J. OPERATOR THEORY
49(2003), 77–84

REAL STRUCTURE IN PURELY INFINITE C^*-ALGEBRAS

P.J. STACEY

Communicated by Norberto Salinas

Abstract. It is shown that if the complexification of a real C^*-algebra A is purely infinite then A shares this property. The converse is also established when the complexification is simple, unital and has real rank zero.

Keywords: *C^*-algebras, purely infinite, real structure, involutory $*$-antiautomorphism.*

MSC (2000): Primary 46L05; Secondary 46L35.

1. INTRODUCTION

A C^*-algebra A is said to be *purely infinite* if each non-zero hereditary C^*-subalgebra B contains an infinite projection p i.e. a projection p for which there is a partial isometry v in B with $vv^* = p$ and $v^*v < p$. For simple C^*-algebras there are many equivalent conditions, as described in [10].

These definitions also make sense for real C^*-algebras, although an examination of the hereditary real C^*-subalgebras of a real C^*-algebra leads naturally to a variant in which not all hereditary C^*-subalgebras are considered. Details are given in Sections 2 and 3 below. It is then natural to conjecture that a real C^*-algebra will be purely infinite if and only if its complexification is or, equivalently, that, for each C^*-algebra A and involutory $*$-antiautomorphism Φ of A, A will be purely infinite if and only if $A_\Phi = \{a \in A : \Phi(a) = a^*\}$ is. The purpose of the present paper is to establish that whenever A is purely infinite, then so is A_Φ. The converse is also established under certain restrictive conditions.

The reader who is unfamiliar with real C^*-algebras is referred to [7] for the basic theory and [15] for more specialised results. Much of the theory closely resembles the complex case but there are significant differences. One such difference, which arises in the present paper, is the looser link between algebraic and order theoretic properties in the real case. For example, the positive cone in a real C^*-algebra need not generate the algebra and there can be more than one algebra with the same positive cone.

2. HEREDITARY SUBALGEBRAS OF REAL C^*-ALGEBRAS

The problem posed above leads to a consideration of the relationship between hereditary subalgebras of A_Φ and A. It is not always true that the complexification of a hereditary subalgebra of A_Φ is hereditary in A, as is shown by taking A to be $M_2(\mathbb{C})$ and Φ to map $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then A_Φ is isomorphic to the algebra $\mathbb{H}$ of quaternions and $\mathbb{R}1$ is a hereditary subalgebra of A_Φ, although its complexification $\mathbb{C}1$ is not hereditary in A. As the following two easy propositions show, this behaviour is related to the fact that both $\mathbb{R}1$ and $\mathbb{H}$ have the same positive cone and only $\mathbb{H}$ is of the form $L \cap L^*$ for some left ideal L in $\mathbb{H}$. The first result is the real version of Theorem 1.5.2 of [11], in which all three maps are bijections.

Proposition 2.1. *Let R be a real C^*-algebra.*

(i) *The mapping $B \mapsto B_+$ is an order preserving surjection from the set of hereditary C^*-subalgebras of R onto the set of closed hereditary cones of R_+.*

(ii) *The mapping $L \mapsto L \cap L^*$ is an order preserving injection from the set of closed left ideals of R into the set of hereditary C^*-subalgebras of R.*

(iii) *The composite mapping $L \mapsto (L \cap L^*)_+$ is an order preserving bijection from the set of closed left ideals of R onto the set of closed hereditary cones of R_+ with inverse $M \mapsto L(M) = \{x \in R : x^*x \in M\}$.*

Proof. The proof of Theorem 1.5.2 of [11] applies without change. ∎

Based on Proposition 2.1, define a *regular* hereditary C^*-subalgebra of a real C^*-algebra R to be one of the form $L \cap L^*$ for some closed left ideal L of R. The simplest example of a non-regular hereditary C^*-subalgebra is $\mathbb{R} \subseteq \mathbb{H}$, which has no non-trivial left ideals. As remarked above, the complexification of this non-regular hereditary subalgebra is not hereditary. However, as the next proposition shows, the complexification of a regular hereditary C^*-subalgebra is always hereditary.

