and unitary operators $u : \mathcal{H} \to e'(L^2(N) \otimes \mathbb{C}^n)$ and $v : \mathcal{H} \to f'(L^2(M) \otimes \mathbb{C}^n)$ such that $N'_{\mathcal{H}} = u^* e'(N' \otimes M_n(\mathbb{C}))e'u$ and $M'_{\mathcal{H}} = v^* f'(M' \otimes M_n(\mathbb{C}))f'v$. This proves successively that $N'_{\mathcal{H}}$, $M'_{\mathcal{H}}$, $M'$ and finally $M$ have the Haagerup property.

If $c_N(\mathcal{H})$ or $c_N(\mathcal{H})^{-1}$ is unbounded, for every integer $m \geqslant 2$, let $z_m \in Z(N)$ be the spectral projection of $c_N(\mathcal{H})$ corresponding to the interval $\left[\frac{1}{m}, m\right]$. Then

$$\frac{z_m}{m} \leqslant c_{Nz_m}(z_m\mathcal{H}) = c_N(\mathcal{H})z_m \leqslant m z_m,$$

hence the reduced algebras $(N'_{\mathcal{H}})_{z_m}$ and $M_{z_m}$ have the Haagerup property for every $m$. As $z_m \to 1$ when $m \to \infty$, we conclude by Theorem 2.3 (ii). ∎

A typical case where conditions of Theorem 2.5 are fulfilled is the case where there exists a conditional expectation $E : M \to N$ with *finite (probabilistic) index* (see [4] and [25]): there exists a positive constant $c$ such that $E(x^*x) \geqslant c\, x^*x$ for every $x \in M$. If this is the case, then $E$ is automatically faithful and normal (see Proposition 1.1 of [26] and the Appendix of the present notes for a slightly different proof). Moreover, the above condition is in fact equivalent to the following apparently stronger property: there exists a positive constant $c'$ such that the map $E - c'\mathrm{Id}$ is completely positive ([13] and [25]), and this implies that $N' \subset B(L^2(M))$ is finite (see for instance [18]). Choose a normal, faithful state $\varphi$ on $M$ such that $\varphi \circ E = \varphi$, and denote by $e$ the extention of $E$ to $L^2(M, \varphi)$. Then $\langle M, e\rangle$ is the von Neumann algebra generated by $M$ and $e$ and it is called the *Jones' basic construction*. As it is equal to $JN'J$, it is also finite. Hence,

Corollary 2.6. *Let $N \subset M$ be such that there exists a conditional expectation $E$ of finite index from $M$ onto $N$. If $N$ has the Haagerup property, then so do $M$ and $\langle M, e\rangle$.*

Following Definition 1.4.3 of [23], let us say that two finite von Neumann algebras $M$ and $N$ are *w-stable equivalent* if there exists a correspondence $\mathcal{H}$ from $M$ to $N$ such that the commutants $M'_{\mathcal{H}}$ and $N'_{\mathcal{H}}$ of $M$ and $N$ in $B(\mathcal{H})$ are finite von Neumann algebras. (Recall that such a correspondence is a left normal $M$-module and a right normal $N$-module.) Thus we have:

COROLLARY 2.7. *If $M$ and $N$ are finite and w-stable equivalent and if one of them has the Haagerup property, then so does the other one.*

In Remark 2.4 of [26], S. Popa gives an example of a pair of finite von Neumann algebras, with atomic centres, with no trace-preserving conditional expectation of finite index, but which has some conditional expectation of finite index: for every integer $k \geqslant 2$, consider an inclusion of locally trivial Jones subfactors $N_k \subset M_k$ with Jones index $\left(\frac{1}{k}\right)^{-1} + \left(1-\frac{1}{k}\right)^{-1} = \frac{k^2}{k-1}$, and set $N = \bigoplus_k N_k \subset \bigoplus_k M_k = M$. Let $E_k : M_k \to N_k$ be the expectation with minimal index: $\mathrm{Ind}(E_k) = 4$. Then $E = \bigoplus_k E_k$ has finite index, but the unique trace preserving conditional expectation $E_N$ does not. If all $M_k$'s have the Haagerup property, then so does $\langle M, e\rangle$, but, as $Z(\langle M, e\rangle)$ is atomic, it is not necessary to apply the Theorem 2.5 because Theorem 2.3 (ii) suffices. We give below a "completely non atomic" analogue of Popa's example where Theorem 2.5 is really necessary to get the conclusion.

For a while, let $M$ be a semifinite von Neumann algebra with a normal, faithful, semifinite trace Tr, and let $L^2(M, \mathrm{Tr})$ be the Hilbert space completion of the $\sigma$-weakly dense ideal $\{x \in M : \mathrm{Tr}(x^*x) < \infty\}$ with respect to the scalar product $\langle x, y\rangle = \mathrm{Tr}(y^*x)$. Set

$$M_{1,1} = \{x \in M : \|x\| \leqslant 1 \text{ and } \mathrm{Tr}(x^*x) \leqslant 1\};$$

set also $P_1(M) = \{p \in M : p^2 = p^* = p \text{ and } \mathrm{Tr}(p) \leqslant 1\}$ and $\mathrm{I}_1(M) = \{u \in M : u^*u \in P_1(M)\}$.

LEMMA 2.8. *$M_{1,1}$ is a complete metric space with respect to the distance function $d(x, y) = \mathrm{Tr}(|x-y|^2)^{1/2}$. Moreover, $P_1(M)$ and $\mathrm{I}_1(M)$ are closed subspaces of $M_{1,1}$. In particular, they are all standard Borel spaces.*

*Proof.* Completeness of $M_{1,1}$ is a straightforward consequence of Proposition 4, Part I, Chapter 5 in [12], and the remaining assertions are immediate. ∎

Now, let $B$ denote the type $\mathrm{I}_\infty$ factor with separable predual and let $P$ be a type $\mathrm{II}_1$ factor with separable predual which has furthermore the following properties:

(1) $P$ has the Haagerup property;

(2) there exists a one parameter group of automorphisms $(\sigma_t)_{t\in\mathbb{R}}$ of $P\overline{\otimes}B$ that scales the trace, namely

$$(\tau \otimes \mathrm{Tr}) \circ \sigma_t = \mathrm{e}^{-t}(\tau \otimes \mathrm{Tr}), \quad \forall t \in \mathbb{R}.$$

One can take for instance $P = R$, the hyperfinite type $\mathrm{II}_1$ factor, more generally, $P = Q\overline{\otimes}R$, where $Q$ is a factor with the Haagerup property, or $P = L(F_\infty)$, where $F_\infty$ denotes the non Abelian free group on countably many generators, by [28]. Let $S$ be the open interval $(0, 1)$, and set for $s \in S$:

$$\beta_s = \sigma_{\log(s/(1-s))},$$

in order that

$$(\tau \otimes \mathrm{Tr}) \circ \beta_s = \frac{1-s}{s}(\tau \otimes \mathrm{Tr}), \quad \forall s \in S.$$

Next, fix a unitary operator $u \in P$ such that $\tau(u^k) = 0$ for every non zero integer $k$, and, following [3], define a family of projections $(e_s)_{s\in S} \subset P$ by

$$e_s = s + \sum_{n\neq 0} \frac{\sin(sn\pi)}{n\pi} u^n.$$

Then $e_s$ has trace $s$ and $s \mapsto e_s$ is $\|\cdot\|_2$-continous. Finally, fix some minimal projection $f \in B$ (so that $\mathrm{Tr}(f) = 1$). The choices made above imply that

$$(\tau \otimes \mathrm{Tr}) \circ \beta_s(e_s \otimes f) = \frac{1-s}{s}(\tau \otimes \mathrm{Tr})(e_s \otimes f) = 1 - s,$$

hence $\beta_s(e_s \otimes f)$ is equivalent to $(1 - e_s) \otimes f$ in $P\overline{\otimes}B$ for every $s$.

