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OPERATORS NEAR COMPLETELY POLYNOMIALLY DOMINATED ONES AND SIMILARITY PROBLEMS

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Abstract. Let T and C be two Hilbert space operators. We prove that if T is near, in a certain sense, to an operator completely polynomially dominated with a finite bound by C, then T is similar to an operator which is completely polynomially dominated by the direct sum of C and a suitable weighted unilateral shift. Among the applications, a refined Banach space version of Rota similarity theorem is given and partial answers to a problem of K. Davidson and V. Paulsen are obtained. The latter problem concerns CAR-valued Foguel-Hankel operators which are generalizations of the operator considered by G. Pisier in his example of a polynomially bounded operator not similar to a contraction.

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1. INTRODUCTION

1.1. Preamble. A good part of the literature concerning similarity problems for operators on a Hilbert space was motivated by a single problem. This problem asks for a simple criterion to determine whether a Hilbert space operator is similar to a contraction. The corresponding problems for similarity to isometries or unitaries have been solved at the late 1940's by Sz.-Nagy ([23]). The conjectured ([6]) characterization: "an operator is similar to a contraction if and only if it is polynomially bounded" was recently shown to be false by G. Pisier ([16]). Recall that T is said to be *polynomially bounded* if there exists a constant M such that

$$\|p(T)\| \leqslant M \sup\{|p(z)| : |z| = 1\} \tag{1.1}$$

for all polynomials p. We refer to [4] for the history of this counterexample.

A positive answer for the similarity problem was given in [10]. The quantitative criterion of V. Paulsen ([10]) asserts that an operator T is similar to a contraction if and only if T is a *completely polynomially bounded* operator, which means that equation (1.1) holds for all matrix-valued polynomials. Moreover, the similarity constant coincides with the smallest possible constant M in the analogue of (1.1). A more general result for similarity of algebra homomorphisms to completely contractive ones was proved in [11] (cf. also [12]).

Paulsen's criteria are consistent with a variety of similarity results in operator theory. They are also consistent with results in some areas of operator algebras and operator spaces theory, areas where completely positive and completely bounded maps have found to be central tools. Generalizations to Banach space operators and to p-complete bounded homomorphisms are given in [14] (see also [15]).

We introduce in this paper the notion of operators T (completely) polynomially dominated with finite bound by a given operator C. For instance, we will say that T is *polynomially dominated* with finite bound by C if there exists $M > 0$ such that

$$\|p(T)\| \leqslant M\|p(C)\|$$

for all polynomials p. Completely polynomially dominated operators with finite bound generalizes completely polynomially bounded operators.

The main goal of this note is to show that an operator T near, in a certain sense, to a Hilbert space operator completely polynomially dominated with a finite bound by a given operator C is similar to an operator which is completely polynomially dominated by the direct sum of C with a suitable weighted unilateral shift. The nearness condition for Hilbert space operators (called here *β-quadratic nearness*) is defined in Section 2. In particular, the class of operators similar to contractions is stable under quadratic nearness. A precursor of results of this type is [8].

Applications to similarity problems for Hilbert space operators include two partial results concerning an open question of K. Davidson and V. Paulsen ([5]). The question mentioned in [5] asks for a characterization of those square summable sequences for which the corresponding CAR-valued Foguel-Hankel operators are similar to contractions. Note that the counterexamples of Pisier ([16]) are operators of this type. It was this question which was the starting point of this note.

Even if the emphasis here will be on Hilbert space operators, we will also consider Banach space operators in Theorem 4.5. As an application, a refined version of Rota's ([20]) similarity result will be obtained. We will show that, given $p > 1$ and a Banach space operator T on X with spectral radius less than one, T is similar to an operator T_1 on a Banach space which, in some sense, "looks like X" such that T_1 is completely polynomially dominated by the unilateral shift S on $\ell_p(X)$. This is related to a conjecture of V.I. Matsaev concerning contractions on L_p-spaces.

We also consider the (easiest) corresponding problem for operators near ones which are similar to unitaries or isometries. We prove that operators asymptotically near operators similar to unitaries/isometries are themselvs similar to unitaries or isometries. There are polynomially bounded operators which are asymptotically near to a contraction without being similar to a contraction.

1.2. Organization of the paper. After this preamble we recall some notation, definitions and known results. We introduce in the next section the notions of completely polynomially dominated operators and of asymptotically near and quadratically near operators. The main results in the Hilbert space situation are stated in Section 3. This section also contains an example of a polynomially bounded operator which is asymptotically near to a contraction without being similar to a contraction. In Section 4 the proof of Theorem 3.3 is reduced to the proof of Theorem 4.1. A more general version of Corollary 4.4 is stated in the Banach space context (Theorem 4.5). Section 5 contains several applications to operators similar to contractions, including a sufficient condition for the similarity to contractions of some CAR-valued Foguel-Hankel operators (Corollary 5.4.1) and a Banach space Rota theorem (Corollary 5.1.1). The proof of Theorem 4.5 is given in Section 6 while the last section contains proofs of the remaining results.

1.3. Preliminaries. We recall now some definitions and results and introduce some notation. We refer to [15] and [12] for more information.

1.3.1. General notation. By H, K (and X, Y, E), with or without subscripts, we will designate complex Hilbert (respectivelly Banach) spaces. We denote by $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. By operator we always mean a bounded linear operator. The adjoint of a Hilbert space operator T is denoted by T^*.

1.3.2. Similarity. Two Hilbert space operators $T_1, T_2 \in \mathcal{B}(H)$ are called *similar* if there exists an invertible operator $L \in \mathcal{B}(H)$ such that $T_2 = L^{-1}T_1L$.

If $\mathcal{A}$ is a class of bounded linear operators, then the *similarity constant* $C_{\mathrm{sim}}(T_1, \mathcal{A})$ of T_1 with respect to $\mathcal{A}$ is defined by

$$C_{\mathrm{sim}}(T_1, \mathcal{A}) = \inf\{\|L^{-1}\| \cdot \|L\| : L \in \mathcal{B}(H), L^{-1}T_1L \in \mathcal{A}\}.$$

We recall that $T \in \mathcal{B}(H)$ is similar to a contraction if and only if there exists a *Hilbertian*, equivalent norm on H with respect to which T is a contraction.

1.3.3. Completely bounded maps. Let $\mathcal{S} \subset \mathcal{B}(H)$ be a subspace. Let $\varphi : \mathcal{S} \to \mathcal{B}(K)$ be a linear map. Let $M_n(\mathcal{S})$ and $M_n(\mathcal{B}(K))$ be the spaces of matrices with entries respectively in $\mathcal{S}$ and $\mathcal{B}(K)$. Endow them with the norm induced respectively by $\mathcal{B}(\ell^2_n(H))$ and $\mathcal{B}(\ell^2_n(K))$. The map φ is called *completely bounded* if there is a constant M such that

$$\sup_n \|I_{M_n} \otimes \varphi : M_n(\mathcal{S}) \to M_n(\mathcal{B}(K))\| \leqslant M.$$

The completely bounded (cb) norm $\|\varphi\|_{\mathrm{cb}}$ is the smallest constant M for which this holds. We call φ *completely contractive* if $\|\varphi\|_{\mathrm{cb}} \leqslant 1$. The map φ is *completely positive* if $I_{M_n} \otimes \varphi$ is a positive map for each n.

