J. OPERATOR THEORY
49(2003), 263–283

FUNCTIONAL CALCULUS, REGULARITY AND RIESZ TRANSFORMS OF WEIGHTED SUBCOERCIVE OPERATORS ON σ-FINITE MEASURE SPACES

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Communicated by Norberto Salinas

Abstract. If H is an n-th order weighted subcoercive operator associated to a continuous representation U of a d-dimensional connected Lie group G in $L_p(\mathcal{M};\mu)$, where $p \in (1,\infty)$ and $(\mathcal{M};\mu)$ is a σ-finite measure space, then we show that $\nu I + \overline{H}$ has a bounded H_∞ functional calculus if $\mathrm{Re}\,\nu$ is large enough.

Moreover, the domain $D((\nu I + \overline{H})^{m/n})$ of the fractional power equals the space of m times differentiable vectors in L_p-sense if $\mathrm{Re}\,\nu$ is large enough and m is in a suitable subset of $[0,\infty)$.

Keywords: *Functional calculus, regularity, Riesz transform, continuous representation, induced representation, weighted subcoercive operator, σ-finite measure space.*

MSC (2000): 47A60, 43A65, 22D30, 35K05, 58C35.

1. INTRODUCTION

During the last two decades there is a steadily growing interest in the following subjects due to their intimate connection with heat kernels, kernels of Poisson and Riesz type, singular integration theory and harmonic analysis in L_p-spaces of manifolds:

– the bounded H_∞ functional calculus of operators, in particular semigroup generators in L_p-spaces with $p \in (1,\infty)$ (cf. [4], [13], [14] and [5]);
– regularity in L_p-spaces with $p \in (1,\infty)$ (cf. [2], [9] and [10]);
– Riesz transforms in L_p-spaces with $p \in (1,\infty)$ (cf. [19] and [17]).

Let H be an n-th order subcoercive operator associated to the left (or right) regular representation of a connected Lie group G in $L_p(G;\mathrm{d}g)$ with $p \in (1,\infty)$, where $\mathrm{d}g$ denotes the left Haar measure on G. Then it was shown that there is a $\theta_C \geqslant 0$ such that for all $\varphi \in (\pi/2 - \theta_C, \pi]$ there is a $\nu_0 \geqslant 0$ such that if $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu \geqslant \nu_0$ then the operator $\nu I + \overline{H}$ has a bounded H_∞ functional calculus in $L_p(G; \mathrm{d}g)$ over F_φ, the space of bounded and holomorphic functions in a sector with angle φ (for definitions and details we refer to [8]). Simple examples of subcoercive operators on the real line already elucidate that the constant ν_0 depends on the angle φ.

As far as the regularity is concerned the following can be remarked. The C^∞-elements of a continuous representation U of a connected Lie group G are exactly the C^∞-elements of a subcoercive operator associated to the representation U (cf. [10], Theorem 2.6.I). However this coincidence is no longer valid in general if one compares the C^m-elements for $m \in \mathbb{N}$. Indeed, the left regular representations in $L_1(\mathbb{R}^2; \mathrm{d}x)$ or $L_\infty(\mathbb{R}^2; \mathrm{d}x)$ already show that the C^m-elements may differ for some $m \in \mathbb{N}$ (cf. [12] and [15]). On the other hand the differential structures are still the same if $p \in (1, \infty)$. More generally, if $p \in (1, \infty)$ and $m \in \mathbb{N}$ then the C^m-elements of the left (respectively right) regular representation of a connected Lie group G in $L_p(G; \mathrm{d}g)$ coincide with the C^m-elements of a subcoercive operator associated to the left (respectively right) regular representation of G in $L_p(G; \mathrm{d}g)$ (cf. [2]). Further let U be a bounded continuous representation of a connected amenable Lie group in $L_p(\mathcal{M}; \mu)$ with $p \in (1, \infty)$ and $(\mathcal{M}, \mu)$ a σ-finite measure space. Then the optimal regularity for n-th order weighted subcoercive operators affiliated to U was established in [6].

The comparison of the differential structures is closely related to the boundedness of the Riesz transforms. For $p \in (1, \infty)$ it was established that if the real part of the zero-order coefficient of a weighted subcoercive operator H is sufficiently large then the Riesz transforms of H are bounded on $L_p(G; \mathrm{d}g)$ (cf. [2] and [11]). Anker ([1]) established boundedness of the Riesz transforms for the Laplace-Beltrami operators in $L_p(X; \mathrm{d}x)$, where X is a non-compact symmetric space obtained by the quotient of a semisimple Lie group G and a maximal compact subgroup K and $\mathrm{d}x$ the G-invariant measure on X. Let H be an n-th order weighted subcoercive operator affiliated to a continuous bounded representation U of an amenable connected Lie group G in $L_p(\mathcal{M}; \mu)$, where $p \in (1, \infty)$ and $(\mathcal{M}, \mu)$ is a σ-finite measure space. In [6] it was shown that there is a $\nu_0 \geqslant 0$ such that if $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu \geqslant \nu_0$ then the Riesz transforms of $\nu I + \overline{H}$ are bounded on $L_p(\mathcal{M}; \mu)$.

Although the techniques used in this paper are rather standard, we are able to generalize most of the above results to a fairly large class of continuous representations. Most of the non trivial results are taken from [8]. In Section 2 we consider n-th order weighted subcoercive operators H with respect to a continuous representation U of a connected Lie group G in $L_p(\mathcal{M}; \mu)$ with $p \in (1, \infty)$ and $(\mathcal{M}, \mu)$ a σ-finite measure space. We show that there is a $\theta_C \geqslant 0$ such that for all $\varphi \in (\pi/2 - \theta_C, \pi]$ there is a $\nu_0 \geqslant 0$ such that if $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu \geqslant \nu_0$ then $\nu I + \overline{H}$ has a bounded H_∞ functional calculus in $L_p(\mathcal{M}; \mu)$ over F_φ, the Riesz transforms of $\nu I + \overline{H}$ are bounded on $L_p(\mathcal{M}; \mu)$ and we deduce optimal regularity for $\nu I + \overline{H}$. Moreover we deduce weak type $(1,1)$-estimates for functional operators generalizing Proposition 3.2 in [8]. This large class of representations U covers amongst others the continuous cocycle representations on homogeneous spaces which include the interesting class of continuous representations induced by a character. We emphasize that the representation U need not be bounded and G

may be non-amenable whereas U was bounded and G amenable in [6]. Finally, we deduce in Section 3 kernel bounds for reduced operator kernels of Riesz transforms and functional operators of strongly elliptic operators on homogeneous spaces.

2. FUNCTIONAL CALCULUS, REGULARITY AND RIESZ TRANSFORMS

In this section we consider n-th order weighted subcoercive operators H with respect to a continuous representation U of a connected Lie group G in $L_p(\mathcal{M};\mu)$ with $p \in (1,\infty)$, where $(\mathcal{M},\mu)$ denotes a σ-finite measure space. We show that there is a $\nu_0 \geqslant 0$ such that if $\nu \in \mathbb{C}$ with $\operatorname{Re}\nu \geqslant \nu_0$ then $\nu I + \overline{H}$ has a bounded H_∞ functional calculus in $L_p(\mathcal{M};\mu)$, the Riesz transforms of $\nu I + \overline{H}$ are bounded on $L_p(\mathcal{M};\mu)$, and optimal regularity is valid for $\nu I + \overline{H}$.