Proposition 2.2. *Let Φ be an involutory $*$-antiautomorphism of a C^*-algebra A and let $A_\Phi = \{x \in A : \Phi(x) = x^*\}$.*

(i) *The mapping $L \mapsto L^{\mathbb{C}} \cap (L^{\mathbb{C}})^*$, where $L^{\mathbb{C}} = \{x + \mathrm{i}y : x, y \in L\}$, is a bijection from the closed left ideals of A_Φ onto the Φ-invariant hereditary C^*-subalgebras of A, with inverse $B \mapsto L(B) \cap A_\Phi = \{x \in A_\Phi : x^*x \in B\}$.*

(ii) *The mapping $B \mapsto B \cap A_\Phi$ is a bijection from the Φ-invariant hereditary C^*-subalgebras of A onto the regular hereditary real C^*-subalgebras of A_Φ, with inverse $C \mapsto C + \mathrm{i}C$.*

Proof. (i) $L^{\mathbb{C}}$ is clearly a closed left ideal of A and so $L^{\mathbb{C}} \cap (L^{\mathbb{C}})^*$ is a hereditary C^*-subalgebra, which is Φ-invariant because $\Phi(L^{\mathbb{C}}) = (L^{\mathbb{C}})^*$. The map $L^{\mathbb{C}} \mapsto L^{\mathbb{C}} \cap (L^{\mathbb{C}})^*$ is injective by Theorem 1.5.2 of [11] and the map $L \mapsto L^{\mathbb{C}}$ is injective because $L^{\mathbb{C}} \cap A_\Phi = L$. If B is a Φ-invariant hereditary C^*-subalgebra of A, then $\{x \in A_\Phi : x^*x \in B\}^{\mathbb{C}} \subseteq \{x \in A : x^*x \in B\}$. Conversely let $(x + \mathrm{i}y)^*(x + \mathrm{i}y) \in B$ with $x, y \in A_\Phi$. From the Φ-invariance of B, $(x^* + \mathrm{i}y^*)(x - \mathrm{i}y) = \Phi(x + \mathrm{i}y)\Phi(x + \mathrm{i}y)^* \in B$, so both $x^*x + y^*y + \mathrm{i}(x^*y - y^*x) \in B$ and $x^*x + y^*y - \mathrm{i}(x^*y - y^*x) \in B$. Thus $x^*x \in B$ and $y^*y \in B$ and so $\{x \in A_\Phi : x^*x \in B\}^{\mathbb{C}} = \{x \in A : x^*x \in B\}$. From Theorem 1.5.2 of [11], it follows that $\{x \in A_\Phi : x^*x \in B\}^{\mathbb{C}} \cap \left(\{x \in A_\Phi : x^*x \in B\}^{\mathbb{C}}\right)^* = B$.

(ii) By part (i) and Proposition 2.1 the map $B \mapsto A_\Phi \cap L(B) \cap L(B)^* = A_\Phi \cap B$ is a bijection from the Φ-invariant hereditary C^*-subalgebras of A onto the regular hereditary real C^*-subalgebras of A_Φ with inverse $D \mapsto L(D)^{\mathbb{C}} \cap \big(L(D)^{\mathbb{C}}\big)^*$. Clearly $D^{\mathbb{C}} \subseteq L(D)^{\mathbb{C}} \cap \big(L(D)^{\mathbb{C}}\big)^*$. Conversely, given $x \in L(D)^{\mathbb{C}} \cap \big(L(D)^{\mathbb{C}}\big)^*$, $x = \frac{1}{2}(x + \Phi(x^*)) - \frac{\mathrm{i}}{2}(\mathrm{i}x + \Phi((\mathrm{i}x)^*))$, where $x + \Phi(x^*)$ and $\mathrm{i}x + \Phi((\mathrm{i}x)^*)$ both belong to $L(D)^{\mathbb{C}} \cap (L(D)^{\mathbb{C}})^* \cap A_\Phi = D$, as required. ∎

It follows from part (ii) of the proposition that, for each positive element x in A_Φ, the closure of $xA_\Phi x$, which is equal to $\overline{xAx} \cap A_\Phi$, is a regular hereditary subalgebra of A_Φ. However the hereditary real C^*-subalgebra generated by x may be a proper subset, as illustrated by the case $x = 1$ and $A_\Phi = \mathbb{H}$.

