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ABELIAN STRICT APPROXIMATION IN MULTIPLIER C^*-ALGEBRAS AND RELATED QUESTIONS

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Dedicated to Professor G.K. Pedersen on his 60^{th} birthday

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Abstract. We prove a general result on the strict approximability of normal elements of the multiplier algebra $M(A)$ of a σ-unital C^*-algebra A from commutative C^*-subalgebras of A. As an application, we reprove a result of L.G. Brown concerning the non-existence of non-zero separable hereditary C^*-subalgebras of the corona algebras of σ-unital C^*-algebras. Subsequently we characterize the situation in which an SAW^*-algebra (whose class contains all corona algebras of σ-unital C^*-algebras) allows non-zero separable hereditary C^*-subalgebras.

Keywords: *C^*-algebra, multiplier algebra, strict topology, hereditary C^*-subalgebra, SAW^*-algebra, AW^*-algebra.*

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INTRODUCTION

The systematic study of the multiplier algebra $M(A)$ of a C^*-algebra A began in the works [8] (for commutative A) and [9], [23], [2] (for general A). In the commutative setting, for A the C^*-algebra $C_0(\Omega)$ of all continuous complex functions on a locally compact Hausdorff topological space Ω, vanishing at infinity, $M(A)$ identifies with the C^*-algebra $C_{\mathrm{b}}(\Omega)$ of all bounded continuous complex functions on Ω. In the general setting $M(A)$ can be represented as the C^*-subalgebra

$$\{x \in A^{**} : xa,\, ax \in A \text{ for all } a \in A\}$$

of the second dual A^{**}. In particular, for A the C^*-algebra $K(H)$ of all compact linear operators on a complex Hilbert space H, we can identify $M(A)$ with the C^*-algebra $B(H)$ of all bounded linear operators on H.

A natural locally convex vector space topology on $M(A)$, called the strict topology β, is defined by the seminorms

$$x \mapsto \|xa\| \quad \text{and} \quad x \mapsto \|ax\|, \quad a \in A.$$

It is complete and compatible with the duality between $M(A)$ and A^*. Hence the strict topology is weaker than the norm-topology on $M(A)$, but stronger than the restriction to $M(A)$ of the weak $*$-topology of A^{**}. Furthermore, A is a strictly dense, norm-closed two-sided ideal of $M(A)$. For Ω as above, on every bounded subset of $C_{\mathrm{b}}(\Omega)$ the strict topology coincides with the topology of the uniform convergence on the compact subsets of Ω. On the other hand, for H a complex Hilbert space, on every bounded subset of $B(H)$ the strict topology coincides with the s^*-topology.

For the basic facts concerning multipliers of C^*-algebras and the strict topology on them we refer to 3.12 in [19] and Chapter 2 in [26].

We recall that if A_0 is a C^*-subalgebra of a C^*-algebra A, which contains an (increasing, positive) approximate unit $(u_\iota)_\iota$ for A, then $M(A_0) \subset M(A)$ (see [2], Proposition 2.6 or [19], 3.12.12). Actually $M(A_0)$ is the strict closure of A_0 in $M(A)$. We notice also that $M(A_0) \cap A = A_0$ because $\|x - u_\iota x\| \to 0$ and $u_\iota x \in A_0$ whenever $x \in M(A_0) \cap A$. Therefore the corona algebra $C(A_0) = M(A_0)/A_0$ is canonically imbedded in $C(A) = M(A)/A$.

We recall also that a C^*-algebra is called *σ-unital* whenever it contains a strictly positive element or, equivalently, it has a countable approximate unit (see [19], 3.10.4, 3.10.5). A C^*-subalgebra A_0 of a σ-unital C^*-algebra A contains an approximate unit for A if and only if it contains a strictly positive element of A. Indeed, if A_0 contains an approximate unit $(u_\iota)_\iota$ for A and a is strictly positive element of A then there is a sequence $\iota_1 \leqslant \iota_2 \leqslant \cdots$ with $\|a - u_{\iota_j} a\| \leqslant \frac{1}{j}$ and it follows that already $(u_{\iota_j})_{j\geqslant 1} \subset A_0$ is approximate unit for A (see e.g. [26], Lemma 2.3.6), so $\sum\limits_{j\geqslant 1} 2^{-j} u_{\iota_j} \in A_0$ is a strictly positive element of A.

Now let A be a C^*-algebra. We say that $x \in M(A)$ belongs to the (atomic) abelian strict closure of A if there exists a commutative C^*-subalgebra C_x of A (generated by a family of mutually orthogonal projections) such that x belongs to the strict closure of C_x in $M(A)$. Every element in the abelian strict closure of A is clearly normal.

Furthermore, we say that $x \in M(A)$ belongs to the strong (atomic) abelian strict closure of A if there exists a commutative C^*-subalgebra C_x of A as above, which additionally contains an approximate unit for A. In this case the strict closure of C_x in $M(A)$ identifies with $M(C_x)$.

Let us assume that the C^*-algebra A is σ-unital and $x \in M(A)$ belongs to the strong atomic abelian strict closure of A. Let further C_x denote a commutative C^*-subalgebra of A, generated by a family of mutually orthogonal projections, containing an approximate unit for A and satisfying $x \in M(C_x)$. Then C_x contains a strictly positive element of A, hence it is generated by a countable family of mutually orthogonal projections, whose strict sum in $M(A)$ is $1_{A^{**}}$. Thus there exists a countable family $(e_j)_j$ of projections in A with

$$\sum_j e_j = 1_{A^{**}} \quad \text{and} \quad x = \sum_j \lambda_j e_j,$$

where $(\lambda_j)_j$ is a bounded family in $\mathbb{C}$ and the series are strictly convergent.

Taking into account the above remark, the celebrated Weyl-von Neumann-Berg-Sikonia theorem claims that, for H a separable complex Hilbert space and $A = K(H)$, every normal element $y \in M(A)$ is of the form $y = b + x$ with $b \in A$ and x belonging to the strong atomic abelian strict closure of A. This property was extensively investigated for general C^*-algebras A of real rank zero (alias satisfying the condition FS of G.K. Pedersen), which seems to be the natural frame for it (see e.g. [17], [7], [27], [12], [14], [15], [16]).

