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C^*-CROSSED PRODUCTS OF C^*-ALGEBRAS WITH THE WEAK BANACH-SAKS PROPERTY

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Abstract. Let (A, G, α) be a C^*-dynamical system. In Section 2, we first treat the discrete group action case. We suppose that G acts freely on the spectrum of A. Then it is shown that A has the weak Banach-Saks property, if and only if G is discrete and the C^*-crossed product $A \times_\alpha G$ has the weak Banach-Saks property.

In Section 3, we shall consider the compact group action case. Let G be a compact group and consider the following conditions (1)–(3):

(1) A has the weak Banach-Saks property;
(2) $A \times_\alpha G$ has the weak Banach-Saks property;
(3) the fixed point algebra A^α of A has the weak Banach-Saks property.

Then it is shown that we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Furthermore we suppose that G is (compact) abelian. Then it is shown that the implication (3) $\Rightarrow$ (2) holds, and that if A is of type I and if α is pointwise unitary, the implication (2) $\Rightarrow$ (1) holds.

Keywords: *C^*-crossed product, weak Banach-Saks property, C^*-dynamical system.*

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1. INTRODUCTION

In [1], Banach and Saks showed that every bounded sequence in $L^p([0,1])$ with $1 < p < \infty$ has a subsequence whose arithmetic means converge in the norm topology. More generally, if every bounded sequence in a Banach space X has a subsequence whose arithmetic means converge in the norm topology, we say that X has *the Banach-Saks property*. It is known that Banach spaces with the Banach-Saks property are reflexive. It hence follows that $L^1([0,1])$ can not have the Banach-Saks property.

Let X be a Banach space. If given any weakly null sequence $\{x_n\}$ in X, one can extract a subsequence $\{x_{n(k)}\}$ such that

$$\lim_{k\to\infty}\frac{1}{k}\|x_{n(1)}+\cdots+x_{n(k)}\|=0,$$

we say that X has *the weak Banach-Saks property.* It was shown by Szlenk ([14]) that $L^1([0,1])$ has the weak Banach-Saks property.

Recently, Chu ([2]) has studied C^*-algebras with the weak Banach-Saks property in detail as a noncommutative extension of characterisations of the Banach space, of complex continuous functions on a compact Hausdorff space, with the weak Banach-Saks property. Actually he has obtained the following characterization of C^*-algebras with the weak Banach-Saks property.

THEOREM. ([2], Theorem 2) *Let A be a C^*-algebra. Then the following conditions are equivalent:*

(1) *A has the weak Banach-Saks property;*

(2) *A is scattered and $c_0(A)$ does not contain an isometric copy of $C_0(\omega^\omega)$ where ω^ω denotes the set $[0,\omega^\omega)$ of ordinals preceding ω^ω with the order topology;*

(3) *A is scattered and does not contain an isometric copy of $C_0(\omega^\omega)$;*

(4) *there exists some natural number k such that $\sigma(a)^{(k)}$ is empty for every self-adjoint $a\in A$, where $\sigma(a)$ denotes the spectrum of a;*

(5) *A is of type* I *and $\widehat{A}^{(k)}$ is empty for some natural number k, where $\widehat{A}^{(0)}=\widehat{A}$, the spectrum of A, and $\widehat{A}^{(n)}$ is the n-th derived set of $\widehat{A}$, consisting of the accumulation points of $\widehat{A}^{(n-1)}$.*

Furthermore, at the end of [2], Chu has shown that a C^*-algebras A has the weak Banach-Saks property if and only if there are closed ideals $I_1\subset I_2\subset\cdots\cdots\subset I_n\subset A$ such that I_1 and all the successive quotients are dual C^*-algebras. We shall use this characterization in order to obtain our main result in Section 2.

Let (A,G,α) be a C^*-dynamical system. By a *C^*-dynamical system,* we mean a triple (A,G,α) consisting of a C^*-algebra A, a locally compact group G and a group homomorphism α from G into the automorphism group of A such that $G\ni t\to\alpha_t(x)$ is continuous for each x in A in the norm topology. Denote by $A\times_\alpha G$ the C^*-crossed product of A by G (see [10] for the details). In this paper, we discuss when $A\times_\alpha G$ has the weak Banach-Saks property provided that A has the weak Banach-Saks property. For this, in Section 2 we shall suppose that the action of G induced by α is free on the spectrum $\widehat{A}$ of A. First we show that if A is a dual C^*-algebra, $A\times_\alpha G$ is also a dual C^*-algebra. In this case, furthermore the topology of G is necessarily determined. In fact, we shall see that G becomes a discrete group. Using such a result on dual C^*-algebras, in the sequel we show that, under the assumption that G should act freely on the spectrum $\widehat{A}$ of A, A has the weak Banach-Saks property if and only if $A\times_\alpha G$ has the weak Banach-Saks property and G is discrete.