The following (Wittstock-Paulsen-Haagerup) factorization theorem for completely bounded maps holds ([15], Chapter 3, and [12], Chapter 7). If $\mathcal{S} \subset \mathcal{B}(H)$ is a subspace and $\varphi : \mathcal{S} \to \mathcal{B}(K)$ is a linear completely bounded map, then there exist a Hilbert space H_π, a unital C^*-algebraic representation $\pi : \mathcal{B}(H) \to \mathcal{B}(H_\pi)$ and operators $V_2 : K \to H_\pi$, $V_1 : H_\pi \to K$, with $\|V_1\| \, \|V_2\| \leqslant \|\varphi\|_{\mathrm{cb}}$, such that $\varphi(a) = V_1\pi(a)V_2$ for any $a \in \mathcal{S}$.

Let $A(\mathbb{D})$ be the disk algebra. For an operator T, let Φ_T be the functional calculus map $p \to p(T)$ defined on polynomials. Then T is completely polynomially bounded if and only if Φ_T extends to a completely bounded map on $A(\mathbb{D})$, if and only if T is similar to a contraction ([10]).

Let $p \geqslant 1$. Similar notions of p-complete bounded maps are defined in the Banach space context ([15]). If $\mathcal{S} \subset \mathcal{B}(X)$ is a subspace, a linear map $\varphi : \mathcal{S} \to \mathcal{B}(Y)$ is p-completely bounded if

$$\|\varphi\|_{\mathrm{pcb}} := \sup_n \|I_{\mathcal{B}(\ell_p^n)} \otimes \varphi : M_n(\mathcal{S}) \to M_n(\mathcal{B}(Y))\| < +\infty,$$

where $M_n(\mathcal{B}(Y))$ and $M_n(\mathcal{S})$ are now equipped with the norms induced by $\mathcal{B}(\ell_p^n(Y))$ and respectively $\mathcal{B}(\ell_p^n(X))$.

We refer to [14] and [15] for more on this, including a factorization theorem.

1.3.4. BANACH SPACES OF CLASS SQ_p. Let $p \geqslant 1$ be a real number. A Banach space E is said to be a *SQ_p-space* if it is a quotient of a subspace of an L_p-space.

Let X be a Banach space. A Banach space E is said to be a $SQ_p(X)$-space if it is (isometric to) a quotient of a subspace of an ultraproduct of spaces of the form $L_p(\Omega, \mu, X)$. Since ultraproducts of L_p-spaces is an L_p-space, the latter definition is consistent with the former. The case $p = 2$ corresponds to the Hilbertian situation.

$SQ_p(X)$-spaces are characterized by a theorem of Hernandez ([7]). See also [14] for a different proof using p-completely bounded maps. Namely, E is a $SQ_p(X)$-space if and only if

$$\|a\|_{p,E} \leqslant \|a\|_{p,X}$$

for each $n \geqslant 1$ and each matrix $a = [a_{ij}] \in M_n(\mathbb{C})$. Here

$$\big\|[a_{ij}]\big\|_{p,Y} = \sup \Big[\Big(\sum_i \Big\| \sum_j a_{ij} y_j \Big\|^p \Big)^{1/p} \Big],$$

where the supremum runs over all n-tuples $(y_1, \dots, y_n)$ in Y which satisfy $\sum \|y_j\|^p \leqslant 1$.

1.3.5. CAR-VALUED FOGUEL-HANKEL OPERATORS. A polynomially bounded operator which is not completely polynomially bounded was found in 1997 by G. Pisier ([16]). The counterexample was a CAR-valued Foguel-Hankel type operator (sometimes called a *CAR-valued Foias-Williams-Peller type* operator).

To be more specific, let Λ be a function from an infinite dimensional Hilbert space H into $\mathcal{B}(H)$ satisfying the *canonical anticommutation relations*: for all $u, v \in H$,

$$\Lambda(u)\Lambda(v) + \Lambda(v)\Lambda(u) = 0$$

and

$$\Lambda(u)\Lambda(v)^* + \Lambda(v)^*\Lambda(u) = (u, v)I.$$

The range of Λ is isometric to Hilbert space. Let $\{e_n\}_{n\geqslant 0}$ be an orthonormal basis for H, and let $C_n = \Lambda(e_n)$ for $n \geqslant 0$. For an arbitrary sequence $\alpha = (\alpha_0, \alpha_1, \dots)$ in ℓ^2, let $Y_\alpha = [\alpha_{i+j} C_{i+j}]$ be a CAR-valued Hankel operator and

$R(Y_\alpha) = \begin{bmatrix} S^{*(\infty)} & Y_\alpha \\ 0 & S^{(\infty)} \end{bmatrix}$ be the corresponding Foguel-Hankel operator ([16], [5]). Here $S^{(\infty)}$ is the unilateral forward shift of multiplicity $\dim H$. The particular choice of α made by Pisier was $\alpha_{2^k-1} = 1$ for $k \geqslant 0$ and $\alpha_i = 0$ otherwise. In this case $R(Y_\alpha)$ is polynomially bounded but not completely polynomially bounded. The following more general result holds ([16], [5]):

1.4. Theorem. (Pisier, Davidson-Paulsen) *Let $\alpha = (\alpha_0, \alpha_1, \ldots)$ be a sequence in ℓ^2 and set*

$$A = \sup_{k \geqslant 0} (k+1)^2 \sum_{i \geqslant k} |\alpha_i|^2 \quad \textit{and} \quad B_2 = \sum_{k \geqslant 0} (k+1)^2 |\alpha_k|^2.$$

The operator $R(Y_\alpha)$ is polynomially bounded if and only if A is finite. If $R(Y_\alpha)$ is similar to a contraction, then B_2 is finite.

It is an open problem if B_2 finite implies $R(Y_\alpha)$ similar to a contraction. A partial answer will be proved in Corollary 5.4.1.

2. Dominance and nearness

2.1. Dominance. We start with several definitions.

2.1.1. Completely polynomially dominated operators. Let T_1 and T_2 be two Hilbert space operators, not necessarily acting on the same space. We say that T_1 is *completely polynomially dominated* by T_2 if

$$\| [p_{ij}(T_1)]_{1 \leqslant i,j \leqslant n} \| \leqslant \| [p_{ij}(T_2)]_{1 \leqslant i,j \leqslant n} \|,$$

for all positive integers n and all $n \times n$ matrices $[p_{ij}]_{1 \leqslant i,j \leqslant n}$ with polynomial entries. Recall that $[p_{ij}(T)]_{1 \leqslant i,j \leqslant n}$ is identified with an operator acting on the direct sum of n copies of the corresponding Hilbert space in a natural way. Let CDOM(T) be the class of all Hilbert space operators completely polynomially dominated by T. Let $M > 0$ be a positive constant. We say that T_1 is *completely polynomially dominated with bound M* by T_2 if

$$\| [p_{ij}(T_1)]_{1 \leqslant i,j \leqslant n} \| \leqslant M \| [p_{ij}(T_2)]_{1 \leqslant i,j \leqslant n} \|,$$

for all positive integers n and all $n \times n$ matrices $[p_{ij}]_{1 \leqslant i,j \leqslant n}$ with polynomial entries. We say that T_1 is *completely polynomially dominated with finite bound* by T_2 if it is completely polynomially dominated with bound M for a suitable M. The least bound of complete dominance of T_1 by T_2 is denoted by $M_{\mathrm{cd}}(T_1, T_2)$. It is the cb norm of the complete bounded map $p(T_2) \to p(T_1)$, $p \in \mathbb{C}[z]$.

Similar notions can be defined in the Banach space context. For instance, we say that $T_1 \in \mathcal{B}(X_1)$ is p-completely dominated with finite bound by $T_2 \in \mathcal{B}(X_2)$ if the map $p(T_2) \to p(T_1)$, $p \in \mathbb{C}[z]$, is p-completely bounded.