3. REAL STRUCTURES IN PURELY INFINITE ALGEBRAS

Following the local version of the usual complex definition, we will say that a real C^*-algebra R is *purely infinite* if, for each non-zero positive element x, the hereditary real C^*-algebra $\overline{xRx}$ contains an infinite projection. Note that it is possible, if somewhat unlikely, that some non regular hereditary real C^*-subalgebra B will not contain a projection which is infinite in B, even though the set of projections of such an algebra is equal to the set of projections of the regular hereditary real C^*–algebra C sharing the same positive cone, and B thus contains a projection which is infinite in C.

The aim of the present section is to establish, using the method of proof of Theorem 3 of [8], that whenever A is purely infinite then so is A_Φ for each involutory $*$-antiautomorphism Φ of A. The first preparatory lemma, which is a minor variant of Lemma 1.1 of [9], shows that involutory $*$-antiautomorphisms share some of the properties of properly outer automorphisms.

Lemma 3.1. *Let A be a C^*-algebra and let Φ be an involutory $*$-antiautomorphism of A. For any non-zero hereditary C^*-subalgebra B of A and for any $a \in A$ with $\Phi(a) = a$,* $\inf\{\|xa\Phi(x)\| : 0 \leqslant x \in B, \|x\| = 1\} = 0$. *When A is non-unital then, in addition,* $\inf\{\|x\Phi(x)\| : 0 \leqslant x \in B, \|x\| = 1\} = 0$.

Proof. Suppose, to obtain a contradiction, that $\inf\{\|xa\Phi(x)\| : 0 \leqslant x \in B,\ \|x\| = 1\} = \delta > 0$ or that $\inf\{\|x\Phi(x)\| : 0 \leqslant x \in B,\ \|x\| = 1\} = \delta > 0$. The first part of the proof of Lemma 1.1 of [9] applies without change to the antilinear automorphism $\alpha = \Phi \circ *$ of A, to produce a pure state φ of B, with GNS representation, $(\pi_\varphi, H_\varphi, \Omega_\varphi)$, and an antiunitary operator V on H_φ with $V\pi_\varphi(x)V^* = \pi_\varphi(\alpha(x))$ for each $x \in A$. If E is the projection onto the closed subspace $[\pi_\varphi(B)\Omega_\varphi] = [\pi_\varphi(B)H_\varphi]$ then, for any unit vector $h \in EH_\varphi$, $|(\pi_\varphi(a)Vh, h)| \geqslant \delta$ or $|(Vh, h)| \geqslant \delta$. For each $x \in A$, $V^2\pi_\varphi(x)V^2 = V\pi_\varphi(\Phi(x^*))V^* = \pi_\varphi(x)$, so $V^2 = \lambda I$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then $V^3 = V^2V = \lambda V$ and also $V^3 = VV^2 = V(\lambda I) = \overline{\lambda}V$, so $\lambda = \overline{\lambda}$ and therefore $\lambda = \pm I$.

When $V^2 = I$ then $(\pi_\varphi(a)Vh, h) = (h, V^*\pi_\varphi(a^*)h) = (h, V\pi_\varphi(a^*)h) = (h, \pi_\varphi(\Phi(a))Vh)$. Thus, using the condition $\Phi(a) = a$, $(\pi_\varphi(a)Vh, h) \in \mathbb{R}$. Also $(Vh, h) \in \mathbb{R}$. Then, after replacing V by $-V$ if necessary, $(E\pi_\varphi(a)VEh, h) =$

$(\pi_\varphi(a)Vh,h) \geqslant \delta$ or $(Vh,h) \geqslant \delta$ for each unit vector $h \in EH_\varphi$. It follows, as in the proof of Lemma 1.1 of [9], that $E\pi_\varphi(a)VE \geqslant \delta E$ or $EVE \geqslant \delta E$ and then that

$$\pi_\varphi(b^*)\pi_\varphi(a)V\pi_\varphi(b) \geqslant \delta\pi_\varphi(b^*b) \quad \text{or} \quad \pi_\varphi(b^*)V\pi_\varphi(b) \geqslant \delta\pi_\varphi(b^*b) \tag{3.1}$$

for each $b \in B$.