For arbitrary C^*-algebra A, not necessarily rich in projections, it seems reasonable to cancel the word "atomic" in the above statement and to look for the elements of $M(A)$ which belong modulo A to the strong abelian strict closure of A. It would be interesting to describe these elements and the main result of the first section can be considered a step toward this goal: we prove that, modulo A, every separable C^*-subalgebra of $M(A)$ can be appropriately decomposed in two C^*-subalgebras of $M(A)$, each one of them having all normal elements in the strong abelian strict closure of A. As an application we reprove a result of L.G. Brown concerning the non-existence of non-zero separable hereditary C^*-subalgebras of the corona algebras of σ-unital C^*-algebras ([5], Corollary 7) by reducing the problem to the commutative case, in which an appropriate classical result of E. Čech can be used.

In the second section we investigate the structure of the separable hereditary C^*-subalgebras of the so called SAW^*-algebras, a class of C^*-algebras containing all corona algebras of σ-unital C^*-algebras (see [21], Theorem 13), but also all quotients of AW^*-algebras by norm-closed two-sided ideals.

1. ABELIAN STRICT APPROXIMATION FOR MULTIPLIERS OF σ-UNITAL C^*-ALGEBRAS

For any subset S of a C^*-algebra A we denote by $\mathrm{Her}_A(S)$ the hereditary C^*-subalgebra of A generated by S. We recall that for a surjective $*$-homomorphism $\pi : A \to B$ between C^*-algebras and $S \subset A$ we have

$$\mathrm{Her}_B(\pi(S)) = \pi(\mathrm{Her}_A(S))$$

(see [19], 1.5.11).

The main result of this section is the following

Theorem 1.1. (On abelian strict approximability) *Let A be a σ-unital C^*-algebra, $0 \leqslant a \leqslant 1_{A^{**}}$ a strictly positive element of A, and $B \subset M(A)$ a separable C^*-subalgebra. Then there are, for $j = 1, 2$:*

(i) *a continuous function $f_j : [0, 1] \to [0, 1]$ vanishing only at 0,*

(ii) *a separable C^*-subalgebra $A_j \subset A$,*

such that $f_j(a)$ is in the centre of A_j (it is necessarily strictly positive in A); and

(iii) *a separable C^*-subalgebra $B_j \subset M(A_j) \cap (A + \mathrm{Her}_{M(A)}(B))$ satisfying $\{x \in B_j : x \text{ normal}\} \subset$ abelian strict closure of A_j (hence $\subset$ strong abelian strict closure of A_j), such that*

$$B \subset A + B_1 + B_2.$$

For the proof we need quasi-central approximate units (see [24], [3], [1]).

More precisely, we shall use the following version, essentially identical to Theorem 2.2 of [22]:

LEMMA 1.2. *If A is a σ-unital C^*-algebra, $0 \leqslant a \leqslant 1_{A^{**}}$ a strictly positive element of A, $(x_k)_{k\geqslant 1}$ a sequence in $M(A)$ and $(\varepsilon_n)_{n\geqslant 1} \subset (0,+\infty)$, then there are:*
(i) *continuous functions $f_n : [0,1] \to [0,1]$, $n \geqslant 1$,*
(ii) $1 > \lambda_1 > \lambda_1' > \lambda_2 > \lambda_2' > \cdots > 0$, $\lambda_n \leqslant \varepsilon_n$,
such that

(a) $f_n(\lambda) = \begin{cases} 1 & \text{for } \lambda \geqslant \lambda_n, \\ 0 & \text{for } \lambda \leqslant \lambda_n'; \end{cases}$

(b) $\|f_n(a)x_k - x_k f_n(a)\| \leqslant \varepsilon_n$ *for all* $1 \leqslant k \leqslant n$.

The following result is actually folklore for the experts (cf. e.g. [18], 2.3, 2.4, 2.5), but we have no reference for it as formulated below:

LEMMA 1.3. *Let A be a σ-unital C^*-algebra, $0 \leqslant a \leqslant 1_{A^{**}}$ a strictly positive element of A, and $B \subset M(A)$ a separable C^*-subalgebra. Then there are:*
(i) *continuous functions $g_n : [0,1] \to [0,1]$, $n \geqslant 1$,*
(ii) $1 = \mu_1 = \mu_1' > \mu_2 > \mu_2' > \cdots > 0$, $\lim\limits_{n\to\infty} \mu_n = 0$,
such that

$$\operatorname{supp} g_n \subset [\mu_{n+1}', \mu_n] \text{ for } n \geqslant 1, \quad \text{and} \quad \sum_{n=1}^{\infty} g_n(\lambda)^2 = 1 \text{ for } \lambda \in (0,1]$$

and, for every $J \subset \{1,2,\ldots\}$,

(a) $\sum\limits_{n\in J} g_n(a)xg_n(a)$ *converges strictly and* $\Big\| \sum\limits_{n\in J} g_n(a)xg_n(a) \Big\| \leqslant \|x\|$
for $x \in M(A)$,

(b) $\sum\limits_{n\in J} g_n(a)xg_n(a)$ *converges in the norm topology for $x \in A$,*

(c) $\sum\limits_{n\in J} g_n(a)xyg_n(a) - x\sum\limits_{n\in J} g_n(a)^2 y \in A$ *for $x, y \in B$.*

In particular, $x - \sum\limits_{n=1}^{\infty} g_n(a)xg_n(a) \in A$ *for $x \in B$.*

Proof. Choose a dense sequence $(x_k)_{k\geqslant 1}$ in the unit ball of B and put $\varepsilon_n = 4^{-n}$, $n \geqslant 1$. Let now $f_n, \lambda_n, \lambda_n'$ be as in Lemma 1.2 and define $g_1 = f_1^{1/2}$, $g_n = (f_n - f_{n-1})^{1/2}$ for $n \geqslant 2$, and $\mu_1 = \mu_1' = 1$, $\mu_n = \lambda_{n-1}$ and $\mu_n' = \lambda_{n-1}'$ for $n \geqslant 2$.