In Section 3, we consider the case where G is a compact group. If G acts freely on $\widehat{A}$, then the stability group at every point in $\widehat{A}$ is trivial. Hence the situation opposite to such a case is that the stability group at every point in $\widehat{A}$ coinsides with G, and as the case where such a situation occurs, we shall pay our attention to the case where the action of G on A is pointwise unitary.

Let (A, G, α) be a C^*-dynamical system and let G be a compact group. We consider the following conditions (1)–(3).

(1) A has the weak Banach-Saks property.
(2) $A \times_\alpha G$ has the weak Banach-Saks property.
(3) The fixed point algebra A^α of A has the weak Banach-Saks property.

Then we shall show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Furthermore we suppose that G is (compact) abelian. Then we shall show that the implication (3) $\Rightarrow$ (2) holds and that, if A is of type I and if α is pointwise unitary, the implication (2) $\Rightarrow$ (1) holds.

2. DISCRETE GROUP ACTION CASE

For a C^*-algebra A, we denote again by $\widehat{A}$ the spectrum of A, that is, the set of (unitary) equivalence classes $[\pi]$ of nonzero irreducible representations π of A equipped with the Jacobson topology. We note that $\widehat{A}$ is a locally compact space, not necessarily a Hausdorff space. However, we will pay our attention later to the case where $\widehat{A}$ is a Hausdorff space. The reader is referred to [3], [10] for the spectrum of a C^*-algebra.

We recall that a C^*-algebra A is called *dual* if and only if it is isomorphic to a C^*-subalgebra of the C^*-algebra of compact operators on some Hilbert space, or equivalently, every maximal abelian subalgebra of A is generated by minimal projections ([3], 4.7.20, or [7]). As is easily seen, A is a type I C^*-algebra with discrete spectrum $\widehat{A}$ if and only if it is a c_0-sum of C^*-algebras of compact operators. Thus the C^*-algebra A is dual if and only if it is a type I C^*-algebra with discrete spectrum $\widehat{A}$ (see [7], Lemma 2.3 and Lemma 2.4).

Let (A, G, α) be a C^*-dynamical system. If A is dual, then it is a C^*-algebra of type I. Hence, type I-ness is necessary for $A \times_\alpha G$ to be a dual C^*-algebra. For this, we need to impose some conditions to α in order to derive type I-ness of $A \times_\alpha G$. Now we exhibit such conditions here. Given a C^*-dynamical system (A, G, α), α induces the natural action of G on $\widehat{A}$ which is defined by

$$(t, [\pi]) \in G \times \widehat{A} \to [\pi \circ \alpha_{t^{-1}}] \in \widehat{A}.$$

This map makes G into a topological transformation group acting on $\widehat{A}$. Throughout this paper, as an action of G on the spectrum of a C^*-algebra, we consider only the natural action of G defined in the above way . For $[\pi] \in \widehat{A}$, we denote by $S_{[\pi]}$ the stability group at $[\pi]$, which is defined by $S_{[\pi]} = \{t \in G \mid [\pi \circ \alpha_{t^{-1}}] = [\pi]\}$. If all stability groups are trivial, i.e., $S_{[\pi]}$ consists only of the identity of G at every $[\pi] \in \widehat{A}$, it is said that G acts *freely* on $\widehat{A}$. If the map

$$(t, [\pi]) \in G \times \widehat{A} \to ([\pi], [\pi \circ \alpha_{t^{-1}}]) \in \widehat{A} \times \widehat{A}$$

is proper in the sense that inverse images of compact sets are compact, it is said that G acts *properly* on $\widehat{A}$.