2.1.2. Example. The following example gives a (generic) class of completely dominated operators. Recall the following useful result ([21]). Let H be a closed subspace of K and let $T = P_H R|H$, $T \in \mathcal{B}(H)$, be the compression of $R \in \mathcal{B}(K)$ to H. Here P_H is the projection onto H. Then R is a dilation of T (that is, $T^n = P_H R^n|H$ for all n) if and only if the subspace H is semi-invariant for R, that is $H = H_1 \ominus H_2$ for two invariant subspaces H_1 and H_2 of R.

Let $T_2 \in \mathcal{B}(H_2)$ be a Hilbert space operator and let $\pi : \mathcal{B}(H_2) \to \mathcal{B}(H_\pi)$ be a unital C^*-representation. Let H_1 be a semi-invariant subspace for $\pi(T_2)$. Let $T_1 \in \mathcal{B}(H_1)$ be the compression of $\pi(T_2)$ on H_1. Then T_1 is completely polynomially dominated by T_2 since π is completely contractive.

The following theorem identifies Hilbert space completely polynomially dominated operators with finite bound.

2.1.3. Theorem. *A Hilbert space operator T_1 is completely polynomially dominated by T_2 if and only if T_1 is unitarily equivalent to the compression of an operator R_2 to a semi-invariant subspace, R_2 being the image of T_2 by a unital C^*-representation. A Hilbert space operator T_1 is completely polynomially dominated by T_2 with finite bound if and only if T_1 is similar to an operator completely polynomially dominated by T_2 and the similarity constant is the least possible bound of dominance.*

Proof. Suppose that $T_1 \in \mathcal{B}(H_1)$ is completely polynomially dominated by T_2. Let φ be the linear map defined on the subspace of the polynomials of $T_2 \in \mathcal{B}(H_2)$ by

$$\varphi(p(T_2)) = p(T_1).$$

The relation of completely polynomially dominance shows that φ is well-defined, unital and completely contractive. Then by Arveson's theorem ([12], Corollary 6.6) φ has an extension $\widetilde{\varphi} : \mathcal{B}(H_2) \to \mathcal{B}(H_1)$ which is a unital completely positive map. By Stinespring's theorem ([12], Theorem 4.1) there are a Hilbert space K_1, an isometry $V : H_1 \to K_1$ and a unital C^*-representation $\pi : \mathcal{B}(H_1) \to \mathcal{B}(K_1)$ such that

$$\widetilde{\varphi} = V^* \pi V.$$

Denote $R_2 = \pi(T_2)$. We obtain

$$T_1^n = \widetilde{\varphi}(T_2^n) = V^* R_2^n V$$

for each $n \geqslant 0$ and so ([21]) T_1 is unitarily equivalent to the compression of R_2 to a semi-invariant subspace.

If T_1 is completely polynomially dominated by T_2 with finite bound, then φ is completely bounded and, by Paulsen similarity theorem ([12], Theorem 8.1) φ is similar to a completely contractive map with the similarity constant given by the complete bounded norm of φ. ∎

Using Paulsen's criterion, $T \in \mathcal{B}(H)$ is completely polynomially bounded (i.e. similar to a contraction) whenever T is completely polynomially dominated with finite bound by a given contraction.

2.2. Nearness. We introduce the following definitions of nearness which will be used in the statement of the main results.

2.2.1. DEFINITION. Two operators T_1 and T_2 acting on the same space are said to be *asymptotically near* if

$$\lim_{n\to\infty} \|T_1^n - T_2^n\| = 0.$$

2.2.2. DEFINITION. Let $\beta : \mathbb{Z}_+ \to \mathbb{R}_+^*$. Two operators T_1 and T_2 are said to be β-*quadratically near* if

$$s := \left[\sup_{N\geqslant 0} \left\| \sum_{n=0}^{N} \frac{1}{\beta(n)^2}(T_1^n - T_2^n)(T_1^n - T_2^n)^* \right\|\right]^{1/2} < +\infty.$$

The two operators are simply called *quadratically near* if this condition holds with $\beta(n) = 1$ for each n.

We denote s in the above definition by $\mathrm{near}_2(T_1, T_2, \beta)$. If $\beta(n) = 1$ for each n, we call s the *nearness* (or the 2-nearness) between T_1 and T_2.

The above definition of β-quadratic nearness uses the row Hilbert space operator structure ([17]). The following result gives an equivalent definition.

2.2.3. LEMMA. *Let $\beta : \mathbb{Z}_+ \to \mathbb{R}_+^*$. T_1 and T_2 are β-quadratically near with* $\mathrm{near}_2(T_1, T_2, \beta) \leqslant s$ *if and only if*

$$\sum_{n=0}^{+\infty} \frac{1}{\beta(n)^2} \|(T_1^n - T_2^n)^* y\|^2 \leqslant s^2 \|y\|^2, \quad y \in H. \tag{2.1}$$

If

$$\sum_{n=0}^{+\infty} \frac{1}{\beta(n)^2} \|T_1^n - T_2^n\|^2 = u^2 < +\infty, \tag{2.2}$$

then T_1 and T_2 are β-quadratically near with $\mathrm{near}_2(T_1, T_2, \beta) \leqslant u$.

Proof. For $N \geqslant 0$ set

$$A_N = \sum_{n=0}^{N} \frac{1}{\beta(n)^2}(T_1^n - T_2^n)(T_1^n - T_2^n)^*.$$

Then T_1 and T_2 are β-quadratically near with $\mathrm{near}_2(T_1, T_2, \beta) \leqslant s$ if and only if $\sup_N \|A_N\| \leqslant s^2$. On the other hand, inequality (2.1) holds if and only if $\sup_N \omega(A_N) \leqslant s^2$, where

$$\omega(A) = \sup\{|\langle Ax|x\rangle| : x \in H, \|x\| = 1\}$$

is the *numerical radius* of A. The stated equivalence follows now from the known fact that $\omega(A) = \|A\|$ for normal operators A.

The second part follows from the fact that (2.2) implies (2.1). ∎

3. MAIN RESULTS: THE HILBERT SPACE CASE

The classes of operators similar to isometries or unitaries are stable under a common nearness condition.

3.1. Proposition. *A Hilbert space operator asymptotically near an operator similar to an isometry (or a unitary) is similar to an isometry (respectively a unitary).*

The following example, build upon work by Pisier and Davidson and Paulsen, shows that there is a polynomially bounded operator which is asymptotically near to a contraction without being similar to a contraction.

3.2. Example. We use the notation recalled in Introduction. Let (α_k) be the sequence in ℓ^2 given by

$$\alpha_k = (k+1)^{-3/2}(\log(k+1))^{-1/2}, \quad k \geqslant 0.$$

Then $\sum_{k\geqslant 0}(k+1)^2|\alpha_k|^2$ diverges and thus $R(Y_\alpha)$ is not similar to a contraction (cf. Theorem 1.4).

On the other hand, for $k > 1$, we have

$$\sum_{i\geqslant k}|\alpha_i|^2 \leqslant \int_k^\infty \frac{1}{t^3\log t}\,\mathrm{d}t \leqslant \frac{1}{\log(k)}\int_k^\infty \frac{1}{t^3}\,\mathrm{d}t \leqslant \frac{1}{2\log(k)}\frac{1}{(k+1)^2}.$$

Therefore $\lim_{k\to\infty}(k+1)^2\sum_{i\geqslant k}|\alpha_i|^2 = 0$ which, using results from [5], implies that $\lim_{k\to\infty}\|R(Y_\alpha)^k - R(0)^k\| = 0$. Thus $R(Y_\alpha)$ is asymptotically near the contraction $R(0) = S^{*(\infty)}\oplus S^{(\infty)}$, without being similar to a contraction. Note also that $R(Y_\alpha)$ is polynomially bounded since quantity A is finite for this (α_k). ∎

The right condition of nearness for the class of operators similar to contractions follows from the following theorem.