Clearly, the functions g_n are continuous, $1 = \mu_1 = \mu_1' > \mu_2 > \mu_2' > \cdots > 0$, $\mu_n \leqslant 4^{1-n}$, $\operatorname{supp} g_n \subset [\mu_{n+1}', \mu_n]$ and $\sum\limits_{n=1}^{\infty} g_n(\lambda)^2 = \lim\limits_{n\to\infty} f_n(\lambda) = 1$ for $\lambda \in (0,1]$. We notice that, according to the proof of Lemma 2.4 from [18], the weak* sum $\sum\limits_{n\in J} g_n(a)xg_n(a)$ exists in A^{**} for every $J \subset \{1,2\ldots\}$ and $x \in A^{**}$, having

$$\Big\| \sum_{n\in J} g_n(a)xg_n(a) \Big\| \leqslant \|x\|. \tag{1.1}$$

Now let $J \subset \{1,2,\ldots\}$ be arbitrary.

For $x \in M(A)$ the sum $\sum\limits_{n\in J} g_n(a)xg_n(a)$ converges strictly. Indeed, by (1.1) and by [26], Lemma 2.3.6, it is enough to prove the norm convergence of $\sum\limits_{n\in J} g_n(a)xg_n(a)a$, which follows from

$$\sum_{n\in J}\|g_n(a)xg_n(a)a\| \leqslant \|x\|\sum_{n\in J}\|g_n(a)a\| \leqslant \|x\|\sum_{n\in J}\mu_n \leqslant \|x\|\sum_{n\in J}4^{1-n} < +\infty.$$

Moreover, for $x \in A$ we have norm convergence. Indeed, by (1.1)

$$\Big\{x \in A : \sum_{n\in J} g_n(a)xg_n(a) \text{ norm convergent}\Big\}$$

is a norm closed linear subspace of A and, by the condition on the supports of the g_n's, it contains every positive $x \in A$ which is majorized by $\sum\limits_{j=1}^{n} g_j(a)^2$ for some n. But $\Big(\sum\limits_{j=1}^{n} g_j(a)^2\Big)^{1/2}$, $n = 1, 2, \ldots$ is an approximate unit for A, so every $0 \leqslant x \leqslant 1_{A^{**}}$ in A is norm limit of $\Big(\sum\limits_{j=1}^{n} g_j(a)^2\Big)^{1/2} x \Big(\sum\limits_{j=1}^{n} g_j(a)^2\Big)^{1/2} \leqslant \sum\limits_{j=1}^{n} g_j(a)^2$, $n = 1, 2, \ldots$.

It remains only to prove the last statement of the lemma. Since

$$\Big\{(x, y) \in B \times B : \sum_{n\in J} g_n(a)xyg_n(a) - x\sum_{n\in J} g(a)^2 y \in A\Big\}$$

is a norm closed cone in $B \times B$, it is enough to prove that it contains (x_k, x_l) for any $k, l \geqslant 1$. Further, this will follow once we prove that

$$\sum_{n>k,l} \|g_n(a)x_kx_lg_n(a) - x_kg_n(a)^2x_l\| < +\infty.$$

But, according to [18], Lemma 2.1, we have for all $n > k, l$

$$\begin{aligned}
\|g_n(a)x_kx_lg_n(a) - x_kg_n(a)^2x_l\| &= \|[g_n(a), x_k]x_lg_n(a) - x_kg_n(a)[g_n(a), x_l]\| \\
&\leqslant \|[g_n(a), x_k]\| + \|[g_n(a), x_l]\| \leqslant \sqrt{2}\big(\|[g_n(a)^2, x_k]\|^{1/2} + \|[g_n(a)^2, x_l]\|^{1/2}\big) \\
&= \sqrt{2}\big(\|[f_n(a), x_k] - [f_{n-1}(a), x_k]\|^{1/2} + \|[f_n(a), x_l] - [f_{n-1}(a), x_l]\|^{1/2}\big) \\
&\leqslant \sqrt{2}\cdot 2(\varepsilon_n + \varepsilon_{n-1})^{1/2} < 4\sqrt{\varepsilon_{n-1}} = 8\cdot 2^{-n}. \quad \blacksquare
\end{aligned}$$

Proof of the Theorem 1.1. Let g_n, μ_n, μ'_n be as in Lemma 1.3. Then the intervals $[\mu'_{n+1}, \mu_n]$, $n \geqslant 1$ odd, are mutually disjoint, hence there exists an increasing continuous function $f_1 : [0, 1] \to [0, 1]$ with

$$f_1(\lambda) = \frac{1}{n} \quad \text{for } \lambda \in [\mu'_{n+1}, \mu_n], \quad n \geqslant 1 \text{ odd}.$$

Similarly, there exists an increasing continuous function $f_2 : [0, 1] \to [0, 1]$ with

$$f_2(\lambda) = \frac{1}{n} \quad \text{for } \lambda \in [\mu'_{n+1}, \mu_n], \quad n \geqslant 1 \text{ even}.$$

Let us consider the separable C^*-subalgebras

$$A_1 = C^*\Big(\{f_1(a)\} \cup \bigcup_{n\geqslant 1\,\mathrm{odd}} g_n(a)Bg_n(a)\Big) \subset A,$$

$$A_2 = C^*\Big(\{f_2(a)\} \cup \bigcup_{n\geqslant 1\,\mathrm{even}} g_n(a)Bg_n(a)\Big) \subset A,$$

$$B_1 = C^*\Big(\Big\{ \sum_{n\geqslant 1\,\mathrm{odd}} g_n(a)xg_n(a) : x \in B\Big\}\Big) \subset M(A),$$

$$B_2 = C^*\Big(\Big\{ \sum_{n\geqslant 1\,\mathrm{even}} g_n(a)xg_n(a) : x \in B\Big\}\Big) \subset M(A),$$

where $C^*(S)$ denotes the C^*-subalgebra of $M(A)$ generated by $S \subset M(A)$.