When $V^2 = -I$ then $(\pi_\varphi(a)Vh,h) = -(h,\pi_\varphi(a)Vh)$ and so $(\pi_\varphi(a)Vh,h) \in \mathrm{i}\mathbb{R}$. Similarly $(Vh,h) \in \mathrm{i}\mathbb{R}$. Again replacing V by $-V$ if necessary, $\mathrm{i}(\pi_\varphi(a)Vh,h) \geqslant \delta$ or $\mathrm{i}(Vh,h) \geqslant \delta$ for each unit vector h in EH_φ, so $\mathrm{i}E\pi_\varphi(a)VE \geqslant \delta E$ or $\mathrm{i}EVE \geqslant \delta E$ and then

$$\mathrm{i}\pi_\varphi(b^*)\pi_\varphi(a)V\pi_\varphi(b) \geqslant \delta\pi_\varphi(b^*b) \quad \text{or} \quad \mathrm{i}\pi_\varphi(b^*)V\pi_\varphi(b) \geqslant \delta\pi_\varphi(b^*b) \tag{3.2}$$

for each $b \in B$.

In both cases ($V^2 = I$ and $V^2 = -I$), the real C^*-algebra generated by $\pi_\varphi(A)$ and V consists of elements of the form $\pi_\varphi(x)+V$ or $\pi_\varphi(x)+\pi_\varphi(y)V$, where $x, y \in A$. The representation of each element is unique because each $\pi_\varphi(x)$ is linear whereas V is antilinear. Hence $\beta : R \to R$ can be defined by $\beta(\pi_\varphi(x)+\pi_\varphi(y)V) = \pi_\varphi(x) - \pi_\varphi(y)V$ and $\beta(\pi_\varphi(x) + V) = \pi_\varphi(x) - V$ and it is easily checked that β is a $*$-automorphism of R. Applying β to both sides of (3.1) or (3.2) then gives the required contradiction. (Although β is not necessarily complex linear, $\beta(\mathrm{i}\pi_\varphi(b^*)) = \beta(\pi_\varphi(\mathrm{i}b^*)) = \pi_\varphi(\mathrm{i}b^*) = \mathrm{i}\pi_\varphi(b^*)$ for $b \in B$.) ∎

Lemma 3.2. *Let A be a purely infinite C^*-algebra, let Φ be an involutory $*$-antiautomorphism of A and let $\varepsilon > 0$. Then there exists a projection $p \in A$ with $\|p\Phi(p)\| < \varepsilon$.*

Proof. By Lemma 3.1 there exists $x \in A$ with $x \geqslant 0$, $\|x\| = 1$ and $\|x\Phi(x)\| < \frac{\varepsilon}{4}$. As described in the proof of Theorem V.5.5 of [4], let

$$f(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 1 - \frac{\varepsilon}{4}, \\ 1 - \frac{4(1-t)}{\varepsilon} & \text{if } 1 - \frac{\varepsilon}{4} < t \leqslant 1. \end{cases}$$

There exists an infinite projection p in the hereditary C^*-subalgebra $\overline{f(x)Af(x)}$ and, as in the proof of Theorem V.5.5 of [4], note that $p \leqslant E_x[1 - \frac{\varepsilon}{4}, 1]$, where the right hand side denotes the spectral projection of x corresponding to the interval $[1 - \frac{\varepsilon}{4}, 1]$. Consideration of the commutative W^*-algebra generated by x reveals that $\|xE_x - E_x\| \leqslant \frac{\varepsilon}{4}$ and thus

$$\begin{aligned} \|E_x\Phi(E_x)\| &\leqslant \|xE_x\Phi(xE_x)\| + \|(E_x - xE_x)\Phi(xE_x)\| + \|E_x(\Phi(E_x) - \Phi(xE_x))\| \\ &\leqslant \|x\Phi(x)\| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon. \end{aligned}$$

Thus, using $pE_x = p = E_x p$,

$$\|p\Phi(p)\| = \|pE_x\Phi(pE_x)\| = \|pE_x\Phi(E_x)\Phi(p)\| \leqslant \|E_x\Phi(E_x)\| \leqslant \varepsilon. \quad ∎$$

The main result can now be proved using the method of proof of Theorem 3 of [8].