LEMMA 2.9. *With notations as above, there exists a Borel map $s \mapsto u_s$ from $S$ to $I_1(P\overline{\otimes}B)$ such that*

$$\beta_s(e_s \otimes f) = u_s^* u_s \quad \text{and} \quad (1 - e_s) \otimes f = u_s u_s^*$$

*for every $s \in S$.*

*Proof.* Set $D_1 = \{(p, q) \in P_1(P\overline{\otimes}B)^2 : \tau \otimes \mathrm{Tr}(p) = \tau \otimes \mathrm{Tr}(q)\}$. Then $D_1$ is a closed subspace of the product metric space $P_1(P\overline{\otimes}B)^2$ because if $(p_n) \subset P_1(P\overline{\otimes}B)$ converges to $p$, then

$$|\mathrm{Tr}(p_n - p)| \leqslant \mathrm{Tr}(|p_n - p|) \leqslant \|p_n - p\|_2 \|p_n + p\|_2 \to 0$$

by Powers-Størmer Inequality.

Define $f : I_1(P\overline{\otimes}B) \to D_1$ by $f(u) = (u^*u, uu^*)$. Then $f$ is continuous and onto. As $D_1$ and $I_1(P\overline{\otimes}B)$ are standard Borel spaces, by von Neumann Selection Theorem, $f$ admits a Borel section $g : D_1 \to I_1(P\overline{\otimes}B)$. Thus define $u_s \in I_1(P\overline{\otimes}B)$ by $u_s = g(\beta_s(e_s \otimes f), (1 - e_s) \otimes f)$. Then

$$(u_s^* u_s, u_s u_s^*) = f(u_s) = (\beta_s(e_s \otimes f), (1 - e_s) \otimes f)$$

for every $s \in S$, and $s \mapsto u_s$ is Borel. ∎

Next, define

$$\theta'_s : (P\overline{\otimes}B)_{e_s\otimes f} = P_{e_s} \otimes \mathbb{C}f \to (P\overline{\otimes}B)_{(1-e_s)\otimes f} = P_{1-e_s} \otimes \mathbb{C}f$$

by $\theta'_s(x) = u_s\beta_s(x)u_s^*$. This gives a family of $*$-isomorphisms $\theta_s : P_{e_s} \to P_{1-e_s}$ such that the map $s \mapsto \theta_s(e_s x(s) e_s)$ is a Borel map for every bounded Borel map $s \mapsto x(s)$ from $S$ to $P$.

At last, set $M = L^\infty(S)\overline{\otimes}P$ and

$$\begin{aligned} N &= \{x \in M : \exists y \in M \text{ s.t. } x(s) = e_s y(s) e_s + \theta_s(e_s y(s) e_s) \text{ a.e.}\} \\ &= \{x \in M : e_s x(s)(1 - e_s) = (1 - e_s)x(s)e_s = 0 \text{ and} \\ &\qquad \theta_s(e_s x(s) e_s) = (1 - e_s)x(s)(1 - e_s) \text{ a.e.}\}. \end{aligned}$$

Then $N$ is a unital von Neumann subalgebra of $M$, it admits a conditional expectation $E : M \to N$ of finite index, namely

$$E(x)(s) = \frac{1}{2}[e_s x(s) e_s + \theta_s^{-1}((1-e_s)x(s)(1-e_s)) + \theta_s(e_s x(s) e_s) + (1-e_s)x(s)(1-e_s)].$$

However, as every finite trace $\tau$ on $M$ is of the form

$$\tau(x) = \int_S \tau_P(x(s))h(s)\,\mathrm{d}s,$$

where $h$ is a positive element of $L^1(S)$, there is a unique trace-preserving conditional expectation $E_N$ from $M$ onto $N$ which is given by

$$\begin{aligned} E_N(x)(s) = s e_s x(s) e_s + (1-s)\theta_s^{-1}((1-e_s)x(s)(1-e_s)) \\ + s\theta_s(e_s x(s) e_s) + (1-s)(1-e_s)x(s)(1-e_s), \end{aligned}$$

and if there existed a positive constant $c$ such that $E_N(x^*x) \geqslant c\,x^*x$ for all $x \in M$, then we would get for almost every $s \in S$:

$$e_s E_N(e_s) e_s = s e_s \geqslant c\, e_s,$$

which forces $c = 0$, a contradiction. This shows that $E_N$ has infinite index. We thus get:

PROPOSITION 2.10. *Let $M = L^\infty(S)\overline{\otimes}P$ and $N \subset M$ be as above. Let $e$ be the projection associated to the conditional expectation $E$ as above. Then $\langle M, e\rangle$ has the Haagerup property.*

## 3. THE CASE OF CROSSED PRODUCTS

We fix first our notations concerning the class of von Neumann algebras that will be discussed in this section. Let $N$ be a finite von Neumann algebra gifted with a finite trace $\tau$ and with a $\tau$-preserving action $\alpha$ of a countable group $\Gamma$. We describe the usual two realizations of the crossed product algebra $M = N \rtimes_\alpha \Gamma$ that will be used here. For $t \in \Gamma$, we denote by $t \mapsto \lambda_t$ (respectively $t \mapsto \rho_t$) the left (respectively right) representation of $\Gamma$ on $\ell^2(\Gamma)$, and by $t \mapsto u_t$ the canonical implementation of the action $\alpha$ on $L^2(N) = L^2(N,\tau)$, i.e. $u_t$ is given by $u_t(x\xi_\tau) = \alpha_t(x)\xi_\tau$ for every $x \in N$. We also set $\lambda(t) = 1 \otimes \lambda_t$, which is a unitary operator acting on $L^2(N) \otimes \ell^2(\Gamma)$. Similarly, for every bounded, complex-valued function $f$ on $\Gamma$, we denote by $m_f$ the associated multiplication operator on $\ell^2(\Gamma)$, and by $m(f)$ the operator $1 \otimes m_f$ on $L^2(N) \otimes \ell^2(\Gamma)$. We denote also by $L(\Gamma)$ the von Neumann algebra acting on $\ell^2(\Gamma)$ and generated by the left regular representation $\lambda$ of $\Gamma$.

The first realization of $M$ is the von Neumann algebra generated by $\pi(N) \cup \{\lambda(t) : t \in \Gamma\}$, where $\pi = \pi_\alpha : N \to B(L^2(N) \otimes \ell^2(\Gamma))$ is defined by

$$(\pi(x)\xi)(g) = \alpha_{g^{-1}}(x)\xi(g)$$

for all $x \in N$, $g \in \Gamma$ and $\xi \in L^2(N) \otimes \ell^2(\Gamma) = \ell^2(\Gamma, L^2(N))$. In this realization, $M$ is a von Neumann subalgebra of $N\overline{\otimes}B$ (where $B = B(\ell^2(\Gamma))$, and more precisely, it is the fixed point algebra under the action $\theta$ of $\Gamma$ defined by $\theta_t = \alpha_t \otimes \mathrm{Ad}\,\rho_t$. The second realization of $N \rtimes_\alpha \Gamma$ also acts on $L^2(N) \otimes \ell^2(\Gamma)$ and it is the von Neumann algebra generated by $\{x u_t \otimes \lambda_t : x \in N,\ t \in \Gamma\}$. The isomorphism between these algebras is

$$\iota : (\pi(N) \cup \lambda(\Gamma))'' \to \{x u_t \otimes \lambda_t : x \in N,\ t \in \Gamma\}''$$

defined by $\iota(X) = wXw^*$ where $w$ is the unitary operator on $L^2(N)\otimes\ell^2(\Gamma)$ given by $(w\xi)(g) = u_g\xi(g)$. One has $\iota(\pi(x)) = x\otimes 1$ for $x\in N$ and $\iota(\lambda(t)) = u_t\otimes\lambda_t$ for $t\in\Gamma$. Finally, we denote by $E_N$ the natural (trace preserving) conditional expectation onto $N$ given by $E_N(\pi(x)\lambda(g)) = \delta_{g,1}\pi(x)$ in the first realization, and by $E_N(xu_g\otimes\lambda_g) = \delta_{g,1}x\otimes 1$ in the second.