Clearly, f_1 and f_2 vanish only at 0. For every odd $n \geqslant 1$ we have

$$g_n(a)f_1(a) = f_1(a)g_n(a) = \frac{1}{n}g_n(a),$$

so $f_1(a)$ commutes with all $g_n(a)xg_n(a)$, $x \in A^{**}$. Consequently $f_1(a)$ belongs to the centre of A_1. Similarly, $f_2(a)$ belongs to the centre of A_2.

Since the sum $\sum\limits_{n\geqslant 1\,\mathrm{odd}} g_n(a)xg_n(a)$ is strictly convergent for any $x \in M(A)$, the C^*-algebra B_1 is contained in the strict closure of A_1 in $M(A)$, which can be identified with $M(A_1)$, as noticed at the beginning of this section. Similarly, $B_2 \subset M(A_2)$.

Let π denote the quotient $*$-homomorphism $M(A) \to C(A) = M(A)/A$.

For every positive element $x \in B$, taking into account that $x - \sum\limits_{n=1}^{\infty} g_n(a)xg_n(a) \in A$, we get successively

$$\pi\Big(\sum_{n\geqslant 1\,\mathrm{odd}} g_n(a)xg_n(a)\Big) \leqslant \pi\Big(\sum_{n=1}^{\infty} g_n(a)xg_n(a)\Big) = \pi(x),$$

$$\pi\Big(\sum_{n\geqslant 1\,\mathrm{odd}} g_n(a)xg_n(a)\Big) \in \mathrm{Her}_{C(A)}(\pi(B)) = \pi(\mathrm{Her}_{M(A)}(B)),$$

$$\sum_{n\geqslant 1\,\mathrm{odd}} g_n(a)xg_n(a) \in \pi^{-1}(\pi(\mathrm{Her}_{M(A)}(B)))$$

and, similarly,

$$\sum_{n\geqslant 1\,\mathrm{even}} g_n(a)xg_n(a) \in \pi^{-1}(\pi(\mathrm{Her}_{M(A)}(B))).$$

Consequently $B_1, B_2 \subset \pi^{-1}(\pi(\mathrm{Her}_{M(A)}(B))) = A + \mathrm{Her}_{M(A)}(B)$. On the other hand, for every $x \in B$ we have $x \in A + \sum\limits_{n=1}^{\infty} g_n(a)xg_n(a) \subset A + B_1 + B_2$, so that

$$B \subset A + B_1 + B_2.$$

Thus it remains only to show that every normal $y \in B_j$ belongs to the abelian strict closure of A_j. We prove this for $j = 1$, the treatment of the case $j = 2$ being completely similar.

Choose for any odd $n \geqslant 1$ a continuous function $h_n : [0,1] \to [0,1]$ such that

$$\begin{array}{ll} h_n(\lambda) = 1 & \text{for } \lambda \in [\mu'_{n+1}, \mu_n], \\ h_n \cdot h_m = 0 & \text{for } n \neq m. \end{array}$$

$D_1 = \Big\{ \sum_{n \geqslant 1 \text{ odd}} g_n(a) x g_n(a) : x \in M(A) \Big\}$ is a $*$-subalgebra of $M(A)$ and, for every odd $n \geqslant 1$, we get a $*$-homomorphism $\pi_n : \overline{D}_1 \to \overline{g_n(a) M(A) g_n(a)} \subset A$ by putting

$$\pi_n(y) = y h_n(a) = h_n(a) y.$$

Moreover, for all $y \in \overline{D}_1$ we have

$$y = \sum_{n \geqslant 1 \text{ odd}} \pi_n(y),$$

where the sum converges strictly. Indeed, the set of all $y \in \overline{D}_1$ for which the above statement holds, is norm closed and plainly contains D_1.

Now $B_1 \subset \overline{D}_1$ and each π_n carries B_1 into A_1. Therefore, for every normal $y \in B_1$, the element $\pi_n(y)$ is normal and $C^*(\{\pi_n(y)\})$ are mutually orthogonal commutative C^*-subalgebras of A_1, so

$$C_y = C^*(\{f_1(a)\} \cup \{\pi_n(y) : n \geqslant 1 \text{ odd}\})$$

is a commutative C^*-subalgebra of A_1, containing the strictly positive element $f_1(a)$ of A. Since $y = \sum_{n \geqslant 1 \text{ odd}} \pi_n(y) \in$ strict closure of C_y, the element y belongs to the abelian strict closure of A_1. ∎

The above theorem implies the following structure result for σ-unital hereditary C^*-subalgebras of corona algebras:

COROLLARY 1.4. *Let A be a σ-unital C^*-algebra, and D a σ-unital hereditary C^*-subalgebra of $C(A) = M(A)/A$. Then there are separable C^*-subalgebras A_1, A_2 of A, whose centers contain strictly positive elements of A, as well as separable C^*-subalgebras $B_1 \subset M(A_1)$, $B_2 \subset M(A_2)$, such that, denoting by π the quotient $*$-homomorphism $M(A) \to C(A)$, $\{x \in B_j : x \text{ normal}\} \subset$ the strong abelian strict closure of A_j, $j = 1, 2$,*

$$D = \mathrm{Her}_{C(A)}(\pi(B_1) \cup \pi(B_2)).$$

In particular, D is generated as hereditary C^*-subalgebra of $C(A)$ by a countable family of elements of the form $\pi(x)$ with x in the strict closure of some separable commutative C^*-subalgebra of A, containing a strictly positive element of A.

