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DISJOINTNESS PRESERVING FREDHOLM LINEAR OPERATORS OF $C_0(X)$

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Abstract. Let X and Y be locally compact Hausdorff spaces. We give a full description of disjointness preserving Fredholm linear operators T from $C_0(X)$ into $C_0(Y)$, and show that T is continuous if either Y contains no isolated point or T has closed range. Our task is achieved by writing T as a weighted composition operator $Tf = h \cdot f \circ \varphi$. Through the relative homeomorphism φ, the structure of the range space of T can be completely analyzed, and X and Y are homeomorphic after removing finite subsets.

Keywords: *Auto-continuity, Fredholm operators, disjointness preserving operators, weighted composition operators.*

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1. INTRODUCTION

A (not necessarily bounded) linear operator S from a Banach space E into a Banach space F is said to be *Fredholm* if it has finite nullity and finite corank; i.e. $\text{nullity}(S) = \dim \ker S < \infty$ and $\text{corank}(S) = \dim F/\operatorname{ran}(S) < \infty$. Fredholm composition operators between $L^2(\mu)$ spaces (see e.g. [23], [14], [24]) and Hilbert spaces of analytic functions (see e.g. [15], [25]) have been well studied and proven to have many applications.

Let X and Y be locally compact Hausdorff spaces. Let $C_0(X)$ and $C_0(Y)$ be Banach spaces of continuous (real- or complex-valued) functions defined on X and Y vanishing at infinity, respectively. A linear operator $T : C_0(X) \to C_0(Y)$ is *disjointness preserving* or *separating* if $Tf \cdot Tg = 0$ whenever $f \cdot g = 0$. If, in addition, T is bounded then T is a (weighted) composition operators $Tf = h \cdot f \circ \varphi$ (see Theorem 2.4).

In this paper, we shall give a full description of the structure of disjointness preserving Fredholm operators $T : C_0(X) \to C_0(Y)$ (Theorem 3.14). When such an operator exists, X and Y are homeomorphic after removing finite subsets.

Moreover, T is bounded if either Y contains no isolated point or T has closed range. These extend the well-known fact that if T is bijective then X and Y are homeomorphic and T is automatically bounded (see [19], [11], [20]). As an application, the information about the range space of T, a finite co-dimensional subspace of $C_0(Y)$, given in Theorem 3.14 is utilized to give a Gleason-Kahane-Zelazko type result (Corollary 4.2). Finally, we remark that our results are very useful in the investigation of shift operators on continuous function spaces [7], [21], which is a subject attracts increasing interests from researchers recently (see e.g. [8], [17], [13], [9], [16], [26], [4]).

2. PRELIMINARIES

Let X_∞ (respectively Y_∞) be the one-point compactification $X \cup \{\infty\}$ (respectively $Y \cup \{\infty\}$) of a locally compact Hausdorff space X (respectively Y). We note that ∞ is an isolated point in X_∞ if and only if X is compact. For each y in Y, let δ_y denote the point evaluation at y, that is, δ_y is the linear functional of $C_0(Y)$ defined by $\delta_y(g) = g(y)$.

We begin with the following two elementary observations. The first of them enables us to assume freely that the underlying field $\mathbb{K}$ is the complex scalars $\mathbb{C}$, while the second suggests us a way to look into the problem of automatic continuity of a linear operator between $C_0(X)$ spaces.

LEMMA 2.1. *Let $T_{\rm r}$ be a real linear operator from the real Banach space $C_0(X,\mathbb{R})$ into $C_0(Y,\mathbb{R})$. Let $T_{\rm c} : C_0(X,\mathbb{C}) \to C_0(Y,\mathbb{C})$ be the complexification of $T_{\rm r}$ defined by*

$$T_{\rm c}(f_1 + {\rm i}f_2) = T_{\rm r}f_1 + {\rm i}T_{\rm r}f_2, \quad f_1, f_2 \in C_0(X,\mathbb{R}).$$

Then, we have:

(i) *$T_{\rm r}$ is bounded if and only if $T_{\rm c}$ is bounded;*
(ii) *$T_{\rm r}$ has closed range if and only if $T_{\rm c}$ has closed range;*
(iii) $\mathrm{nullity}(T_{\rm r}) = \mathrm{nullity}(T_{\rm c})$ *and* $\mathrm{corank}(T_{\rm r}) = \mathrm{corank}(T_{\rm c})$*;*
(iv) *$T_{\rm r}$ is disjointness preserving if and only if $T_{\rm c}$ is disjointness preserving.*

Proof. Most of the arguments are straightforward. We just mention that if $f = f_1 + {\rm i}f_2$, $g = g_1 + {\rm i}g_2$ with f_1, f_2, g_1 and g_2 in $C_0(X,\mathbb{R})$ then $f \cdot g = 0$ is equivalent to $f_j \cdot g_k = 0$ for $j, k = 1, 2$. ∎

LEMMA 2.2. *Let T be a linear operator from $C_0(X)$ into $C_0(Y)$. Then T is bounded if and only if $\delta_y \circ T$ is bounded for all y in Y.*

Proof. It is an easy consequence of the Uniform Boundedness Principle. Alternatively, one can make use of the Closed Graph Theorem. ∎

DEFINITION 2.3. In view of Lemma 2.2, we divide Y into three disjoint parts, the *nullity part* Y_0, the *continuous part* Y_c and the *discontinuous part* Y_d, where

$$Y_0 = \{y \in Y : \delta_y \circ T \equiv 0\},$$
$$Y_c = \{y \in Y : \delta_y \circ T \text{ is nonzero and continuous}\}$$

and

$$Y_d = \{y \in Y : \delta_y \circ T \text{ is discontinuous}\}.$$

Accordingly, T is bounded if and only if $Y_d = \emptyset$.

THEOREM 2.4. ([19], [20]; see also [1], [10]) *Let T be a disjointness preserving linear operator from $C_0(X)$ into $C_0(Y)$. Then:*

(i) *Y_0 is closed and Y_d is open;*

(ii) *a unique continuous map φ from $Y_c \cup Y_d$ into X_∞ exists such that*

$$\varphi(y) \notin \operatorname{supp}(f) \Rightarrow T(f)(y) = 0, \quad \forall f \in C_0(X);$$

(iii) *$\varphi(Y_c) \subseteq X$ and $\varphi(Y_d)$ is a finite set of non-isolated points;*

(iv) *a unique continuous non-vanishing scalar function h on Y_c exists such that*

$$Tf_{|Y_c} = h \cdot f \circ \varphi, \quad Tf_{|Y_0} \equiv 0.$$

Theorem 2.4 can be improved if T is also Fredholm. Notations in Theorem 2.4 will be used throughout this paper. Recall that a bounded linear operator T from a Banach space E into a Banach space F is called an *injection* if there is an $r > 0$ such that $\|Tx\| \geqslant r\|x\|$. It follows from the Open Mapping Theorem that T is an injection if and only if T is injective and has closed range. See [2] for more information.