LEMMA 2.1. *Let (A, G, α) be a C^*-dynamical system. Suppose that G acts freely on $\widehat{A}$. If there exists a point $[\pi]$ in $\widehat{A}$ such that $\{[\pi]\} \subset \widehat{A}$ is an open subset, then G is a discrete group. In particular, if $\widehat{A}$ is discrete, G is a discrete group.*

Proof. It suffices to show that the identity e of G is an open subset. When we fix $[\pi]$, the map $t \in G \to [\pi \circ \alpha_{t^{-1}}] \in \widehat{A}$ is continuous. Since the identity e of G is just the inverse image of $[\pi]$ by the above map, $\{e\}$ is an open subset in G. ∎

LEMMA 2.2. *Let (A, G, α) be a C^*-dynamical system. If $\widehat{A}$ is discrete and if G acts freely on $\widehat{A}$, then G acts properly on $\widehat{A}$.*

Proof. Since G acts freely on $\widehat{A}$, we can easily check that the map

$$(t, [\pi]) \in G \times \widehat{A} \to ([\pi], [\pi \circ \alpha_{t^{-1}}]) \in \widehat{A} \times \widehat{A}$$

is injective. So the inverse image of a finite subset by this map is also a finite set. Since the product topology of $\widehat{A} \times \widehat{A}$ is discrete, every compact subset in $\widehat{A} \times \widehat{A}$ is a finite set. Hence the inverse image of any compact subset of $\widehat{A} \times \widehat{A}$ is compact. ∎

We are ready to mention when the C^*-crossed product of a type I C^*-algebra becomes a type I C^*-algebra. Let (A, G, α) be a C^*-dynamical system and let A be a type I C^*-algebra with Hausdorff spectrum. It is seen from the proof of Theorem 1.1 (1) in [12] that if G acts freely and properly on $\widehat{A}$, then $A \times_\alpha G$ is of type I. However, in the case where $\widehat{A}$ is discrete, if we assume only that G acts freely on $\widehat{A}$, G automatically does properly on $\widehat{A}$ by Lemma 2.2. Then $A \times_\alpha G$ becomes a type I C^*-algebra.

THEOREM 2.3. *Let (A, G, α) be a C^*-dynamical system. Suppose that G acts freely on $\widehat{A}$. Then the following conditions are equivalent:*
(i) *A is a dual C^*-algebra;*
(ii) *G is discrete and $A \times_\alpha G$ is a dual C^*-algebra.*

Proof. (i) ⇒ (ii) Since A is a dual C^*-algebra, $\widehat{A}$ is discrete. Hence it follows from Lemma 2.1 that G is discrete. Since A is of type I and G acts freely on $\widehat{A}$, $A \times_\alpha G$ is a type I C^*-algebra. Furthermore, it follows from Theorem 1.1 (1) of [12] that $(A \times_\alpha G)\widehat{\ }$ is homeomorphic to the G-orbit space $\widehat{A}/G$ of $\widehat{A}$ by G. Since we easily see that $\widehat{A}/G$ is also discrete, $A \times_\alpha G$ is a dual C^*-algebra.

(ii) ⇒ (i) Since G is discrete, A is a C^*-subalgebra of $A \times_\alpha G$. Since any C^*-subalgebra of a dual C^*-algebra is dual by definition, A is a dual C^*-algebra. ∎

Let (A, G, α) be a C^*-dynamical system and let I be an α-invariant ideal of A. Then $I \times_\alpha G$ is a closed ideal of $A \times_\alpha G$. Note that the converse also holds. In fact, if, for an α-invariant C^*-subalgebra B of A, $B \times_\alpha G$ is a closed ideal of $A \times_\alpha G$, then B is an ideal of A (see [8]).

For each $x \in A$, we denote by $[x]$ the image of x under the canonical quotient map from A onto A/I. Define an action $\overline{\alpha}$ of G on A/I by

$$\overline{\alpha}_t([x]) = [\alpha_t(x)]$$

for $x \in A$. Thus we obtain the C^*-dynamical system $(A/I, G, \overline{\alpha})$, and $\overline{\alpha}$ induces the natural action of G on $\widehat{A/I}$. It is well-known that the quotient $(A \times_\alpha G)/(I \times_\alpha G)$ is isomorphic to $(A/I) \times_{\overline{\alpha}} G$ (for example, [5], Proposition 12). The following lemma plays an important role in the proof of Theorem 2.6.