Let $\beta : \mathbb{Z}_+ \to \mathbb{R}_+^*$. We denote by $S_{w(\beta)}$ the forward weighted shift on ℓ_2, $S_w e_n = w_n e_{n+1}$, with weights

$$w(\beta)_n = w_n = \frac{\beta(n+1)}{\beta(n)}, \quad n \geqslant 0.$$

Then $S = S_{w(1)}$ is the unilateral forward shift on ℓ_2 obtained for $\beta(n) = 1$, $n \geqslant 0$.

3.3. Theorem. *Let $T, R \in \mathcal{B}(H)$ and $C \in \mathcal{B}(H_{\mathrm{c}})$. Suppose that R is completely polynomially dominated with finite bound by C. Let $M = M_{\mathrm{cd}}(R, C)$ be the least possible bound for this dominance. Let $\beta : \mathbb{Z}_+ \to \mathbb{R}_+^*$ and suppose that T is β-quadratically near R. Let $s = \mathrm{near}_2(T, R, \beta)$. Then T is similar to an operator completelly polynomially dominated by $C \oplus S_{w(\beta)}$. Moreover, the similarity constant satisfies*

$$C_{\mathrm{sim}}(T, \mathrm{CDOM}(C \oplus S_{w(\beta)})) \leqslant M + \beta(0)s.$$

If $\beta(n) = 1$ for each n we obtain the following consequence.

3.4. Corollary. *Let $T, R \in \mathcal{B}(H)$ and $C \in \mathcal{B}(H_{\rm c})$. Suppose that T is quadratically near R and that R is completelly polynomially dominated with finite bound by C. Then T is similar to the compression of $\pi(C \oplus S)$ to a semi-invariant subspace, where π is a unital C^*-representation defined on $\mathcal{B}(H_{\rm c} \oplus \ell_2)$.*

For similarity to contractions we have

3.5. Corollary. *Suppose $R \in \mathcal{B}(H)$ is similar to a contraction. Let $T \in \mathcal{B}(H)$ and suppose that there exists $C > 0$ such that*

$$\sum_{n \geqslant 0} \|(T^n - R^n)x\|^2 \leqslant C\|x\|^2$$

for each $x \in H$. Then T is similar to a contraction.

Indeed, according to Lemma 2.2.3, T^* is quadratically near R^*. Note also that T is similar to a contraction if and only if T^* is.

3.6. Remark. Operators having their spectrum in the open unit disk are quadratically near 0 (the null operator). Therefore operators with spectral radius smaller than 1 are similar to contractions (Rota's theorem, [20]). The relation of quadratic nearness is an equivalence relation. It is easy to see that the equivalence class of the null operator is the class of all operators having their spectrum in the open unit disk.

4. A reduction of Theorem 3.3 and a Banach space extension

The main result Theorem 3.3 is a consequence of the following result. It is a generalization of a result of Holbrook ([8]).

4.1. Theorem. *Let $T \in \mathcal{B}(H)$ and suppose that there exist Hilbert space K, operators $V_2 : H \to K$, $V_1 : K \to H$, $C_1 \in \mathcal{B}(K)$, and a function $\beta : \mathbb{Z}_+ \to \mathbb{R}_+^*$ such that*

$$\sup_{N \geqslant 0} \Big\| \sum_{n=0}^{N} \frac{1}{\beta(n)^2} (T^n - V_1 C_1^n V_2)(T^n - V_1 C_1^n V_2)^* \Big\| = s^2 < +\infty. \tag{4.1}$$

Then T is similar to an operator completely polynomially dominated by $C_1 \oplus S_{w(\beta)} \in \mathcal{B}(K \oplus \ell_2)$. Moreover, the similarity constant satisfies

$$C_{\rm sim}(T, {\rm CDOM}(C_1 \oplus S_{w(\beta)})) \leqslant \|V_1\| \, \|V_2\| + \beta(0)s.$$

4.2. Remarks. (i) If $s = 0$ in the above theorem, then S_w can be omitted in the direct sum.

(ii) For an *arbitrary* T and any *finite* N, there are operators V_1, V_2 and C_1 like in Theorem 4.1 such that $T^n = V_1 C_1^n V_2$ for $n = 0, 1, \ldots, N$ (cf. [6], p. 910).

4.3. Theorem 4.1 implies Theorem 3.3. Suppose that R is completely polynomially dominated with finite bound by $C \in \mathcal{B}(H_{\rm c})$ and let $M = M_{\rm cd}(R, C)$ be the least possible bound for this dominance. Let $\mathcal{S} \subset \mathcal{B}(H_{\rm c})$ be the subspace of all operators $p(C)$, $p \in \mathbb{C}[z]$. Consider the map $\Phi : \mathcal{S} \to \mathcal{B}(H)$ defined by $\Phi(p(C)) = p(R)$. Since R is completely polynomially dominated with finite bound by C, the map Φ is completely bounded with $\Phi(I) = I$. According to the factorization theorem, there is a Hilbert space K, a unital C^*-algebraic representation $\pi : \mathcal{B}(H_c) \to B(K)$ and operators $V_2 : H \to K$, $V_1 : K \to H$ with $\|V_1\|\,\|V_2\| \leqslant M$ such that $\Phi(p(C)) = V_1\pi(p(C))V_2$ for each polynomial p. Set $C_1 = \pi(C)$. We obtain

$$R^n = \Phi(C^n) = V_1\pi(C^n)V_2 = V_1 C_1^n V_2$$

with $\|V_1\|\,\|V_2\| \leqslant M$. Since π is completely contractive, Theorem 4.1 implies Theorem 3.3. ∎

We also obtain the follwing result.

4.4. Corollary. *Let $T \in \mathcal{B}(H)$ and suppose that there exist Hilbert space K, operators $V_2 : H \to K$, $V_1 : K \to H$, $C_1 \in \mathcal{B}(K)$, and a function $\beta : \mathbb{Z}_+ \to \mathbb{R}_+^*$ such that*

$$\sum_{n=0}^{+\infty} \frac{1}{\beta(n)^2}\|T^n - V_1 C_1^n V_2\|^2 = u^2 < +\infty.$$

Then T is similar to an operator completely polynomially dominated by $C_1 \oplus S_{w(\beta)} \in \mathcal{B}(K \oplus \ell_2)$. Moreover, the similarity constant satisfies

$$C_{\mathrm{sim}}(T, \mathrm{CDOM}(C_1 \oplus S_{w(\beta)})) \leqslant \|V_1\|\,\|V_2\| + \beta(0)u.$$

In fact the following Banach space version of Corollary 4.4 holds (for simplicity, we will not deal with estimates of the similarity constant here).