Let G be a connected d-dimensional Lie group with Haar measure $\mathrm{d}g$. Suppose that $U : G \to \mathcal{L}(\mathcal{X})$ is a continuous representation of G in a Banach space $\mathcal{X}$ endowed with the norm $\|\cdot\|$. Let $a_1,\ldots,a_{d'}$ be an algebraic basis for $\mathfrak{g}$, i.e., a finite sequence of linearly independent elements of $\mathfrak{g}$ which generate $\mathfrak{g}$. This means that one can find an integer r such that $a_1,\ldots,a_{d'}$ together with all the multi-commutators $(\operatorname{ad} a_{j_1})\cdots(\operatorname{ad} a_{j_{n-1}})(a_{j_n})$, with $j_1,\ldots,j_n \in \{1,\ldots,d'\}$ and $n \leqslant r$, establish a basis for $\mathfrak{g}$. Next let $w_1,\ldots,w_{d'}$ denote a set of weights in $[1,\infty)$. Then the algebraic basis $a_1,\ldots,a_{d'}$ and the weights $w_1,\ldots,w_{d'}$ induce in a natural way a modulus $g \mapsto |g|'$ on the connected Lie group G. For a detailed description and definition we refer to [11], Section 6. Let $B'_\varepsilon = \{g \in G : |g|' < \varepsilon\}$ be the corresponding ball for all $\varepsilon > 0$. The modulus function $|\cdot|'$ in turn defines a unique local dimension D' such that there is a $C \geqslant 1$ such that

$$C^{-1}\rho^{D'} \leqslant \mathrm{Vol}_G(B'_\rho) \leqslant C\rho^{D'}$$

for all $\rho \in (0,1]$, where Vol_G denotes the volume with respect to the left Haar measure $\mathrm{d}g$.

For all $i \in \{1,\ldots,d'\}$ denote by $A_i = \mathrm{d}U(a_i)$ the infinitesimal generator of the one parameter group $t \mapsto U(\exp(-ta_i))$. We also need multi-index notation. Let $J(d') = \bigoplus_{k=0}^{\infty}\{1,\ldots,d'\}^k$ denote the set of all multi-indices over the index set $\{1,\ldots,d'\}$. If $\alpha = (i_1,\ldots,i_k) \in J(d')$ then we set $A^\alpha = A_{i_1}\circ\cdots\circ A_{i_k}$ and we denote by $\|\alpha\| = \sum_{j=1}^{k} w_{i_j}$ the weighted length of the multi-index α.

The C^m-subspace $\mathcal{X}'_m$ is the weighted space defined by

$$\mathcal{X}'_m = \bigcap_{\substack{\alpha\in J(d')\\ \|\alpha\|\leqslant m}} D(A^\alpha)$$

endowed with the norm

$$\|u\|'_m = \max_{\substack{\alpha\in J(d')\\ \|\alpha\|\leqslant m}} \|A^\alpha u\|.$$

An n-th order form is a function $C : J(d') \to \mathbb{C}$ such that $C(\alpha) = 0$ for $\|\alpha\| > n$ and there is an $\alpha \in J(d')$ with $\|\alpha\| = n$ such that $C(\alpha) \neq 0$. We consider the n-th order operator H

$$H = \mathrm{d}U(C) = \sum_{\substack{\alpha \in J(d') \\ \|\alpha\| \leqslant n}} c_\alpha A^\alpha \tag{2.1}$$

with domain $D(H) = \bigcap_{\substack{\alpha \in J(d') \\ \|\alpha\| \leqslant n}} D(A^\alpha)$ and $c_\alpha = C(\alpha)$ for all $\alpha \in J(d')$ with $\|\alpha\| \leqslant n$. In the sequel we denote the zero-order coefficient c_α with $\|\alpha\| = 0$ also by c_0.

Let $\widetilde{A}_1, \ldots, \widetilde{A}_{d'}$ be the infinitesimal generators with respect to the left regular representation L_G of G in $L_2(G; \mathrm{d}g)$ and the directions $a_1, \ldots, a_{d'}$. Then we say that C is an n-th order G-weighted subcoercive form if $n/w_i \in 2\mathbb{N}$ for each $i \in \{1, \ldots, d'\}$ and the operator $\mathrm{d}L_G(C)$ satisfies the following inequality: there is a $\mu > 0$ and $\nu \in \mathbb{R}$ such that

$$\mathrm{Re}(v, \mathrm{d}L_G(C)v) \geqslant \mu \Big(\max_{\substack{\alpha \in J(d') \\ \|\alpha\| = n/2}} \|\widetilde{A}^\alpha v\|_2 \Big)^2 - \nu \|v\|_2^2$$

for all $v \in C_c^\infty(V)$, where V is some open neighbourhood of the identity $e \in G$. Moreover, the corresponding operator $H = \mathrm{d}U(C)$ is called an *n-th order weighted subcoercive operator associated to* U. Then by Theorem 1.1.IV of [11] the operator $\overline{H}$ generates a holomorphic semigroup S in an open representation independent sector $\Lambda(\theta_C)$, where

$$\Lambda(\varphi) = \{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \varphi \}$$

for all $\varphi \in (0, \pi]$. Moreover, it follows from Theorem 1.1.IV of [11] that the semigroup S has a representation independent, fast decreasing, Lie group kernel K such that

$$A^\alpha S_z u = \int_G (\widetilde{A}^\alpha K_z)(g) U(g) u \, \mathrm{d}g$$

for all $\alpha \in J(d')$, $z \in \Lambda(\theta_C)$ and $u \in \mathcal{X}$. For all $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu$ sufficiently large the fractional powers of the resolvent $(\nu I + \overline{H})^{-\delta}$ are defined for all $\delta > 0$ by the Laplace transforms

$$(\nu I + \overline{H})^{-\delta} = \Gamma(\delta)^{-1} \int_0^\infty \mathrm{e}^{-\nu t} t^{\delta - 1} S_t \, \mathrm{d}t.$$

Let $\theta \in (0, \theta_C)$. Since S is analytic in $\Lambda(\theta)$ it follows that there exist $M \geqslant 1$ and $\omega \geqslant 0$ such that $\|S_z u\|_p \leqslant M \mathrm{e}^{\omega |z|} \|u\|_p$ for all $z \in \Lambda(\theta)$ and $u \in \mathcal{X}$. Therefore $(\nu I + \overline{H})^{-\delta}$ are defined for all $\delta > 0$, $\nu \in \mathbb{C}$ such that $\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\varphi} \nu) > \omega$ for some $\varphi \in (-\theta, \theta)$ and

$$(\nu I + \overline{H})^{-\delta} = \Gamma(\delta)^{-1} \mathrm{e}^{\mathrm{i}\delta\varphi} \int_0^\infty \mathrm{e}^{-\mathrm{e}^{\mathrm{i}\varphi} \nu t} t^{\delta - 1} S_{\mathrm{e}^{\mathrm{i}\varphi} t} \, \mathrm{d}t.$$

Next for $\varphi \in [0,\theta]$ set $\Gamma(\varphi;\omega) = \Gamma_+(\varphi;\omega) \cup \Gamma_-(\varphi;\omega)$ with $\Gamma_\pm(\varphi;\omega) = \{z \in \mathbb{C} : \mathrm{Re}(z\mathrm{e}^{\pm \mathrm{i}\varphi}) > \omega\}$. Then the fractional powers of the resolvents are defined by the above procedure for all $\nu \in \Delta(\theta;\omega)$, where $\Delta(\theta;\omega) = \bigcup_{\varphi\in[0,\theta]} \Gamma(\varphi;\omega)$.