The first result of this section is already known for semidirect product groups $H\rtimes_\alpha\Gamma$ where $H$ has the Haagerup property and $\Gamma$ is amenable (see [7], Chapter 6); moreover, it follows directly from Proposition 4.17 of [1] that if $\Gamma$ is amenable and if $N$ has the Haagerup property, then $N\rtimes_\alpha\Gamma$ has the *compact approximation property* (see Definition 4.13 of [1]). We do not know how to deduce directly from this that $N\rtimes_\alpha\Gamma$ has the Haagerup property. Instead, our proof is inspired by that of Theorem 3.11 in [11]:

Proposition 3.1. *If $\Gamma$ is amenable and if $N$ has the Haagerup property, then $N\rtimes_\alpha\Gamma$ has also the Haagerup property.*

*Proof.* We use the first realization of $N\rtimes_\alpha\Gamma$. Let $f$ be a finitely supported, nonnegative-valued function on $\Gamma$ such that $\sum_{t\in\Gamma} f(t)^2 = 1$. By Lemmas 3.7 and 3.9 of [11], the mapping $\Psi_f : N\overline{\otimes}B\to N\rtimes_\alpha\Gamma$ defined by

$$\Psi_f(X) = \sum_{t\in\Gamma}\theta_t(m(f)Xm(f))$$

is well defined, normal, and obviously completely positive. By Lemma 3.8 of [11], we have moreover

$$\Psi_f(1) = \sum_{t\in\Gamma} f^2(t)\pi(\alpha_t(1)) = 1.$$

Now, let $\Phi : N\to N$ be a completely positive normal map such that $\tau\circ\Phi\leqslant\tau$ and which is $L^2$-compact. We define $\Phi_f$ on $N\rtimes_\alpha\Gamma$ by

$$\Phi_f(X) = \Psi_f\circ\Phi\otimes i_B(X)$$

for all $X\in N\rtimes_\alpha\Gamma$, where $i_B$ denotes the identity map on $B = B(\ell^2(\Gamma))$. $\Phi_f$ is clearly completely positive and normal. We need the following formula, for $x\in N$ and $g\in\Gamma$:

$$\Phi_f(\pi(x)\lambda(g)) = \sum_{t\in S(f)} f(t)f(g^{-1}t)\pi(\alpha_t\circ\Phi\circ\alpha_{t^{-1}}(x))\lambda(g) \tag{3.1}$$

where $S(f)$ denotes the support of $f$. Denote also by $(e_{u,v})_{u,v\in\Gamma}$ the system of matrix units associated with the canonical basis $(\delta_t)_{t\in\Gamma}$ of $\ell^2(\Gamma)$: $e_{u,v}(\xi) = \langle\xi,\delta_v\rangle\delta_u$ for every $\xi\in\ell^2(\Gamma)$. We compute first $\Phi_f(\pi(x)m(\chi_F)\lambda(g))$ for any finite subset $F$ of $\Gamma$ containing $S(f)$. By definition, $\pi(x)m(\chi_F) = \sum_{u\in F}\alpha_{u^{-1}}(x)\otimes e_{u,u}$; moreover, one has for $y\in N$, $g\in\Gamma$ and any finitely supported function $h$ on $\Gamma$:

$$\sum_{t\in\Gamma}\theta_t(m(f)y\otimes m_h\lambda_g m(f)) = \sum_{t\in\Gamma}\varphi_{g,h}(t)\pi(\alpha_t(y))\lambda(g) \tag{3.2}$$

where $\varphi_{g,h}(t) = f(t)h(t)f(g^{-1}t)$. This follows from Lemma 3.8 and the proof of Lemma 3.9 of [11], but the referee suggested that we give a quick proof for the reader's convenience. First, for fixed $t \in \Gamma$, one has:

$$\begin{aligned}\theta_t(m(f)y \otimes m_h\lambda_g m(f)) &= \theta_t(1 \otimes m_f \cdot y \otimes m_h\lambda_g \cdot 1 \otimes m_f)\\ &= \theta_t(y \otimes (m_{fh}\lambda_g m_f)) = \theta_t(y \otimes (m_{\varphi_{g,h}}))\lambda(g)\end{aligned}$$

since $\operatorname{Ad}\rho_t(\lambda_g) = \lambda_g$. Next, we claim that $\sum_{t\in\Gamma} \theta_t(y \otimes m_{\varphi_{g,h}}) = \sum_{t\in\Gamma} \varphi_{g,h}(t)\pi(\alpha_t(y))$. Indeed, one has for every finitely supported $L^2(N)$-valued function $\xi$:

$$\begin{aligned}&\sum_{t\in\Gamma}\langle\theta_t(y\otimes m_{\varphi_{g,h}})\xi,\xi\rangle\\ &\quad= \sum_{t\in\Gamma}\langle\alpha_t(y)\otimes\rho_t m_{\varphi_{g,h}}\rho_t^{-1}\xi,\xi\rangle = \sum_{s,t\in\Gamma}\varphi_{g,h}(st)\langle\alpha_t(y)\xi(s),\xi(s)\rangle\\ &\quad= \sum_{u,s\in\Gamma}\varphi_{g,h}(u)\langle\alpha_{s^{-1}}(\alpha_u(y))\xi(s),\xi(s)\rangle = \sum_{t\in\Gamma}\varphi_{g,h}(t)\langle\pi(\alpha_t(y))\xi,\xi\rangle.\end{aligned}$$

This proves our claim and ends the proof of (3.2).

Now, as

$$\begin{aligned}\Phi\otimes i_B(\pi(x)m(\chi_F)\lambda(g)) &= \sum_{u\in F}\Phi\otimes i_B(\alpha_{u^{-1}}(x)\otimes(e_{u,u}\lambda_g))\\ &= \sum_{u\in F}(\Phi\circ\alpha_{u^{-1}}(x))\otimes(e_{u,u}\lambda_g),\end{aligned}$$

we get

$$\begin{aligned}\Phi_f(\pi(x)m(\chi_F)\lambda(g)) &= \sum_{u\in F}\sum_{t\in\Gamma}\theta_t\left(m(f)\Phi\circ\alpha_{u^{-1}}(x)\otimes e_{u,u}\lambda_g m(f)\right)\\ &= \sum_{u\in F}\sum_{t\in\Gamma}\varphi_u(t)\pi(\alpha_t\circ\Phi\circ\alpha_{u^{-1}}(x))\lambda(g),\end{aligned}$$

with $\varphi_u(t) = f(t)\delta_u(t)f(g^{-1}t)$ (because $e_{u,u} = m_{\delta_u}$). Thus