LEMMA 2.4. *Let (A, G, α) be a C^*-dynamical system and let I be an α-invariant closed ideal of A. If G acts freely on $\widehat{A}$, then G acts freely on $\widehat{I}$ and on $\widehat{A/I}$, respectively.*

Proof. Note that there are a canonical homeomorphism from $\widehat{I}$ onto $\widehat{A} \setminus \mathrm{hull}(I)$ and a canonical one from $\widehat{A/I}$ onto $\widehat{A} \setminus \widehat{I}$ ([10], Theorem 4.1.11). It is easy to check that such homeomorphisms are G-equivariant. Hence, when we regard $\widehat{I}$ and $\widehat{A/I}$ as subsets of $\widehat{A}$, a straightforward discussion shows that the stability group of any point $[\pi]$ in $\widehat{I}$ (respectively $\widehat{A/I}$) is equal to that of $[\pi]$ in $\widehat{A} \setminus \mathrm{hull}(I) \subset \widehat{A}$ (respectively $\widehat{A} \setminus \widehat{I} \subset \widehat{A}$). Thus, freeness of the action of G on $\widehat{A}$ yields that G acts freely on $\widehat{I}$ and on $\widehat{A/I}$, respectively. ∎

Let X be a topological space. Then we recall that the n-th derived set $X^{(n)}$ of X is defined as follows: Put $X^{(0)} = X$ and define $X^{(n)}$ as the set of all accumulation points of $X^{(n-1)}$.

Now we mention a remark regarding Lemma 1 in [2] and adopt the notation used therein. Suppose that a locally compact group G acts on X as a homeomorphism group. Suppose that the n-th derived set $X^{(n)}$ is empty for some natural number n. Since the image of an accumulation point by any homeomorphism is an accumulation point again, $X^{(k)}$ is G-invariant for each natural number k. Hence the open subsets $Y_{n-1} \subset Y_{n-2} \subset \cdots \subset Y_1 \subset X$ given in Lemma 1 in [2] are G-invariant, which is easily seen from the proof of Lemma 1 in [2] and $Y_{n-1}, Y_{n-2} \setminus Y_{n-1}, \ldots, X \setminus Y_1$ are all discrete in the relative topology.

The following proposition is a generalization of Chu's characterization following his theorem mentioned in the introduction, which plays an essential role in proving Theorem 2.6. In fact, if we take the trivial group as G, Proposition 2.5 below is nothing but Chu's characterization.

PROPOSITION 2.5. *Let (A, G, α) be a C^*-dynamical system. Then the following conditions are equivalent:*

(i) *A has the weak Banach-Saks property.*

(ii) *There is a finite chain of α-invariant ideals $I_1 \subset I_2 \subset \cdots \subset I_n \subset A$ such that $I_1, I_2/I_1, I_3/I_2, \ldots, A/I_n$ are dual C^*-algebras.*

Proof. We have only to show the implication (i) $\Rightarrow$ (ii) The corresponding obsevation in [2] is valid for the proof. But, for the convenience of the reader, we will give the proof here.

Suppose that $\widehat{A}^{(n+1)}$ is empty for some natural number n. By the above remark, there exist G-invariant open subsets $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \widehat{A}$ such that $\Omega_1, \Omega_2 \setminus \Omega_1, \Omega_3 \setminus \Omega_2, \ldots, \widehat{A} \setminus \Omega_n$ are all discrete in the relative topology. There then exist α-invariant closed ideals $I_1 \subset I_2 \subset \cdots \subset I_n \subset A$ such that $\widehat{I_1} = \Omega_1, \widehat{I_2} = \Omega_2, \ldots, \widehat{I_n} = \Omega_n$. In fact, each I_k is given by taking the intersection of those $\ker\pi$ with $[\pi] \in \widehat{A} \setminus \Omega_k$. Since A is of type I, I_1 and quotients $I_2/I_1, I_3/I_2, \ldots, A/I_n$ are also of type I. Since $\widehat{I_1}, \widehat{I_2} \setminus \widehat{I_1}, \widehat{I_3} \setminus \widehat{I_2}, \ldots, \widehat{A} \setminus \widehat{I_n}$ are discrete in the relative topology, I_1 and the quotients $I_2/I_1, I_3/I_2, \ldots, A/I_n$ are dual C^*-algebras. ∎

We are now in a position to establish the main result in this section.

Theorem 2.6. *Let (A, G, α) be a C^*-dynamical system. Suppose that G acts freely on $\widehat{A}$. Then the following conditions are equivalent:*

(i) *A has the weak Banach-Saks property;*

(ii) *G is discrete and $A \times_\alpha G$ has the weak Banach-Saks property.*

Proof. (i) $\Rightarrow$ (ii) Since A has the weak Banach-Saks property, it follows from Proposition 2.5 that there exist α-invariant closed ideals $I_1 \subset \cdots \subset I_n \subset A$ such that I_1 and all the successive quotients are dual C^*-algebras.

