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ON C^*-ALGEBRAS ASSOCIATED WITH SOFIC SHIFTS

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ABSTRACT. We show that for every sofic shift Λ that satisfies a certain condition, Matsumoto's C^*-algebra $\mathcal{O}_\Lambda$ is isomorphic to the Cuntz-Krieger algebra of the left Krieger cover graph of Λ.

KEYWORDS: *C^*-algebras, Sofic shifts, Cuntz-Krieger algebra, Matsumoto algebra, left Krieger cover.*

MSC (2000): 46L05, 37B10.

1. INTRODUCTION

In [2] Cuntz and Krieger defined the Cuntz-Krieger algebras. It is natural to see them as C^*-algebras associated with topological Markov shifts. In [5] (see also [1]) Matsumoto associated to each subshift a C^*-algebra in such a way that if the subshift is a topological Markov shift, the Matsumoto algebra associated to it is the Cuntz-Krieger algebra associated to it. Furthermore in [6] Matsumoto proved that for a sofic shift the associated Matsumoto algebra has the same K_0 and K_1 as the Cuntz-Krieger algebra for the left Krieger cover graph of the shift. It is therefore natural to ask whether the Matsumoto algebra associated to a sofic shift is isomorphic to the Cuntz-Krieger algebra of the left Krieger cover graph of the shift. In this paper we prove that if the sofic shift satisfies the condition (1.1) defined below, then indeed it is.

We will construct the isomorphism by using the universal properties of the Cuntz-Krieger algebra and the Matsumoto algebra to construct $*$-homomorphisms between them and then prove that these $*$-homomorphisms are each other's inverse.

Let Λ be a subshift defined on a finite alphabet $\mathfrak{A}$. We will follow the notation used in [5], [6], [7] and [8]. That is we denote by X_Λ the set of all right infinite sequences that appear in Λ, and we let for each $k \in \mathbb{N}$, Λ^k be the set of all words with length k appearing in some $x \in \Lambda$. We set $\Lambda_l = \bigcup_{k=0}^{l} \Lambda^k$ and $\Lambda^* = \bigcup_{k=0}^{\infty} \Lambda^k$, where Λ^0 denotes the empty word $\emptyset$.

We will by $\mathcal{O}_\Lambda$ denote the C^*-algebra defined in [5] by Matsumoto. Then $\mathcal{O}_\Lambda$ is generated by partial isometries $S_i, i \in \mathfrak{A}$. For $\mu \in \Lambda^*$ we define $S_\mu = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_{|\mu|}}$. Following [5], [6], [7] and [8] we let $\mathcal{A}_\Lambda$ be the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu^* S_\mu$, $\mu \in \Lambda^*$, and $\mathcal{D}_\Lambda$ the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu S_\nu^* S_\nu S_\mu^*$, $\mu, \nu \in \Lambda^*$. For $\mu \in \Lambda^*$ we denote by U_μ the cylinder set for μ:

$$U_\mu = \{x \in X_\Lambda \mid x_1 = \mu_1, \ldots, x_{|\mu|} = \mu_{|\mu|}\}.$$

We will need the following known facts about $\mathcal{O}_\Lambda$:

Lemma 1.1. (Lemma 3.1 of [5]) *If $i, j \in \mathfrak{A}$ are different, then*

$$S_i^* S_j = 0.$$

For a subshift Λ consider the following condition defined in [1]:

(1.1)
For every $l \in \mathbb{N}$ and every infinite subset F of Λ^* such that
$\{\nu \in \Lambda_l \mid \nu\mu_1 \in \Lambda^*\} = \{\nu \in \Lambda_l \mid \nu\mu_2 \in \Lambda^*\}$ for every $\mu_1, \mu_2 \in F$,
there exists a right infinite sequence $x \in X_\Lambda$ such that
$\{\nu \in \Lambda_l \mid \nu x \in X_\Lambda\} = \{\nu \in \Lambda_l \mid \nu\mu \in \Lambda^*\}$ for $\mu \in F$.

We denote by $\mathcal{B}(X_\Lambda)$ the C^*-algebra of all bounded functions on X_Λ.