$$\begin{aligned}\Phi_f(\pi(x)m(\chi_F)\lambda(g)) &= \sum_{u\in F} f(u)f(g^{-1}u)\pi(\alpha_u\circ\Phi\circ\alpha_{u^{-1}}(x))\lambda(g)\\ &= \sum_{t\in S(f)} f(t)f(g^{-1}t)\pi(\alpha_t\circ\Phi\circ\alpha_{t^{-1}}(x))\lambda(g),\end{aligned}$$

which proves (3.1) because of the $\sigma$-weak continuity of $\Phi_f$. Let us still denote by $\tau$ the trace on $N \rtimes_\alpha \Gamma$ induced by that on $N$; we prove that $\tau\circ\Phi_f \leqslant \tau$ : for that, let $X = \sum \pi(x_g)\lambda(g) \in N \rtimes_\alpha \Gamma$ with $x_g = 0$ except for finitely many $g$'s. One has $X^*X = \sum_s \Big(\sum_h \pi(\alpha_{h^{-1}}(x_h^* x_{hs}))\Big)\lambda(s)$, which gives, taking account of (3.1),

$$\Phi_f(X^*X) = \sum_{s,h}\sum_{t\in S(f)} f(t)f(s^{-1}t)\pi(\alpha_t\circ\Phi\circ\alpha_{t^{-1}h^{-1}}(x_h^* x_{hs}))\lambda(s),$$

and

$$\tau \circ \Phi_f(X^*X) = \sum_{t\in S(f)} \sum_h f(t)^2 \tau \circ \Phi(\alpha_{t^{-1}h^{-1}}(x_h^* x_h))$$
$$\leqslant \sum_{t\in S(f)} \sum_h f(t)^2 \tau(x_h^* x_h) = \tau(X^*X).$$

In order to check that $\Phi_f$ is $L^2$-compact, recall that $\xi_\tau \otimes \delta_1$ is the vector associated to $\tau$, and observe that for $x \in N$ and $g \in \Gamma$, one has

$$\pi(\alpha_t \circ \Phi \circ \alpha_{t^{-1}}(x))\xi_\tau \otimes \delta_g = u_{g^{-1}t} T_\Phi(u_{t^{-1}} x \xi_\tau) \otimes \delta_g,$$

which gives

$$\Phi_f(\pi(x)\lambda(g))\xi_\tau \otimes \delta_1 = \sum_{t\in S(f)} f(t) f(g^{-1}t) u_{g^{-1}t} T_\Phi u_{t^{-1}}(x\xi_\tau) \otimes \delta_g.$$

If $\xi = \sum_g \xi(g) \otimes \delta_g \in L^2(N) \otimes \ell^2(\Gamma)$ is arbitrary, we have:

$$T_{\Phi_f}(\xi) = \sum_{g\in\Gamma} \sum_{t\in S(f)} f(t) f(g^{-1}t) u_{g^{-1}t} T_\Phi u_{t^{-1}}(\xi(g)) \otimes \delta_g$$
$$= \sum_{g\in S(f)S(f)^{-1}} \sum_{t\in S(f)} f(t) f(g^{-1}t) u_{g^{-1}t} T_\Phi u_{t^{-1}}(\xi(g)) \otimes \delta_g.$$

This proves that $T_{\Phi_f}$ is a finite sum of compact operators, hence it is also compact. Finally, in order to complete the proof, fix finite subsets $K \subset N$ and $L \subset \Gamma$, and $\varepsilon > 0$. Since $\Gamma$ is amenable, there exists a finitely supported function $f$ as above such that $|\langle \lambda_g f, f\rangle - 1| \leqslant \frac{\varepsilon}{C}$ $\forall g \in L$, where $C = 2\big(1 + \max_{x\in K} \|x\|\big)$. Next, choose $\Phi$ such that $\|\Phi\circ\alpha_{t^{-1}}(x) - \alpha_{t^{-1}}(x)\|_2 \leqslant \frac{\varepsilon}{2}$ $\forall t \in S(f), x \in K$. Then it is straightforward to check that

$$\|\Phi_f(\pi(x)\lambda(g)) - \pi(x)\lambda(g)\|_2 \leqslant \varepsilon, \quad \forall x \in K, g \in L$$

and this ends the proof. ∎

Our next result relies on Proposition 3.1 above, on Proposition 3 in [5] and also on the deep Theorem 1.1 of [21] which states that if $(\beta, u)$ is a centrally free cocycle crossed action (see below) of an amenable countable group $Q$ on a von Neumann algebra $M$ and if its restriction to $Z(M)$ preserves the restriction of some normal, faithful state, then the cocycle $u$ is a coboundary. In particular, the cocycle crossed product algebra $N \rtimes_{\beta,u} Q$ is $*$-isomorphic to an ordinary crossed product algebra. Let us recall some definitions from [21], for instance.

A *central sequence* in a finite von Neumann algebra $N$ is a bounded sequence $(x_n) \subset N$ such that $\|x_n x - x x_n\|_2 \to 0$ as $n \to \infty$ for every $x \in N$. An automorphism $\theta$ of $N$ is called *centrally trivial* if $\|\theta(x_n) - x_n\|_2 \to 0$ for every central sequence $(x_n)$ of $N$. The set of all centrally trivial automorphisms of $N$ is denoted by $\mathrm{Ct}(N)$ and it is a normal subgroup of $\mathrm{Aut}(N)$; $\theta$ is called *properly centrally non trivial* if $\theta|pN$ is not centrally trivial for any non zero $\theta$-invariant projection

$p \in Z(N)$. A *cocycle crossed action* of a group $Q$ on $N$ is a pair $(\beta, u)$ where $\beta : Q \to \mathrm{Aut}(N)$ and $u : Q \times Q \to U(N)$ satisfy for all $q, r, s \in Q$

$$\beta_q \beta_r = \mathrm{Ad}(u(q,r))\beta_{qr},$$
$$u(q,r)u(qr,s) = \beta_q(u(r,s))u(q,rs),$$
$$u(1,q) = u(q,1) = 1.$$

The cocycle crossed action is then called *centrally free* if $\beta_q$ is properly centrally non trivial for every $q \neq 1$.

THEOREM 3.2. *Let $1 \to H \to \Gamma \to Q \to 1$ be a short exact sequence of countable groups and let $\alpha$ be an action of $\Gamma$ on $N$ which preserves some trace $\tau$. Assume that:*

(i) *the crossed product $N \rtimes_{\alpha|H} H$ has the Haagerup property;*

(ii) *$Q$ is an amenable group.*

*Then the crossed product $N \rtimes_\alpha \Gamma$ has the Haagerup property.*

*Proof.* By Proposition 3 of [5], there exists a cocycle crossed action $(\beta, u)$ of $Q$ on $N \rtimes_{\alpha|H} H$ such that $N \rtimes_\alpha \Gamma$ is $*$-isomorphic to the cocycle crossed product $(N \rtimes H) \rtimes_{\beta,u} Q$. If $(\beta, u)$ is centrally free, then it follows from Theorem 1.1 of [21] that $u$ is a coboundary, hence that there exists an action $\beta'$ of $Q$ on $N \rtimes_{\alpha|H} H$ such that $(N \rtimes H) \rtimes_{\beta'} Q$ is $*$-isomorphic to $N \rtimes_\alpha \Gamma$, and Proposition 3.1 above applies. However, it may well happen that $(\beta, u)$ be not centrally free. In order to deal with this case, choose an outer action $\theta : Q \to \mathrm{Aut}(R)$, where $R$ denotes the hyperfinite type $\mathrm{II}_1$ factor. Since $\mathrm{Ct}(R)$ is the subgroup of inner automorphisms of $R$, $\theta$ is a centrally free action.