LEMMA 2.5. *Let $Tf(y) = h(y)f(\varphi(y))$ be a weighted composition operator from $C_0(X)$ into $C_0(Y)$. Here, h is a continuous non-vanishing scalar-valued function on Y and φ is a continuous map from Y into X. Then T is continuous. If, in addition, T has closed range then there exist positive constants r and R such that*

$$0 < r \leqslant \sup_{y \in \varphi^{-1}(\{x\})} |h(y)| \leqslant R \quad \text{for all } x \text{ in } \varphi(Y). \tag{2.1}$$

Proof. Since $\delta_y \circ T = h(y) \cdot \delta_{\varphi(y)}$ for all y in Y, it follows from Lemma 2.2 that T is continuous. Moreover, if T is injective and has closed range then T is an injection. So there are constants $r, R > 0$ such that $r\|f\| \leqslant \|Tf\| \leqslant R\|f\|$. It is obvious that $\sup_{y \in \varphi^{-1}(\{x\})} |h(y)| \leqslant R$ for all x in $\varphi(Y)$. On the other hand, for each x_0 in $\varphi(Y)$ let U and V be open neighborhoods of x_0 with $U \subseteq V$. Let $0 \leqslant f_{UV} \leqslant 1$ in $C_0(X)$ satisfy the conditions that $f_{UV|U} = 1 = \|f_{UV}\|$, and $f_{UV}(x) = 0$ if $x \notin V$. Then

$$r = r\|f_{UV}\| \leqslant \|Tf_{UV}\| = \sup_{y \in Y} |Tf_{UV}(y)| = \sup_{y \in Y} |h(y) f_{UV}(\varphi(y))| \leqslant \sup_{y \in \varphi^{-1}(V)} |h(y)|.$$

Therefore, we are able to choose a net $\{y_\lambda\}$ from Y and $\varepsilon > 0$ such that

$$\varphi(y_\lambda) \to x_0 \quad \text{and} \quad \big|h(y_\lambda)\big| > r - \varepsilon > 0.$$

By passing to a subnet if necessary, we can assume that $\{y_\lambda\}$ converges in Y_∞. Since $\big|Tf_{UV}(y_\lambda)\big| = \big|h(y_\lambda)\big| > r - \varepsilon > 0$ eventually for all neighborhoods U and V of x_0 with $U \subseteq V$, we have $\{y_\lambda\}$ converges to some $y_0 \neq \infty$. Clearly, $\varphi(y_0) = x_0$ and we have

$$r - \varepsilon \leqslant \big|h(y_0)\big| \leqslant \sup_{y \in \varphi^{-1}(\{x_0\})} \big|h(y)\big|.$$

Since ε can be arbitrary small, the desired inequality follows.

Finally, suppose that T has closed range but not necessarily injective. Let $\overline{\varphi(Y)}$ be the closure of $\varphi(Y)$ in X. Consider the disjointness preserving linear operator $\widetilde{T} : C_0(\overline{\varphi(Y)}) \to C_0(Y)$ defined by $\widetilde{T}\widetilde{f}(y) = Tf(y) = h(y) \cdot f(\varphi(y))$, where f is any Tietze extension in $C_0(X)$ of $\widetilde{f}$. It is plain that $\widetilde{T}$ is injective and $\operatorname{ran}(\widetilde{T}) = \operatorname{ran}(T)$ is closed. The desired assertion thus follows from the first part of the proof. ∎

REMARK 2.6. Note that "sup" cannot be dropped in (2.1). We would like to thank Martin Stanev for providing us a counterexample for this. Lemma 2.5 fixes a related bug in Proposition 4 of [20].

The following statement is a consequence of the Open Mapping Theorem. Its proof can be found in, for example, 28A of [5].

PROPOSITION 2.7. *Let T be a bounded linear operator from a Banach space E into a Banach space F with finite corank. Then T has closed range.*

3. MAIN RESULTS

In the following lemmas, we always assume that T is a disjointness preserving linear operator from $C_0(X)$ into $C_0(Y)$.

LEMMA 3.1. *Let T have finite corank. Then there exist positive scalars r and R such that*

$$0 < r \leqslant \sup_{y \in \varphi^{-1}(\{x\}) \cap Y_c} |h(y)| \leqslant R \quad \text{for all } x \text{ in } \varphi(Y_c).$$

Proof. Since T has finite corank, there exist $g_1, \ldots, g_n$ in $C_0(Y)$ such that $C_0(Y)$ is the linear span of $g_1, \ldots, g_n$ and $\operatorname{ran}(T)$. So, for every g in $C_0(Y)$, there are scalars $\lambda_1, \ldots, \lambda_n$ and an f in $C_0(X)$ such that $g = \lambda_1 g_1 + \cdots + \lambda_n g_n + Tf$.

Let $T' : C_0(X) \to C_0(Y_c \cup Y_0)$ be the linear operator defined by $T'f = Tf_{|Y_c \cup Y_0}$. Then T' is continuous by Lemma 2.2. Since Y_d is an open set in Y, $Y_c \cup Y_0 \cup \{\infty\} = Y_\infty \setminus Y_d$ is closed in Y_∞. By Tietze's extension theorem, every continuous function g' in $C_0(Y_c \cup Y_0)$ can be extended to a g in $C_0(Y)$. Consequently,

$$\begin{aligned} g' = g_{|Y_c \cup Y_0} &= \lambda_1 g_{1|Y_c \cup Y_0} + \cdots + \lambda_n g_{n|Y_c \cup Y_0} + Tf_{|Y_c \cup Y_0} \\ &= \lambda_1 g_{1|Y_c \cup Y_0} + \cdots + \lambda_n g_{n|Y_c \cup Y_0} + T'f, \end{aligned}$$

for some scalars $\lambda_1, \ldots, \lambda_n$ and an f in $C_0(X)$. This shows that T' has finite corank. By Proposition 2.7, $\operatorname{ran}(T')$ is closed. Since $T' f_{|Y_0} \equiv 0$, the induced map $\widetilde{T} : C_0(X) \to C_0(Y_c)$ defined by $\widetilde{T} f = T' f_{|Y_c}$ has closed range as well. Then, Lemma 2.5 applies. ∎

Lemma 3.2. *Let T have finite nullity m. Then $\overline{\varphi(Y_c)} \cap X = \overline{\varphi(Y_c \cup Y_d)} \cap X$, and*

$$X \setminus \overline{\varphi(Y_c)} = \{x_1, \ldots, x_m\}$$

consisting of exactly m isolated points, where the closure is taken in X_∞. Moreover,

$$\ker T = \operatorname{span}\{\chi_{\{x_1\}}, \chi_{\{x_2\}}, \ldots, \chi_{\{x_m\}}\},$$

where $\chi_{\{x_i\}}$ is the characteristic functions of $\{x_i\}$ for $i = 1, 2, \ldots, m$.