Let $\delta > 0$. For $\nu \in \Delta(\theta;\omega)$ define $R_{\nu,\delta} : G \to \mathbb{C}$ by

$$R_{\nu,\delta}(g) = \Gamma(\delta)^{-1} \int_0^\infty \mathrm{e}^{-\nu t} t^{\delta-1} K_t(g)\,\mathrm{d}t, \quad g \in G.$$

Then similarly to the proof of Theorem A.1 in [8], for all $\alpha \in J(d')$ there exist $a, b > 0$, independent of δ, such that

$$|(\widetilde{A}^\alpha R_{\nu,\delta})(g)| \leqslant a\rho^{(D'+\|\alpha\|-n\delta)/n} F_{\|\alpha\|,\delta}(\rho^{1/n}|g|')\mathrm{e}^{-b\rho^{1/n}|g|'} \tag{2.2}$$

for all $g \in G$ with $g \neq e$ and $\nu \in \Delta(\theta;\omega)$, where $\rho = \rho(\nu;\Delta)$ denotes the distance from ν to the boundary of $\Delta(\theta;\omega)$ and

$$F_{k,\delta}(x) = \begin{cases} x^{-(D'+k-n\delta)} & \text{if } D'+k > n\delta, \\ 1+\log^+ x^{-1} & \text{if } D'+k = n\delta, \\ 1 & \text{if } D'+k < n\delta, \end{cases}$$

with $\log^+ y = \log y$ if $y \geqslant 1$ and $\log^+ y = 0$ if $y \leqslant 1$.

Now, let $p \in [1,\infty]$ and suppose that $U : G \to \mathcal{L}(L_p(\mathcal{M};\mu))$ is a continuous representation of G in $L_p(\mathcal{M};\mu)$, where $(\mathcal{M},\mu)$ denotes a σ-finite measure space. Let H be an n-th order weighted subcoercive operator associated to U. Since there exist $C, \eta > 0$ such that $\|U(g)u\|_p \leqslant C\mathrm{e}^{\eta|g|'}\|u\|_p$ for all $u \in L_p(\mathcal{M};\mu)$ and $g \in G$ it follows that there is a $\nu_0 \geqslant 0$ such that

$$\int_G \|(\widetilde{A}^\alpha R_{\nu,\delta})(g)U(g)u\|_p\,\mathrm{d}g < \infty$$

for all $\alpha \in J(d')$ with $\|\alpha\| < n\delta$, $u \in L_p(\mathcal{M};\mu)$ and $\nu \in \Delta(\theta;\omega)$ with $\mathrm{Re}\,\nu \geqslant \nu_0$. Therefore if $\nu \in \Delta(\theta;\omega)$ and $\mathrm{Re}\,\nu \geqslant \nu_0$ then

$$A^\alpha(\nu I + \overline{H})^{-\delta}u = \Gamma(\delta)^{-1}\int_0^\infty \mathrm{e}^{-\nu t}t^{\delta-1}(A^\alpha S_t)u\,\mathrm{d}t = \int_G (\widetilde{A}^\alpha R_{\nu,\delta})(g)U(g)u\,\mathrm{d}g$$

for all $\alpha \in J(d')$ with $\|\alpha\| < n\delta$ and $u \in L_p(\mathcal{M};\mu)$. Note that if $\|\alpha\| = m$ and $\delta = m/n$ then the Lie group kernel $\widetilde{A}^\alpha R_{\nu,\delta}$ has a logarithmic singularity in the identity $e \in G$ and the integral operator is not norm-convergent for $p \in [1,\infty]$ in general.

Now we discuss the functional calculus of the n-th order weighted subcoercive operator H (cf. also [4], [13] and [14]). It follows from Theorem 1.1.III of [11] that there exist $M \geqslant 1$ and $\omega \geqslant 0$ such that $\|S_z\| \leqslant M\mathrm{e}^{\omega|z|}$ for all $z \in \Lambda(\theta)$. Therefore if we replace H by $\nu I + H$ with $\mathrm{Re}\,\nu$ sufficiently large then S is uniformly bounded in the sector $\Lambda(\theta)$, i.e., there is an $M \geqslant 1$ such that $\|S_z\| \leqslant M$ for all $z \in \Lambda(\theta)$. Then $(-\lambda I + \overline{H})^{-1}$ is defined and satisfies bounds $\|(-\lambda I + \overline{H})^{-1}\| \leqslant M|\lambda|^{-1}$ for all non-zero $\lambda \in \mathbb{C}$ with $|\arg \lambda| \geqslant \pi/2 - \theta$.

Next for $0 < \varphi \leqslant \pi$ consider the class

$$F_\varphi = \{f : \Lambda(\varphi) \to \mathbb{C} : f \text{ is bounded and holomorphic}\}.$$

Then it is clear that F_φ is a Banach space with respect to the norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in \Lambda(\varphi)\}.$$

For technical reasons we also need the following subspaces

$$\Phi_{\varphi,\xi} = \{f \in F_\varphi : |f(z)| \leqslant c|z|^\xi (1+|z|)^{-2\xi} \text{ for some } c > 0 \text{ and all } z \in \Lambda(\varphi)\},$$

where $\xi > 0$. Furthermore, set

$$\Phi_\varphi = \bigcup_{\xi > 0} \Phi_{\varphi,\xi}.$$

If $f \in \Phi_\varphi$ with $\varphi \in (\pi/2 - \theta, \pi]$ one can define an operator $f(\overline{H})$ by the familiar complex Cauchy representation formula

$$\text{(2.3)} \quad f(\overline{H}) = (2\pi \mathrm{i})^{-1} \int_{\Gamma_\chi} f(\lambda)(-\lambda I + H)^{-1} \, \mathrm{d}\lambda = \int_{\Gamma_\chi} f(\lambda) \int_G R_{-\lambda,1}(g) U(g) \, \mathrm{d}g \, \mathrm{d}\lambda,$$

where Γ_χ is the contour determined by the function

$$\Gamma_\chi(t) = \begin{cases} t\mathrm{e}^{\mathrm{i}\chi} & \text{if } t \in [0,\infty), \\ -t\mathrm{e}^{-\mathrm{i}\chi} & \text{if } t \in (-\infty, 0], \end{cases}$$

with $\pi/2 - \theta < \chi < \varphi$. The integral (2.3) is norm-convergent, independent of the particular choice of contour and the operator $f(\overline{H})$ is bounded.

However for $f \in F_\varphi$ one can define $f(\overline{H})$ by a similar Cauchy formula but the contour integral in (2.3) is not necessarily norm-convergent anymore. Therefore the integral definition for $f(\overline{H})$ in (2.3) is to be interpreted in the strong or weak* topology according to whether S is strongly or weakly* continuous and the domain of $f(\overline{H})$ is the subspace of $\mathcal{X}$ on which the integral is convergent. In this way we obtain closed operators $f(\overline{H})$. We say that H has a bounded H_∞ functional calculus in $L_p(\mathcal{M};\mu)$ over F_φ if all the operators $\{f(\overline{H}) : f \in F_\varphi\}$ are bounded and if $f \mapsto f(\overline{H})$ is a continuous map from the Banach space of bounded holomorphic functions F_φ into the Banach algebra $\mathcal{L}(L_p(\mathcal{M};\mu))$ of bounded operators on $L_p(\mathcal{M};\mu)$, i.e., if there is a $c_p > 0$ such that

$$\|f(\overline{H})u\|_p \leqslant c_p \|f\|_\infty \|u\|_p$$

for all $u \in L_p(\mathcal{M};\mu)$ and $f \in F_\varphi$.