We introduce some notation. Consider the space $\ell_p(\beta, X)$ of elements $z = (z_0, z_1, \ldots)$, $z_k \in X$, endowed with the norm

$$\|z\|_{\ell_p(\beta,X)} = \Big(\sum_k \beta(k)^p \|z_k\|^p\Big)^{1/p}.$$

The shift operator S acts on $\ell_p(\beta, X)$ by

$$S(z_0, z_1, \ldots) = (0, z_0, z_1, \ldots).$$

4.5. Theorem. *Let p and q be real numbers greater than 1 such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $T \in \mathcal{B}(X)$ and suppose that there exist a $SQ_p(X)$-space Y, operators $V_1 : Y \to X$, $V_2 : X \to Y$, and $C_1 \in \mathcal{B}(Y)$, and a function $\beta : \mathbb{Z}_+ \to \mathbb{R}_+^*$ such that*

$$\sum_{n=0}^{+\infty} \frac{1}{\beta(n)^q}\|T^n - V_1 C_1^n V_2\|^q = s^q < +\infty.$$

Then there is a Banach space E which is a $SQ_p(X)$-space and an isomorphism $L : E \to X$ such that, if $T_1 = L^{-1}TL \in \mathcal{B}(E)$, then T_1 is p-completely polynomially dominated by $C_1 \oplus S \in \mathcal{B}(E \oplus \ell_p(\beta, X))$.

4.6. Remark. As was communicated to the author by V. Paulsen, it is possible to prove in a different way Corollary 4.4 using Theorem 2.1.3. We have chosen to present a direct proof of its Banach space version because of the applications of Theorem 4.5 which are of independent interest. A Banach space version of Theorem 3.3 can be given using Theorem 4.5 and the factorization theorem for p-completed bounded maps of Pisier ([14], [15]). We will not develop this idea here.

5. SEVERAL APPLICATIONS

We present now briefly several applications of the main results.

5.1. A BANACH SPACE ROTA THEOREM. It has already been mentioned that Rota's theorem is a consequence of Corollary 3.5. The following application of Theorem 4.5 is a refined Banach space version of Rota theorem.

5.1.1. COROLLARY. *Let X be a Banach space and suppose that $T \in \mathcal{B}(X)$ has a spectral radius smaller than 1. Then, for every $p > 1$, there exist a Banach space E which is a quotient of $\ell_p(X)$ and an isomorphism $L : E \to X$ such that, if $T_1 = L^{-1}TL \in \mathcal{B}(E)$, then*

$$\|p(T_1)\|_{\mathcal{B}(E)} \leqslant \|p(S)\|_{\mathcal{B}(\ell_p(X))} \tag{5.1}$$

for each analytic polynomial p; even more generally,

$$\big\|[p_{ij}(T_1)]\big\|_{\mathcal{B}(\ell_p^n(E))} \leqslant \big\|[p_{ij}(S)]\big\|_{\mathcal{B}(\ell_p^n(X))}$$

for all matrices of polynomials.

Equation (5.1) shows in particular that T_1 is a contraction. It was conjectured in 1966 by V.I. Matsaev (see [13]) that

$$\|p(T_1)\| \leqslant \|p(S)\|_{\mathcal{B}(\ell_p)}$$

holds for all contractions T_1 on an infinite dimensional L_p-space. Several partial results are now known ([13]) but the conjecture is still open. The above theorem shows that if the spectral radius $r(T)$ of $T \in \mathcal{B}(X)$ is smaller than one, then T is similar to an operator on a quotient E of $\ell_p(X)$ completely polynomially dominated by S on $\ell_p(X)$.

If we ask only for a $SQ_p(X)$-space E and not for a quotient of $\ell_p(X)$, the proof of Corollary 5.1.1 follows easily from Theorem 4.5. Indeed, if $r(T) < 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{n\geqslant 0} \|T^n\|^q < +\infty$$

and thus Theorem 4.5 is applicable with $C_1 = 0$. We postpone the proof of Corollary 5.1.1 (with E a quotient of $\ell_p(X)$) to the last Section.

5.2. OPERATORS OF CLASS C_ρ. Let $\rho > 0$. Operators of class C_ρ are defined as operators having ρ-dilations: $T \in \mathcal{B}(H)$ is in C_ρ if there exists a larger Hilbert space $K \supset H$ and a unitary operator U on K such that

$$T^n h = \rho P_H U^n h, \quad h \in H.$$

Thus contractions are operators of class C_1. An operator T is in C_2 if and only if $\omega(T) \leqslant 1$. We refer to [24] for more information on operators of class C_ρ.

A more general class of operators can be constructed as follows ([19]). Let $(\rho_n)_{n\geqslant 1}$ be a sequence of positive numbers. We say that $T \in \mathcal{B}(H)$ is of class $C_{\rho_1,\rho_2,\ldots}$ if there exists a larger Hilbert space $K \supset H$ and a unitary operator U on K such that

$$T^n h = \rho_n P_H U^n h, \quad h \in H, \tag{5.2}$$

for all $n \geqslant 1$. The operator T satisfies (5.2) if and only if the spectrum of T is in the closed unit disc and

$$\mathrm{Re}\Big[I + \sum_{n\geqslant 1} \frac{2\lambda^n}{\rho_n} T^n\Big] \geqslant 0, \quad |\lambda| < 1.$$

5.2.1. COROLLARY. (Rácz) *Let $(\rho_n)_{n\geqslant 1}$ be a sequence of positive numbers. Suppose that there exist $k \geqslant 1$ and $M > 0$ such that*

$$\sum_{n=1}^{\infty} (\rho_{nk} - M)^2 < \infty.$$

Then every operator of class $C_{\rho_1,\rho_2,\dots}$ is similar to a contraction.

For the proof, denote $S = T^k$. Then $S^n = \rho_{nk} V^* U^{nk} V$ with a suitable isometry V and a unitary U. It follows that

$$\|S^n - MV^*U^{nk}V\| \leqslant \|S^n - \rho_{nk}V^*U^{nk}V\| + |\rho_{nk} - M|.$$

Using Theorem 4.1, with $C_1 = U^k$, it follows that $S = T^k$ is similar to a contraction and thus T has the same property (cf. [6]).

If $M = \rho_1 = \rho_2 = \cdots = \rho$, we obtain the following result originally proved by Sz.-Nagy and Foias in 1967.

5.2.2. COROLLARY. (Sz.-Nagy–Foias) *Every operator of class C_ρ is similar to a contraction.*

5.3. COMPLETELY BOUNDED MAPS ON $z^d A(\mathbb{D})$. Let $d \geqslant 1$ be an integer and let $z^d A(\mathbb{D})$ be the non-unital subalgebra of the disc algebra $A(\mathbb{D})$ consisting of all functions $f \in A(\mathbb{D})$ such that $f(0) = f'(0) = \cdots = f^{(d-1)}(0) = 0$.

What happens if the inequality of complete dominance with finite bound holds only for polynomials in $z^d A(\mathbb{D})$? We consider for simplification only Hilbert space operators. We refer to [15], p. 80, and to [9] for related results in the Banach space situation.

5.3.1. COROLLARY. *Let $T \in \mathcal{B}(H)$ and $C \in \mathcal{B}(H_c)$ be two Hilbert space operators such that*

$$\| [p_{ij}(T)]_{1\leqslant i,j\leqslant n} \| \leqslant M \big\| [p_{ij}(C)]_{1\leqslant i,j\leqslant n} \big\|,$$

for all positive integers n and all $n \times n$ matrices of polynomials p_{ij} in $z^d A(\mathbb{D})$. Then T is similar to an operator completely polynomially dominated by $C \oplus S \in \mathcal{B}(H_c \oplus \ell_2)$.