Proof. Suppose that there were distinct points $x_1, x_2, \ldots, x_{m+1}$ in $X \setminus \overline{\varphi(Y_c \cup Y_d)}$. Let $V_1, V_2, \ldots, V_{m+1}$ be disjoint compact neighborhoods of $x_1, x_2, \ldots, x_{m+1}$ in $X \setminus \overline{\varphi(Y_c \cup Y_d)}$, respectively. For each $i = 1, 2, \ldots, m+1$, let $0 \leqslant f_i \leqslant 1$ in $C_0(X)$ satisfy that $f_i(x_i) = 1$ and $f_i = 0$ outside V_i. Then $\varphi(y) \notin \operatorname{supp}(f_i)$, and thus $Tf_i(y) = 0$, for all y in $Y_c \cup Y_d$ by Theorem 2.4. Note that Tf vanishes on Y_0 for all f in $C_0(X)$. Hence $f_i \in \ker T$. Since $\{f_1, f_2, \ldots, f_{m+1}\}$ is linearly independent, we have $\dim(\ker T) \geqslant m + 1$, a contradiction. So the open set $X \setminus \overline{\varphi(Y_c \cup Y_d)} = \{x_1, x_2, \ldots, x_k\}$ consists of isolated points, and $k \leqslant m$.

As a finite set, $\varphi(Y_d)$ is closed (Theorem 2.4). Therefore,

$$X \setminus \overline{\varphi(Y_c)} \subseteq (X \setminus \overline{\varphi(Y_c \cup Y_d)}) \cup \varphi(Y_d). \tag{3.1}$$

Since both $X \setminus \overline{\varphi(Y_c \cup Y_d)}$ and $\varphi(Y_d)$ are finite, $X \setminus \overline{\varphi(Y_c)}$ is a finite open subset of X. This implies that $X \setminus \overline{\varphi(Y_c)}$ consists of isolated points. Since $\varphi(Y_d)$ contains only non-isolated points in X_∞ (Theorem 2.4), $\varphi(Y_d) \cap X \subseteq \overline{\varphi(Y_c)}$, and thus

$$X \setminus \overline{\varphi(Y_c)} = X \setminus \overline{\varphi(Y_c \cup Y_d)} = \{x_1, x_2, \ldots, x_k\} \tag{3.2}$$

by (3.1). Consequently, $\overline{\varphi(Y_c)} \cap X = \overline{\varphi(Y_c \cup Y_d)} \cap X$.

Finally, we prove that $X \setminus \overline{\varphi(Y_c)}$ consists of exactly m isolated points whose characteristic functions span $\ker T$. Since $Tf = h \cdot f \circ \varphi$ on Y_c and h is non-vanishing, we have $f_{|\overline{\varphi(Y_c)}} = 0$ if $f \in \ker T$. Conversely, if f vanishes on $\overline{\varphi(Y_c)}$ then $f = \sum_{i=1}^{k} \lambda_i \chi_{\{x_i\}}$. Therefore, $\operatorname{supp}(f) \cap \varphi(Y_d) = \{x_1, \ldots, x_k\} \cap \varphi(Y_d) = \emptyset$ by (3.2). By Theorem 2.4 again, $Tf_{|Y_d} = 0$. As $Tf_{|Y_c \cup Y_0} = 0$, we have $f \in \ker T$. It follows that $\ker T = \operatorname{span}\{\chi_{\{x_1\}}, \ldots, \chi_{\{x_k\}}\}$. Since $\{\chi_{\{x_i\}}\}_{i=1}^{k}$ is linearly independent and the dimension of $\ker T$ is m, we have $k = m$. ∎

Remark 3.3. In the proof of Lemma 3.2, we have shown that

$$Tf_{|Y_c} = 0 \quad \text{implies } Tf = 0,$$

provided T has finite nullity.

The following result ensures that T is bounded whenever Y contains no isolated point.

LEMMA 3.4. *Let T be Fredholm. Then:*

(i) Y_{d} *consists of finitely many isolated points. In fact, the cardinality of* Y_{d} *is less than or equal to n, the corank of T;*

(ii) $\varphi(Y_{\mathrm{c}})$ *is closed in X, and*

$$\varphi(Y_{\mathrm{c}}) = \overline{\varphi(Y_{\mathrm{c}} \cup Y_{\mathrm{d}})} \cap X.$$

Proof. (i) We first claim that if $\mathrm{supp}(g) \subseteq Y_{\mathrm{d}}$ then $g \notin \mathrm{ran}(T)$, unless $g = 0$. In fact, it is a direct consequence of Remark 3.3. Now, suppose there were distinct $y_1, y_2, \ldots, y_{n+1}$ in Y_{d}. Since Y_{d} is open (Theorem 2.4), there are disjoint neighborhoods $V_1, V_2, \ldots, V_{n+1}$ of $y_1, y_2, \ldots, y_{n+1}$ in Y_{d}, respectively, which are open in Y. Let U_i be a compact neighborhood of y_i contained in V_i and let $g_i \neq 0$ be in $C_0(Y)$ with $\mathrm{supp}(g_i) \subseteq U_i \subseteq V_i$ for $i = 1, 2, \ldots, n+1$. Since $\dim(C_0(Y)/\mathrm{ran}(T)) = n$, there are some not all zero scalars $\lambda_1, \ldots, \lambda_{n+1}$ such that $0 \neq \sum_{i=1}^{n+1} \lambda_i g_i \in \mathrm{ran}(T)$. But $\mathrm{supp}\left(\sum_{i=1}^{n+1} \lambda_i g_i\right) \subseteq \bigcup_{i=1}^{n+1} V_i \subseteq Y_{\mathrm{d}}$. Consequently, $\sum_{i=1}^{n+1} \lambda_i g_i \notin \mathrm{ran}(T)$, a contradiction. Hence the cardinality of Y_{d} is at most n. Being finite and open, Y_{d} consists of isolated points.

(ii) By Lemma 3.2, it suffices to show that $\varphi(Y_{\mathrm{c}})$ is closed in X. To this end, let $x_0 \in \overline{\varphi(Y_{\mathrm{c}})} \cap X$. First, we note that $x_0 \neq \infty$. If x_0 is isolated in $\overline{\varphi(Y_{\mathrm{c}})} \cap X$ then $x_0 \in \varphi(Y_{\mathrm{c}})$. So we assume there exist y_λ in Y_{c} such that $\varphi(y_\lambda) \neq x_0$ and $\varphi(y_\lambda) \to x_0$. It follows from Lemma 3.1 that we can assume $h(y_\lambda)$ is away from zero. By passing to a subnet, if necessary, we can also assume $y_\lambda \to y_0$ in Y_∞. Since all points in Y_{d} are isolated by (i), we have $y_0 \notin Y_{\mathrm{d}}$. Suppose that $y_0 \in Y_0$ or $y_0 = \infty$. We have

$$0 = \lim_{\lambda \to \infty} Tf(y_\lambda) = \lim_{\lambda \to \infty} h(y_\lambda) f(\varphi(y_\lambda)) \quad \forall f \in C_0(X).$$

Since $h(y_\lambda)$ is away from zero,

$$f(x_0) = \lim_{\lambda \to \infty} f(\varphi(y_\lambda)) = 0 \quad \forall f \in C_0(X).$$

This implies $x_0 = \infty$, a contradiction. So $y_0 \in Y_{\mathrm{c}}$, and then $x_0 = \varphi(y_0) \in \varphi(Y_{\mathrm{c}})$. Hence $\varphi(Y_{\mathrm{c}}) = \overline{\varphi(Y_{\mathrm{c}})} \cap X$ is closed in X. ∎

Let $\#(S)$ denote the cardinality of a set S.