Let $K_f : G \to \mathbb{C}$ be defined by

$$K_f(g) = (2\pi \mathrm{i})^{-1} \int_{\Gamma_\chi} f(\lambda) R_{-\lambda,1}(g) \, \mathrm{d}\lambda$$

for all $g \in G$. Then it follows from a similar argument as used in the proof of Corollary A.2 in [8] that for all $\alpha \in J(d')$ there exist $a, b > 0$, independent of the zero-order coefficient c_0 but depending on φ, and $c > 0$, linearly dependent of $\operatorname{Re} c_0$ with a positive coefficient depending on φ, such that

$$|(\widetilde{A}^\alpha K_f)(g)| \leqslant a\|f\|_\infty(|g|')^{-(D'+\|\alpha\|)}\mathrm{e}^{-bc^{1/n}|g|'} \tag{2.4}$$

for all $g \in G$ with $g \neq e$ and $f \in F_\varphi$. Moreover, it follows from Section 3 of [8], that if $f \in \Phi_\varphi$ then there is a $C > 0$ such that $\|K_f\|_\infty \leqslant C$. Therefore, if $f \in \Phi_\varphi$ then there is a $\nu_0 \geqslant 0$ such that if $\operatorname{Re} c_0 \geqslant \nu_0$ then

$$\int_G \|K_f(g)U(g)u\|_p \,\mathrm{d}g < \infty$$

for all $u \in L_p(\mathcal{M};\mu)$. Hence if $\operatorname{Re} c_0 \geqslant \nu_0$ then

$$f(\overline{H})u = \int_G K_f(g)U(g)u \,\mathrm{d}g$$

for all $u \in L_p(\mathcal{M};\mu)$. However, if $f \in F_\varphi$ then K_f may have a logarithmic singularity in the identity $e \in G$ and $f(\overline{H})$ is not norm-convergent for $p \in [1,\infty]$ in general.

The key ingredient in the proof of the first main theorem of this paper is the following proposition.

PROPOSITION 2.1. *Let $p \in (1,\infty)$ and let $\widetilde{H}$ be an n-th order weighted subcoercive operator associated to the left (or right) regular representation of G in $L_p(G;\mathrm{d}g)$. Then for all $\varphi \in (\pi/2 - \theta_C, \pi]$ there is a $\nu_0 \geqslant 0$ such that for all $\nu \in \mathbb{C}$ with $\operatorname{Re}\nu \geqslant \nu_0$ the operator $\nu I + \widetilde{H}$ has a bounded H_∞ functional calculus in $L_p(G;\mathrm{d}g)$ over F_φ.*

Proof. For the left regular representation in $L_p(G;\mathrm{d}g)$ the derivation of the bounded H_∞ functional calculus in $L_p(G;\mathrm{d}g)$ over F_φ for all $\varphi \in (\pi/2 - \theta_C, \pi]$ is completely analogous to the proof of Theorem 3.1 from [8]. The result for the right regular representation follows from a similar duality argument as used in the proof of Lemma 2.1 in [8]. ∎

Now we are able to state the first main result of this paper.

THEOREM 2.2. *Let $p \in (1,\infty)$. Suppose that $U : G \to \mathcal{L}(L_p(\mathcal{M};\mu))$ is a continuous representation of a connected Lie group G in $L_p(\mathcal{M};\mu)$, where $(\mathcal{M},\mu)$ denotes a σ-finite measure space. Let H be an n-th order weighted subcoercive operator. Then the following three statements hold:*

(i) *For all $\varphi \in (\pi/2 - \theta_C, \pi]$ there is a $\nu_0 \geqslant 0$ such that for all $\nu \in \mathbb{C}$ with $\operatorname{Re}\nu \geqslant \nu_0$ the operator $\nu I + \overline{H}$ has a bounded H_∞ functional calculus in $L_p(\mathcal{M};\mu)$ over F_φ.*

(ii) *There is a $\nu_0 \geqslant 0$ such that if $\nu \in \mathbb{C}$ with $\operatorname{Re}\nu \geqslant \nu_0$ and $m > 0$ then the space $(\nu I + \overline{H})^{-m/n}(L_p(\mathcal{M};\mu))$ is continuously embedded in $D(A^\alpha)$ for all $\alpha \in J(d')$ with $\|\alpha\| = m$ and the Riesz transform*

$$A^\alpha(\nu I + \overline{H})^{-m/n}$$

is bounded on $L_p(\mathcal{M};\mu)$.

(iii) *Let* $w = \min\{x \in [1,\infty) : x \in w_i\mathbb{N}$ *for all* $i \in \{1,\ldots,d'\}\}$. *If* $m \in \{nw : n \in \mathbb{N}\}$ *then there is a* $\nu_0 \geqslant 0$, *independent of* m, *such that for all* $\nu \in \mathbb{C}$ *with* $\mathrm{Re}\,\nu \geqslant \nu_0$ *the spaces* $D((\nu I + \overline{H})^{m/n})$ *and* $L'_{p;m}$ *coincide as sets and there is a* $C_{p,m,\nu} \geqslant 1$ *such that*

$$C_{p,m,\nu}^{-1}\|u\|'_{p;m} \leqslant \|(\nu I + \overline{H})^{m/n}u\|_p \leqslant C_{p,m,\nu}\|u\|'_{p;m}$$

for all $u \in L'_{p;m}$.

Proof. The proof is based on a transference method inspired by [3]. First let $\varphi \in (\pi/2 - \theta_C, \pi]$ and $f \in \Phi_\varphi$. Let $\chi : G \to [0,1]$ be a C^∞ cut-off function such that $\chi(g) = 1$ for all $g \in B'_{1/2}$ and $\chi(g) = 0$ for all $g \in G \setminus B'_1$. If R denotes the right regular representation of G in $L_p(G;\mathrm{d}g)$ then there exist C, $\eta > 0$ such that

$$\|R(g)v\|_p \leqslant C\mathrm{e}^{\eta|g|'}\|v\|_p \tag{2.5}$$

for all $v \in L_p(G;\mathrm{d}g)$ and $g \in G$ and, moreover,

$$\|U(g)u\|_p \leqslant C\mathrm{e}^{\eta|g|'}\|u\|_p \tag{2.6}$$

for all $u \in L_p(\mathcal{M};\mu)$ and $g \in G$. Note that $\eta > 0$ depends on p and U.

Let $\widetilde{H} = \mathrm{d}R(C)$ be the n-th order weighted subcoercive operator associated to R. Then it follows from (2.4) and Proposition 2.1 that there exist a $\gamma > 0$ and a $C_{1,p} > 0$, independent of f but depending on φ and η, such that if $\mathrm{Re}\,c_0 \geqslant \gamma$ then

$$\begin{aligned}\int_G |K_f(g)(1-\chi(g))|\,\|U(g)u\|_p\,\mathrm{d}g &\leqslant C\int_G |K_f(g)(1-\chi(g))|\mathrm{e}^{\eta|g|'}\,\mathrm{d}g\|u\|_p \\ &\leqslant C_{1,p}\|f\|_\infty\|u\|_p\end{aligned} \tag{2.7}$$

for all $u \in L_p(\mathcal{M};\mu)$,

$$\begin{aligned}\int_G |K_f(g)(1-\chi(g))|\,\|R(g)v\|_p\,\mathrm{d}g &\leqslant C\int_G |K_f(g)(1-\chi(g))|\mathrm{e}^{\eta|g|'}\,\mathrm{d}g\|v\|_p \\ &\leqslant C_{1,p}\|f\|_\infty\|v\|_p\end{aligned} \tag{2.8}$$

for all $v \in L_p(G;\mathrm{d}g)$ and, moreover,

$$\|f(\widetilde{H})v\|_p \leqslant C_{1,p}\|f\|_\infty\|v\|_p \tag{2.9}$$

for all $v \in L_p(G;\mathrm{d}g)$.