For the proof, note that the map $P(C) \to P(T)$ defined on the subspace

$$\{P(C) : P \in z^d A(\mathbb{D}), P \text{ polynomial}\}$$

is completely bounded. By the factorization theorem ([15], Theorem 3.6) we can write

$$P(T) = V_1 \pi(P(C)) V_2, \quad P \in z^d A(\mathbb{D})$$

with suitable operators V_1, V_2 and a unital C^*-algebraic representation π on $\mathcal{B}(H_c)$. Let $C_1 = \pi(C)$. We obtain

$$T^k = V_1 C_1^k V_2, \quad k \geqslant d.$$

This shows that T is quadratically near C_1. The conclusion follows now from Corollary 4.4.

5.3.2. COROLLARY. (Paulsen criterion for $z^dA(\mathbb{D})$) *Let $d \geqslant 1$. Let $T \in \mathcal{B}(H)$ and suppose that*

$$\| [p_{ij}(T)]_{1\leqslant i,j\leqslant n} \| \leqslant M \sup_{|z|=1} \| [p_{ij}(z)]_{1\leqslant i,j\leqslant n} \|,$$

for all positive integers n and all $n \times n$ matrices of polynomials p_{ij} in $z^dA(\mathbb{D})$. Then T is similar to a contraction.

5.4. CAR-VALUED FOGUEL-HANKEL OPERATORS. We use notation as above.

5.4.1. COROLLARY. *Let $\alpha = (\alpha_0, \alpha_1, \ldots)$ be a sequence in ℓ^2 such that*

$$B_3 := \sum_{k\geqslant 0} (k+1)^3 |\alpha_k|^2 < +\infty.$$

Then $R(Y_\alpha)$ is similar to a contraction.

Proof. Set $R(0) = S^{*(\infty)} \oplus S^{(\infty)}$. Using the notations of [5], we have $\|R(Y_\alpha)^n - R(0)^n\| \leqslant \|\mathcal{Y}_\alpha(z^n)\|$. It was proved in [5] that

$$\|\mathcal{Y}_\alpha(z^n)\| \leqslant (n+1) \left[\sum_{i\geqslant n} |\alpha_i|^2 \right]^{1/2}.$$

We obtain

$$\sum_{n\geqslant 0} \|R(Y_\alpha)^n - R(0)^n\|^2 \leqslant \sum_{n\geqslant 0} (n+1)^2 \left[\sum_{i\geqslant n} |\alpha_i|^2 \right].$$

By a Abel summation method, the series $\sum_{n\geqslant 0} (n+1)^2 \left[\sum_{i\geqslant n} |\alpha_i|^2 \right]$ is convergent if

$$\sum_{n\geqslant 0} \left[\sum_{0\leqslant i\leqslant n} (i+1)^2 \right] |\alpha_n|^2$$

it is. It is indeed convergent because of our assumption on B_3. Therefore $R(Y_\alpha)$ is quadratically near the contraction $R(0)$ and thus similar to a contraction. ∎

We still don't know if B_2 finite implies $R(Y_\alpha)$ similar to a contraction. Nevertheless, the following similarity result holds.

5.4.2. COROLLARY. *Let $\alpha = (\alpha_0, \alpha_1, \ldots)$ be a sequence in ℓ^2 such that*

$$B_2 := \sum_{k\geqslant 0} (k+1)^2 |\alpha_k|^2 < +\infty.$$

Then $R(Y_\alpha)$ is similar to an operator completely polynomially dominated by $R(0) \oplus D$, where $D \in \mathcal{B}(\ell_2)$ is the Dirichlet shift, i.e. the weighted unilateral shift with weights $w_n = \sqrt{(n+2)/(n+1)}$.

Note that $R(0)$ is a contraction while the Dirichlet shift is expansive; it is however a 2-isometry ([1]), that is $I - 2D^*D + D^{*2}D^2 = 0$.

The proof is similar to the proof of the precedent corollary: if $\beta(n) = \sqrt{n+1}$, then

$$\frac{1}{\beta(n)}\|R(Y_\alpha)^n - R(0)^n\| \leqslant \sqrt{n+1}\Big[\sum_{i\geqslant n}|\alpha_i|^2\Big]^{1/2}.$$

This shows that

$$\sum_{n\geqslant 0}\frac{1}{n+1}\|R(Y_\alpha)^n - R(0)^n\|^2 \leqslant \sum_{n\geqslant 0}(n+1)\Big[\sum_{i\geqslant n}|\alpha_i|^2\Big]$$

and the right hand side is convergent if $B_2 < +\infty$. Apply Corollary 4.4 with $\beta(n) = \sqrt{n+1}$ and $C_1 = R(0)$.

5.4.3. Remark. Corollary 5.4.1 was obtained as a particular case of a general theorem. Using other methods, Vern Paulsen and the author improved Corollary 5.4.1 as follows: $R(Y_\alpha)$ is similar to a contraction if there exists $\varepsilon > 0$ such that

$$B_{2+\varepsilon} := \sum_{k\geqslant 0}(k+1)^{2+\varepsilon}|\alpha_k|^2 < +\infty.$$

Details will be given elsewhere ([2]). A different sufficient condition for the similarity to contractions of operator-valued Foguel-Hankel operators was given by G. Blower ([3]).

6. PROOF OF THEOREM 4.5

Put, for simplicity, $C_1 = C$. Let γ be a positive constant. We will chose this constant in the proof of Theorem 4.1 in the next section when estimating the similarity constant.

Set

$$|x|^p = \inf\Big\{\gamma^p\Big\|\sum_{n\geqslant 0}C^nV_2x_n\Big\|_Y^p + \sum_{n\geqslant 0}\beta(n)^p\|x_n\|^p : x = \sum_{k\geqslant 0}T^kx_k\Big\}, \tag{6.1}$$

the inf being taken over all (finite) decompositions of x as sums of powers of T applied to elements of X.

6.1. $|\cdot|$ is a seminorm. Take two decompositions

$$x = \sum_{k=0}^{d}T^kx_k \quad \text{and} \quad y = \sum_{k=0}^{d}T^ky_k$$

for fixed x and y in X. By adding eventually $x_k = 0$ or $y_k = 0$, we may assume that decompositions have the same length $d+1$. This will be always used in the sequel without any further comment.

Using the triangle inequality $\|a+b\| \leqslant \|a\| + \|b\|$ in $\ell_p^{d+1}(X)$ for

$$a = \Big(\gamma\sum_{n=0}^{d}C^nV_2x_n, \beta(0)x_0, \beta(1)x_1, \ldots, \beta(p)x_p\Big)$$

and

$$b = \Big(\gamma \sum_{n=0}^{d} C^n V_2 y_n, \beta(0)y_0, \beta(1)y_1, \dots, \beta(p)y_p\Big)$$

and taking the infimum over all representations of x and y, we get

$$|x+y| \leqslant |x| + |y|.$$

The proofs of the inequality $|\lambda x| \leqslant |\lambda|\,|x|$ and its converse are left to the reader.