DEFINITION 3.5. We define an equivalence relation $\sim$ on Y_{c} such that $y \sim y'$ if and only if $\varphi(y') = \varphi(y)$; or equivalently, $\ker \delta_y \circ T = \ker \delta_{y'} \circ T$ (Theorem 2.4). Let y be a point in Y_{c}. Denote by $[y]$ the equivalence class in Y_{c} represented by y. We call y a *merging point* of T if $[y]$ contains more than one points. In this case, we call $\varphi(y)$ the *merged point* of T in X for the class $[y]$. Let M be the set of all merging points of T and

$$m(T) = \sum \{\#([y]) - 1 : [y] \in Y_{\mathrm{c}}/\sim\} = \#(M) - \#(\varphi(M)).$$

We call $\#(Y_{\mathrm{d}})$ the *discontinuity index*, $\#(Y_0)$ the *vanishing index* and $m(T)$ the *merging index* of T, respectively.

REMARK 3.6. It is easy to see that if $g \in \mathrm{ran}(T)$ then

(i) $g(y) = 0$ for all y in Y_0;
(ii) $g(y) = 0$ for some y in Y_c if and only if $g(y') = 0$ for all y' in $[y]$.

LEMMA 3.7. *Let T be continuous and Fredholm and have finite corank n. Then the sum of the merging and vanishing indices of T is equal to n, i.e.,*

$$m(T) + \#(Y_0) = n.$$

Proof. Suppose first that the inequality

$$m(T) + \#(Y_0) \leqslant n \tag{3.3}$$

does not hold, i.e., there exist $y_1^{(0)}, \dots, y_{l_0}^{(0)}$ in Y_0 and merged points $x_1, \dots, x_k$ in $\varphi(Y_c)$ with corresponding merging points $y_1^{(i)}, \dots, y_{l_i}^{(i)}$ in $\varphi^{-1}(x_i) \cap Y_c$ for $i = 1, \dots, k$, such that

$$\sum_{i=1}^{k}(l_i - 1) + l_0 \geqslant n + 1.$$

For $i = 0, 1, 2, \dots, k$, let $g_j^{(i)}$ be in $C_0(Y)$ such that $g_j^{(i)}(y_j^{(i)}) = 1$ and $g_j^{(i)}(y_{j'}^{(i')}) = 0$ whenever $i' \neq i$ or $j' \neq j$ for $1 \leqslant j \leqslant l_i - 1$ ($1 \leqslant j \leqslant l_0$ when $i = 0$). Without loss of generality, we can assume all $g_j^{(i)}$'s have disjoint supports. Then we have at least $n+1$ such $g_j^{(i)}$ in $C_0(Y)$. By Remark 3.6, all $g_1^{(0)}, \dots, g_{l_0}^{(0)}$, $g_1^{(1)}, \dots, g_{l_1-1}^{(1)}$, $g_1^{(2)}, \dots, g_{l_2-1}^{(2)}, \dots, g_1^{(k)}, \dots, g_{l_k-1}^{(k)}$ are not in $\operatorname{ran}(T)$. Moreover, they are linear independent in $C_0(Y)$ modulo $\operatorname{ran}(T)$. In fact, if $g = \sum \lambda_j^{(i)} g_j^{(i)} \in \operatorname{ran}(T)$, we shall show $\lambda_j^{(i)} = 0$ for all i, j. Note that $g(y_{l_i}^{(i)}) = 0$ for $1 \leqslant i \leqslant k$. Then, by Remark 3.6 again, $\lambda_j^{(0)} = g(y_j^{(0)}) = 0$ for $j = 1, \dots, l_0$, and $\lambda_j^{(i)} = g(y_j^{(i)}) = 0$ for all indices (i, j) with $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l_i - 1$. So $\dim(C_0(Y)/\operatorname{ran}(T)) \geqslant n + 1$. This contradiction says that inequality (3.3) holds.

Since T is continuous, $Y_d = \emptyset$. Let $Y_0 = \{y_1^{(0)}, \dots, y_{l_0}^{(0)}\}$, and $\varphi(M) = \{x_1, \dots, x_k\}$ with $\varphi^{-1}(x_i) \cap Y_c = \{y_1^{(i)}, \dots, y_{l_i}^{(i)}\}$ for $i = 1, \dots, k$. Let $g_j^{(i)}$ be defined as above. Let $\mathcal{A}$ be the span of $g_j^{(i)}$'s. As shown in the first paragraph, $\operatorname{ran}(T) \cap \mathcal{A} = \{0\}$. We will show that $C_0(Y) = \operatorname{ran}(T) \oplus \mathcal{A}$, and thus $m(T) + \#(Y_0) = \dim(\mathcal{A}) = n$.

For each g in $C_0(Y)$, choose $\lambda_j^{(i)}$'s in $\mathbb{C}$ such that $g' = g - \sum_{i,j} \lambda_j^{(i)} g_j^{(i)}$ satisfying $g'(y) = 0$ for all y in Y_0, and $\frac{g'(y_1^{(i)})}{h(y_1^{(i)})} = \dots = \frac{g'(y_{l_i}^{(i)})}{h(y_{l_i}^{(i)})}$ for $i = 1, \dots, k$. Define a scalar-valued function f' by setting $f'(\varphi(y)) = \frac{g'(y)}{h(y)}$ for all y in Y_c. Then f' is continuous and well-defined on $\varphi(Y_c)$ which is closed in X by Lemma 3.4 (ii). By Tietze's Extension Theorem, there is a continuous function f in $C_0(X)$ such that $f(x) = f'(x)$ for all x in $\varphi(Y_c)$. Note that $Tf_{|Y_0} = 0$ and $Tf(y) = h(y)f(\varphi(y)) = g'(y)$ for y in Y_c. Since $Y_d = \emptyset$, we have $g' = Tf$. That is,

$$g = Tf + \sum_{i,j} \lambda_j^{(i)} g_j^{(i)} \in \operatorname{ran}(T) \oplus \mathcal{A}. \quad \blacksquare$$

LEMMA 3.8. *Let T be Fredholm (but not necessarily continuous) and have finite corank n. Then*

$$m(T) + \#(Y_0) + \#(Y_{\mathrm{d}}) = n. \tag{3.4}$$

Proof. By Lemma 3.4 (i), $Y_{\mathrm{d}} = \{y_1, \ldots, y_k\}$ consists of $k \leqslant n$ isolated points. Let $\widetilde{T} : C_0(X) \to C_0(Y_0 \cup Y_{\mathrm{c}})$ be a disjointness preserving linear operator defined by

$$\widetilde{T}f = Tf_{|Y_0 \cup Y_{\mathrm{c}}}.$$

Then $\widetilde{T}$ is continuous. By Remark 3.3, $\ker T = \ker \widetilde{T}$. We claim that $\widetilde{T}$ has finite corank. Since $y_1, \ldots, y_k$ are isolated points in Y, all $\chi_{\{y_1\}}, \ldots, \chi_{\{y_k\}}$ are functions in $C_0(Y)$. By Remark 3.3, $\chi_{\{y_i\}} \notin \operatorname{ran}(T)$ for all $i = 1, \ldots, k$ and they are linear independent. Without lose of generality, we can assume that $g_1, \ldots, g_n$ form a basis of $C_0(Y)$ modulo $\operatorname{ran}(T)$, where $g_i = \chi_{\{y_i\}}$ for all $i = 1, \ldots, k$. In particular, for all g in $C_0(Y)$, $g = \lambda_1 g_1 + \cdots + \lambda_n g_n + Tf$ for some f in $C_0(X)$ and scalars λ_i. Now, for all $\widetilde{g}$ in $C_0(Y_0 \cup Y_{\mathrm{c}})$,