Consider the C^*-dynamical system (I_1, G, α). Then Lemma 2.4 shows that G acts freely on $\widehat{I_1}$. Thus it follows from Theorem 2.3 that G is a discrete group and $I_1 \times_\alpha G$ is a dual C^*-algebra.

Consider the C^*-crossed product $I_k \times_\alpha G$ of I_k by G for $k = 1, 2, \ldots, n$. Then we obtain a sequence of ideals $I_1 \times_\alpha G \ \subset\ I_2 \times_\alpha G \ \subset\ \cdots\ \subset I_n \times_\alpha G \ \subset A \times_\alpha G$. To complete the proof, we have only to show that quotients

$$(I_2 \times_\alpha G)/(I_1 \times_\alpha G), (I_3 \times_\alpha G)/(I_2 \times_\alpha G), \ldots, (A \times_\alpha G)/(I_n \times_\alpha G)$$

are dual C^*-algebras. Put $I_{n+1} = A$. Since G acts freely on $\widehat{A}$, it follows from Lemma 2.4 that G acts freely on $\widehat{I_{k+1}}$ for any k, hence from Lemma 2.4 again that G acts freely on $(I_{k+1}/I_k)\widehat{\ }$. Since I_{k+1}/I_k is a dual C^*-algebra, $(I_{k+1}/I_k) \times_{\overline{\alpha}} G$ is a dual C^*-algebra by Theorem 2.3. Since $(I_{k+1} \times_\alpha G)/(I_k \times_\alpha G)$ is isomorphic to $(I_{k+1}/I_k) \times_{\overline{\alpha}} G$, $(I_{k+1} \times_\alpha G)/(I_k \times_\alpha G)$ is a dual C^*-algebra.

(ii) $\Rightarrow$ (i) Since G is discrete, A is a C^*-subalgebra of $A \times_\alpha G$. Since any C^*-subalgebra of a C^*-algebra with the weak Banach-Saks property has the weak Banach-Saks property ([2], Theorem 2) A has the weak Banach-Saks property. ∎

In the above theorem, the assumption that G should act freely on $\widehat{A}$ is necessary to show the implication (i) $\Rightarrow$ (ii). Even though G is discrete, Condition (i) does not necessarily imply Condition (ii) in general. For example, consider $A = \mathbb{C} \cdot 1$ and $G = \mathbb{Z}$, where we denote here by $\mathbb{Z}$ the set of all integers. Then we see that $A \times_\alpha G = C^*(\mathbb{Z}) = C(\mathbb{T})$, where we denote by $\mathbb{T}$ the one-dimensional torus group which is the dual group of $\mathbb{Z}$ and $C(\mathbb{T})$ denotes the C^*-algebra of all continuous functions on $\mathbb{T}$. Since the spectrum of $A \times_\alpha G$ is homeomorphic to $\mathbb{T}$, the n-th derived set of $(A \times_\alpha G)\widehat{\ }$ is $(A \times_\alpha G)\widehat{\ }$ itself for any natural number n. Thus we see that $A \times_\alpha G$ does not have the weak Banach-Saks property (see [2], Theorem 2).

3. COMPACT GROUP ACTION CASE

In Theorem 2.6 above, the group which acts on A as an automorphism group is discrete and $S_{[\pi]}$ consists only of the identity of the group at every $[\pi] \in \widehat{A}$. Hence, given a C^*-dynamical system (A, G, α), the situation opposite to that of Theorem 2.6 is that G is compact and $S_{[\pi]} = G$ at every $[\pi] \in \widehat{A}$. In the main theorem below, we shall suppose that G is a compact group and we treat the case where the situation that $S_{[\pi]} = G$ at every $[\pi] \in \widehat{A}$ occurs.

Let X be a topological space. We denote again by $X^{(n)}$ the n-th derived set of X for each natural number n. We first need the following lemma on derived sets of a topological space to show the main theorem below.