Proposition 1.2. (Lemma 3.1 of [7]) *For every subshift Λ that satisfies condition* (1.1)*, the correspondence Φ defined by*

$$\Phi(S_\mu S_\nu^* S_\nu S_\mu^*) = 1_{U_\mu \cap \sigma^{-|\mu|}(\sigma^{|\nu|}(U_\nu))}, \quad \mu, \nu \in \Lambda^*$$

gives rise to an isomorphism from the commutative C^*-algebra $\mathcal{D}_\Lambda$ onto the C^*-subalgebra $C^*(1_{U_\mu \cap \sigma^{-|\mu|}(\sigma^{|\nu|}(U_\nu))}; \mu, \nu \in \Lambda^*)$ of $\mathcal{B}(X_\Lambda)$. Its restriction to $\mathcal{A}_\Lambda$ yields an isomorphism between $\mathcal{A}_\Lambda$ and $C^*(1_{\sigma^{|\nu|}(U_\nu)}; \nu \in \Lambda^*)$.

In Proposition 1.2 of [7] was stated without the requirement that the subshift satisfies condition (1.1), but in [1] there is an example of a subshift Λ that does not satisfies condition (1.1), and for which the correspondence considered in Proposition 1.2 does not give rise to an isomorphism from the commutative C^*-algebra $\mathcal{D}_\Lambda$ onto the C^*-subalgebra $C^*(1_{U_\mu \cap \sigma^{-|\mu|}(\sigma^{|\nu|}(U_\nu))}; \mu, \nu \in \Lambda^*)$ of $\mathcal{B}(X_\Lambda)$. So we need the subshifts to satisfy condition (1.1).

We will from now on assume that the subshifts we consider satisfy condition (1.1).

Lemma 1.3. *For each $f \in C^*(1_{\sigma^{|\nu|}(U_\nu)}; \nu \in \Lambda^*)$ and each $i \in \mathfrak{A}$,*

$$\Phi(S_i \Phi^{-1}(f) S_i^*) = \sigma^\star(f) 1_{U_i},$$

where $\sigma^\star$ is defined by $\sigma^\star(f)(x) = f(\sigma(x))$, $x \in X_\Lambda$.

Proof. Let

$$A = \{f \in C^*(1_{\sigma^{|\nu|}(U_\nu)}; \nu \in \Lambda^*) \mid \forall i \in \mathfrak{A} : \Phi(S_i \Phi^{-1}(f) S_i^*) = \sigma^\star(f) 1_{U_i}\}.$$

We want to show that $A = C^*(1_{\sigma^{|\nu|}(U_\nu)}; \nu \in \Lambda^*)$.

It is easy to see that A is a closed subset and that it is closed under addition and conjugation.

Let $f, g \in A$ and $i \in \mathfrak{A}$. Then

$$\begin{aligned}\Phi(S_i\Phi^{-1}(fg)S_i^*) &= \Phi(S_i\Phi^{-1}(f)\Phi^{-1}(g)S_i^*S_iS_i^*) = \Phi(S_i\Phi^{-1}(f)S_i^*)\Phi(S_i\Phi^{-1}(g)S_i^*)\\ &= \sigma^\star(f)1_{U_i}\sigma^\star(g)1_{U_i} = \sigma^\star(fg)1_{U_i}\end{aligned}$$

so $fg \in A$. Hence A is also closed under multiplication. So it is a C^*-subalgebra of $C^*(1_{\sigma^{|\nu|}(U_\nu)}; \nu \in \Lambda^*)$.

Since

$$\begin{aligned}\Phi(S_i\Phi^{-1}(1_{\sigma^{|\nu|}(U_\nu)})S_i^*) &= \Phi(S_iS_\nu^*S_\nu S_i^*) = 1_{U_i\cap\sigma^{-1}(\sigma^{|\nu|}(U_\nu))}\\ &= 1_{U_i}1_{\sigma^{-1}(\sigma^{|\nu|}(U_\nu))} = \sigma^\star(1_{\sigma^{|\nu|}(U_\nu)})1_{U_i},\end{aligned}$$

$1_{\sigma^{|\nu|}(U_\nu)} \in A$ for each $\nu \in \Lambda^*$.