THEOREM 3.3. *Let A be a purely infinite C^*-algebra, let Φ be an involutory $*$-antiautomorphism of A and let $A_\Phi = \{x \in A : \Phi(x) = x^*\}$. Then A_Φ is purely infinite.*

Proof. Let $0 < \varepsilon < \frac{1}{10}$, let B be a regular hereditary real C^*-subalgebra of A_Φ and let $B^{\mathbb{C}} = B + \mathrm{i}B$ be the Φ-invariant hereditary C^*-subalgebra of A associated with B as in Proposition 2.2 (ii). By Lemma 3.2 applied to $B^{\mathbb{C}}$ there exists a projection p in $B^{\mathbb{C}}$ with $\|p\Phi(p)\| < \varepsilon$. By Lemma 2.7 of [6] $\widetilde{p} = p \vee \Phi(p) \in B^{\mathbb{C}}$ and hence, by Proposition 2.2 (ii), $\widetilde{p} \in B^{\mathbb{C}}_+ \cap A_\Phi = B_+$. From the purely infinite property of $B^{\mathbb{C}}$ there exists a partial isometry, v in $B^{\mathbb{C}}$ with $v^*v = p,\ vv^* = e < p,\ v = ev$ and $v = vp$. As in the proof of Theorem 3 of [8], put $w = v + \Phi(v^*)$. Then $w^*w = v^*v + \Phi(v^*v) + v^*\Phi(v^*) + \Phi(v)v = p + \Phi(p) + v^*\Phi(v^*) + \Phi(v)v$. Note that $\|v^*\Phi(v^*)\| = \|\Phi(v)v\| = \|\Phi(ev)ev\| = \|\Phi(v)\Phi(e)ev\| \leqslant \|\Phi(e)e\| = \|\Phi(pe)pe\| = \|\Phi(e)\Phi(p)pe\| \leqslant \|\Phi(p)p\| < \varepsilon$ and that, by Lemma 2 of [8], $\|p + \widetilde{\Phi}(p) - \widetilde{p}\| < 4\varepsilon/\sqrt{1-2\varepsilon} < 8\varepsilon$. Thus $\|w^*w - \widetilde{p}\| < 10\varepsilon < 1$. However, using $vp = v$ and $\Phi(v^*)\Phi(p) = \Phi(v^*)$, $w\widetilde{p} = [v + \Phi(v^*)](p \vee \Phi(p)) = v(p \vee \Phi(p)) + \Phi(v^*)(p \vee \Phi(p)) = v + \Phi(v^*) = w$. Thus $w^*w\widetilde{p} = w^*w$ and hence also $\widetilde{p}w^*w = w^*w$, so $w^*w \in B^{\mathbb{C}}_{\widetilde{p}} = \widetilde{p}B^{\mathbb{C}}\widetilde{p}$. From $\|w^*w - \widetilde{p}\| < 1$, w^*w is invertible in $B^{\mathbb{C}}_{\widetilde{p}}$. Let its inverse be y and let $u = wy^{1/2}$. Then $u^*u = y^{1/2}w^*wy^{1/2} = \widetilde{p}$ and $uu^* = wyw^*$. As observed above, $\|\Phi(e)e\| < \varepsilon$ so $\widetilde{e} = e \vee \Phi(e) \in B^{\mathbb{C}}$ and $\widetilde{e}w = (e \vee \Phi(e))(v + \Phi(v^*)) = v + \Phi(v^*) = w$. Hence $w^*\widetilde{e} = w^*$ and $\widetilde{e}uu^* = uu^* = uu^*\widetilde{e}$.

If $\widetilde{e} = \widetilde{p}$ then $(p - e)\widetilde{e} = p - e$ but

$$\|(p-e)\widetilde{e}\| \leqslant \|\widetilde{e} - e - \Phi(e)\| + \|(p-e)(e + \Phi(e)\| < \frac{4\varepsilon}{\sqrt{1-2\varepsilon}} + \|(p-e)\Phi(e)\|$$

$$= \frac{4\varepsilon}{\sqrt{1-2\varepsilon}} + \|(p-e)p\Phi(p)\Phi(e)\| < \frac{4\varepsilon}{\sqrt{1-2\varepsilon}} + \varepsilon < 1.$$

Hence $uu^* \leqslant \widetilde{e} < \widetilde{p}$, showing that $\widetilde{p}$ is an infinite projection in $B^{\mathbb{C}}$. From the definition of $w, \Phi(w) = w^*$ so $w \in B^{\mathbb{C}} \cap A_\Phi$ which, by the regularity of B, is equal to B and therefore $y \in B$ and $u \in B$. Thus $\widetilde{p}$ is an infinite projection in B. ∎