Next let $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu \geqslant \max(\gamma - \mathrm{Re}\,c_0, 0)$ and replace H by $\nu I + H$. Then one can write

$$f(\overline{H})u = \int_G K_f(g)U(g)u\,\mathrm{d}g = f(\overline{H})_1u + f(\overline{H})_2u$$

for all $u \in L_p(\mathcal{M};\mu)$, where

$$f(\overline{H})_1 = \int_G K_f(g)\chi(g)U(g)\,\mathrm{d}g$$

and

$$f(\overline{H})_2 = \int_G K_f(g)(1-\chi(g))U(g)\,\mathrm{d}g.$$

Set $k_f = K_f\chi$. Then $k_f \in L_1(G;\mathrm{d}g)$ with compact support. Suppose that one has shown that $f(\overline{H})_1$ maps $L_p(\mathcal{M};\mu)$ into $L_p(\mathcal{M};\mu)$ and, moreover, that there is a $C' > 0$, independent of p and f, such that

$$\|f(\overline{H})_1 u\|_p \leqslant C' N_p(k_f)\|u\|_p \tag{2.10}$$

for all $u \in L_p(\mathcal{M};\mu)$, where $N_p(k_f) > 0$ denotes the $\mathcal{L}(L_p(G;\mathrm{d}g))$-norm of the operator

$$T_f v = \int_G k_f(g)R(g)v\,\mathrm{d}g.$$

It follows from (2.8) that

$$\|(T_f - f(\widetilde{H}))v\|_p \leqslant C_{1,p}\|f\|_\infty\|v\|_p$$

for all $v \in L_p(G;\mathrm{d}g)$, and hence from (2.9) that

$$N_p(k_f) \leqslant 2C_{1,p}\|f\|_\infty.$$

Therefore, if (2.10) is valid then

$$\|f(\overline{H})_1 u\|_p \leqslant 2C'C_{1,p}\|f\|_\infty\|u\|_p$$

for all $u \in L_p(\mathcal{M};\mu)$.

Now we deduce (2.10). The operator $f(\overline{H})_1$ is a well defined object for $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$ because k_f has compact support. Since $L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$ is a dense subset of $L_p(\mathcal{M};\mu)$ it suffices to show (2.10) for all $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$. Since U is uniformly bounded on $(B_1')^{-1}$ there is a $c > 0$ such that

$$\|f(\overline{H})_1 u\|_p \leqslant c\|U(h)(f(\overline{H})_1 u)\|_p$$

for all $h \in B_1'$ and $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$. Then

$$\|f(\overline{H})_1 u\|_p^p \leqslant c^p \mathrm{Vol}_G(B_1')^{-1} \int_{B_1'} \|U(h)(f(\overline{H})_1 u)\|_p^p\,\mathrm{d}h$$

for all $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$. Observe that

$$U(h)(f(\overline{H})_1 u) = \int_G k_f(g)U(hg)u\,\mathrm{d}g \tag{2.11}$$

for all $h \in B_1'$ and $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$.

Next let $\widetilde{\chi} : G \to \{0,1\}$ be the characteristic function $\widetilde{\chi} = 1_{B_2'}$. Then it follows from (2.11) and Fubini's theorem that

$$\begin{aligned}
\|f(\overline{H})_1 u\|_p^p &\leqslant c^p \mathrm{Vol}_G(B_1')^{-1} \int_{\mathcal{M}} \int_{B_1'} \left| \int_G k_f(g)\widetilde{\chi}(hg)(U(hg)u)(x)\,\mathrm{d}g \right|^p \mathrm{d}h\,\mathrm{d}\mu(x) \\
&\leqslant c^p \mathrm{Vol}_G(B_1')^{-1} \int_{\mathcal{M}} \int_G \left| \int_G k_f(g)\widetilde{\chi}(hg)(U(hg)u)(x)\,\mathrm{d}g \right|^p \mathrm{d}h\,\mathrm{d}\mu(x)
\end{aligned}$$

for all $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$. It follows from Fubini's theorem again that

$$\begin{aligned}\|f(\overline{H})_1 u\|_p^p &\leqslant c^p \mathrm{Vol}_G(B_1')^{-1} (N_p(k_f))^p \int_{\mathcal{M}} \int_G \widetilde{\chi}(w) |(U(w)u)(x)|^p \, \mathrm{d}w \, \mathrm{d}\mu(x) \\ &= (cN_p(k_f))^p \mathrm{Vol}_G(B_1')^{-1} \int_G \widetilde{\chi}(w) \int_{\mathcal{M}} |(U(w)u)(x)|^p \, \mathrm{d}\mu(x) \, \mathrm{d}w\end{aligned}$$

for all $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$. Hence

$$\|f(\overline{H})_1 u\|_p^p \leqslant (cC\mathrm{e}^{2\eta} N_p(k_f))^p \mathrm{Vol}_G(B_1')^{-1} \mathrm{Vol}_G(B_2') \|u\|_p^p$$

for all $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$ and (2.10) is proved.

Now we consider the second operator $f(\overline{H})_2$. It follows from (2.7) that

$$\|f(\overline{H})_2 u\|_p \leqslant C_{1,p} \|f\|_\infty \|u\|_p$$

for all $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$. Therefore, a combination of the results above gives

$$\|f(\overline{H}) u\|_p \leqslant (2C' + 1) C_{1,p} \|f\|_\infty \|u\|_p$$

for all $u \in L_p(\mathcal{M};\mu)$. This result extends to all $f \in F_\varphi$ by McIntosh's convergence theorem ([13], Section 5) and the proof of (i) is complete.

Next we prove (ii) and (iii). We use a similar approximation procedure as used in [2] and we show that that there is a $\nu_0 \geqslant 0$ such that if $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu \geqslant \nu_0$ then the space $(\nu I + \overline{H})^{-m/n}(L_p(\mathcal{M};\mu))$ is continuously embedded in $D(A^\alpha)$ for all $\alpha \in J(d')$ with $\|\alpha\| = m$ and the Riesz transform

$$A^\alpha (\nu I + \overline{H})^{-m/n}$$

is bounded on $L_p(\mathcal{M};\mu)$. Fix $N \in \mathbb{N}$, $N > D'$ and for all $j \in \mathbb{N}$ with $j > 2\,\mathrm{Re}\,\nu$ and $\mathrm{Re}\,\nu$ sufficiently large consider the operators

$$X_j = j^N (jI + \overline{H})^{-N} (\nu I + \overline{H})^{-m/n}. \tag{2.12}$$

Then for $\|\alpha\| = m$ with $m > 0$ one expects to find a $\nu_0 \geqslant 0$ such that the operators $A^\alpha X_j$ converge to $A^\alpha(\nu I + \overline{H})^{-m/n}$ as j tends to infinity and $\mathrm{Re}\,\nu \geqslant \nu_0$. We prove that for $\mathrm{Re}\,\nu \geqslant \nu_0$ the operators $A^\alpha X_j$ are bounded uniformly in j on $L_p(\mathcal{M};\mu)$ and it follows from this result that $(\nu I + \overline{H})^{-m/n}$ maps into the domain of A^α and $A^\alpha(\nu I + \overline{H})^{-m/n}$ is bounded on $L_p(\mathcal{M};\mu)$. The uniformity with respect to j is obtained from the uniform upper bounds with respect to j for the Lie group kernels of $A^\alpha X_j$ for large $|g|'$, established similarly as in [2].