So $A = C^*(1_{\sigma^{|\nu|}(U_\nu)}; \nu \in \Lambda^*)$. ∎

THEOREM 1.4. (Theorem 4.9 of [5][1]) *Let A be a unital C^*-algebra. Suppose that there is a unital $*$-homomorphism ψ from $\mathcal{A}_\Lambda$ to A and there are partial isometries s_i, $i \in \mathfrak{A}$ satisfying the following relations:*

(i) $\sum_{i\in\mathfrak{A}} s_is_i^* = 1$;

(ii) $s_\mu^*s_\mu s_\nu = s_\nu s_{\mu\nu}^*s_{\mu\nu}$ *for all* $\mu, \nu \in \mathfrak{A}^{(\mathbb{N})}$;

(iii) $s_\mu^*s_\mu = \psi(S_\mu^*S_\mu)$ *for all* $\mu \in \mathfrak{A}^{(\mathbb{N})}$;

where $s_\mu = s_{\mu_1}\cdots s_{\mu_{|\mu|}}$, $\mu = (\mu_1, \ldots, \mu_{|\mu|}) \in \Lambda^*$. *Then ψ extends to a unital $*$-homomorphism from $\mathcal{O}_\Lambda$ to A such that $\psi(S_i) = s_i$ for all $i \in \mathfrak{A}$.*

2. SOFIC SHIFTS

As in [6] and [8] we put for each $x \in X_\Lambda$ and each $l \in \mathbb{N}$

$$\Lambda_l(x) = \{\mu \in \Lambda_l \mid \mu x \in X_\Lambda\}.$$

Two points $x, y \in X_\Lambda$ are said to be l-past equivalent if $\Lambda_l(x) = \Lambda_l(y)$. It is easy to see that this is an equivalence relation. We write this equivalence as $x \sim_l y$. Let $\mathcal{E}_i^l$, $i = 1, 2, \ldots, m(l)$ be the set of all l-past equivalence classes of X_Λ. We denote by $\Omega_l = X_\Lambda/{\sim_l}$ the quotient space of the l-past equivalence classes of X_Λ.

Sofic shifts is a class of subshifts characterized by the following: A subshift Λ is *sofic* if and only if there exists $l \in \mathbb{N}$, such that $\Omega_k = \Omega_l$ for all $k \geqslant l$ (cf. [11] and [6]). In this case we will let $\Omega_\Lambda = \Omega_l$, $m_\Lambda = m(l)$ and $\mathcal{E}_i = \mathcal{E}_i^l$.

For a subset $\mathcal{E} \subseteq X_\Lambda$ and a $\mu \in \mathfrak{A}^{(\mathbb{N})}$ (the set of finite words over the alphabet $\mathfrak{A}$) we let

$$\mu\mathcal{E} = \{\mu x \in X_\Lambda \mid x \in \mathcal{E}\}.$$

Notice that if $\mu\mathcal{E}_i \neq \emptyset$ and $x \in \mathcal{E}_i$, then $\mu x \in X_\Lambda$.

When Λ is a sofic shift we define the left Krieger cover graph of Λ to be the labeled graph with vertex set $\{1, 2, \ldots, m_\Lambda\}$ and where there for each vertex i and

[1] We remark that it is necessary to include all finite words $\mathfrak{A}^{\mathbb{N}}$ and not just Λ^* in condition (ii) and (iii) to rule out the existence of a $*$-homomorphism from $\mathcal{O}_\Lambda$ to $\mathcal{O}_n$ (where n is the number of letters in the alphabet) sending the generators to the generators.

each $j \in \mathfrak{A}$ such that $j\mathcal{E}_i \neq \emptyset$ is an edge labeled j going from k to i, where k is the unique element of $\{1, 2, \dots, m_\Lambda\}$ such that $j\mathcal{E}_i \subseteq \mathcal{E}_k$ (cf. [6] and [3]). Notice that this graph is left-resolving (i.e. all edges ending at the same vertex have different labels).

For an edge e we will by $s(e)$, $r(e)$ and $\mathcal{L}(e)$ denote the source, range and label of e.

We let B_Λ be the matrix over the edge set $\mathfrak{E}_\Lambda$ defined by

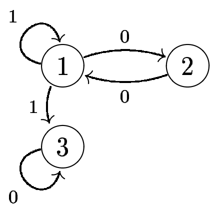
$$B_\Lambda(e, f) = \begin{cases} 1 & \text{if } r(e) = s(f), \\ 0 & \text{else.} \end{cases}$$

Then B_Λ is a $\{0, 1\}$-matrix with no zero-row or -column.