6.2. $|\cdot|$ IS AN EQUIVALENT NORM. The representation $x = x_0 + Tx_1$ with $x_0 = x$ and $x_1 = 0$, gives

$$|x|^p \leqslant \gamma^p \|V_2 x\|^p + \beta(0)^p \|x\|^p \leqslant (\gamma^p \|V_2\|^p + \beta(0)^p)\|x\|^p$$

and therefore

$$|x| \leqslant [\gamma^p \|V_2\|^p + \beta(0)^p]^{1/p} \ \|x\|. \tag{6.2}$$

For the converse inequality, suppose that $x = x_0 + Tx_1 + \dots + T^d x_d$. We have

$$\begin{aligned} \|x\| &= \Big\| \sum_{k=0}^{d} V_1 C^k V_2 x_k + \sum_{k=0}^{d} (T^k - V_1 C^k V_2) x_k \Big\| \\ &\leqslant \frac{1}{\gamma} \|V_1\| \gamma \Big\| \sum_{k=0}^{d} C^k V_2 x_k \Big\| + \sum_{k=0}^{d} \frac{1}{\beta(k)} \|T^k - V_1 C^k V_2\| \beta(k) \|x_k\|. \end{aligned}$$

By using the Hölder inequality, the last quantity is less or equal than

$$\Big[\frac{1}{\gamma^q}\|V_1\|^q + \sum_{k=0}^{d} \frac{1}{\beta(k)^q} \|T^k - V_1 C^k V_2\|^q\Big]^{1/q} \Big[\gamma^p \Big\| \sum_{k=0}^{d} C^k V_2 x_k \Big\|^p + \sum_{k=0}^{d} \beta(k)^p \|x_k\|^p\Big]^{1/p}.$$

Taking the infimum over all representations of x, we obtain

$$\|x\| \leqslant \Big[\frac{\|V_1\|^q}{\gamma^q} + s^q\Big]^{1/q} |x|. \tag{6.3}$$

Thus $|\cdot|$ is a norm equivalent to the original one and, using (6.2) and (6.3), we have

$$\Big[\frac{\|V_1\|^q}{\gamma^q} + s^q\Big]^{-1/q} \|x\| \leqslant |x| \leqslant \Big[\gamma^p \|V_2\|^p + \beta(0)^p\Big]^{1/p} \|x\|. \tag{6.4}$$

We denote by E the Banach space X endowed with the new norm $|\cdot|$.

6.3. THE BANACH SPACE E IS A $SQ_p(X)$-SPACE. Let $x_j \in X$, $j = 1, \dots, n$, with their decompositions

$$x_j = \sum_{k \geqslant 0} T^k x_j^{(k)}.$$

Let $a = [a_{ij}] \in M_n(\mathbb{C})$ be a matrix such that $\|a\|_{p,X} \leqslant 1$. This means that

$$\sum_i \Big\| \sum_j a_{ij} y_j \Big\|^p \leqslant \sum_j \|y_j\|^p \tag{6.5}$$

for all $y_j \in X$, $j = 1, \dots, n$. We will then have

$$\sum_{j=1}^n a_{ij} x_j = \sum_k T^k \Big(\sum_j a_{ij} x_j^{(k)} \Big).$$

By Hernandez theorem we have to prove that $\|a\|_{p,E} \leqslant \|a\|_{p,X}$. Recall that Y is a $SQ_p(X)$-space. We have

$$\begin{aligned}
\sum_i \Big| \sum_j a_{ij} x_j \Big|^p &\leqslant \sum_i \Big(\gamma^p \Big\| \sum_k C^k V_2 \Big(\sum_j a_{ij} x_j^{(k)} \Big) \Big\|_Y^p + \sum_k \beta(k)^p \Big\| \sum_j a_{ij} x_j^{(k)} \Big\|^p \Big) \\
&= \gamma^p \sum_i \Big\| \sum_j a_{ij} \Big(\sum_k C^k V_2 x_j^{(k)} \Big) \Big\|_Y^p + \sum_k \beta(k)^p \sum_i \Big\| \sum_j a_{ij} x_j^{(k)} \Big\|^p \\
&\leqslant \gamma^p \sum_j \Big\| \sum_k C^k V_2 x_j^{(k)} \Big\|_Y^p + \sum_k \beta(k)^p \sum_j \|x_j^{(k)}\|^p \\
&\qquad \text{(by using equation (6.5) for } X \text{ and } Y) \\
&= \sum_j \Big(\gamma^p \Big\| \sum_k C^k V_2 x_j^{(k)} \Big\|^p + \sum_k \beta(k)^p \|x_j^{(k)}\|^p \Big).
\end{aligned}$$

By taking infimum over all possible decompositions we get

$$\sum_i \Big| \sum_j a_{ij} x_j \Big|^p \leqslant \sum_j |x_j|^p$$

and therefore $E = (X, |\cdot|)$ is a $SQ_p(X)$-space.

6.4. THE OPERATOR T WITH RESPECT TO $|\cdot|$. Let x be decomposed as $x = \sum_{k \geqslant 0} T^k x_k$ and let $P(z) = \sum_{s=0}^d a_s z^s$ be a fixed polynomial. Then $P(T)x = \sum_k T^k \Big(\sum_{i+j=k} a_i x_j \Big)$ is a decomposition of $P(T)x$. We obtain $|P(T)x|^p \leqslant \Sigma_1 + \Sigma_2$, where the two sums are given by

$$\Sigma_1 = \gamma^p \Big\| \sum_k C^k V_2 \Big(\sum_{i+j=k} a_i x_j \Big) \Big\|^p \quad \text{and} \quad \Sigma_2 = \sum_k \beta(k)^p \Big\| \sum_{i+j=k} a_i x_j \Big\|^p.$$

6.4.1. THE FIRST SUM. Since

$$\sum_k C^k V_2 \Big(\sum_{i+j=k} a_i x_j \Big) = \sum_m a_m C^m \Big(\sum_n C^n V_2 x_n \Big),$$

we have

$$\Sigma_1 = \gamma^p \Big\| P(C) \Big(\sum_n C^n V_2 x_n \Big) \Big\|^p \leqslant \gamma^p \|P(C)\|_{\mathcal{B}(Y)}^p \Big\| \sum_n C^n V_2 x_n \Big\|^p.$$

6.4.2. The second sum. The shift operator on $\ell_p(\beta, X)$, also denoted by S, acts by

$$S(z_0, z_1, \ldots) = (0, z_0, z_1, \ldots).$$

Denote $\widetilde{x} = (x_0, x_1, \ldots) \in \ell_p(\beta, X)$, where x_k are the elements occuring in the (finite) decomposition of x. The nth component of $P(S)\widetilde{x} \in \ell_p(\beta, X)$ is $\sum\limits_{i+j=n} a_i x_j$; hence

$$\begin{aligned}\Sigma_2 &= \sum_k \beta(k)^p \Big\| \sum_{i+j=k} a_i x_j \Big\|^p = \|P(S)\widetilde{x}\|^p_{\ell_p(\beta,X)} \\ &\leqslant \|P(S)\|^p_{\mathcal{B}(\ell_p(\beta,X))} \Big(\sum_{n \geqslant 0} \beta(n)^p \|x_n\|^p \Big).\end{aligned}$$

Combining now the estimates for the two sums, we obtain

$$|P(T)x|^p \leqslant \max\Big(\|P(C)\|^p, \|P(S)\|^p_{\mathcal{B}(\ell_p(\beta,X))} \Big) \Big(\gamma^p \Big\| \sum_{n\geqslant 0} C^n V_2 x_n \Big\|^p + \sum_{n\geqslant 0} \beta(n)^p \|x_n\|^p \Big).$$

Taking the infimum over all representations of x we get

$$|P(T)x| \leqslant \max\big(\|P(C)\|_{\mathcal{B}(Y)}, \|P(S)\|_{\mathcal{B}(\ell_p(\beta,X))} \big) |x|.$$

Therefore

$$\|P(T)\|_{B(E)} \leqslant \max\big(\|P(C)\|_{\mathcal{B}(Y)}, \|P(S)\|_{\mathcal{B}(\ell_p(\beta,X))} \big).$$

In an analogous way it can be proved that

$$\big\| [P_{ij}(T)] \big\|_{B(\ell_p^n(E))} \leqslant \max\big(\big\| [P_{ij}(C)] \big\|_{\mathcal{B}(\ell_p^n(Y))}, \big\| [P_{ij}(S)] \big\|_{\mathcal{B}(\ell_p^n(\beta,X))} \big)$$

for all polynomials with matrix coefficients. We omit the details.