Next, let $\alpha \otimes \theta : \Gamma \to \mathrm{Aut}(N \overline{\otimes} R)$ be the action defined by $(\alpha \otimes \theta)_g = \alpha_g \otimes \theta_{\psi(g)}$, where $\psi$ is the canonical projection from $\Gamma$ onto $Q$. We intend to show that the crossed product $(N \overline{\otimes} R) \rtimes_{\alpha \otimes \theta} \Gamma$ has the Haagerup property; as it obviously contains $N \rtimes_\alpha \Gamma$, the proof will be complete.

Remark that $(N \overline{\otimes} R) \rtimes_{\alpha \otimes \theta|H} H$ is naturally isomorphic to the tensor product $(N \rtimes_{\alpha|H} H) \overline{\otimes} R$, which has the Haagerup property. Let us still denote by $(\beta, u)$ the cocycle crossed action of $Q$ on $M = (N \overline{\otimes} R) \rtimes_{\alpha \otimes \theta|H} H$ constructed in the proof of Proposition 3 of [5]. We need to check that it is centrally free. To do that, let us recall how it is defined: For every $q \in Q$, choose $n_q \in \Gamma$ (with $n_1 = 1$) such that $\psi(n_q) = q$, define $\sigma : Q \to \mathrm{Aut}(H)$ and $v : Q \times Q \to H$ by

$$\sigma_q(h) = n_q h n_q^{-1} \quad \text{and} \quad v(q,r) = n_q n_r n_{qr}^{-1},$$

and set $\gamma = (\alpha \otimes \theta)|H$. Then $u(q,r) = \lambda(v(q,r))$ and $\beta : Q \to \mathrm{Aut}(M)$ is characterized by

(a) $\beta_q(\pi_\gamma(x)) = \pi_\gamma((\alpha \otimes \theta)_{n_q}(x))$ for every $x \in N$;

(b) $\beta_q(\lambda(h)) = \lambda(\sigma_q(h))$ for every $h \in H$.

Let us fix $q \in Q$, $q \neq 1$ and a non zero $\beta_q$-invariant projection $p \in Z(M) = Z(N \rtimes H) \otimes 1$; hence write $p = z \otimes 1$ with $z \in Z(N \rtimes H)$. It follows that $\beta_q(z \otimes x) = z \otimes \theta_q(x)$ for every $x \in R$. As $\theta_q$ is not centrally trivial, there exists a central sequence $(x_n) \subset R$ such that

$$\liminf \|\theta_q(x_n) - x_n\|_2 > 0.$$

Set $y_n = z \otimes x_n$ for all $n$. Then it is a central sequence in $M$ and $\|\beta_q(y_n) - y_n\|_2$ does not tend to 0 as $n \to \infty$, which proves that that $(\beta, u)$ is centrally free. The first part of the proof shows that $(N\overline{\otimes}R) \rtimes_{\alpha\otimes\theta} \Gamma$ has the Haagerup property. ∎

We also mention the following result which is a consequence of Theorems 2.3 and 2.5, and the fact that if an action of $\Gamma$ is trivial on some von Neumann subalgebra $P$ of $N$ then the crossed product $P \rtimes \Gamma$ is equal to the tensor product $P\overline{\otimes}L(\Gamma)$:

Proposition 3.3. *Assume that $N$ and $\Gamma$ have the Haagerup property, and let $\alpha$ be an action of $\Gamma$ on $N$ that preserves some trace $\tau$. Then the crossed product algebra $N \rtimes_\alpha \Gamma$ has the Haagerup property if the triple $(N, \Gamma, \alpha)$ satisfies at least one of the following conditions:*

(i) *There exists a Hilbert space $\mathcal{H}$ such that $N \subset B(\mathcal{H})$ and the commutant $(N^\Gamma)'_{\mathcal{H}}$ of the fixed point subalgebra is finite.*

(ii) *There exists a sequence of $\alpha$-invariant von Neumann subalgebras*

$$1 \in N_1 \subset N_2 \subset \cdots \subset N$$

*whose union is $\|\cdot\|_2$-dense in $N$ and such that every crossed product algebra $N_n \rtimes_\alpha \Gamma$ has the Haagerup property.*

Remark 3.4. Assume that $(N, \Gamma, \alpha)$ satisfies conditions (i) above. Assume also that there exists a unitary representation $v$ of $\Gamma$ on $\mathcal{H}$ which implements $\alpha$ on $N$ and that the action is *quasi-free* in the sense of [2]: for every $g \neq 1$, the condition

$$x \in N,\ xy = \alpha_g(y)x, \quad \forall y \in N$$

implies $x = 0$. Then it follows from Section II of [2] that $\Gamma$ acts by $\alpha' = \mathrm{Ad}(v)$ on the commutant $N'_{\mathcal{H}}$ of $N$ and that the commutant $(N^\Gamma)'_{\mathcal{H}}$ is $*$-isomorphic to the crossed product $N'_{\mathcal{H}} \rtimes_{\alpha'} \Gamma$. In particular, the latter crossed product algebra has also the Haagerup property.

Let us describe a case where condition (ii) above is satisfied: suppose that $(A_n, \tau_n)_{n\geqslant 1}$ is a sequence of finite von Neumann algebras, each one being gifted with a trace-preserving action $\alpha^{(n)}$ of a group $\Gamma$ such that all crossed products $A_n \rtimes_{\alpha^{(n)}} \Gamma$ have the Haagerup property. Let $(N, \tau)$ be the infinite tensor product von Neumann algebra $\Big(\bigotimes_{n\geqslant 1} A_n, \otimes_n \tau_n\Big)$ and let $\alpha = \bigotimes_n \alpha^{(n)}$ be the corresponding action. Then the crossed product $N \rtimes_\alpha \Gamma$ has the Haagerup property.

We will see now how to exploit the following fact: Assume that the group $\Gamma$ admits a unitary representation $\rho : \Gamma \to \mathcal{N}_M(N)$ where $M$ is a finite von Neumann algebra containing $N$ and where $\mathcal{N}_M(N) = \{u \in U(M) : uNu^* = N\}$ is the normalizer of $N$ in $M$. This gives an action $\alpha$ on $N$ by $\alpha_g(x) = \mathrm{Ad}\,\rho(g)(x) = \rho(g)x\rho(g^{-1})$. Then the crossed product $N \rtimes_\alpha \Gamma$ is contained in $M \rtimes_\alpha \Gamma$ and this one is isomorphic to the tensor product $M\overline{\otimes}L(\Gamma)$. (Thus, $N \rtimes_\alpha \Gamma$ has the Haagerup property if $M$ and $\Gamma$ do.) Indeed, denote by $\beta_g$ the automorphism $\mathrm{Ad}\,\rho(g)$ on $M$ and by $J$ the canonical antilinear involution on $L^2(M, \tau)$ defined by $J(x\xi_\tau) = x^*\xi_\tau$

for $x \in M$. Then the canonical implementation of the automorphism $\beta_g$ is $u_g = \rho(g)J\rho(g)J$ for every $g \in \Gamma$. Define $V$ from $L^2(M) \otimes \ell^2(\Gamma)$ to itself by

$$V \sum_{g\in\Gamma} \xi(g) \otimes \delta_g = \sum_{g\in\Gamma} J\rho(g)J\xi(g) \otimes \delta_g.$$

Then $V$ is a unitary operator and we have for every $g \in \Gamma$ and every $x \in M$:

$$V(x\rho(g) \otimes \lambda_g)V^* = xu_g \otimes \lambda_g.$$

This proves that $M\overline{\otimes}L(\Gamma)$ is isomorphic to the crossed product $M \rtimes_\beta \Gamma$.