EXAMPLE 2.1. Let $\mathfrak{A} = \{0, 1\}$ and $\Lambda \subseteq \mathfrak{A}^{\mathbb{Z}}$ be the set of all sequences such that between two 1's there are an even number of 0's. Then Λ is a sofic shift called *the even shift* (cf. [4]). One can show that $m_\Lambda = 3$ and

$$\begin{aligned} \mathcal{E}_1 &= \{0^{2n}1x \mid n \in \mathbb{N}_0,\ x \in X_\Lambda\}, \\ \mathcal{E}_2 &= \{0^{2n+1}1x \mid n \in \mathbb{N}_0,\ x \in X_\Lambda\}, \\ \mathcal{E}_3 &= \{0^\infty\}. \end{aligned}$$

The left Krieger cover graph of Λ is given by:



$$B_\Lambda = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The following result is a key element to the results which will be proved in this paper.

PROPOSITION 2.2. *Let Λ be a sofic shift. Then there exists mutually orthogonal projections E_i, $i = 1, 2, \dots, m_\Lambda$ in $\mathcal{A}_\Lambda$ such that:*

(i) *$\mathcal{A}_\Lambda$ is generated by $E_i, i = 1, 2, \dots, m_\Lambda$;*

(ii) *for each $\mu \in \mathfrak{A}^{(\mathbb{N})}$*

$$S_\mu^* S_\mu = \sum_{\mu\mathcal{E}_i \neq \emptyset} E_i;$$

(iii) *for each $i \in \{1, 2, \dots, m_\Lambda\}$*

$$E_i = \sum_{s(e)=i} S_{\mathcal{L}(e)} E_{r(e)} S_{\mathcal{L}(e)}^*.$$

Proof. First notice that for each $\mu \in \mathfrak{A}^{(\mathbb{N})}$

$$\sigma^{|\mu|}(U_\mu) = \bigcup_{\mu\mathcal{E}_i \neq \emptyset} \mathcal{E}_i. \tag{2.1}$$

From this we get that for each $i \in \{1,2,\dots,m_\Lambda\}$

$$\mathcal{E}_i = \Big[\bigcap_{\mu\mathcal{E}_i \neq \emptyset} \sigma^{|\mu|}(U_\mu)\Big] \cap \Big[\bigcap_{\mu\mathcal{E}_i = \emptyset} X_\Lambda \setminus \sigma^{|\mu|}(U_\mu)\Big].$$

By (2.1) we see that there is only a finite number of different sets $\sigma^{|\mu|}(U_\mu)$. So we can for each $i \in \{1,2,\dots,m_\Lambda\}$ choose finite sets $M_i \subseteq \Lambda^*$ and $N_i \subseteq \Lambda^*$ such that

$$\mathcal{E}_i = \Big[\bigcap_{\mu \in M_i} \sigma^{|\mu|}(U_\mu)\Big] \cap \Big[\bigcap_{\mu \in N_i} X_\Lambda \setminus \sigma^{|\mu|}(U_\mu)\Big].$$

From this we get

$$1_{\mathcal{E}_i} = \Big[\prod_{\mu \in M_i} 1_{\sigma^{|\mu|}(U_\mu)}\Big]\Big[\prod_{\mu \in N_i} (1 - 1_{\sigma^{|\mu|}(U_\mu)})\Big].$$

So $1_{\mathcal{E}_i} \in C^*(1_{\sigma^{|\nu|}(U_\nu)}; \nu \in \Lambda^*)$ for each $i \in \{1,2,\dots,m_\Lambda\}$.

We can therefore define E_i by

$$E_i = \Phi^{-1}(1_{\mathcal{E}_i})$$

where Φ is as in Proposition 1.2. Since the $\mathcal{E}_i$'s are mutually disjoint the E_i's are mutually orthogonal projections. By (1.2) we have that

$$S_\mu^* S_\mu = \sum_{\mu\mathcal{E}_i \neq \emptyset} E_i.$$

So $\mathcal{A}_\Lambda = C^*(S_\mu^* S_\mu; \mu \in \Lambda^*)$ is generated by E_i, $i = 1,2,\dots,m_\Lambda$. Since for each $i \in \{1,2,\dots,m_\Lambda\}$