Set

$$k_j(g) = \int_0^\infty f_j(t) K_t(g) \, \mathrm{d}g$$

for all $g \in G$ and $j \in \mathbb{N}$, where

$$f_j(t) = j^N (N-1)!^{-1} \Gamma(m/n)^{-1} \int_0^t x^{N-1} \mathrm{e}^{-jx} (t-x)^{m/n-1} \mathrm{e}^{-\nu(t-x)} \, \mathrm{d}x.$$

Let $m > 0$ and $\alpha \in J(d')$ with $\|\alpha\| = m$. It follows from [2] that

$$A^\alpha X_j u = \int_G k_{j,1}(g) U(g) u \, \mathrm{d}g + \int_G k_{j,2}(g) U(g) u \, \mathrm{d}g$$

for all $u \in L_p(\mathcal{M};\mu)$ and $j \in \mathbb{N}$, where

$$k_{j,1} = (\widetilde{A}^\alpha k_j)\chi \quad \text{and} \quad k_{j,2} = (\widetilde{A}^\alpha k_j)(1-\chi).$$

Let $N_p(k_{j,1}) > 0$ be the $\mathcal{L}(L_p(G;\mathrm{d}g))$-norm of the operator

$$W_j v = \int_G k_{j,1}(g) R(g) v \, \mathrm{d}g$$

for all $j \in \mathbb{N}$. Let $N_p(\widetilde{A}^\alpha k_j)$ denote the $\mathcal{L}(L_p(G;\mathrm{d}g))$-norm of the operator

$$V_j v = \int_G (\widetilde{A}^\alpha k_j)(g) R(g) v \, \mathrm{d}g$$

for all $j \in \mathbb{N}$. Then it follows from a similar argument as used in the proof of Theorem A.1 of [8] that there exists a $c > 0$ such that

$$|k_{j,2}(g)| \leqslant |(\widetilde{A}^\alpha k_j)(g)| \leqslant \mathrm{e}^{-c\nu^{1/n}|g|'} \tag{2.13}$$

for all $g \in G \setminus B'_1$ and $j \in \mathbb{N}$. Therefore it follows from (2.5), (2.6) and a similar argument as used in the proof of Theorem 2.3 from [2] that there is a $\nu_0 \geqslant 0$ such that if $\operatorname{Re}\nu \geqslant \nu_0$ then there is a $C_{2,p} > 0$, independent of j and ν_0, such that

$$\|(W_j - V_j)v\|_p \leqslant C_{2,p}\|v\|_p$$

for all $v \in L_p(G;\mathrm{d}g)$ and $j \in \mathbb{N}$,

$$N_p(\widetilde{A}^\alpha k_j) \leqslant C_{2,p}$$

for all $j \in \mathbb{N}$ and, further,

$$\Big(\int_{\mathcal{M}} \Big|\int_G k_{j,2}(g)(U(g)u)(x)\,\mathrm{d}g\Big|^p \mathrm{d}\mu(x)\Big)^{1/p} \leqslant C_{2,p}\|u\|_p$$

for all $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$ and $j \in \mathbb{N}$. Hence if $\operatorname{Re}\nu \geqslant \nu_0$ then

$$N_p(k_{j,1}) \leqslant 2C_{2,p}$$

for all $j \in \mathbb{N}$. Next it follows from a similar transference argument as used above that if $\operatorname{Re}\nu \geqslant \nu_0$ then there is a $C > 0$, independent of ν_0, such that

$$\int_{\mathcal{M}} \Big|\int_G k_{j,1}(g)(U(g)u)(x)\,\mathrm{d}g\Big|^p \mathrm{d}\mu(x) \leqslant (CN_p(k_{j,1}))^p \|u\|_p^p$$

for all $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$ and $j \in \mathbb{N}$. It follows that

$$\|A^\alpha X_j u\|_p \leqslant (2C+1)C_{2,p}\|u\|_p$$

for all $u \in L_\infty(\mathcal{M};\mu) \cap L_p(\mathcal{M};\mu)$ and $j \in \mathbb{N}$. Finally the Statements (ii) and (iii) follow as in the proof of Theorem 2.3 in [2] (see also [11], Section 9). ∎

In the following theorem we formulate a rather weak condition which ensures the existence of weak type $(1,1)$-estimates. The theorem generalizes the weak type $(1,1)$-estimates deduced in [2] and [8], Proposition 3.2.

Theorem 2.3. *Let $(\mathcal{M},\mu)$ be a σ-finite measure space. Let G be a connected Lie group and suppose that U is a continuous representation of G such that there exist $c_1, c_2 > 0$ such that*

$$\mu\big(\{x \in \mathcal{M} : |(U(g)u)(x)| > \tau\}\big) \leqslant c_1 \mu\big(\{x \in \mathcal{M} : |u(x)| > c_2\tau\}\big)$$

for all $g \in B'_1$, $u \in L_1(\mathcal{M};\mu)$ and $\tau > 0$. Let H be an n-th order weighted subcoercive operator. Then the following two statements hold:

(i) *For all $\varphi \in (\pi/2 - \theta_C, \pi]$ there is a $\nu_0 \geqslant 0$ such that if $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu \geqslant \nu_0$ then there is a $c > 0$, independent of ν, such that*

$$\mu\big(\{x \in \mathcal{M} : |(f(\nu I + \overline{H})u)(x)| > \tau\}\big) \leqslant c\tau^{-1}\|f\|_\infty \|u\|_1$$

for all $u \in L_1(\mathcal{M};\mu)$, $\tau > 0$ and $f \in \Phi_\varphi$.

(ii) *There is a $\nu_0 \geqslant 0$ such that if $m > 0$ and $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu \geqslant \nu_0$ then there is a $c > 0$, independent of ν, such that*

$$\mu\big(\{x \in \mathcal{M} : |(A^\alpha X_j u)(x)| > \tau\}\big) \leqslant c\tau^{-1}\|u\|_1$$

for all $u \in L_1(\mathcal{M};\mu)$, $\tau > 0$, $j \in \mathbb{N}$ and $\alpha \in J(d')$ with $\|\alpha\| = m$. The operators X_j for $j \in \mathbb{N}$ are the operators as defined in (2.12) in the proof of Theorem 2.2.

Proof. We only prove the first statement. The second statement can be proved analogously.

Let $\varphi \in (\pi/2 - \theta_C, \pi]$, $f \in \Phi_\varphi$, $u \in L_1(\mathcal{M};\mu)$ and $\tau > 0$. Let $\chi : G \to [0,1]$ be a C^∞ cut-off function such that $\chi(g) = 1$ for all $g \in B'_{1/2}$ and $\chi(g) = 0$ for all $g \in G \setminus B'_1$. If R denotes the right regular representation of G in $L_1(G;\mathrm{d}g)$ then there exist $C, \eta > 0$ such that

$$\|U(g)u\|_1 \leqslant C\mathrm{e}^{\eta|g|'}\|u\|_1 \tag{2.14}$$

for all $u \in L_1(\mathcal{M};\mu)$ and $g \in G$ and, moreover,

$$\|R(g)v\|_1 \leqslant C\mathrm{e}^{\eta|g|'}\|v\|_1 \tag{2.15}$$

for all $v \in L_1(G;\mathrm{d}g)$ and $g \in G$. Note that $\eta > 0$ depends on U.

Next let $\widetilde{H} = \mathrm{d}R(C)$ denote the n-th order weighted subcoercive operator associated to R. Then it follows from (2.4) and a similar argument as used in the proof of Proposition 3.2 in [8] that there exist a $\gamma > 0$ and a $c_3 > 0$, independent of f but depending on φ and η, such that if $\mathrm{Re}\,c_0 \geqslant \gamma$ then

$$\int_G |K_f(g)(1-\chi(g))|\,\|U(g)u\|_1\,\mathrm{d}g \leqslant c_3\|f\|_\infty\|u\|_1 \tag{2.16}$$

for all $u \in L_1(\mathcal{M};\mu)$ and

$$\int_G |K_f(g)(1-\chi(g))|\,\|R(g)v\|_1\,\mathrm{d}g \leqslant c_3\|f\|_\infty\|v\|_1 \tag{2.17}$$

for all $v \in L_1(G;\mathrm{d}g)$ and, moreover,

$$\mathrm{Vol}_G\big(\{g \in G : |(f(\widetilde{H})v)(g)| > \sigma\}\big) \leqslant c_3\sigma^{-1}\|f\|_\infty\|v\|_1 \tag{2.18}$$

for all $\sigma > 0$ and $v \in L_1(G;\mathrm{d}g)$.