4. THE COMPLEXIFICATION OF CERTAIN PURELY INFINITE REAL ALGEBRAS

It is natural to conjecture that the converse of Theorem 3.3 is also true. However a natural difficulty arises when trying to establish the existence of an infinite projection in a hereditary C^*-subalgebra B of A with $B \cap A_\Phi = \{0\}$. The algebra A can be regarded as a subalgebra of $M_2(A_\Phi)$ but there are difficulties establishing that a projection in A which is infinite in $M_2(A_\Phi)$ is infinite in A, just as there are differences between the K-theory of a real algebra A_Φ and its complexification A. In the present section a partial converse will be obtained by making the restrictive assumptions that A is simple, unital and has real rank zero. It is natural to conjecture that the latter property can be deduced from the fact, to be noted below, that A_Φ has real rank zero. However, in an analogous way to the discrete crossed product situation discussed after Proposition 3.6 of [3], it is not obvious how to link the real rank of an algebra and its complexification.

The method to be employed in establishing the partial converse is to observe that Rørdam's proof, that if A is a purely infinite simple C^*-algebra then $M(A \otimes K)/A \otimes K$ is simple, can be carried over to the real setting. The first step is to observe the equivalence in the real case of the definition of purely infinite used in [14] and that given above.

Proposition 4.1. *A unital simple real C^*-algebra A different from $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ is purely infinite if and only if for all non-zero positive $x \in A$ there exists $r \in A$ with $r^*xr = 1$.*

Proof. The proof of Theorem V.5.5 of [4] (together with the proofs of its antecedents Theorem V.5.1, Exercise V.9 and Lemma V.5.4) applies without substantial change to the real case. It should however be noted that the proof relies on the regularity of the appropriate hereditary subalgebra B in order to show that a given partial isometry SP belongs to B: this is necessary to show that the projection P is infinite in B. ∎

Other results which hold in the real case by the same proofs as in the complex case (from Theorem 2.6 of [3] and Proposition 14 of [12]) are the equivalence of the conditions HP, FS and real rank zero. Furthermore, a purely infinite real C^*-algebra has the real rank zero for the same reasons as in the complex case, described in Lemma 1.1, Corollary 1.3 and Theorem 1.2 of [18], together with the underlying Theorem 4.23 of [2] and Theorem 2.1, Lemmas 2.2 to 2.5, Corollary 2.6 and Theorem 3.1 of [1]. (The constructions in these results are compatible with the actions of the appropriate antiautomorphisms.) It also follows as in the complex case, described in Theorems 2.10 and 2.5 of [3], that the $n \times n$ matrix algebra over a real C^*-algebra with the property FS also has this property. This then leads to the following real version of a well-known result.

Lemma 4.2. *If R is a purely infinite real C^*-algebra then so is $M_n(R)$.*

Proof. Let A be the complexification of R and identify R with $A_\Phi = \{x \in A : \Phi(x) = x^*\}$ for an appropriate involutory $*$-antiautomorphism Φ of A. By Lemma 1.2 of [16], if p, q are projections in $M_n(A)$ then there are projections e, f in $M_n(A)$ such that $[f] \leqslant [p]$, $[e] \leqslant [1-p]$ and $q = e + f$, where 1 is the identity of $M(M_n(A))$. An inspection of the proofs shows that if $\widetilde{\Phi}$ is the antiautomorphism of $M_n(A)$ corresponding to $M_n(A_\Phi)$ and if $\widetilde{\Phi}(p) = p$ and $\widetilde{\Phi}(q) = q$ then, using the FS property of A_Φ, e and f can be chosen with $\widetilde{\Phi}(e) = e$ and $\widetilde{\Phi}(f) = f$. Also the partial isometries $v = (qpq)^{-1/2}(qp)$ and $w = (e(1-p)e)^{-1/2}e(1-p)$ satisfy $\widetilde{\Phi}(v) = v^*$ and $\widetilde{\Phi}(w) = w^*$, as required to show that $[q] \leqslant [p]$ and $[e] \leqslant [1-p]$ in $M(M_n(A_\Phi))$. Let B be a regular hereditary real C^*-subalgebra of $M_n(A_\Phi)$. Then, by the FS property, B contains a non-zero projection q. If r is a non-zero projection in A_Φ and $1 - p = \operatorname{diag}(r, 0, 0, \dots, 0)$ then, as described above there exists $e \leqslant q$ in $M_n(A_\Phi)$ with $[e] \leqslant [1-p]$. If $f \leqslant 1-p$ with $[e] = [f]$ then f is of the form $\operatorname{diag}(s, 0, \dots, 0)$ where, by the purely infinite property of A_Φ, s is infinite. Thus f is infinite in $M_n(A_\Phi)$. Hence so is e and therefore q. It then follows from the regularity of B that q is infinite in B. ∎

The other main input needed for the real analogue of Rørdam's result is a real version of Theorem 2.8 of [14] and hence a real version of Theorem 3.1 of [5]. The following lemma supplies this.