$$\begin{aligned}\mathcal{E}_i &= \bigcup_{j \in \mathfrak{A}} \{jx \mid jx \in \mathcal{E}_i\} = \bigcup_{j \in \mathfrak{A}} \bigcup_{k=1}^{m_\Lambda} \{jx \mid jx \in \mathcal{E}_i, x \in \mathcal{E}_k\} \\ &= \bigcup_{j \in \mathfrak{A}} \bigcup_{j\mathcal{E}_k \subseteq \mathcal{E}_i} j\mathcal{E}_k = \bigcup_{s(e)=i} U_{\mathcal{L}(e)} \cap \sigma^{-1}(\mathcal{E}_{r(e)})\end{aligned}$$

we have by Lemma 1.3 that

$$E_i = \sum_{s(e)=i} S_{\mathcal{L}(e)} E_{r(e)} S_{\mathcal{L}(e)}^*. \quad \blacksquare$$

Example 2.3. If we let Λ be as in Example 2.1, then we get:

$$\begin{aligned}E_1 &= S_1^* S_1 (1 - S_{10}^* S_{10}), \\ E_2 &= S_{10}^* S_{10} (1 - S_1^* S_1), \\ E_3 &= S_1^* S_1 S_{10}^* S_{10}.\end{aligned}$$

3. THE ISOMORPHISM

DEFINITION 3.1. For a matrix A over a finite set Σ, with $A(i,j) \in \{0,1\}$ and where every row and column of A is non-zero, we define (cf. [2]) the *Cuntz-Krieger algebra for A* to be the universal C^*-algebra $\mathcal{O}_A$ generated by partial isometries s_i, $i \in \Sigma$ such that

(a) $s_i s_i^* s_j s_j^* = 0$ for $i \neq j$,

(b) $s_i^* s_i = \sum_{j \in \Sigma} A(i,j) s_j s_j^*$.

We notice that $\sum_{i \in \Sigma} s_i s_i^* = 1_{\mathcal{O}_A}$.

PROPOSITION 3.2. *Let Λ be a sofic shift. Then there exists a $*$-homomorphism from $\mathcal{O}_{B_\Lambda}$ to $\mathcal{O}_\Lambda$ sending s_e to $S_{\mathcal{L}(e)} E_{r(e)}$, where E_i is as in Proposition* 2.2.

Proof. Let $\widetilde{S}_e = S_{\mathcal{L}(e)} E_{r(e)}$. By Proposition 2.2 $S_j^* S_j = \sum_{j\mathcal{E}_i \neq \emptyset} E_i$ for each $j \in \mathfrak{A}$, and since $\mathcal{L}(e)\mathcal{E}_{r(e)} \neq \emptyset$, we have $E_{r(e)} \leqslant S_{\mathcal{L}(e)}^* S_{\mathcal{L}(e)}$. So

$$\widetilde{S}_e^* \widetilde{S}_e = E_{r(e)} S_{\mathcal{L}(e)}^* S_{\mathcal{L}(e)} E_{r(e)} = E_{r(e)}.$$

Hence $\widetilde{S}_e$ is a partial isometry.

Since the left Krieger cover graph is left-resolving, we have that if $e \neq f$ either $\mathcal{L}(e) \neq \mathcal{L}(f)$ or $r(e) \neq r(f)$. If $\mathcal{L}(e) \neq \mathcal{L}(f)$

$$S_{\mathcal{L}(e)}^* S_{\mathcal{L}(f)} = 0$$

by Lemma 1.2, and if $r(e) \neq r(f)$

$$E_{r(e)} S_{\mathcal{L}(e)}^* S_{\mathcal{L}(f)} E_{r(f)} = E_{r(e)} E_{r(f)} = 0.$$

So

$$\widetilde{S}_e \widetilde{S}_e^* \widetilde{S}_f \widetilde{S}_f^* = S_{\mathcal{L}(e)} E_{r(e)} S_{\mathcal{L}(e)}^* S_{\mathcal{L}(f)} E_{r(f)} S_{\mathcal{L}(f)}^* = 0$$

for $e \neq f$.