LEMMA 3.1. *Let X be a topological space and let $\{\mathcal{O}_i\}_{i\in I}$ be a family of open subsets in X. Suppose that $X = \bigcup_{i\in I} \mathcal{O}_i$. Then we have $X^{(k)} = \bigcup_{i\in I} \mathcal{O}_i^{(k)}$ for each natural number $k \in \mathbb{N}$.*

Proof. First we show that $X^{(k)} \subset \bigcup_{i\in I} \mathcal{O}_i^{(k)}$. Take any element x from $X^{(k)}$. If x belongs to $\mathcal{O}_{i_0}^{(k)}$ for some i_0, then we see that $X^{(k)} \subset \bigcup_{i\in I} \mathcal{O}_i^{(k)}$. Hence we assume that there exists x in $X^{(k)}$ such that $x \notin \mathcal{O}_i^{(k)}$ for all i. Since we have $X^{(k)} \subset \mathcal{O}_i^{(k)} \cup (X \setminus \mathcal{O}_i)$ for each i (see [2], Lemma 2), we conclude that $x \in X \setminus \mathcal{O}_i$ for all i. Then we see that

$$x \in \bigcap_{i\in I}(X \setminus \mathcal{O}_i) = X \setminus \Big(\bigcup_{i\in I} \mathcal{O}_i\Big) = \emptyset,$$

which is a contradiction. Thus we obtain the desired incusion.

The reverse inclusion is trivial. In fact, since the inclusion $X \supset \mathcal{O}_i$ shows that $X^{(k)} \supset \mathcal{O}_i^{(k)}$, we see that $X^{(k)} \supset \bigcup_{i\in I} \mathcal{O}_i^{(k)}$. Thus we complete the proof. ∎

For a C^*-dynamical system (A, G, α), we say that α is *pointwise unitary* if for every irreducible representation (π, H_π) of A, there exists a strongly continuous unitary representation u of G on the Hilbert space H_π such that

$$\pi(\alpha_t(x)) = u_t \pi(x) u_t^*$$

for all $x \in A$ and $t \in G$. In this case, we easily see that $S_{[\pi]} = G$ at every $[\pi] \in \widehat{A}$. We denote by $C(G)$ the set of all continuous functions on G and by A^α the fixed point algebra of A, respectively, which is defined by

$$A^\alpha = \{x \in A \mid \alpha_t(x) = x \text{ for all } t \in G\}.$$

Now we are ready to establish the main theorem for compact group action.

THEOREM 3.2. *Let (A, G, α) be a C^*-dynamical system and let G be a compact group. Consider the following conditions:*

(i) *A has the weak Banach-Saks property;*

(ii) *$A \times_\alpha G$ has the weak Banach-Saks property;*

(iii) *A^α has the weak Banach-Saks property.*

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Furthermore we suppose that G is (compact) abelian. Then the implication (iii) $\Rightarrow$ (ii) *holds. If A is of type* I *and if α is pointwise unitary, the implication* (ii) $\Rightarrow$ (i) *holds.*

Proof. Let $C(L^2(G))$ be the C^*-algebra of all compact operators on $L^2(G)$. It is easily seen that A is of type I if and only if so is $A \otimes C(L^2(G))$, and that $\widehat{A}$ is homeomorphic to the spectrum of $A \otimes C(L^2(G))$. It hence follows from Theorem 2 of [2] that A has the weak Banach-Saks property if and only if $A \otimes C(L^2(G))$ does so. We will repeatedly employ this fact in the proof.

(i) $\Rightarrow$ (ii) It follows from Imai-Takai's duality ([6]) that there exists an injective homomorphism β on $A \times_\alpha G$ such that the crossed product $(A \times_\alpha G) \times_\beta G$ by β is isomorphic to $A \otimes C(L^2(G))$. Since G is compact, $C(G)$ has the identity. Since $(A \times_\alpha G) \times_\beta G$ is generated by $(1 \otimes C(G))\beta(A \times_\alpha G)$, $A \times_\alpha G$ is identified with a C^*-subalgebra of $(A \times_\alpha G) \times_\beta G$. Since $A \otimes C(L^2(G))$ has the weak Banach-Saks property and since every C^*-subalgebra of a C^*-algebra with the weak Banach-Saks property has the same property, $A \times_\alpha G$ has the weak Banach-Saks property.

(ii) $\Rightarrow$ (iii) Since A^α is isomorphic to a hereditary C^*-subalgebra of $A \times_\alpha G$ (see [13]), A^α has the weak Banach-Saks property.

From now on, we assume that G is abelian.

(iii) $\Rightarrow$ (ii) By [13], there exists a projection p in the multiplier algebra of $A \times_\alpha G$ such that the hereditary C^*-subalgebra $p(A \times_\alpha G)p$ is isomorphic to A^α. Let B be the closed ideal of $A \times_\alpha G$ generated by $p(A \times_\alpha G)p$. We will identify A^α with $p(A \times_\alpha G)p$ unless there is confusion. Since A^α is of type I, so is $A \times_\alpha G$ (see [4], Theorem 3.2). Hence B is also of type I. For every nonzero irreducible representation (π, H) of B, the restriction of π to A^α is not zero. Hence the map $\pi \to \pi|A^\alpha$ induces a homeomorphism from $\widehat{B}$ onto $\widehat{A^\alpha}$. Since $\widehat{A^\alpha}^{(k)}$ is empty for some natural number k, $\widehat{B}^{(k)}$ is also empty. Thus it follows from Theorem 2 of [2] that B has the weak Banach-Saks property.