Lemma 4.3. *Let A be a unital C^*-algebra, let Ψ be an involutory $*$-antiautomorphism of A and let $\widetilde{\Psi} = \Psi \otimes \mathrm{Tr}$ be the associated antiautomorphism of $A \otimes K$, where* Tr *denotes the transpose map on K given by some choice of orthonormal basis of the underlying space. For every $Y \in M(A \otimes K)$ with $\widetilde{\Psi}(Y) = Y^*$ there is a diagonal element $X \in M(A \otimes K)^+$ with $\widetilde{\Psi}(X) = X^*$ such that $I(X) + A \otimes K = I(Y) + A \otimes K$.*

Proof. Each stage of the construction in the proof of Theorem 3.1 of [5] can be made compatible with the given $*$-antiautomorphisms. ∎

The proof of Theorem 3.2 of [14] can now be applied to yield the following result.

Theorem 4.4. *Let A be a simple C^*-algebra, let Ψ be an involutory $*$-antiautomorphsim of A and let $\widetilde{\Psi} = \Psi \otimes \mathrm{Tr}$ be the associated antiautomorphism of $A \otimes K$. If A_Ψ is purely infinite then $M((A \otimes K)_{\widetilde{\Psi}})/(A \otimes K)_{\widetilde{\Psi}}$ is simple.*

Proof. Let I be an ideal of $M(A \otimes K)_{\widetilde{\Psi}}$ properly containing $(A \otimes K)_{\widetilde{\Psi}}$, so that its complexification $I^{\mathbb{C}}$ is a $\widetilde{\Psi}$-invariant ideal of $M(A \otimes K)$ properly containing $A \otimes K$. There exists $Y \in I^{\mathbb{C}}$ with $Y \notin A \otimes K$ and $\widetilde{\Psi}(Y) = Y^*$. Then, by Lemma 4.3, there exists a positive diagonal $X = \mathrm{Diag}(x_1, x_2, \ldots)$ with $\widetilde{\Psi}(X) = X^*$ and $X \notin A \otimes K$. The construction in the proof of Theorem 3.2 of [14], using the result of Lemma 4.2 that each matrix algebra over A_Ψ is purely infinite, shows that $I^{\mathbb{C}} \supseteq I(X) = M(A \otimes K)$ and so $I = M((A \otimes K)_{\widetilde{\Psi}}$. ∎

It follows from Theorem 4.4 that the centre of $M((A \otimes K)_{\widetilde{\Psi}})/(A \otimes K)_{\widetilde{\Psi}}$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. The second case seems unlikely and is ruled out under restrictive conditions in the following proposition.

Proposition 4.5. *Let A be a σ-unital simple C^*-algebra with real rank zero and let Φ be an involutory $*$-antiautomorphism of A with extension $\widetilde{\Phi}$ to $M(A)/A$. Then if $(M(A)/A)_{\widetilde{\Phi}}$ is simple its centre is isomorphic to $\mathbb{R}$.*

Proof. If the centre of $(M(A)/A)_{\widetilde{\Phi}}$ is isomorphic to $\mathbb{C}$, then $M(A)/A$ is isomorphic to $(M(A)/A)_{\widetilde{\Phi}} \oplus (M(A)/A)_{\widetilde{\Phi}}^{\mathrm{op}}$ via the isomorphism $(x, y) \mapsto (x + y^*)/2 + \mathrm{i}(x - y^*)/2I$ (where I is used for the element of the center of $(M(A)/A)_{\widetilde{\Phi}}$ corresponding to the imaginary unit). However this contradicts Corollary 6.3 of [17], which states that, under the hypotheses of the proposition, the intersection of any finite number of nonzero closed ideals of $M(A)/A$ is non-zero. ∎

The partial converse to Theorem 3.3 can now be obtained.