$$\begin{aligned} \widetilde{g} = g_{|Y_0 \cup Y_{\mathrm{c}}} &= \lambda_1 g_{1|Y_0 \cup Y_{\mathrm{c}}} + \cdots + \lambda_n g_{n|Y_0 \cup Y_{\mathrm{c}}} + Tf_{|Y_0 \cup Y_{\mathrm{c}}} \\ &= \lambda_{k+1} g_{k+1|Y_0 \cup Y_{\mathrm{c}}} + \cdots + \lambda_n g_{n|Y_0 \cup Y_{\mathrm{c}}} + \widetilde{T}f, \end{aligned}$$

where g is any Tietze extension of $\widetilde{g}$. This implies $\operatorname{corank}(\widetilde{T}) \leqslant n - \#(Y_{\mathrm{d}})$. By Lemma 3.7, we have $\operatorname{corank}(\widetilde{T}) = m(\widetilde{T}) + \#(Y_0) = m(T) + \#(Y_0)$. Hence $m(T) + \#(Y_0) + \#(Y_{\mathrm{d}}) \leqslant n$. The equality (3.4) then follows in the same manner as in the proof of Lemma 3.7. ∎

LEMMA 3.9. *Let T be Fredholm. Then h is bounded and away from zero, that is, there exist positive constants r and R such that*

$$0 < r \leqslant |h(y)| \leqslant R \quad \text{for all } y \text{ in } Y_{\mathrm{c}}.$$

Proof. It follows from Lemma 3.8 that the merging index $m(T)$ of T is finite. By Lemma 3.1, we see that the non-vanishing scalar function h is bounded and away from zero. ∎

Recall that a map φ from Y into X is said to be *proper* if preimages of compact subsets of X under φ are compact in Y. It is obvious that φ is proper if and only if $\lim_{y\to\infty} \varphi(y) = \infty$. As a consequence, a proper continuous map φ from Y onto X is a quotient map, i.e. $\varphi^{-1}(O)$ is open in Y if and only if O is open in X.

LEMMA 3.10. *Let T be Fredholm. Then φ is proper. More precisely, φ has a continuous extension from Y_∞ to X_∞ by setting $\varphi_{|Y_0} \equiv \infty$ and $\varphi(\infty) = \infty$. If, in addition, X is compact then the finite set Y_0 consists of isolated points.*

Proof. It is enough to show that if $y_\lambda \in Y_{\mathrm{c}}$ such that $y_\lambda \to p \in Y_0 \cup \{\infty\}$ then $\varphi(y_\lambda) \to \infty$ in X_∞. For f in $C_0(X)$, we have

$$0 = Tf(p) = \lim Tf(y_\lambda) = \lim_{\lambda\to\infty} h(y_\lambda) f(\varphi(y_\lambda)).$$

Since h is away from zero by Lemma 3.9, we have

$$\lim_{\lambda\to\infty} f(\varphi(y_\lambda)) = 0, \quad \forall f \in C_0(X).$$

This implies $\varphi(y_\lambda) \to \infty$ in X_∞. Finally, we note that ∞ is an isolated point in X_∞ when X is compact. Thus, $\varphi^{-1}(\infty)$ is open in Y_∞ in this case. Since $Y_0 \subseteq \varphi^{-1}\{\infty\} \subseteq Y_{\mathrm{d}} \cup Y_0 \cup \{\infty\}$ are all finite (the last inclusion is provided by Theorem 2.4), the open set $\varphi^{-1}\{\infty\}$ consists of isolated points. ∎

In the following example, we shall see that Y_0 may contain non-isolated points when X is not compact.

EXAMPLE 3.11. Let c_0 (respectively c) be the Banach space of null (respectively convergent) sequences. In other words, $c_0 = C_0(\mathbb{N})$ and $c = C(\mathbb{N}_\infty)$. Let T be the canonical embedding from c_0 into c. In this case, $X = \mathbb{N}$, $Y = \mathbb{N}_\infty$, $Y_{\mathrm{c}} = \mathbb{N}$, $Y_0 = \{\infty\}$ and $\varphi : \mathbb{N} \to \mathbb{N}$ is the identity map. We note that ∞ is the unique cluster point in $\mathbb{N}_\infty$. ∎

With a little more efforts, we have a similar example in which $X = Y$ and φ is a homeomorphism.

EXAMPLE 3.12. Let X be the disjoint union in $\mathbb{R}^2$ of $I_n^+ = \{(n,t) : 0 < t \leqslant 1\}$ and $I_n^- = \{(n,t) : -1 < t < 0\}$ for $n = 1, 2, \ldots$. Let p be the point $(1,1)$ and let $X_1 = X \setminus \{p\}$. Let φ be the homeomorphism from X_1 onto X by sending the intervals $I_1^+ \setminus \{p\}$ onto I_1^-, I_{n+1}^+ onto I_n^+, and I_n^- onto I_{n+1}^- in a canonical way for $n = 1, 2, \ldots$. Then the corank one disjointness preserving linear isometry $Tf = f \circ \varphi$ from $C_0(X)$ into $C_0(X)$ has exactly one vanishing point, i.e., p. We note that p is not an isolated point in X. In a similar manner, one can even construct an example in which X is connected (by adjoining each $I_n^\pm$ a common base point, for example). ∎

In view of Theorem 2.4 and Lemmas 3.2, 3.4 and 3.8, the following result implies that X and Y are homeomorphic after removing finite subsets. This extends the well-known fact that if there is a disjointness preserving bijective linear operator between $C_0(X)$ and $C_0(Y)$ then X and Y are homeomorphic (see [19], [11], [20]).

LEMMA 3.13. *Let T be Fredholm. Then $\varphi : (Y_{\mathrm{c}}, M) \to (\varphi(Y_{\mathrm{c}}), \varphi(M))$ is a relative proper homeomorphism. More precisely, $\varphi : Y_{\mathrm{c}} \setminus M \to \varphi(Y_{\mathrm{c}}) \setminus \varphi(M)$ is a proper homeomorphism, and the induced map $\widetilde{\varphi} : Y_{\mathrm{c}}/\!\sim\, \to \varphi(Y_{\mathrm{c}})$ is also an homeomorphism, where "$\sim$" is the equivalence relation such that $y_1 \sim y_2$ if and only if $\varphi(y_1) = \varphi(y_2)$.*

Proof. It follows from Lemma 3.10 that $\varphi : Y_{\mathrm{c}} \setminus M \to \varphi(Y_{\mathrm{c}}) \setminus \varphi(M)$ is bijective, proper and continuous. We claim that $\varphi^{-1} : \varphi(Y_{\mathrm{c}}) \setminus \varphi(M) \to Y_{\mathrm{c}} \setminus M$ is continuous, i.e., $y_\lambda \to y_0$ in $Y_{\mathrm{c}} \setminus M$ whenever $\varphi(y_\lambda) \to \varphi(y_0)$ in $\varphi(Y_{\mathrm{c}}) \setminus \varphi(M)$. Without loss of generality, we can assume that y_λ converge to a non-isolated point y' in Y_∞. If $y' \in Y_0$ or $y' = \infty$, then

$$0 = \lim_{\lambda\to\infty} Tf(y_\lambda) = \lim_{\lambda\to\infty} h(y_\lambda) f(\varphi(y_\lambda)), \quad f \in C_0(X).$$

Since h is away from zero (Lemma 3.9), we have $f(\varphi(y_\lambda)) \to 0$ as $\lambda \to \infty$ for all f in $C_0(X)$. This implies that $\varphi(y_0) = \infty$. It is impossible as $\varphi(Y_{\mathrm{c}}) \subseteq X$ (Theorem 2.4). By Lemma 3.4 (i), we thus have $y' \in Y_{\mathrm{c}}$ and $\varphi(y_0) = \varphi(y')$. Since y_0 is not a merging point of T in Y_{c}, we have $y_0 = y'$. Hence φ^{-1} is continuous, as asserted.