Next let $\nu \in \mathbb{C}$ be such that $\mathrm{Re}\,\nu \geqslant \max(\gamma - \mathrm{Re}\,c_0, 0)$ and replace H by $\nu I + H$. We first estimate

$$\mu\bigg(\bigg\{x \in \mathcal{M} : \bigg|\int_G k_f(g)(U(g)u)(x)\,\mathrm{d}g\bigg| > \tau\bigg\}\bigg),$$

where $k_f = K_f\chi$. Let $\widetilde{\chi} : G \to \{0,1\}$ be the characteristic function $\widetilde{\chi} = 1_{B_2'}$ again. If $h \in B_1'$ then

$$\begin{aligned}\int_G k_f(g)U(h)(U(g)u)\,\mathrm{d}g &= \int_{B_1'} k_f(g)U(h)(U(g)u)\,\mathrm{d}g = \int_{B_1'} k_f(g)\widetilde{\chi}(hg)U(h)(U(g)u)\,\mathrm{d}g\\ &= \int_G k_f(g)\widetilde{\chi}(hg)U(hg)u\,\mathrm{d}g.\end{aligned}$$

Therefore

$$\begin{aligned}&\mu\bigg(\bigg\{x \in \mathcal{M} : \bigg|\int_G k_f(g)(U(g)u)(x)\,\mathrm{d}g\bigg| > \tau\bigg\}\bigg)\\ &\quad= \mu\bigg(\bigg\{x \in \mathcal{M} : \bigg|\bigg(U(h^{-1})\bigg(\int_G k_f(g)U(hg)u\,\mathrm{d}g\bigg)\bigg)(x)\bigg| > \tau\bigg\}\bigg)\\ &\quad= \mu\bigg(\bigg\{x \in \mathcal{M} : \bigg|\bigg(U(h^{-1})\bigg(\int_G k_f(g)\widetilde{\chi}(hg)U(hg)u\,\mathrm{d}g\bigg)\bigg)(x)\bigg| > \tau\bigg\}\bigg)\\ &\quad\leqslant c_1\mu\bigg(\bigg\{x \in \mathcal{M} : \bigg|\int_G k_f(g)\widetilde{\chi}(hg)(U(hg)u)(x)\,\mathrm{d}g\bigg| > c_2\tau\bigg\}\bigg)\end{aligned}$$

for all $h \in B_1'$. Moreover, if

$$T_f v = \int_G K_f(g)(1 - \chi(g))R(g)v\,\mathrm{d}g$$

for all $v \in L_1(G;\mathrm{d}g)$ then it follows from (2.17) that

$$\|T_f v\|_1 \leqslant c_3\|f\|_\infty\|v\|_1$$

for all $v \in L_1(G;\mathrm{d}g)$. Therefore, it follows from (2.18) that

$$\mathrm{Vol}_G\big(\{g \in G : |((f(\widetilde{H}) - T_f)v)(g)| > \sigma\}\big) \leqslant 2c_3\sigma^{-1}\|f\|_\infty\|v\|_1$$

for all $v \in L_1(G; \mathrm{d}g)$ and $\sigma > 0$. Then it follows from (2.14) and Fubini's theorem that there exist $b, c > 0$, independent of f, u and τ, such that

$$\mu\Big(\Big\{x \in \mathcal{M} : \Big|\int_G k_f(g)(U(g)u)(x)\,\mathrm{d}g\Big| > \tau\Big\}\Big)$$

$$\leqslant c_1 \mathrm{Vol}_G(B_1')^{-1} \int_{B_1'} \mu\Big(\Big\{x \in \mathcal{M} : \Big|\int_G k_f(g)\widetilde{\chi}(hg)(U(hg)u)(x)\,\mathrm{d}g\Big| > c_2\tau\Big\}\Big)\mathrm{d}h$$

$$= c_1 \mathrm{Vol}_G(B_1')^{-1} \int_{\mathcal{M}} \mathrm{Vol}_G\Big(\Big\{h \in B_1' : \Big|\int_G k_f(g)\widetilde{\chi}(hg)(U(hg)u)(x)\,\mathrm{d}g\Big| > c_2\tau\Big\}\Big)\mathrm{d}\mu(x)$$

$$\leqslant c_1 \mathrm{Vol}_G(B_1')^{-1} \int_{\mathcal{M}} \mathrm{Vol}_G\Big(\Big\{h \in G : \Big|\int_G k_f(g)\widetilde{\chi}(hg)(U(hg)u)(x)\,\mathrm{d}g\Big| > c_2\tau\Big\}\Big)\mathrm{d}\mu(x)$$

$$\leqslant c_1 \mathrm{Vol}_G(B_1')^{-1} \int_{\mathcal{M}} c\|f\|_\infty \tau^{-1} \int_G |\widetilde{\chi}(g)(U(g)u)(x)|\,\mathrm{d}g\,\mathrm{d}\mu(x)$$

$$= c_1 c \tau^{-1} \|f\|_\infty \mathrm{Vol}_G(B_1')^{-1} \int_G \widetilde{\chi}(g) \int_{\mathcal{M}} |(U(g)u)(x)|\,\mathrm{d}\mu(x)\,\mathrm{d}g$$

$$\leqslant c_1 b c \tau^{-1} \|f\|_\infty \mathrm{Vol}_G(B_1')^{-1} \mathrm{Vol}_G(B_2')\|u\|_1.$$

So the inequality for the operator with kernel k_f is proved. Finally

$$\int_{\mathcal{M}} \Big|\int_G (K_f(g)(1-\chi(g))(U(g)u)(x)\,\mathrm{d}g\Big|\,\mathrm{d}\mu(x) \leqslant c_3\|f\|_\infty\|u\|_1.$$

In particular the operator with kernel $K_f(1-\chi)$ also satisfies a weak type $(1,1)$-estimate and the theorem follows immediately. ∎

Now we discuss a class of representations affiliated to cocycles and quasi-invariant measures. This class embraces the class of continuous unitary representations induced by a character.

Example 2.4. Let $X = G/M$ with G a connected Lie group and M a σ-compact Lie subgroup. Let $S : X \times G \to \mathbb{C}$ be a continuous cocycle, i.e.,

$$S(x, e) = 1 \quad \text{and} \quad S(x, gh) = S(hx, g)S(x, h)$$

for all $g, h \in G$ and $x \in X$.