Proof. Let $0 \leqslant a \leqslant 1_{A^{**}}$ be a strictly positive element of A, and $0 \leqslant x \in M(A)$ with $\pi(x)$ strictly positive in D, so that

$$D = \mathrm{Her}_{C(A)}(\{\pi(x)\}) = \pi(\mathrm{Her}_{M(A)}(\{x\})).$$

Putting $B = C^*(\{x\})$, let $f_1, f_2, A_1, A_2, B_1, B_2$ be as in Theorem 1.1. Since

$$B_1 \cup B_2 \subset A + \mathrm{Her}_{M(A)}(B) = A + \mathrm{Her}_{M(A)}(\{x\}),$$

we have

$$\mathrm{Her}_{C(A)}(\pi(B_1) \cup \pi(B_2)) \subset D.$$

On the other hand, since $B \subset A + B_1 + B_2$, so $\pi(B) \subset \pi(B_1) + \pi(B_2)$, we have also

$$D = \mathrm{Her}_{C(A)}(\pi(B)) \subset \mathrm{Her}_{C(A)}(\pi(B_1) \cup \pi(B_2)). \quad ∎$$

The above result allows us to give an alternate proof for Corollary 7 of [5] by using reduction to the commutative case, much in spirit of the proof of Theorem 2.7 in [2]:

COROLLARY 1.5. (L.G. Brown) *Let A be a σ-unital C^*-algebra, and D a separable hereditary C^*-subalgebra of $C(A)$. Then $D = \{0\}$.*

Proof. Let us assume that $D \neq \{0\}$. By Corollary 1.4 there exists a commutative C^*-subalgebra $A_0 \subset A$, containing a strictly positive element of A, and $x \in M(A_0)$, such that the canonical image $\pi(x)$ of x in $C(A)$ is non-zero and belongs to D. Therefore $D_0 = \mathrm{Her}_{C(A_0)}(\{\pi(x)\}) \subset D$ is a non-zero separable hereditary C^*-subalgebra of $C(A_0)$.

Now let Ω be the Gelfand spectrum of A_0, and $\beta\Omega$ its Stone-Čech compactification. Then $C(A_0)$ is $*$-isomorphic to $C(\beta\Omega \setminus \Omega)$ and D_0 corresponds to $C_0((\beta\Omega \setminus \Omega) \setminus F_0)$ with F_0 some closed subset of $\beta\Omega \setminus \Omega$. Since D_0 is non-zero and separable, $(\beta\Omega \setminus \Omega) \setminus F_0$ is non-empty and metrizable. Let ω_0 be any element of $(\beta\Omega \setminus \Omega) \setminus F_0$. Then $\{\omega_0\}$ is a G_δ-set in $(\beta\Omega \setminus \Omega) \setminus F_0$, hence, F_0 being compact, also in $\beta\Omega \setminus \Omega$. But the σ-unitality of A_0 means that Ω is σ-compact, or equivalently, that $\beta\Omega \setminus \Omega$ is a G_δ-set in $\beta\Omega$. Consequently $\{\omega_0\}$ is a G_δ-set in $\beta\Omega$. This contradicts a classical result of E. Čech, claiming that no point in the corona of a completely regular topological space can be G_δ-set in the Stone-Čech compactification (see e.g. [11], Corollary 9.6 or [25], Corollary 3.7). ∎

The above result yields immediately, as already noticed by L.G. Brown (and in the commutative case by E. Čech), that any separable C^*-algebra A is the greatest separable two-sided ideal of $M(A)$. Consequently, if A and B are separable C^*-algebras then any $*$-isomorphism of $M(A)$ onto $M(B)$ carries A onto B. Here the separability of both A and B is essential. Indeed, the separable C^*-algebra $c_0(\mathbb{Z})$ and $l^\infty(\mathbb{Z}) = M(c_0(\mathbb{Z}))$ are not $*$-isomorphic, but their multiplier algebras are identical. The separability is essential even in the case $A = B$. Indeed, putting $A = \Big\{(\lambda_n)_n \in l^\infty(\mathbb{Z}) : \lim\limits_{n\to+\infty} \lambda_n = 0\Big\}$, the map $(\lambda_n)_n \mapsto (\lambda_{-n})_n$ is a $*$-automorphism of $M(A) = l^\infty(\mathbb{Z})$ which does not carry A into A.

2. SEPARABLE HEREDITARY C^*-ALGEBRAS OF GENERAL SAW^*-ALGEBRAS

In this section we investigate the separable hereditary C^*-subalgebras of the so called SAW^*-algebras, considered by G.K. Pedersen in [21]. We recall that an SAW^*-algebra is a C^*-algebra A such that for any positive $x, y \in A$ with $xy = 0$ there is a positive $e \in A$ with $ex = x$ (i.e. e is a local unit for x) and $ey = 0$. Defining $f : [0, +\infty) \mapsto [0, 1]$ by

$$f(\lambda) = \begin{cases} \lambda & \text{for } \lambda \leqslant 1, \\ 1 & \text{for } \lambda \geqslant 1, \end{cases}$$

we have

$$f(e)x = f(1)x = x \quad \text{and} \quad f(e)y = f(0)y = 0,$$

so in the above definition we always can choose $e \leqslant 1_{A^{**}}$.

Corona algebras of σ-unital C^*-algebras are SAW^*-algebras (see [21], Theorem 13 or [18], 3.2).

For any Borel set $S \subset \mathbb{R}$ we denote by χ_S its characteristic function. Thus, for a a self-adjoint element of a C^*-algebra A, the symbol $\chi_S(a)$ will stand for the spectral projection of a in A^{**} corresponding to S.

First we complete the list of the basic facts about SAW^*-algebras in [21] by showing that adjoining a unit to an SAW^*-algebra we get still an SAW^*-algebra:

Lemma 2.1. *Let A be an SAW^*-algebra. For every $0 \leqslant x \in A$ and $y^* = y \in A$ with $xy = x$ there is $0 \leqslant e \leqslant 1_{A^{**}}$ in A such that $xe = x$ and $ey = e$. Therefore the C^*-algebra $\widetilde{A}$ generated by A and $1_{A^{**}}$ is an SAW^*-algebra.*

Proof. Let $f_n : \mathbb{R} \to [0, 1], n \geqslant 1$ be continuous functions such that $f_n \nearrow \chi_{\mathbb{R}\setminus\{0,1\}}$. Then

$$f_n(y) \nearrow \chi_{\mathbb{R}\setminus\{0,1\}}(y) = s(y) - \chi_{\{1\}}(y),$$

where $s(y)$ denotes the support projection of y in A^{**}.