By Proposition 2.2

$$\begin{aligned} \widetilde{S}_e^* \widetilde{S}_e = E_{r(e)} &= \sum_{s(f)=r(e)} S_{\mathcal{L}(f)} E_{r(f)} S_{\mathcal{L}(f)}^* \\ &= \sum_{s(f)=r(e)} \widetilde{S}_f \widetilde{S}_f^* = \sum_{f \in \mathfrak{E}_\Lambda} B(e,f) \widetilde{S}_f \widetilde{S}_f^*. \end{aligned}$$

So the partial isometries $\widetilde{S}_e$, $e \in \mathfrak{E}_\Lambda$, satisfy the Cuntz-Krieger relations and therefore there exists a $*$-homomorphism from $\mathcal{O}_{B_\Lambda}$ to O_Λ sending s_e to $\widetilde{S}_e = S_{\mathcal{L}(e)} E_{r(e)}$. ∎

LEMMA 3.3. *Let Λ be a sofic shift. For $\mu \in \mathfrak{A}^{(\mathbb{N})}$ and $i \in \{1, 2, \ldots, m_\Lambda\}$ the following are equivalent:*

(i) *$\mu\mathcal{E}_i \neq \emptyset$;*

(ii) *there exists a path α on the left Krieger cover graph of Λ such that $\mathcal{L}(\alpha) = \mu$ and $r(\alpha) = i$.*

The path α is unique, and furthermore it fulfills that $\mu\mathcal{E}_i \subseteq \mathcal{E}_{s(\alpha)}$.

Proof. We will prove the statement by induction over the length of μ. First assume that $\mu \in \mathfrak{A}$. Then the statement follows directly from the definition of the left Krieger cover graph.

Assume next that we have proved the statement for $\mu \in \mathfrak{A}^k$, and that $\nu \in \mathfrak{A}^{k+1}$. Let $\mu = (\nu_2, \nu_3, \ldots, \nu_{|\nu|})$.

If $\nu\mathcal{E}_i \neq \emptyset$, then $\mu\mathcal{E}_i \neq \emptyset$. So there exists a unique path α such that $\mathcal{L}(\alpha) = \mu$ and $r(\alpha) = i$ and furthermore $\mu\mathcal{E}_i \subseteq \mathcal{E}_{s(\alpha)}$. Since $\nu\mathcal{E}_i = \nu_1\mu\mathcal{E}_i \subseteq \nu_1\mathcal{E}_{s(\alpha)}$ and $\nu\mathcal{E}_i \neq \emptyset$, $\nu_1\mathcal{E}_{s(\alpha)} \neq \emptyset$. Thus there exists an unique edge e, such that $\mathcal{L}(e) = \nu_1$ and $r(e) = s(\alpha)$ and furthermore $\nu_1\mathcal{E}_{s(\alpha)} \subseteq \mathcal{E}_{s(e)}$. Since $r(e) = s(\alpha)$, $e\alpha$ is a path on the left Krieger cover graph and $\mathcal{L}(e\alpha) = \nu$, $r(e\alpha) = i$ and $\nu\mathcal{E}_i \subseteq \mathcal{E}_{s(e\alpha)}$. If α' is another path such that $\mathcal{L}(\alpha') = \nu$ and $r(\alpha') = i$, then $\mathcal{L}((\alpha'_2, \ldots, \alpha'_{|\alpha'|})) = \mu$, $r((\alpha'_2, \ldots, \alpha'_{|\alpha'|})) = i$, $\mathcal{L}(\alpha'_1) = \nu_1$ and $r(\alpha'_1) = s((\alpha'_2, \ldots, \alpha'_{|\alpha'|}))$. So $(\alpha'_2, \ldots, \alpha'_{|\alpha'|}) = \alpha$ and $\alpha'_1 = e$. Hence $\alpha' = e\alpha$.

If there exists a path β such that $\mathcal{L}(\beta) = \nu$ and $r(\beta) = i$, then $\gamma = (\beta_2, \beta_3, \ldots, \beta_{|\beta|})$ is a path such that $\mathcal{L}(\gamma) = \mu$ and $r(\beta) = i$, and β_1 is an edge such that $\mathcal{L}(\beta_1) = \nu_1$ and $r(\beta_1) = s(\gamma)$. So $\emptyset \neq \mu\mathcal{E}_i \subseteq \mathcal{E}_{s(\gamma)}$ and $\nu_1\mathcal{E}_{s(\gamma)} \neq \emptyset$. Hence $\nu\mathcal{E}_i = \nu_1\mu\mathcal{E}_i \neq \emptyset$. ∎

PROPOSITION 3.4. *Let Λ be a sofic shift. Then there exists a $*$-homomorphism from $\mathcal{O}_\Lambda$ to $\mathcal{O}_{B_\Lambda}$ sending S_i to $\sum_{\mathcal{L}(e)=i} s_e$ and $E_{r(e)}$ to $s_e^* s_e$, where E_i is as in Proposition* 2.2.