7. Remaining proofs

7.1. Proof of Theorem 4.1. Set again $C_1 = C$. Consider the equivalent norm $|\cdot|$ as defined in the previous proof ($p = q = 2$, $X = H$ and γ to be precised later on). Since the class of Hilbert spaces is stable by taking subspaces, quotients and ultraproducts of spaces of the form $L_2(\mu; H)$, E is Hilbertian, that is, the new norm $|\cdot|$ comes from an inner product. Also, the unilateral shift S on $\ell_2(\beta)$ is unitarily equivalent to the weighted shift $S_{w(\beta)}$ on ℓ_2 ([22]). The other parts of the preceding proofs, excepting the inequality corresponding to (6.3), are the same. The proof of the inequality

$$\|x\| \leqslant \left[\frac{\|V_1\|^2}{\gamma^2} + s^2 \right]^{1/2} |x|$$

runs as follows.

Suppose $x = x_0 + Tx_1 + \cdots + T^d x_d$. We have

$$\begin{aligned}\|x\| &= \Big\| \sum_{k=0}^d V_1 C^k V_2 x_k + \sum_{k=0}^d (T^k - V_1 C^k V_2) x_k \Big\| \\ &\leqslant \frac{1}{\gamma} \|V_1\| \Big\| \sum_{k=0}^d \gamma C^k V_2 x_k \Big\| + \Big\| \sum_{k=0}^d (T^k - V_1 C^k V_2) x_k \Big\|.\end{aligned}$$

Let $y \in H$. It follows from Lemma 2.2.3 that $\sum_{n=0}^{+\infty} \frac{1}{\beta(n)^2} \|(T^n - V_1 C_1^n V_2)^* y\|^2 \leqslant s^2\|y\|^2$. We obtain

$$\begin{aligned}\Big|\Big\langle \sum_{k=0}^{d} (T^k - V_1 C^k V_2) x_k, y \Big\rangle\Big| &= \Big| \sum_{k=0}^{d} \langle \beta(k) x_k, \frac{1}{\beta(k)} (T^k - V_1 C^k V_2)^* y\rangle \Big| \\ &\leqslant \Big[\sum_k \beta(k)^2 \|x_k\|^2)^{1/2}\Big] \Big[\sum_{n=0}^{d} \frac{1}{\beta(n)^2} \|(T^n - V_1 C_1^n V_2)^* y\|^2 \Big]^{1/2} \\ &\leqslant \Big[\sum_k \beta(k)^2 \|x_k\|^2)^{1/2}\Big] s\|y\|.\end{aligned}$$

Therefore $\Big\| \sum_{k=0}^{d} (T^k - V_1 C^k V_2) x_k \Big\| \leqslant s \Big[\sum_k \beta(k)^2 \|x_k\|^2 \Big]^{1/2}$. Another application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}\|x\| &\leqslant \frac{1}{\gamma} \|V_1\| \Big\| \sum_{k=0}^{d} \gamma C^k V_2 x_k \Big\| + s \Big[\sum_k \beta(k)^2 \|x_k\|^2 \Big]^{1/2} \\ &\leqslant \Big[\frac{1}{\gamma^2} \|V_1\|^2 + s^2 \Big]^{1/2} \Big[\Big\| \sum_k \gamma C^k V_2 x_k \Big\|^2 + \sum_k \beta(k)^2 \|x_k\|^2 \Big]^{1/2}.\end{aligned}$$

Taking the infimum over all representations of x, we obtain

$$\|x\| \leqslant \left[\frac{\|V_1\|^2}{\gamma^2} + s^2 \right]^{1/2} |x|.$$

This gives the similarity statement.

We prove now the estimate for the similarity constant. From Equation (6.4) and the proof given above we have

$$C_{\rm sim}(T, \mathrm{CDOM}(C \oplus S_{w(\beta)})) \leqslant \left[\frac{\|V_1\|^2}{\gamma^2} + s^2 \right]^{1/2} \Big[\gamma^2 \|V_2\|^2 + \beta(0)^2 \Big]^{1/2}.$$

By assuming $C = 0$ if necessary, we may assume that V_2 is not the null operator. If $s \neq 0$, choose

$$\gamma = \left[\frac{\beta(0)\|V_1^*\|}{s\|V_2\|} \right]^{1/2}.$$

We then have

$$C_{\rm sim}(T, \mathrm{CDOM}(C \oplus S_{w(\beta)}))^2 \leqslant (\|V_1^*\| \, \|V_2\| + \beta(0)s)^2.$$

If $s = 0$, then $T^n = V_1 C^n V_2$ and thus T is completely polynomially dominated by C with bound $\|V_1\| \cdot \|V_2\|$. Apply now Theorem 2.1.3. Note that in this case $S_{w(\beta)}$ is absent from the direct sum. The proof of Theorem 4.1 is now complete. ∎

7.2. PROOF OF COROLLARY 5.1.1. The proof of this version of Rota theorem is similar to the proof of Theorem 4.5. Indeed, if $C = 0$, then the new norm $|\cdot|$ is given by

$$|x|^p = \inf\Big\{\sum_{n\geqslant 0}\beta(n)^p\|x_n\|^p : x = \sum_{k\geqslant 0}T^k x_k\Big\},$$

the inf being taken over all (finite) decompositions of x as sums of powers of T applied to elements of X. This is the quotient norm of $\ell_p(X)/\mathrm{Ker}(\psi)$, where the onto map ψ is given by

$$\ell_p(X) \ni (x_0, x_1, \ldots) \mapsto \psi(x_0, x_1, \ldots) = \sum_k T^k x_k \in X.$$

Take E to be X with this new norm. The rest of the proof is the same. ∎

7.3. PROOF OF PROPOSITION 3.1. For the first part of the theorem, it is sufficient to prove that an operator asymptotically near an isometry is similar to an isometry. Indeed, if we suppose that $\lim\limits_{n\to\infty}\|T^n - L^{-1}V^nL\| = 0$, with V an isometry, then $\|(LTL^{-1})^n - V^n\| = \|L(T^n - L^{-1}V^nL)L^{-1}\|$ tends to 0 as n goes to infinity and so we will obtain the similarity of LTL^{-1}, so of T, to an isometry.

Now, if T is asymptotically near an isometry V, then for each $r \in]0,1[$ there exists $k \in \mathbb{Z}_+$ such that $\sup\limits_{n\geqslant k}\|T^n - V^n\| \leqslant r$. Set $R = T^k$ and $W = V^k$ (W is an isometry). We obtain $\sup\limits_{m\geqslant 1}\|R^m - W^m\| \leqslant r < 1$. This implies that, for each x and each $m \geqslant 1$,

$$(1-r)\|x\| = \|W^m x\| - r\|x\| \leqslant \|W^m x\| - \|R^m x - W^m x\| \leqslant \|R^m x\| \leqslant (1+r)\|x\|.$$

By a theorem of Sz.-Nagy ([23]), $R = T^k$ is similar to an isometry and this implies (Corollary 4.2, [18]) that T is similar to an isometry.

Suppose now that T is asymptotically near a unitary U. By the first part of the proof, T is similar to an isometry. Therefore we can write $V^* = L^{-1}T^*L$, with V an isometry, for a suitable invertible operator L. But T^* is asymptotically near the isometry U^* and so T^* is similar to an isometry. This implies that T^* and V^* are injective and so the isometry V is also onto. Therefore V is unitary and so T is similar to a unitary. ∎

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