Since $xy = x$, we have $xf(y) = f(1)x$ for any continuous function $f : \mathbb{R} \to \mathbb{R}$, so $xf_n(y) = 0$ for all $n \geqslant 1$. Therefore x and $\sum\limits_{n=1}^{\infty} 2^{-n} f_n(y)$ are orthogonal positive elements of the SAW^*-algebra A and it follows that there exists a positive element $a \in A$ satisfying $xa = x$ and

$$a \sum_{n=1}^{\infty} 2^{-n} f_n(y) = 0 \Longleftrightarrow af_n(y) = 0 \text{ for all } n \geqslant 1 \Longleftrightarrow as(y) = a\chi_{\{1\}}(y).$$

In particular, $ay = a\chi_{\{1\}}(y)$, hence $ay^2 = ay$. Thus $b = yay$ is a positive element of A and

$$xb = xyay = xay = xy = x,$$
$$by = yay^2 = yay = b.$$

Now, defining $f : [0, +\infty] \to [0, 1]$ by

$$f(\lambda) = \begin{cases} \lambda & \text{for } \lambda \leqslant 1, \\ 1 & \text{for } \lambda \geqslant 1, \end{cases}$$

and putting $e = f(b) \in A$, we have $0 \leqslant \mathrm{e} \leqslant 1_{A^{**}}$ and $xe = x$, $ey = e$.

In order to prove that $\widetilde{A}$ is an SAW^*-algebra, let $x, y \in \widetilde{A}$ be arbitrary positive elements with $xy = 0$. Then either x or y, say x, must belong to A. If also y belongs to A, we have nothing to prove, so let us assume that $y = \lambda_0 1_{A^{**}} - y_0$ with $0 \neq \lambda_0 \in \mathbb{R}$ and $y_0^* = y_0 \in A$. Then $\frac{1}{\lambda_0} y_0$ is a self-adjoint element of A and

$$x\Big(\frac{1}{\lambda_0} y_0\Big) = x - \frac{1}{\lambda_0} xy = x.$$

By the first part of the proof there exists $0 \leqslant e \leqslant 1_{A^{**}}$ in A with $xe = x$ and $e\Big(\frac{1}{\lambda_0} y_0\Big) = e$, hence $ey = \lambda_0 e - e y_0 = 0$. We have also

$$0 \leqslant 1_{A^{**}} - e \leqslant 1_{A^{**}}, \quad x(1_{A^{**}} - e) = 0, \quad y(1_{A^{**}} - e) = y. \quad ∎$$

We notice that $\widetilde{A}$ can be SAW^*-algebra without A being SAW^*-algebra. For example, if M is an atomless, countably decomposable, commutative W^*-algebra and A is a maximal ideal of M, then $\widetilde{A} = M$ is an SAW^*-algebra, while A is not. Indeed, if $e_1, e_2, \ldots$ is a maximal family of mutually orthogonal non-zero projections in A then $\sum\limits_{n \geqslant 1} 2^{-n} e_n \in A$ has no local unit in A, so A is not an SAW^*-algebra according to [21], Proposition 4.

Now we characterize SAW^*-algebras in terms of the existence of *almost spectral projections*:

THEOREM 2.2. (On the characterization of SAW^*-algebra) *Let A be a C^*-algebra. Then A is an SAW^*-algebra if and only if, for every $a^* = a \in A$ and every open set $D \subset \mathbb{R}$ not containing 0, there exists $e \in A$ with $\chi_D(a) \leqslant e \leqslant \chi_{\overline{D}}(a)$. Moreover, A is an SAW^*-algebra of real rank zero if and only if in the above situation e always can be chosen as a projection.*

Proof. Let $f_n, g_n : \mathbb{R} \to [0,1], n \geqslant 1$ be continuous functions such that $f_n \nearrow \chi_D$ and $g_n \nearrow \chi_{\mathbb{R} \setminus \overline{D}}$, so that

$$f_n(a) \nearrow \chi_D(a) \quad \text{and} \quad g_n(a) \nearrow \chi_{\mathbb{R} \setminus \overline{D}}(a) \text{ in } A^{**}.$$

Putting

$$x = \sum_{n=1}^{\infty} 2^{-n} f_n(a) \in A, \quad y = \sum_{n=1}^{\infty} 2^{-n} g_n(a) \in \widetilde{A},$$

we have $x, y \geqslant 0$ and $xy = 0$. Therefore there exists $0 \leqslant e \leqslant 1_{A^{**}}$ in A with $ex = x$ and $ey = 0$. Indeed, if $y \in A$ then we have just to use the fact that A is an SAW^*-algebra, while if $y \in \widetilde{A} \setminus A$ then we get by Lemma 2.1 an element $0 \leqslant e \leqslant 1_{A^{**}}$ in $\widetilde{A}$ with the above properties and notice that

$$y \in \widetilde{A} \setminus A, \quad ey = 0 \Longrightarrow e \in A.$$

Now $ex = x$, that is $(1_{A^{**}} - e)\, x = 0$ implies successively that

$$(1_{A^{**}} - e) f_n(a) = 0 \quad \text{for all } n \geqslant 1,$$
$$(1_{A^{**}} - e) \chi_D(a) = 0 \quad \text{and} \quad \chi_D(a) = e\chi_D(a) \leqslant e.$$

On the other hand, $ey = 0$ implies similarly that

$$e g_n(a) = 0 \text{ for all } n \geqslant 1, \quad e\chi_{\mathbb{R} \setminus \overline{D}}(a) = 0, \quad e = e\chi_{\overline{D}}(a) \leqslant \chi_{\overline{D}}(a).$$

Let us next additionally assume that A is of real rank zero. Then, the compact projection $\chi_{\{1\}}(e) \in A^{**}$ and the closed projection $\chi_{[0,1/2]}(e) \in A^{**}$ being orthogonal, by [6], Theorem 1 there is a projection $p \in A$ with $\chi_{\{1\}}(e) \leqslant p \leqslant \chi_{(\frac{1}{2},1]}(e)$. But $\chi_D(a) \leqslant e \leqslant 1_{A^{**}}$ implies that $\chi_D(a) \leqslant \chi_{\{1\}}(e) \leqslant p$, while $e \leqslant \chi_{\overline{D}}(a)$ implies that

$$p \leqslant \chi_{\left(\frac{1}{2},1\right]}(e) \leqslant s(e) \leqslant \chi_{\overline{D}}(a).$$