A typical (and rather trivial) case where hypotheses above are satisfied is when $N$ is finite dimensional: $\alpha$ is implemented by a unitary representation $u$ on the finite dimensional space $L^2(N)$, hence we take $M = B(L^2(N))$ which is finite dimensional, too. Another case which may be of interest is the case where $M = L(\Gamma)$, $N$ is an $\mathrm{Ad}\,\lambda_g$-invariant von Neumann subalgebra ($\forall g \in \Gamma$) and, of course, $\alpha = \mathrm{Ad}\,\lambda$.

The first case allows us to give a family of properly outer actions of groups $\Gamma$ with the Haagerup property on the hyperfinite type $\mathrm{II}_1$ factor $R$ such that the crossed products $R \rtimes \Gamma$ have the Haagerup property. We realize $R$ as the von Neumann algebra obtained by the GNS construction of the CAR $C^*$-algebra with respect to its unique normalized trace $\tau$. Let us recall the definition: given any separable Hilbert space $\mathcal{H}$ (whose scalar product is assumed to be linear in the second variable), there exists a unique unital, simple $C^*$-algebra $\mathrm{CAR}(\mathcal{H})$ with an isometric linear map $a : \mathcal{H} \to \mathrm{CAR}(\mathcal{H})$ such that

$$a^*(\xi)a(\eta) + a(\eta)a^*(\xi) = \langle \xi, \eta \rangle 1 \quad \text{and} \quad a(\xi)a(\eta) + a(\eta)a(\xi) = 0$$

for all $\xi, \eta \in \mathcal{H}$, and such that $a(\mathcal{H})$ generates $\mathrm{CAR}(\mathcal{H})$ as a $C^*$-algebra. It admits a unique trace $\tau$ which is characterized by

$$\tau(a^*(\xi_m) \cdots a^*(\xi_1)a(\eta_1) \cdots a(\eta_n)) = 2^{-n}\delta_{m,n} \det(\langle \xi_j, \eta_k \rangle_{j,k})$$

for all $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n \in \mathcal{H}$. Denote by $\pi_\tau$ the representation of the corresponding GNS construction. If $\mathcal{H}$ is infinite-dimensional, then the von Neumann algebra $\pi_\tau(\mathrm{CAR}(\mathcal{H}))''$ is the hyperfinite type $\mathrm{II}_1$ factor. Furthermore, every unitary operator $u$ on $\mathcal{H}$ defines a $\tau$-preserving automorphism $\mathrm{Bog}(u)$ on $\mathrm{CAR}(\mathcal{H})$ (and hence on $R$) characterized by

$$\mathrm{Bog}(u)(a(\xi)) = a(u\xi)$$

for all $\xi \in \mathcal{H}$. Such an automorphism is called a *Bogoliubov automorphism*. Recall also that $\mathrm{Bog}(u)$ is inner in $\mathrm{Aut}(R)$ if and only if $u-1$ is a Hilbert-Schmidt operator on $\mathcal{H}$.

Let now $\Gamma$ be a countable group with the Haagerup property that acts (on the left) on some infinite countable set $X$ with the following two properties:

(H1) each orbit $\Gamma x$ is finite;
(H2) for every $g \in \Gamma$, $g \neq 1$, the set $\{x \in X : gx \neq x\}$ is infinite.

Denote by $\pi_X$ the representation of $\Gamma$ on $\ell^2(X)$ given by

$$(\pi_X(g)\xi)(x) = \xi(g^{-1}x),$$

and by $\alpha^X$ the action on $R$ given by

$$\alpha_g^X = \mathrm{Bog}(\pi_X(g))$$

for every $g$. Then condition (H1) implies that the triple $(R, \Gamma, \alpha^X)$ satisfies condition (ii) in Proposition 3.3. Moreover, condition (H2) implies that $\alpha_g^X$ is an outer automorphism of $R$ for every $g \neq 1$, thus $\alpha^X$ is a properly outer action. Observe that such a set $X$ always exists if $\Gamma$ is residually finite: indeed, in this case, $\Gamma$ contains a decreasing sequence $(\Gamma_n)$ of normal subgroups of finite index with trivial intersection. One can take $X = \coprod_n \Gamma/\Gamma_n$.

To end this section, let us recall Definition 3.1 of [6] on the Haagerup property for an inclusion of finite von Neumann algebras: Consider a pair $1 \in N \subset M$ of finite von Neumann algebras (with separable preduals, say) with a fixed finite trace $\tau$ on $M$. Denote by $E_N$ the unique $\tau$-preserving conditional expectation from $M$ onto $N$ and by $e_N$ the associated projection from $L^2(M)$ onto $L^2(N)$. Let $\mathcal{F}_N(M)$ be the set of operators $T \in N' \cap B(L^2(M))$ which are finite sums of the form $T = \sum_{i \in F} a_i e_N b_i$ where $F$ is a finite set and $a_i, b_i \in M$ for all $i$. Denote also by $\mathcal{K}_N(M)$ the norm closure of $\mathcal{F}_N(M)$ in $B(L^2(M))$. Then we say that the *inclusion* $N \subset M$ has the *Haagerup property* if there exists a sequence $(\Phi_n)_{n \geqslant 1}$ of $E_N$-preserving, $N$-bimodules, unital, normal, completely positive maps from $M$ to itself such that:

(1) $\lim_{n \to \infty} \|\Phi_n(x) - x\|_2 = 0$ for every $x \in M$;

(2) $T_{\Phi_n} \in \mathcal{K}_N(M)$ for all $n$.

Then we have (compare with Corollary 3.20 of [6]):

PROPOSITION 3.5. *Let $N$ be a finite von Neumann algebra with a finite trace $\tau$ and let $\alpha$ be a $\tau$-preserving action of a countable group $\Gamma$ on $N$. Then the inclusion $N \subset N \rtimes_\alpha \Gamma$ has the Haagerup property if and only if $\Gamma$ does.*

*Proof.* Suppose that the inclusion $N \subset N \rtimes_\alpha \Gamma$ has the Haagerup property, and let $(\Phi_n)_{n \geqslant 1}$ be as above. Define $\varphi_n : \Gamma \to \mathbb{C}$ by

$$\varphi_n(g) = \tau(\Phi_n(\lambda(g))\lambda(g^{-1})).$$

It is straightforward to check that $\varphi_n$ is a normalized, positive definite function on $\Gamma$. Moreover, for each fixed $g \in \Gamma$, one has

$$|\varphi_n(g) - 1| = |\tau(\{\Phi_n(\lambda(g)) - \lambda(g)\}\lambda(g^{-1}))| \underset{n \to \infty}{\longrightarrow} 0.$$

It remains to see that for every $n \geqslant 1$, $\varphi_n \in C_0(\Gamma)$. Fix $n$ and $\varepsilon > 0$; there exist $a_1, \ldots, a_m, b_1, \ldots, b_m \in N \rtimes_\alpha \Gamma$ such that

$$\left\| T_{\Phi_n} - \sum_{i=1}^m a_i e_N b_i \right\| \leqslant \frac{\varepsilon}{2}.$$

In particular,

$$\sup_{g \in \Gamma} \left\| \Phi_n(\lambda(g)) - \sum_{i=1}^m a_i E_N(b_i \lambda(g)) \right\|_2 \leqslant \frac{\varepsilon}{2}.$$

But for fixed $a, b \in N \rtimes_\alpha \Gamma$, the function $g \mapsto \|aE_N(b\lambda(g))\|_2$ belongs to $C_0(\Gamma)$, hence there exists a finite subset $F$ of $\Gamma$ such that

$$\Big\| \sum_{i=1}^{m} a_i E_N(b_i\lambda(g)) \Big\|_2 \leqslant \frac{\varepsilon}{2}$$

for every $g \notin F$. This implies that $|\varphi_n(g)| \leqslant \varepsilon$ if $g \notin F$.