Proof. Observe that $B(e, g) = B(f, g)$ for all $g \in \mathfrak{E}_\Lambda$ if $r(e) = r(f)$, and that $B(e, g)B(f, g) = 0$ for all $g \in \mathfrak{E}_\Lambda$ if $r(e) \neq r(f)$. So $s_e^* s_e = s_f^* s_f$ if $r(e) = r(f)$ and $s_e^* s_e s_f^* s_f = 0$ if $r(e) \neq r(f)$.

Since $\mathcal{A}_\Lambda$ is generated by E_i, $i = 1, 2, \ldots, m_\Lambda$ and $E_i E_j = 0$ for $i \neq j$ there exists a $*$-homomorphism ψ from $\mathcal{A}_\Lambda$ to $\mathcal{O}_{B_\Lambda}$ sending $E_r(e)$ to $s_e^* s_e$.

For each $\mu \in \Lambda^*$ define $\widetilde{s}_\mu$ by

$$\widetilde{s}_\mu = \sum_{\mathcal{L}(\alpha)=\mu} s_{\alpha_1} \cdots s_{\alpha_{|\alpha|}}.$$

Since

$$\begin{aligned}
\widetilde{s}_\mu \widetilde{s}_\nu &= \sum_{\mathcal{L}(\alpha)=\mu} s_{\alpha_1} \cdots s_{\alpha_{|\alpha|}} \sum_{\mathcal{L}(\beta)=\nu} s_{\beta_1} \cdots s_{\beta_{|\beta|}} \\
&= \sum_{\substack{\mathcal{L}(\alpha)=\mu \\ \mathcal{L}(\beta)=\nu \\ r(\alpha)=s(\beta)}} s_{\alpha_1} \cdots s_{\alpha_{|\alpha|}} s_{\beta_1} \cdots s_{\beta_{|\beta|}} = \sum_{\mathcal{L}(\gamma)=\mu\nu} s_{\gamma_1} \cdots s_{\gamma_{|\gamma|}} = \widetilde{s}_{\mu\nu},
\end{aligned}$$

we have that $\widetilde{s}_\mu = \widetilde{s}_{\mu_1} \cdots \widetilde{s}_{\mu_{|\mu|}}$ for each μ.

Since the left Krieger cover graph is left-resolving we have

$$\widetilde{s}_i \widetilde{s}_i^* \widetilde{s}_i = \sum_{\mathcal{L}(e)=i} s_e \sum_{\mathcal{L}(f)=i} s_f^* \sum_{\mathcal{L}(g)=i} s_g = \sum_{\mathcal{L}(e)=i} s_e s_e^* s_e = \sum_{\mathcal{L}(e)=i} s_e = \widetilde{s}_i,$$

so $\widetilde{s}_i$ is a partial isometry.

We see that

$$\sum_{i \in \mathfrak{A}} \widetilde{s}_i \widetilde{s}_i^* = \sum_{i \in \mathfrak{A}} \sum_{\mathcal{L}(e)=i} s_e s_e^* = \sum_{e \in \mathfrak{E}_\Lambda} s_e s_e^* = 1.$$

If $\nu\mu \notin \Lambda^*$,

$$\widetilde{s}_\nu^* \widetilde{s}_\nu \widetilde{s}_\mu = 0 = \widetilde{s}_\mu \widetilde{s}_{\nu\mu}^* \widetilde{s}_{\nu\mu},$$

and if $\nu\mu \in \Lambda^*$,

$$\widetilde{s}_\nu^* \widetilde{s}_\nu \widetilde{s}_\mu = \widetilde{s}_\mu = \widetilde{s}_\mu \widetilde{s}_{\nu\mu}^* \widetilde{s}_{\nu\mu},$$

so $\widetilde{s}_\nu^* \widetilde{s}_\nu \widetilde{s}_\mu = \widetilde{s}_\mu \widetilde{s}_{\nu\mu}^* \widetilde{s}_{\nu\mu}$ for all $\nu, \mu \in \mathfrak{A}^{(\mathbb{N})}$.