Next, we claim that $\widetilde{\varphi} : Y_c/\sim \to \varphi(Y_c)$ is an open map. Let $\widetilde{U}$ be an open set in $Y_c/\sim$, which lifts to an open set U in Y_c. Since $\varphi : Y_c \setminus M \to \varphi(Y_c) \setminus \varphi(M)$ is a homeomorphism, it suffices to show that if $c \in \widetilde{\varphi}(\widetilde{U})$ for some merged point c of T in X, then c is an interior point of $\widetilde{\varphi}(\widetilde{U})$. Suppose not, and there were z_λ in $\varphi(Y_c) \setminus \widetilde{\varphi}(\widetilde{U})$ such that z_λ converge to the non-isolated point c in $\varphi(Y_c)$. Without lose of generality, we can assume that all z_λ's are not in the finite set $\varphi(M)$. Let $y_\lambda = \varphi^{-1}(z_\lambda)$ in Y_c. As the equivalence classes $[y_\lambda] \notin \widetilde{U}$ imply $y_\lambda \notin U$, there exists a convergent subnet y_{λ_α} of y_λ in Y_∞ such that $y' = \lim y_{\lambda_\alpha} \notin U$. If $y' \in Y_c$ then $y' \notin U$ implies $[y'] \notin \widetilde{U}$. But, $\widetilde{\varphi}([y']) = \varphi(y') = \lim_{\lambda\to\infty} \varphi(y_\lambda) = \lim_{\lambda\to\infty} z_\lambda = c \in \widetilde{\varphi}(\widetilde{U})$, a contradiction. It is also plain that $y' \notin Y_d$ since Y_d contains only isolated points by Lemma 3.4 (i). So $y' \in Y_0$ or $y' = \infty$. Therefore,

$$0 = \lim Tf(y_{\lambda_\alpha}) = \lim h(y_{\lambda_\alpha}) f(\varphi(y_{\lambda_\alpha})).$$

Since h is away from zero by Lemma 3.9, we have $f(z_{\lambda_\alpha}) = f(\varphi(y_{\lambda_\alpha})) \to 0$. Thus, $f(c) = 0$ for all f in $C_0(X)$. This is a contradiction again. So c is an interior point of $\widetilde{\varphi}(\widetilde{U})$. This shows that $\widetilde{\varphi}(\widetilde{U})$ is an open set. Consequently, $\widetilde{\varphi} : Y_c/\sim \to \varphi(Y_c)$ is an homeomorphism as asserted. ∎

Now we are ready to state the main result in this paper.

THEOREM 3.14. *Let T be a disjointness preserving Fredholm linear operator from $C_0(X)$ into $C_0(Y)$ with nullity m and corank n. Then Y is a disjoint union*

$$Y = Y_c \cup Y_d \cup Y_0,$$

where the continuous part Y_c is open, the discontinuous part Y_d is finite and consists of isolated points, and the nullity part Y_0 is finite (and consists of isolated points when X is compact). There is a unique bounded and away from zero continuous scalar function h on Y_c and a unique continuous map φ from Y_∞ into X_∞ with $\varphi(\infty) = \infty$ such that:

(i) *$Tf = h \cdot f \circ \varphi$ on Y_c and Tf vanishes on Y_0, $\forall f \in C_0(X)$.*

(ii) *$\varphi(Y_0) = \{\infty\}$, $\varphi(Y_c) \subseteq X$ and $X \setminus \varphi(Y_c) = \{x_1, x_2, \ldots, x_m\}$ consists of m isolated points.*

(iii) $\ker(T) = \operatorname{span}\{\chi_{\{x_1\}}, \chi_{\{x_2\}}, \ldots, \chi_{\{x_m\}}\}$ *is a closed ideal of $C_0(X)$ of dimension m.*

(iv) *Let M be the finite set of all merging points of T in Y_c. Then*

$$\varphi : (Y_c, M) \to (\varphi(Y_c), \varphi(M))$$

is a relative proper homeomorphism. The induced map

$$\widetilde{\varphi} : Y_c/\sim \to \varphi(Y_c)$$

is an homeomorphism, where $y \sim y'$ if and only if $\varphi(y) = \varphi(y')$.

(v) *$n = m(T) + \#(Y_0) + \#(Y_d)$, where the merging index $m(T) = \#(M) - \#(\varphi(M))$.*

(vi) *If, in addition, Y contains no isolated point or T has closed range then T is bounded. Moreover, if T is bounded then T has closed range*

$$\operatorname{ran}(T) = \Big\{ g \in C_0(Y) : g(p_1) = g(p_2) = \cdots = g(p_k) = 0 \text{ and } \frac{g(a_1^{(i)})}{h(a_1^{(i)})} = \frac{g(a_2^{(i)})}{h(a_2^{(i)})} = \cdots = \frac{g(a_{l_i}^{(i)})}{h(a_{l_i}^{(i)})},\ \forall i = 1, 2, \ldots, j \Big\}.$$

Here $Y_0 = \{p_1, p_2, \ldots, p_k\}$, and $\varphi(M) = \{c_1, c_2, \ldots, c_j\}$ is the set of merged points of T in X when $\varphi^{-1}(c_i) = \{a_1^{(i)}, a_2^{(i)}, \ldots, a_{l_i}^{(i)}\}$ consists of the corresponding merging points of T in Y_c for $i = 1, 2, \ldots, j$.

Proof. We first note that assertions (i), (ii), (iii), (iv) and (v) are included in previous lemmas. Moreover, the description of the range space of T in (vi) follows from the continuity of T and Lemma 3.7 (and its proof). When Y contains no isolated points, T is automatically bounded by Lemma 3.4 (i). By Lemma 2.2, it is then enough to verify $Y_d = \emptyset$ when T has closed range.