Let $\widehat{B}^{(k)}$ be empty for some integer k. Since $\widehat{B}$ is an open subset in $(A \times_\alpha G)^\frown$, there exists the largest open subset Ω in $(A \times_\alpha G)^\frown$ such that $\Omega^{(k)}$ is empty. In fact, consider the family of closed ideals

$$\{J_i \mid J_i \text{ is a closed ideal of } A \times_\alpha G \text{ and } \widehat{J}_i^{(k)} = \emptyset\}.$$

Denote by J the closed ideal generated by $\bigcup_i J_i$. Since we see that $\widehat{J} = \bigcup_i \widehat{J}_i$, it follows from Lemma 3.1 that $\widehat{J}^{(k)} = \bigcup_i \widehat{J}_i^{(k)}$. Since every open subset $\mathcal{O}$ of $(A \times_\alpha G)^\frown$ is given by $\mathcal{O} = \widehat{I}$ with some closed ideal I of $A \times_\alpha G$, $\widehat{J}$ is the largest of all open subsets $\mathcal{O}$ with $\mathcal{O}^{(k)} = \emptyset$. Thus we have only to take $\Omega = \widehat{J}$.

We have already mentioned above that $A \times_\alpha G$ is of type I. Hence, in order to obtain Condition (ii), by Theorem 2 of [2], it suffices to show that the k-th derived set of $(A \times_\alpha G)^\wedge$ is empty. For this, we have only to show that $(A \times_\alpha G)^\wedge = \Omega$. To derive a contradiction, we assume that $(A \times_\alpha G)^\wedge \neq \Omega$. Then we see that $J \neq A \times_\alpha G$ because $\Omega = \widehat{J}$. We claim that J is $\widehat{\alpha}$-invariant, where $\widehat{\alpha}$ denotes the dual action of $\widehat{G}$ on $A \times_\alpha G$. Since Ω is the largest of all open subsets in $(A \times_\alpha G)^\wedge$ whose k-th derived sets are empty, Ω is invariant under every homeomorphism of $(A \times_\alpha G)^\wedge$; in particular, invariant under the action of $\widehat{G}$ on $(A \times_\alpha G)^\wedge$ induced by $\widehat{\alpha}$, from which it easily follows that J is $\widehat{\alpha}$-invariant.

Since it is easy to check that J is a G-product (see, for example, [10], 7.8.2 for the details of a G-product), it follows from [10], 7.8.8 that there exists a nonzero α-invariant closed ideal I of A such that $J = I \times_\alpha G$. Then $J \neq A \times_\alpha G$ yields that $I \neq A$. But this is impossible by the proof of Theorem 3.2 in [4] because $J \supset B$. Thus we have reached a contradiction.

(ii) $\Rightarrow$ (i) Assume that A is of type I and α is pointwise unitary. Recall that $A \times_\alpha G$ is the enveloping C^*-algebra of $L^1(A, G)$, where $L^1(A, G)$ denotes the Banach*-algebra of all Bochner integrable A-valued functions on G, and that given a covariant representation (π, u, H) of A, one can construct the representation $(\pi \times u, H)$ of $A \times_\alpha G$ (see [10], 7.6 for the details).

First of all we assert that the action of $\widehat{G}$ induced by the dual action $\widehat{\alpha}$ of $\widehat{G}$ on $A \times_\alpha G$ is free on $(A \times_\alpha G)^\wedge$. For any $x \in L^1(A, G)$, we have

$$(\pi \times u)(\widehat{\alpha}_\gamma^{-1}(x)) \equiv \int_G \pi(\widehat{\alpha}_\gamma^{-1}(x(t)))u_t \,\mathrm{d}t \equiv \int_G \pi(x(t))\langle t, \gamma\rangle u_t \,\mathrm{d}t = \int_G \pi(x(t))(\gamma u)_t \,\mathrm{d}t$$