By Proposition 2.2 and Lemma 3.3 we have that

$$\psi(S_\mu^* S_\mu) = \psi\Bigg(\sum_{\mu\mathcal{E}_i \neq \emptyset} E_i\Bigg) = \sum_{\mathcal{L}(\alpha)=\mu} s_{\alpha_{|\alpha|}}^* s_{\alpha_{|\alpha|}} = \sum_{\mathcal{L}(\alpha)=\mu} s_{\alpha_{|\alpha|}}^* \cdots s_{\alpha_1}^* s_{\alpha_1} \cdots s_{\alpha_{|\alpha|}} = \widetilde{s}_\mu^* \widetilde{s}_\mu$$

for all $\mu \in \mathfrak{A}^{(\mathbb{N})}$.

So according to Theorem 1.4 ψ extends to a $*$-homomorphism from $\mathcal{O}_\Lambda$ to $\mathcal{O}_{B_\Lambda}$ sending $E_{r(e)}$ to $s_e^* s_e$ and S_i to $\widetilde{s}_i = \sum_{\mathcal{L}(e)=i} s_e$. ∎

THEOREM 3.5. *Let Λ be a sofic shift. Then $\mathcal{O}_\Lambda \simeq \mathcal{O}_{B_\Lambda}$.*

Proof. According to Proposition 3.2 there exists a $*$-homomorphism $\varphi : \mathcal{O}_{B_\Lambda} \to \mathcal{O}_\Lambda$ such that $\varphi(s_e) = S_{\mathcal{L}(e)} E_{r(e)}$, and according to Proposition 3.4 there exists a $*$-homomorphism $\psi : \mathcal{O}_\Lambda \to \mathcal{O}_{B_\Lambda}$ such that $\psi(S_i) = \sum_{\mathcal{L}(e)=i} s_e$ and $\psi(E_{r(e)}) = s_e^* s_e$.

We have that

$$\varphi(\psi(S_i)) = \varphi\Bigg(\sum_{\mathcal{L}(e)=i} s_e\Bigg) = \sum_{\mathcal{L}(e)=i} \varphi(s_e) = \sum_{\mathcal{L}(e)=i} S_{\mathcal{L}(e)} E_{r(e)} = \sum_{\mathcal{L}(e)=i} S_i E_{r(e)} = S_i,$$

where we for the last equality use that $\sum_{j=1}^{m_\Lambda} E_j = 1$, and that $S_i E_j = 0$ if there does not exists an edge with range j and label i. So $\varphi \circ \psi = \mathrm{id}_{\mathcal{O}_\Lambda}$, and since

$$\psi(\varphi(s_e)) = \psi(S_{\mathcal{L}(e)}E_{r(e)}) = \sum_{\mathcal{L}(f)=\mathcal{L}(e)} s_f s_e^* s_e = s_e,$$

$\psi \circ \varphi = \mathrm{id}_{\mathcal{O}_{B_\Lambda}}$. Thus ψ and φ are each other's inverse and $\mathcal{O}_\Lambda \simeq \mathcal{O}_{B_\Lambda}$. ∎

REMARK 3.6. For a sofic shift Λ that does not satisfy condition (1.1), we can not be sure that the correspondence considered in Proposition 1.2 gives rise to an isomorphism from the commutative C^*-algebra $\mathcal{D}_\Lambda$ onto the C^*-subalgebra $C^*(1_{U_\mu \cap \sigma^{-|\mu|}(\sigma^{|\nu|}(U_\nu))}; \mu, \nu \in \Lambda^*)$ of $\mathcal{B}(X_\Lambda)$, and thus that $\mathcal{O}_\Lambda \simeq \mathcal{O}_{B_\Lambda}$. But it is still true that $\mathcal{O}_\Lambda$ is isomorphic to a Cuntz-Krieger algebra $\mathcal{O}_{A_\Lambda}$ for another matrix A_Λ that is a bit more complicated to describe than B_Λ.

REMARK 3.7. After this paper was completed, the author received Kengo Matsumoto's preprint ([9]), where there is a result from which Theorem 3.5 follows in case the sofic shift Λ satisfies a certain condition (I) and condition (1.1).

REMARK 3.8. In [1], Matsumoto and the author have considered another C^*-algebra associated with a subshift. By using exactly the same methods used in this paper one can show that for a sofic shift that satisfies a certain condition (I) (but not necessarily condition (1.1)) this C^*-algebra is isomorphic to the Cuntz-Krieger algebras of the left Krieger cover graph of the sofic shift.

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