For the first converse statement take arbitrary positive elements $x, y \in A$ with $xy = 0$. Then $a = x - y$ is self-adjoint in A and if $e \in A$ satisfies $\chi_{(0,+\infty)}(a) \leqslant e \leqslant \chi_{[0,+\infty)}(a)$, then

$$x^2 = x\chi_{(0,+\infty)}(a)x \leqslant xex \leqslant x\chi_{[0,+\infty)}(a)x = x^2, \quad x(1_{A^{**}} - e)x = 0, \quad ex = x$$

and

$$0 = y\chi_{(0,+\infty)}(a)y \leqslant yey \leqslant y\chi_{[0,+\infty)}(a)y = 0, \quad ey = 0.$$

The second converse statement follows from the first one and from (iv) ⇒ (i) of Theorem 2.6 in [7]. ∎

We notice that if in the above $D = (\lambda, +\infty)$, then $0 \leqslant e - \chi_{(\lambda,+\infty)}(a) \leqslant \chi_{\{\lambda\}}(a)$, so

$$a\left(e - \chi_{(\lambda,+\infty)}(a)\right) = \lambda\left(e - \chi_{(\lambda,+\infty)}(a)\right)$$

is self-adjoint and it follows that e commutes with a. Thus the above theorem implies that a C^*-algebra A is an SAW^*-algebra of real rank zero if and only if it satisfies the so called *spectral axiom* considered in Section 2 of Chapter III from [28], p. 1048:

$$\text{(S)} \quad \begin{cases} \textit{for every } a^* = a \in A \textit{ and every } \lambda \geqslant 0 \textit{ there exists a projection } e \in A \\ \textit{commuting with } a \textit{ and such that} \\ \qquad ae \geqslant \lambda e, \quad a\left(1_{A^{**}} - e\right) \leqslant \lambda\left(1_{A^{**}} - e\right). \end{cases}$$

Corollary 2.3. *Let A be an SAW^*-algebra.*

If $0 \leqslant a \in A$ with spectrum $\sigma(a)$ generates a separable hereditary C^*-subalgebra of A then $\sigma(a) \cap (\varepsilon, +\infty)$ is finite for every $\varepsilon > 0$.

If a projection $e \in A$ generates a separable hereditary C^*-subalgebra of A then eAe is finite-dimensional.

In particular, any separable hereditary C^*-subalgebra of A is the norm closed linear span of the minimal projections of A contained in it. Therefore A contains non-zero separable hereditary C^*-subalgebras if and only if it contains minimal projections.

Proof. Assume that $\sigma(a) \cap (\varepsilon, +\infty)$ contains infinitely many distinct $\lambda_1, \lambda_2, \ldots$. Passing to a subsequence, if necessary, we can assume that the sequence $\lambda_1, \lambda_2, \ldots$ is monotone, so we can choose mutually disjoint open sets $D_1, D_2, \ldots \subset (\varepsilon, +\infty)$ with $\lambda_j \in D_j, j \geqslant 1$. By Theorem 2.2, for every set $J \subset \{1, 2, \ldots\}$ there exists in A some e_J

$$\chi_{\bigcup\limits_{j\in J} D_j}(a) \leqslant e_J \leqslant \chi_{\overline{\bigcup\limits_{j\in J} D_j}}(a) \leqslant \chi_{[\varepsilon,+\infty)}(a) \leqslant \frac{1}{\varepsilon}a$$

and then $e_J \in \mathrm{Her}_A(\{a\})$.

Let $J_0, J \subset \{1, 2, \ldots\}$ be such that there exists $j_0 \in J_0 \setminus J$. Then D_{j_0} and $\overline{\bigcup_{j \in J} D_j}$ are disjoint, so $\chi_{D_{j_0}}(a) \cdot e_J = 0$, and it follows that

$$(e_{J_0} - e_J)^2 \geqslant (e_{J_0} - e_J)\chi_{D_{j_0}}(a)(e_{J_0} - e_J) = e_{J_0} \cdot \chi_{D_{j_0}}(a) \cdot e_{J_0} = \chi_{D_{j_0}}(a),$$
$$\|e_{J_0} - e_J\|^2 \geqslant \|\chi_{D_{j_0}}(a)\| = 1.$$

Thus $\{x \in \mathrm{Her}_A(\{a\}) : \|x - e_J\| < 1/2\}$, $J \subset \{1, 2, \ldots\}$ are uncountably many disjoint non-empty open sets in $\mathrm{Her}_A(\{a\})$, which therefore can not be separable.

For the second statement we have only to notice that $\mathrm{Her}_A(\{e\}) = eAe$ is an SAW^*-algebra (see [21], Proposition 4), so its separability implies its finite-dimensionality (see [21], Corollary 2). ∎

Using the above result, we can give a somewhat simpler variant of our proof for Corollary 1.5, without using the result of E. Čech on the remainder points of Stone-Čech compactifications:

Indeed, according to Corollary 2.3 we have to prove that, for every σ-unital C^*-algebra A, the corona algebra $C(A)$ does not contain any minimal projection e. Let us assume the contrary. Applying Corollary 1.4 with $D = \mathbb{C} \cdot e$, it follows that there exists a commutative C^*-subalgebra $A_0 \subset A$, containing a strictly positive element of A, and $x \in M(A_0)$, such that the canonical image $\pi(x)$ of x in $C(A)$ is equal to e. Therefore $C(A_0) \subset C(A)$ contains the minimal projection e. Denoting by Ω the Gelfand spectrum of A_0, and by $\beta\Omega$ its Stone-Čech compactification, x corresponds to some $f \in C(\beta\Omega)$, and e to some isolated point ω_0 of $\beta\Omega \setminus \Omega$, such that

$$f(\omega_0) = 1 \quad \text{and} \quad f(\omega) = 0 \quad \text{for } \omega_0 \neq \omega \in \beta\Omega \setminus \Omega.$$