In view of Lemma 3.4 (i), we might suppose, on the contrary, that the open set

$$Y_d = \{y_1, \ldots, y_l\} \neq \emptyset$$

and $l \leqslant n$. By Lemma 3.4 (ii), either there exists a y_i' in Y_c such that $\varphi(y_i) = \varphi(y_i') \neq \infty$, or $\varphi(y_i) = \infty$ in which case we set $y_i' = \infty$, for each $i = 1, 2, \ldots, l$. For those $y_i' = \infty$, we let V_i be any open set in Y_c such that $Y_c \setminus V_i$ is compact and V_i is disjoint from the finite set M. For the others, we let V_i be any open set in Y_c containing $\varphi^{-1}(\varphi(y_i)) \cap Y_c$. It follows from Lemma 3.13 that $\varphi(V_i)$ is open in $\varphi(Y_c) = X \setminus \{x_1, x_2, \ldots, x_m\}$, and thus open in X since $x_1, x_2, \ldots, x_m$ are isolated points in X, for $i = 1, 2, \ldots, l$.

Claim 1. Let g in $C_0(Y)$ satisfy that g vanishes on $\bigcup_{i=1}^{l} V_i$ and Y_0, and $\frac{g(z_1)}{h(z_1)} = \frac{g(z_2)}{h(z_2)}$ whenever $z_1 \sim z_2$ in Y_c. Then $g \in \operatorname{ran}(T)$ if and only if g vanishes on Y_d.

Let $g = Tf \in \operatorname{ran}(T)$. Note that g vanishes on $\bigcup_{i=1}^{l} V_i$, and h is away from zero by Lemma 3.9. So f vanishes on the open set $\bigcup_{i=1}^{l} \varphi(V_i)$, and thus, $\varphi(y_i) \notin \operatorname{supp}(f)$ for $i = 1, 2, \ldots, l$. By Theorem 2.4, we have $g(y_i) = Tf(y_i) = 0$.

On the other hand, if g vanishes on Y_d we define $f(\varphi(y)) = \frac{g(y)}{h(y)}$, $\forall y \in Y_c$, and $f(x) = 0$, $\forall x \in X \setminus \varphi(Y_c)$. By Lemmas 3.2 and 3.4, $f \in C_0(X)$ and $Tf(y) = g(y)$, $\forall y \in Y_c \cup Y_0$. Note that Tf vanishes on $\bigcup_{i=1}^{l} V_i$. By an argument similar to above, $Tf(y_i) = 0$ for all $i = 1, 2, \ldots, l$. Hence $g = Tf \in \operatorname{ran}(T)$.

Claim 2. Let g be in $C_0(Y)$ such that g vanishes on $Y_d \cup Y_0$, and $\frac{g(z_1)}{h(z_1)} = \frac{g(z_2)}{h(z_2)}$ whenever $z_1 \sim z_2$ in Y_c. Suppose that $g(y) = 0$ whenever $y \in Y_c$ and $\varphi(y) = \varphi(y_i)$ for some $i = 1, 2, \ldots, l$. Then $g \in \operatorname{ran}(T)$.

For all $\varepsilon > 0$, we note that $U_\varepsilon = \{y \in Y_\infty : |g(y)| < \varepsilon\}$ is an open subset of Y_∞. We choose some open sets V_i, described as in the paragraph just before Claim 1, such that $V_i \subseteq U_{\frac{\varepsilon}{2}}$ for $i = 1, 2, \ldots, l$. Let $0 \leqslant h_\varepsilon \leqslant 1$ be a continuous function in Y such that $h_\varepsilon \equiv 1$ outside U_ε and $h_\varepsilon(y) = 0$ for y in $U_{\frac{\varepsilon}{2}}$. Let $g_\varepsilon = g \cdot h_\varepsilon$ in $C_0(Y)$. Then g_ε vanishes on $U_{\frac{\varepsilon}{2}}$ and Y_0. Since the merging index $m(T)$ is finite, we can even assume that $\frac{g_\varepsilon(z_1)}{h(z_1)} = \frac{g_\varepsilon(z_2)}{h(z_2)}$, whenever $z_1 \sim z_2$ in Y_c, if ε is small enough. By Claim 1, we have $g_\varepsilon \in \operatorname{ran}(T)$. Clearly, $\|g_\varepsilon - g\| \leqslant 2\varepsilon$. Since $\operatorname{ran}(T)$ is closed, we have $g \in \operatorname{ran}(T)$.

Claim 3. Let f be in $C_0(X)$ such that $Tf(y'_i) = 0$ for all $i = 1, 2, \ldots, l$. Then $Tf(y_i) = 0$ for all $i = 1, 2, \ldots, l$.

Note that the assumption Tf vanishes at y'_i implies that $Tf(y) = h(y)f(\varphi(y))$ also vanishes at all other y in Y_c with $\varphi(y) = \varphi(y'_i) = \varphi(y_i)$ for $i = 1, 2, \ldots, l$. Define a scalar-valued function g on Y by $g(y) = Tf(y)$ for $y \neq y_1, y_2, \ldots, y_l$, and $g(y) = 0$ for $y = y_1, y_2, \ldots, y_l$. Note that $y_1, y_2, \ldots, y_l$ are isolated points (Lemma 3.4 (i)). So $g \in C_0(Y)$. By Claim 2, we have $g \in \operatorname{ran}(T)$, and $\sum_{i=1}^{l} \lambda_i \chi_{\{y_i\}} = Tf - g \in \operatorname{ran}(T)$ for some scalars λ_i. But $\sum_{i=1}^{l} \lambda_i \chi_{\{y_i\}} \notin \operatorname{ran}(T)$ unless all λ_i's are zero by Remark 3.3. Therefore, $Tf(y_i) = 0$ for all $i = 1, 2, \ldots, l$.

We now define linear operators $S_1, S_2 : C_0(X) \to \mathbb{F}^l$ by

$$S_1(f) = \begin{pmatrix} Tf(y_1) \\ \vdots \\ Tf(y_l) \end{pmatrix} \quad \text{and} \quad S_2(f) = \begin{pmatrix} Tf(y'_1) \\ \vdots \\ Tf(y'_l) \end{pmatrix},$$

where $\mathbb{F}$ is the underlying scalar field. Let $A : \mathbb{F}^l \to \mathbb{F}^l$ be a linear operator satisfying that

$$A \begin{pmatrix} Tf(y'_1) \\ \vdots \\ Tf(y'_l) \end{pmatrix} = \begin{pmatrix} Tf(y_1) \\ \vdots \\ Tf(y_l) \end{pmatrix}.$$

Since $\bigcap \ker(\delta_{y'_i} \circ T) \subseteq \bigcap \ker(\delta_{y_i} \circ T)$ by Claim 3, such a linear operator A exists. Moreover, A can be represented as an $l \times l$ matrix $(a_{ij})_{l \times l}$, and $S_1 = AS_2$. As a result, $\delta_{y_i} \circ T = \sum_j a_{ij} \cdot \delta_{y'_j} \circ T$ for all $i = 1, 2, \ldots, l$. Note that $y'_j \in Y_c \cup \{\infty\}$ for all $j = 1, 2, \ldots, l$. Thus, $\delta_{y_i} \circ T$ is continuous, but $y_i \in Y_d$ for $i = 1, 2, \ldots, l$. This contradiction says that $Y_d = \emptyset$. ∎

As a consequence of Proposition 2.7 and Theorem 3.14, we have

COROLLARY 3.15. *Let T be a disjointness preserving Fredholm linear operator from $C_0(X)$ into $C_0(Y)$. Then T is bounded if and only if T has closed range.*

REMARK 3.16. One can find an example of an unbounded disjointness preserving linear operator in [19]. Consequently, there is an unbounded disjointness preserving linear functional ϱ on some $C_0(X)$ (Lemma 2.2). Let $Y = X \cup \{y\}$ be a disjoint union. Define $T : C_0(X) \to C_0(Y)$ by setting $Tf_{|X} = f$ and $Tf(y) = \varrho(f)$

for each f in $C_0(X)$. Then, T is an injective disjointness preserving linear map of corank 1. But, T is unbounded. It is easy to see that T does not have closed range. For example, let f_n be a null sequence in $C_0(X)$ such that $\varrho(f_n)$ approaches 1. The Cauchy sequence $\{Tf_n\}$ does not converge in $\operatorname{ran}(T)$. Note also that Y contains an isolated point in this case (cf. Lemma 3.4).