Assume that $\Gamma$ has the Haagerup property, and let $(\varphi_n)_{n \geqslant 1} \subset C_0(\Gamma)$ be a sequence of normalized, positive definite functions that tends to 1 pointwise on $\Gamma$. By Lemma 4.10 of [19], for instance, every $\varphi_n$ extends to a normal, completely positive map $\Phi_n$ on $N \rtimes_\alpha \Gamma$ satisfying

$$\Phi_n(axb) = a\Phi_n(x)b \quad \text{and} \quad \Phi_n(\lambda(g)) = \varphi_n(g)\lambda(g)$$

for all $a, b \in N$, $x \in N \rtimes_\alpha \Gamma$ and $g \in \Gamma$. Moreover, it is easy to see that $\Phi_n$ is $E_N$-preserving and, as $\varphi_n(g) \to 1$ as $n \to \infty$ for every $g$, we see that $\|\Phi_n(x) - x\|_2 \to 0$ as $n \to \infty$ for every $x$. Finally, we show that $T_{\Phi_n} \in \mathcal{K}_N(M)$: let $1 \in F_1 \subset F_2 \subset \cdots \subset \Gamma$ be an increasing sequence of finite sets whose union equals $\Gamma$, and set for every $m \geqslant 1$

$$T_m = \sum_{g \in F_m} \varphi_n(g)\lambda(g)e_N\lambda(g^{-1}).$$

Since the projections $\lambda(g)e_N\lambda(g^{-1})$ commute with $N$ and since $T_{\Phi_n} = 1 \otimes m_{\varphi_n}$, we have

$$\|T_{\Phi_n} - T_m\| = \|m_{\varphi_n} - m_{\varphi_n \chi_{F_m}}\| = \sup_{g \notin F_m} |\varphi_n(g)|,$$

where $\chi_{F_m}$ denotes the characteristic function of $F_m$, which shows that

$$\|T_{\Phi_n} - T_m\| \to 0$$

as $m \to \infty$ for $\varphi_n \in C_0(\Gamma)$. ∎

Consider two inclusions $1 \in N_j \subset M_j$, $j = 1, 2$, of finite von Neumann algebras, and suppose that they are isomorphic: there exists a $*$-isomorphism $\theta : M_1 \to M_2$ such that $\theta(N_1) = N_2$. Obviously, if one inclusion has the Haagerup property then so does the other one. As an application of that fact, consider two countable groups $\Gamma_1$ and $\Gamma_2$ that are *Orbit Equivalent*: for $j = 1, 2$ there exists an essentially free measure-preserving action of $\Gamma_j$ on some probability space $(S_j, \mu_j)$ and a measurable bijection (modulo null sets) $\theta : S_1 \to S_2$ that is measure-class preserving and such that for almost all $s \in S_1$, $\theta(s\Gamma_1) = \theta(s)\Gamma_2$. Put $M_j = L^\infty(S_j) \rtimes \Gamma_j$ for $j = 1, 2$. It is well-known that the inclusions $L^\infty(S_1) \subset M_1$ and $L^\infty(S_2) \subset M_2$ are isomorphic (see for instance [10], p. 226). Thus we get from Proposition 3.4:

COROLLARY 3.6. *If $\Gamma_1$ and $\Gamma_2$ are Orbit Equivalent and if one of them has the Haagerup property, then so does the other one.*

## 4. APPENDIX

Let $1 \in N \subset M$ be a pair of arbitrary (i.e. not necessarily $\sigma$-finite) von Neumann algebras and let $E : M \to N$ be a conditional expectation. Recall from [25], [22] or [4] that $E$ has *finite index* if there exists a number $c > 0$ such that

$$E(x^*x) \geqslant c\, x^*x$$

for every $x \in M$.

The aim of this appendix is to give a proof of the following useful technical result of S. Popa which appeared in [26] :

PROPOSITION 4.1. *Every conditional expectation $E : M \to N$ with finite index is normal.*

Let $M$ and $N$ be as above. We need to recall a few facts about the decomposition of every element $\varphi$ of the dual $M^*$ of $M$ into its normal and singular parts (see [29], Chapter III, Theorem 2.14): Consider $M$ as a $*$-subalgebra of its bidual $M^{**}$; there exists a unique central projection $z \in M^{**}$ such that the predual $M_*$ of $M$ is

$$M_* = M^*z = \{\varphi z : \varphi \in M^*\}\,,$$

and thus every $\varphi \in M^*$ has a unique decomposition

$$\varphi = \varphi_n + \varphi_s$$

with $\varphi_n = \varphi z|M \in M_*$ is the *normal part* of $\varphi$ and $\varphi_s = \varphi(1-z)|M$ is its *singular part*. Moreover, $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$. It is also clear that the map $\varphi \mapsto \varphi_n$ from $M^*$ to $M_*$ is linear and that

$$(a\varphi b)_n = a\varphi_n b$$

for all $\varphi \in M^*$ and $a, b \in M$. (Recall that $(a\varphi b)(x) = \varphi(bxa)$ for all $a, b, x \in M$.)

The proof of Proposition 4.1 requires two lemmas, the first of which giving a more useful description of $\varphi_n$ for positive $\varphi$; it is inspired by the classical (i.e. commutative) case of finite measures.

For $x \in M_+$, denote by $\mathcal{D}(x)$ the set of families $(x_i)_{i\in I} \subset M_+$ such that $\sum_i x_i = x$, the series converging $\sigma$-weakly.

LEMMA 4.2. *For every $\varphi \in M_+^*$ and every $x \in M_+$ we have*

$$\varphi_n(x) = \inf\left\{\sum_{i\in I} \varphi(x_i) : (x_i)_{i\in I} \in \mathcal{D}(x)\right\}.$$

*Proof.* For simplicity, for $\varphi \in M_+^*$ and $x \in M_+$, set

$$\nu(x) = \inf\left\{\sum_{i\in I} \varphi(x_i) : (x_i) \in \mathcal{D}(x)\right\}.$$

The following two properties follow readily from the definition of $\nu$:

(1) $\nu(\lambda x) = \lambda\nu(x)$ for $\lambda \in \mathbb{R}_+$ and $x \in M_+$ ;
(2) $\nu(x+y) \leqslant \nu(x) + \nu(y)$ for $x, y \in M_+$.