where $(\gamma u)_t \equiv \langle t, \gamma\rangle u_t$ and we adopted here that $\widehat{\alpha}_\gamma(x(t)) = \overline{\langle t, \gamma\rangle} x(t)$, as the definition of $\widehat{\alpha}$. Thus we obtain that $(\pi \times u) \circ \widehat{\alpha}_\gamma^{-1} = \pi \times (\gamma u)$. Let $(\pi \times u, H)$ be an irreducible representation of $A \times_\alpha G$. Since A is of type I, it follows from Proposition 2.1 of [11] that π is also irreducible. Suppose that $[(\pi \times u) \circ \widehat{\alpha}_\gamma^{-1}] = [\pi \times u]$ for some $\gamma \in \widehat{G}$, that is, $(\pi \times u) \circ \widehat{\alpha}_\gamma^{-1}$ is unitarily equivalent to $\pi \times u$. Then there exists a unitary V on H_π such that

$$(\pi \times u) \circ \widehat{\alpha}_\gamma^{-1}(\,\cdot\,) = V(\pi \times u)(\,\cdot\,)V^*.$$

Hence we see that

$$\pi \times (\gamma u) = (V\pi(\,\cdot\,)V^*) \times (VuV^*).$$

Since $(V\pi(\,\cdot\,)V^*, VuV^*, H_\pi)$ is a covariant representation, we conclude that $\pi(\,\cdot\,) = V\pi(\,\cdot\,)V^*$ and $\gamma u = VuV^*$. Then $\pi(\,\cdot\,) = V\pi(\,\cdot\,)V^*$ implies that $V \in \pi(A)' = \mathbb{C}\cdot 1$. Thus, we have $V = \lambda \cdot 1$ with $\lambda \in \mathbb{C}$. Hence we obtain that $\gamma u = VuV^* = u$, from which it follows that γ must be the identity element of $\widehat{G}$. Thus we see that $\widehat{G}$ acts freely on $(A \times_\alpha G)^\wedge$.

Applying Theorem 2.6 to $(A \times_\alpha G, \widehat{G}, \widehat{\alpha})$, it then follows that $(A \times_\alpha G) \times_{\widehat{\alpha}} \widehat{G}$ has the weak Banach-Saks property. Since $(A \times_\alpha G) \times_{\widehat{\alpha}} \widehat{G}$ is isomorphic to $A \otimes C(L^2(G))$ by Takai's duality ([10], 7.9.3), $A \otimes C(L^2(G))$ has the weak Banach-Saks property. Therefore A has the weak Banach-Saks property. ∎

We end this paper by giving some remarks concerning Thoerem 3.2.

REMARKS 3.3. (1) We remark that the weak Banach-Saks property in C^*-algebras is preserved under (strong) Morita equivalence ([9]). In the the proof of the implication (iii) $\Rightarrow$ (ii), we have shown that A^α has the weak Banach-Saks property if and only if the closed ideal B of $A \times_\alpha G$ generated by A^α does so. This will also follows from the well-known fact that A^α and B are (strongly) Morita equivalent (cf. [13]).

(2) In the above thoerem, even though A is of type I and G is abelian, the implication (ii) $\Rightarrow$ (i) does not necessarily hold in general. Hence the assumption that α be pointwise unitary is necessary to show (ii) $\Rightarrow$ (i). For example, take $A = C(\mathbb{T})$ as a C^*-algebra of type I and $G = \mathbb{T}$, where $C(\mathbb{T})$ denotes the set of all continuous functions on the one-dimensional torus group $\mathbb{T}$. We consider the translation on $\mathbb{T}$ as α. Then the Stone-von Neumann theorem shows that $A \times_\alpha G = C(\mathbb{T}) \times_\alpha \mathbb{T} \cong C(L^2(\mathbb{T}))$. Hence $A \times_\alpha G$ has the weak Banach-Saks property. Since $\widehat{A}$ is homeomorphic to $\mathbb{T}$, we obtain that $\widehat{A}^{(n)} = \widehat{A}$ for all $n \in \mathbb{N}$. Thus A does not have the weak Banach-Saks property.

(3) Note that there are unital C^*-algebras of non-type I which admit ergodic actions of compact abelian groups. Hence for such C^*-algebras A of non-type I, $A^\alpha(= \mathbb{C} \cdot 1)$ has the weak Banach-Saks property. But A does not have the weak Banach-Saks property because A is not of type I.

(4) Let (A, G, α) be a C^*-dynamical system and let G be a finite group. Then it follows from Therem 3.2 (and Theorem 2.6) that A has the weak Banach-Saks property if and only if $A \times_\alpha G$ has the weak Banach-Saks property.

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