In [3], Araujo showed that an injective disjointness preserving linear operator on a class of continuous vector-valued function spaces is bounded if it has closed range. Compare this with the following result.

COROLLARY 3.17. *Let T be a disjointness preserving linear operator from $C_0(X)$ into $C_0(Y)$ with finite corank. Then T is bounded if and only if T has closed range and the kernel of T is a closed ideal of $C_0(X)$. In this case, the conclusions in Theorem* 3.14, *except for possibly* (ii) *and* (iii), *are valid for T. Instead, we have $\varphi(Y_c)$ is closed in X and $\ker T = \{f \in C_0(X) : f$ vanishes on $\varphi(Y_c)\}$.*

Proof. By Lemma 2.1, we can assume that the underlying field is the complex scalars. The necessity follows from Theorem 2.4 and Proposition 2.7. For the sufficiency, we note that if $\ker T$ is a closed ideal then it must be in the form of $\{f \in C_0(X) : f \text{ vanishes on } \widetilde{X}\}$ for some closed subset $\widetilde{X}$ of X. Consequently, the quotient algebra of $C_0(X)$ by $\ker T$ is isomorphic to $C_0(\widetilde{X})$. The induced injective linear operator $\widetilde{T}$ from $C_0(\widetilde{X})$ into $C_0(Y)$ also has closed range and finite corank. We shall show that $\widetilde{T}$ is disjointness preserving. To this end we note that if $\widetilde{f}$ and $\widetilde{g}$ are non-negative functions in $C_0(\widetilde{X})$ with $\widetilde{f} \cdot \widetilde{g} = 0$, we can extend them to f and g in $C_0(X)$ with $f \cdot g = 0$, too. In fact, we can set $\widetilde{h} = \widetilde{f} - \widetilde{g}$ and extend it to an h in $C_0(X)$ by Tietze's Extension Theorem. Then the desired disjoint extensions are $f = \max\{h, 0\}$ and $g = \max\{-h, 0\}$, respectively. Consequently, $\widetilde{T}\widetilde{f} \cdot \widetilde{T}\widetilde{g} = Tf \cdot Tg = 0$ for all non-negative $\widetilde{f}$, $\widetilde{g}$ in $C_0(\widetilde{X})$ with $\widetilde{f} \cdot \widetilde{g} = 0$. By writing each function in $C_0(\widetilde{X})$ as a linear sum of at most four non-negative functions, we can conclude that $\widetilde{T}$ is disjointness preserving (cf. the proof of Lemma 2.1). The other assertions follows from Theorem 3.14. In particular, $\widetilde{X} = \varphi(Y_c)$ in this case. ∎

4. AN APPLICATION

The results of this paper are important tools in the study of shift operators ([7], [21]). Besides, we provide a supplement to the following interesting Gleason-Kahane-Zelazko type result of Chang-Pao Chen ([6]; see also Jarosz [18]) as an application. Recall that a subspace A of $C_0(Y)$ is said to be an *Z_n-subspace* if every g in A has at least n distinct zeroes in Y. If every (closed) n-codimensional Z_n-subspace of $C_0(Y)$ is of the form $\bigcap\{\ker \delta_{y_i} : i = 1, 2, \ldots, n\}$ for n distinct points $y_1, y_2, \ldots, y_n$ in Y then $C_0(Y)$ is said to have the (*closed*) *I_n-property.* If Y is compact then the classical Gleason-Kahane-Zelazko Theorem ([12], [22]) states that $C(Y)$ has I_1-property.

THEOREM 4.1. ([6], Corollary 4.4) *Let Y be a locally compact Hausdorff space and let n be a positive integer greater than 1. Then every closed Z_n-subspace of $C_0(Y)$ is the intersection of n maximal ideals of $C_0(Y)$ if and only if Y is σ-compact and every point in Y is a G_δ set.*

The σ-compactness and G_δ conditions on Y ensure that for every point y in Y_∞ there is an f in $C_0(Y)$ vanishing exactly at y (and ∞). As a consequence of this fact and Theorem 3.14 (without assuming Theorem 4.1, though), we have

COROLLARY 4.2. *Let A be an n-codimensional subspace of $C_0(Y)$, which is the range of a bounded disjointness preserving linear operator T. If A is an ideal then A is a closed Z_n-subspace. In this case, A is the intersection of n maximal ideals of $C_0(Y)$. The converse holds if Y is σ-compact and every point in Y is a G_δ set.*

Proof. In view of the proof of Corollary 3.17, we may assume that T is injective. By Theorem 3.14 (vi), we see that $A = \operatorname{ran}(T)$ is a closed Z_n-subspace of $C_0(Y)$ if $\operatorname{ran}(T)$ is an ideal of $C_0(Y)$. In fact, $\operatorname{ran}(T)$ is the intersection of n maximal ideals of $C_0(Y)$ in this case. Conversely, suppose that $\operatorname{ran}(T)$ is an Z_n-subspace of $C_0(Y)$. We let g be a continuous scalar function on Y vanishing exactly at $p_1, p_2, \ldots, p_k$ and ∞. In particular, g does not vanish in a neighborhood of each $a_j^{(i)}$ in the notation of Theorem 3.14 (vi). Note that there exists an everywhere positive continuous function f in $C_0(Y)$. For each $a_j^{(i)}$, one by one, let $f_j^{(i)}$ be a non-negative continuous function on Y such that $f_j^{(i)}(a_j^{(i)}) = \frac{1}{2}\frac{h(a_j^{(i)})}{g(a_j^{(i)})}$ and $f_j^{(i)}$ vanishes outside a neighborhood of $a_j^{(i)}$, which does not contain any of $p_1, \ldots, p_k$ or other $a_{j'}^{(i')}$. Replace g by $g \cdot \left(f_j^{(i)} + \frac{1}{2}\frac{h(a_j^{(i)})}{g(a_j^{(i)})}\frac{f}{f(a_j^{(i)})}\right)$, recursively. Then $g(p_1) = \cdots = g(p_k) = 0$ and $\frac{g(a_j^{(i)})}{h(a_j^{(i)})} = 1$. In this way, we can redefine g locally at each $a_j^{(i)}$ such that g satisfies the remaining $n - k$ linear equations stated in Theorem 3.14 (vi) without introducing additional zeroes to g. Then g is in the range of T having exactly k zeroes. This implies that $n = k$ and thus $\operatorname{ran}(T)$ is an ideal. ∎

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