Moreover, we have

$$\varphi_n(x) \leqslant \nu(x) \leqslant \varphi(x)$$

because $\varphi_n \leqslant \varphi$, and since $\nu(x)$ is an infimum. In particular, $\nu(x) \leqslant \|\varphi\| \, \|x\|$ for every $x \in M_+$. We claim that

$$\nu(p) = \varphi_n(p)$$

for every projection $p \in M$: indeed, as $\varphi_n \leqslant \nu$, we only need to check that $\nu(p) \leqslant \varphi_n(p)$. But it follows from Theorem III.3.8 of [29] that for every projection $p$ in $M$, there exists a family of pairwise orthogonal projections $(p_i)_{i\in I} \subset M$ with sum $p$ and such that $\varphi_s(p_i) = 0$ for every $i \in I$. Hence $\varphi(p_i) = \varphi_n(p_i)$ for every $i$, and, as $\varphi_n$ is normal,

$$\nu(p) \leqslant \sum_{i\in I} \varphi(p_i) = \sum_{i\in I} \varphi_n(p_i) = \varphi_n(p).$$

Using moreover (1) and (2), we get for all $\lambda_1, \ldots, \lambda_n \geqslant 0$ and all pairwise orthogonal projections $p_1, \ldots, p_n \in M$:

$$\varphi_n\Big(\sum_j \lambda_j p_j\Big) \leqslant \nu\Big(\sum_j \lambda_j p_j\Big) \leqslant \sum_j \lambda_j \nu(p_j) = \sum_j \lambda_j \varphi_n(p_j) = \varphi_n\Big(\sum_j \lambda_j p_j\Big).$$

If $x \in M_+$ is arbitrary and if $\varepsilon > 0$, there exist non negative numbers $\lambda_1, \ldots, \lambda_n$ and pairwise orthogonal projections $p_1, \ldots, p_n \in M$ such that

$$\sum_j \lambda_j p_j \leqslant x \quad \text{and} \quad \Big\| x - \sum_j \lambda_j p_j \Big\| \leqslant \varepsilon.$$

Then

$$\begin{aligned} \nu(x) &\leqslant \nu\Big(x - \sum_j \lambda_j p_j\Big) + \nu\Big(\sum_j \lambda_j p_j\Big) \\ &\leqslant \|\varphi\| \cdot \varepsilon + \Big|\varphi_n\Big(\sum_j \lambda_j p_j - x\Big)\Big| + \varphi_n(x) \leqslant 2\|\varphi\| \cdot \varepsilon + \varphi_n(x), \end{aligned}$$

which shows that $\nu(x) \leqslant \varphi_n(x)$. ∎

REMARK 4.3. Observe that if $M$ is $\sigma$-finite, then

$$\varphi_n(x) = \inf\Big\{ \sum_{k\in\mathbb{N}} \varphi(x_k) : (x_k)_{k\in\mathbb{N}} \subset M_+,\ \sum_k x_k = x \Big\}.$$

From now on we fix a conditional expectation $E : M \to N$. We define its *normal part* $E_n$ as follows: For $x \in M$, $E_n(x)$ is the element of $N$ characterized by

$$\varphi(E_n(x)) = (\varphi \circ E)_n(x)$$

for every $\varphi \in N_*$.

LEMMA 4.4. *The map $E_n : M \to N$ has the following properties:*

(i) $E_n(x+y) = E_n(x) + E_n(y)$ *for all* $x, y \in M$.

(ii) $E_n$ *is a bi-$N$-module map, i.e.* $E_n(axb) = aE_n(x)b$ *for all* $x \in M$ *and* $a, b \in N$.

(iii) $E_n$ *is positive, i.e.* $E_n(x) \geqslant 0$ *for every* $x \in M_+$.

(iv) $E_n$ *is a normal map, i.e.* $\varphi \circ E_n \in M_*$ *for every* $\varphi \in N_*$.

(v) *If there exists* $c > 0$ *such that* $E(x) \geqslant c\,x$ *for every* $x \in M_+$, *then* $E_n(x) \geqslant c\,x$ *for every* $x \in M_+$.

*Proof.* Properties (i), (ii) and (iii) are straightforward consequences of the reminded properties of the map $\varphi \mapsto \varphi_n$.

(iv) If $(x_i)_{i\in I}$ is an increasing net in $M_+$ which converges $\sigma$-weakly to $x \in M_+$, we have for positive $\varphi \in N_*$:

$$\varphi(E_n(x)) = (\varphi \circ E)_n(x) = \sup_i (\varphi \circ E)_n(x_i) = \sup_i \varphi(E_n(x_i)),$$

which shows that $E_n$ is normal.

(v) We assume that $N \subset M \subset B(H)$ for some Hilbert space $H$. Fix $x \in M_+$ and $\xi \in H$. For any element $(x_i) \in \mathcal{D}(x)$, we have

$$\langle c\,x\xi, \xi\rangle = \sum_i \langle c\,x_i\xi, \xi\rangle \leqslant \sum_i \langle E(x_i)\xi, \xi\rangle,$$

hence, by Lemma 4.2,

$$\langle c\,x\xi, \xi\rangle \leqslant \langle E_n(x)\xi, \xi\rangle.$$

This ends the proof of the lemma. ∎

Now, if $E$ is of finite index, its normal part $E_n$ is a finite, normal operator-valued weight with finite index from $M$ to $N$. As in [4], we associate to $E_n$ the Hilbert $N$-module $X = (M, \langle\,\cdot\,,\,\cdot\,\rangle_{E_n})$, where $X = M$ as a right $N$-module with $N$-valued inner product $\langle x, y\rangle_{E_n} = E_n(x^*y)$. Since $E_n$ is of finite index, $X$ is selfdual by Proposition 3.3 of [4], and as $E : M \to N$ is bounded and $N$-linear, there exists $h \in M$ such that

$$E(x) = E_n(hx)$$

for every $x \in M$. But this implies that $E$ is normal, and this ends the proof of Proposition 4.1.

In order to see the usefulness of that result, we state a consequence that is inspired by a private communication of S. Popa:

PROPOSITION 4.5. *Let* $1 \in N \subset M$ *be von Neumann algebras such that there exists a conditional expectation of finite index* $E : M \to N$ *and let* $1 \in A \subset N' \cap M$ *be an injective von Neumann subalgebra. Then there exists a normal, faithful conditional expectation of* $M$ *onto* $A' \cap M$.

*Proof.* By Corollary 1.7 of [18] for instance, the relative commutant $N' \cap M$ is finite, hence $A$ is finite and injective. Thus, by [15], there exists a mean $m$ on the unitary group $U(A)$ such that

$$\int_{U(A)} V(au^*, u)\,\mathrm{d}m(u) = \int_{U(A)} V(u^*, ua)\,\mathrm{d}m(u) \tag{4.1}$$

for every separately $\sigma$-continuous bilinear form $V$ on $A \times A$ and for every $a \in A$. Define $E' : M \to A' \cap M$ by

$$\varphi(E'(x)) = \int_{U(A)} \varphi(uxu^*)\,\mathrm{d}m(u)$$

for all $\varphi \in M_*$ and all $x \in M$. This defines a conditional expectation from $M$ onto $A' \cap M$, and we will check that it is faithful and normal. For fixed $\varphi \in M_*$ and $x \in M$, set $V(a,b) = \varphi(bxa)$ and apply (4.1): one gets $\varphi(E'(xa)) = \varphi(E'(ax))$, hence $E'(uxu^*) = E'(x)$ for all $x \in M$ and all $u \in U(A)$. As $U(A)$ is contained in $N' \cap M$, and as $E$ has finite index, there exists a positive constant $c$ such that

$$E(uxu^*) = u^* E(uxu^*) u \geqslant c\,x$$

for all $x \in M_+$ and all $u \in U(A)$. This implies that

$$E \circ E'(x) \geqslant c\,x$$

for every $x \in M_+$. By Proposition 4.1, $E \circ E'$ has finite index, hence it is faithful and normal. Thus $E'$ is faithful and finally we prove that it is normal: let $(x_i)_{i\in I} \subset M_+$ be a decreasing, bounded generalized sequence which converges $\sigma$-weakly to 0. Set $y = \lim_{i\in I} E'(x_i) = \inf_i E'(x_i) \geqslant 0$. As $E \circ E'$ is normal, we have $0 = \lim E \circ E'(x_i) \geqslant E(y)$, hence $y = 0$ since $E$ is faithful. ∎

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