On the other hand, since A_0 is σ-unital, there exist relatively compact open subsets $U_1, U_2, \ldots$ of Ω such that $\overline{U}_n \subset U_{n+1}$ and $\bigcup_{n \geqslant 1} U_n = \Omega$. We can construct by induction a sequence $(\omega_k)_{k \geqslant 1}$ in Ω and a sequence $1 = n_1 < n_2 < \cdots$ of natural numbers such that, for all $k \geqslant 1$, $\omega_k \in U_{n_{k+1}} \setminus \overline{U}_{n_k}$, $|1 - f(\omega_k)| < 1/k$. Every limit point ω of $(\omega_k)_{k \geqslant 1}$ belongs to $\beta\Omega \setminus \Omega$ and, since $f(\omega)$ is limit point of $(f(\omega_k))_{k \geqslant 1}$, hence $f(\omega) = 1$, it follows that $\omega = \omega_0$. Thus, by the compactness of $\beta\Omega$, we have $\omega_k \to \omega_0$. Now let $g_k : \Omega \to [0, 1]$ be a continuous function with $g_k(\omega_k) = 1$ and with support contained in $U_{n_{k+1}} \setminus \overline{U}_{n_k}$. Then, for every bounded sequence $(\lambda_k)_{k \geqslant 1}$ in $\mathbb{C}$, the function $g = \sum_{k \geqslant 1} \lambda_k g_k$ belongs to $C_{\mathrm{b}}(\Omega)$ and satisfies $g(\omega_k) = \lambda_k$ for all $k \geqslant 1$. Since g extends by continuity to $\beta\Omega$ and $\omega_k \to \omega_0$, it follows that the sequence $(\lambda_k)_{k \geqslant 1}$ converges. But this is obviously not true for every bounded sequence $(\lambda_k)_{k \geqslant 1}$ in $\mathbb{C}$.

Corollary 2.3 enables us to prove the lack of non-zero separable hereditary C^*-subalgebras also in corona algebras of a wide class of non σ-unital C^*-algebras.

Let M be an AW^*-algebra (for their theory, which will be freely used, we refer to [4]), and A an essential, norm closed, two-sided ideal of M. Then M can be naturally identified with $M(A)$ (see [13] or [20]). Using [1], Proposition 2.3, it is easy to verify that $C(A) = M/A$ is a unital SAW^*-algebra of real rank zero. We notice that, for example, if M is a type II_∞ factor and A is the norm closed linear span of all finite projections of M, then A is not σ-unital (see [1], Proposition 4.5).

Actually, since in the proof only the orthogonal additivity of the trace is used, the above statement holds assuming only that M is a type II_∞ AW^*-factor.

COROLLARY 2.4. *Let M be an AW^*-algebra, and A an essential, norm closed, two-sided ideal of M. Then every separable hereditary C^*-subalgebra of M/A is the norm closed linear span of the minimal projections of M/A contained in it. Moreover, any minimal projection of M/A is the canonical image of an abelian projection e of M, for which $A \cap eMe$ is a maximal ideal of eMe.*

Proof. The first statement follows immediately from Corollary 2.3.

Let π denote the quotient $*$-homomorphism $M \to M/A$. By a well known result, every projection in M/A lifts to a projection in M (see e.g. [28], Chapter III, Corollary 2.5). Let $e_0 \in M$ be a projection such that $\pi(e_0)$ is a minimal projection of M/A.

According to the geometry of projections in AW^*-algebras, there are orthogonal central projections p_0 and $p_\aleph$, $\aleph \geqslant 1$ cardinal number, in M such that $e_0Me_0p_0$ is continuous, $e_0Me_0p_\aleph$ is of type $I_\aleph$, $\aleph \geqslant 1$ and $p_0 \vee \bigvee_{\aleph \geqslant 1} p_\aleph = 1_M$.

It is easy to see that there are decompositions in mutually orthogonal projections

$$\begin{aligned} e_0p_0 &= f_0 + g_0, & f_0 &\sim g_0, \\ e_0p_\aleph &= f_\aleph + g_\aleph + h_\aleph, & f_\aleph &\sim g_\aleph \succ h_\aleph, \quad \aleph \geqslant 2 \end{aligned}$$

(actually we can take $h_\aleph = 0$ unless $\aleph$ is an odd natural number $\geqslant 3$). Then

$$e = e_0p_1, \quad f_0 \vee \bigvee_{\aleph \geqslant 2} f_\aleph, \quad g = g_0 \vee \bigvee_{\aleph \geqslant 2} g_\aleph, \quad h = \bigvee_{\aleph \geqslant 2} h_\aleph$$

are mutually orthogonal and

$$e_0 = e + f + g + h, \quad f \sim g \succ h.$$

Since $\pi(e_0)$ is minimal, we have $\pi(f) = \pi(g) = \pi(h) = 0$, so $\pi(e_0) = \pi(e)$.

But e is an abelian projection of M and the codimension of $\ker(\pi|eMe) = A \cap eMe$ in eMe is one. ∎

COROLLARY 2.5. *Let M be an AW^*-factor, and A an essential, norm closed, two sided ideal of M. Then $C(A) = M/A$ does not contain any non-zero separable hereditary C^*-subalgebra.*

Proof. Let us assume the contrary. Then Corollary 2.4 implies the existence of an abelian projection e of M, for which the codimension of $A \cap eMe$ in eMe is one. Since M is factor, e is minimal, so $A \cap eMe = \{0\}$. But then e is orthogonal to A, in contradiction with the essentiality of A. ∎

In contrast to Corollary 1.5, in the proof of Corollary 2.5 we did not make use of abelian strict approximation. Instead we used the geometry of projections available in AW^*-algebras. We notice that we were forced to do this, because in relevant cases commutative C^*-subalgebras of C^*-algebras can have very poor strict closures. For example, if M is a type II_∞ factor and A is the norm closed linear span of all finite projections of M, then any commutative C^*-subalgebra of A is strictly closed in $M(A) = M$ (see [10]). This is surprising since, assuming M to be additionally of countable type, every normal element $y \in M$ is of the form $y = b + x$ with $b \in A$ and x belonging to what we could call the "strong atomic abelian s^*-closure" of A (see [29]).

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