Theorem 4.6. *Let A be a simple, unital C^*-algebra with real rank zero and let Φ be an involutory $*$-antiautomorphism of A. If A_Φ is purely infinite then so is A.*

Proof. By Theorem 4.4, $M((A \otimes K)_{\widetilde{\Phi}})/(A \otimes K)_{\widetilde{\Phi}}$ is simple, where $\widetilde{\Phi}$ is the antiautomorphism $\Phi \otimes \mathrm{Tr}$ and, by Proposition 4.5, its centre is isomorphic to $\mathbb{R}$. Thus $M((A \otimes K)_\Phi)/(A \otimes K)_{\widetilde{\Phi}}$ is a central simple $\mathbb{R}$-algebra (as defined in Section 12.4 of [13]). It follows from Lemma 6 in Section 12.4 of [13] that $M(A \otimes$

$K)/(A \otimes K)$, which is isomorphic to $[M((A \otimes K)_{\widetilde{\Phi}})/(A \otimes K)_{\widetilde{\Phi}}] \otimes_{\mathbb{R}} \mathbb{C}$, is simple. Hence, by Theorem 3.2 of [14], A is either isomorphic to $M_n(\mathbb{C})$ for some n, which contradicts the conditions on A_Φ, or is purely infinite, as required. ∎

REFERENCES

1. L.G. Brown, Stable isomorphism of hereditary subalgebras of C^*-algebras, *Pacific J. Math.* **71**(1977), 335–348.
2. L.G. Brown, Semicontinuity and multipliers of C^*-algebras, *Canadian J. Math.* **40**(1988), 865–988.
3. L.G. Brown, G.K. Pedersen, C^*-algebras of real rank zero, *J. Funct. Anal.* **99** (1991), 131–149.
4. K.R. Davidson, *C^*-Algebras by Example*, Fields Inst. Monogr., vol. 6, Amer. Math. Soc., Providence, RI, 1996.
5. G.A. Elliot, Derivations of matroid C^*-algebras. II, *Ann. of Math.* **100**(1974), 407–422.
6. G.A. Elliot, Some simple C^*-algebras constructed as crossed products with discrete outer automorphism groups, *Publ. Reas. Inst. Math. Sci.* **16**(1980), 299–311.
7. K.R. Goodearl, *Notes on Real and Complex C^*-Algebras*, Shiva, (town??), 1982.
8. J.A. Jeong, Purely infinite simple C^*-crossed products, *Proc. Amer. Math. Soc.* **123**(1995), 3075–3078.
9. A. Kishimoto, Outer automorphisms and reduced crossed products of simple C^*-algebras, *Comm. Math. Pyhs.* **81**(1981), 429–435.
10. H. Lin, S. Zhang, On infinite simple C^*-algebras, *J. Funct. Anal.* **100**(1991), 221–231.
11. G.K. Pedersen, *C^*-Algebras and their Automorphism Groups*, Academic Press, London-New York-San Francisco, 1979.
12. G.K. Pedersen, The linear span of projections in simple C^*-algebras, *J. Operator Theory,* **4**(1980), 289–296.
13. R.S. Pierce, *Associative Algebras*, Grad. Texts in Math., vol. 88, Springer-Verlag, New York-Heidelberg-Berlin, 1982.
14. M. Rørdam, Ideals in the multipliers algebra of a stable C^*-algebra, *J. Operator Theory* **25**(1991), 283–298.
15. H. Schröder, *K-theory for Real C^*-Algebras and Applications*, Pitman Res. Notes Math. Ser., vol. 290, Longman, 1993.
16. S. Zhang, A Riesz decomposition property and ideal structure of multiplier algebras, *J. Operator Theory* **24**(1990), 209–225.
17. S. Zhang, C^*-algebras with real rank zero and the internal structure of their corona and multiplier algebra. III, *Canad. J. Math.* **42**(1990), 159–190.
18. S. Zhang, Certain C^*-algebras with real rank zero and their corona and multiplier algebras. I, *Pacific J. Math.* **155**(1992), 169–197.

P.J. STACEY
Department of Mathematics
La Trobe University
Victoria, 3086
AUSTRALIA
E-mail: P.Stacey@latrobe.edu.au

Received August 8, 2000.