Moreover, let $\mathrm{d}x$ be a quasi-invariant non-zero positive regular Borel measure on X. By the Radon-Nikodym theorem there exists a function $R : X \times G \to (0, \infty)$ such that for each $g \in G$ the function $x \mapsto R(x, g)$ is Borel measurable, for all $\varphi \in C_c(X)$ the function $x \mapsto \varphi(x)R(x, g)$ belongs to $L_1(X; \mathrm{d}x)$ and

$$\int_X \varphi(g^{-1}x)\,\mathrm{d}x = \int_X \varphi(x)R(x, g)\,\mathrm{d}x.$$

Suppose in addition that R is continuous and that there exist $C_R, C_S \geqslant 1$ such that

$$C_R^{-1} \leqslant R(x, g) \leqslant C_R \tag{2.19}$$

and

$$C_S^{-1} \leqslant S(x,g) \leqslant C_S \tag{2.20}$$

for all $x \in X$ and $g \in B_1'$. Let $p \in [1,\infty]$ and consider the representation U of G in $L_p(X;\mathrm{d}x)$ given by

$$(U(g)u)(x) = S(x,g^{-1})^{-1}R(x,g^{-1})^{1/2}u(g^{-1}x), \quad \text{a.e. } x \in X \tag{2.21}$$

for all $\varphi \in L_p(X;\mathrm{d}x)$ and $g \in G$. Then it follows from Proposition 2.7 of [18] that U is a strongly continuous representation of G in $L_p(X;\mathrm{d}x)$ if $p \in [1,\infty)$ and weakly* continuous if $p = \infty$. Therefore if $p \in (1,\infty)$ then it follows from Theorem 2.2 that for each weighted subcoercive operator H associated to U and $\varphi \in (\pi/2 - \theta_C, \pi]$ there is a $\nu_0 \geqslant 0$ such that if $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu \geqslant \nu_0$ then $\nu I + \overline{H}$ has a bounded H_∞ functional calculus in $L_p(X;\mathrm{d}x)$ over F_φ, and the Riesz transforms of $\nu I + \overline{H}$ are bounded on $L_p(X;\mathrm{d}x)$. Moreover, if $\mu = \mathrm{d}x$, $g \in B_1'$ and $u \in L_1(X;\mathrm{d}x)$ then

$$\begin{aligned}
&\mu\big(\{x \in X : |(U(g)u)(x)| > \tau\}\big)\\
&\quad = \mu\big(\{x \in X : |S(x,g^{-1})^{-1}R(x,g^{-1})^{1/2}u(g^{-1}x)| > \tau\}\big)\\
&\quad \leqslant \mu\big(\{x \in X : |u(g^{-1}x)| > C_S^{-1}C_R^{-1/2}\tau\}\big)\\
&\quad = \int_X 1_{\{x\in X : |u(x)| > C_S^{-1}C_R^{-1/2}\tau\}}(y) R(y,g)\,\mathrm{d}y\\
&\quad \leqslant C_R \int_X 1_{\{x\in X : |u(x)| > C_S^{-1}C_R^{-1/2}\tau\}}(y)\,\mathrm{d}y\\
&\quad = c_1\mu\big(\{x \in X : |u(x)| > c_2\tau\}\big)
\end{aligned}$$

for all $\tau > 0$, where $c_1 = C_R \geqslant 1$ and $c_2 = C_S^{-1}C_R^{-1/2} > 0$. Therefore it follows from Theorem 2.3 that weak type $(1,1)$-estimates are valid for the functional operators and Riesz transforms. ∎

In the following example we consider continuous representations of a connected Lie group G induced by a representation of G by measurable bijections from $(\mathcal{M},\mu)$ onto $(\mathcal{M},\mu)$ with $(\mathcal{M},\mu)$ a σ-finite measure space (generalizing the left Haar measure preserving left regular representation of G). See also Theorem 2.6 in [3].

Example 2.5. For a σ-finite measure space $(\mathcal{M},\mu)$ let T be a representation of a connected Lie group G by measurable bijections from $(\mathcal{M},\mu)$ onto $(\mathcal{M},\mu)$ such that there exist $c_1, c_2 > 0$ such that

$$\mu\big(T(g)\{x \in \mathcal{M} : |u(x)| > \tau\}\big) \leqslant c_1\mu\big(\{x \in \mathcal{M} : |u(x)| > c_2\tau\}\big)$$

for all $g \in B_1'$, $\tau > 0$ and $u \in L_1(\mathcal{M};\mu)$. Set $U(g)u = u \circ T(g^{-1})$ for all $u \in L_1(\mathcal{M};\mu)$ and $g \in G$ and assume that U is a continuous representation in $L_1(\mathcal{M};\mu)$. Then

$$\begin{aligned}
\mu\big(\{x \in \mathcal{M} : |(U(g)u)(x)| > \tau\}\big) &= \mu\big(T(g)\{x \in \mathcal{M} : |u(x)| > \tau\}\big)\\
&\leqslant c_1\mu\big(\{x \in \mathcal{M} : |u(x)| > c_2\tau\}\big)
\end{aligned}$$

for all $g \in B_1'$, $\tau > 0$ and $u \in L_1(\mathcal{M};\mu)$, and weak type $(1,1)$-estimates hold for any weighted subcoercive operator associated to U, by Theorem 2.3.

If T is a representation of G by measure preserving measurable bijections from $(\mathcal{M},\mu)$ onto $(\mathcal{M},\mu)$ then the corresponding U need not be continuous. One easily verifies, however, that there exist $C,\eta > 0$ such that

$$\int\limits_{\mathcal{M}} |u(T(g)x)| \,\mathrm{d}\mu(x) \leqslant C\mathrm{e}^{\eta|g|'}\|u\|_1$$

for all $u \in L_1(\mathcal{M};\mu)$ and $g \in G$ and the proof of Theorem 2.3 still works. ∎

3. REDUCED OPERATOR KERNELS

We show in this section that the Riesz transforms and functional operators of a strongly elliptic operator H, affiliated to U, given by (2.21), are reduced kernel operators, and we derive upper bounds for these reduced kernels. Before we can introduce resolvent reduced kernels, we need the following lemma. In the sequel, $R_{\nu,\delta}$ and K_f denote the Lie group kernels corresponding to the strongly elliptic operator H, associated to U, given by (2.21). By Lemma 2.4.IV in [18], there is a unique function $\rho : G \to (0,\infty)$ such that

$$R(\dot{k},g) = \rho(gk)\rho(k)^{-1}$$

for all $g,k \in G$, and, further,

$$\int\limits_G \varphi(g)\rho(g)\,\mathrm{d}g = \int\limits_{G/M}\int\limits_M \varphi(gm)\,\mathrm{d}m\,\mathrm{d}\dot{g}$$

for all $\varphi \in C_{\mathrm{c}}(G)$. Moreover, let $\Delta_G : G \to (0,\infty)$ denote the modular function of G. Let $\Delta(\theta;\omega)$ and $\rho(\nu;\Delta)$, with $\theta \in (0,\theta_C)$, $\omega > 0$ and $\nu \in \Delta(\theta;\omega)$, be as in Section 2.

LEMMA 3.1. *For all $\delta > 0$ there exists an $\omega_0 \geqslant \omega$ such that the integral*

$$\int\limits_M R_{\nu,\delta}(gm^{-1}k^{-1})S(\dot{k},gm^{-1}k^{-1})(\rho(g)\rho(km))^{-1/2}\Delta_G(m^{-1}k^{-1})\,\mathrm{d}m$$

exists for all $g,k \in G$ with $\dot{g} \neq \dot{k}$, uniformly with respect to $\nu \in \Delta(\theta;\omega)$ with $\mathrm{Re}\,\nu > \omega_0$.

Proof. Let $\delta > 0$. By Theorem A.1 of [8] there exist $a,b > 0$ such that

$$\begin{aligned} |R_{\nu,\delta}(gm^{-1}k^{-1})| &\leqslant a\mathrm{e}^{-b\rho(\nu;\Delta)^{1/n}|gm^{-1}k^{-1}|} \\ &\leqslant a\mathrm{e}^{b\rho(\nu;\Delta)^{1/n}(|g|+|k|)}\mathrm{e}^{-b\rho(\nu;\Delta)^{1/n}|m|} \end{aligned} \tag{3.1}$$

for all $m \in M$ and $g,k \in G$ such that $|gm^{-1}k^{-1}| > \widetilde{C}$ for some $\widetilde{C} > 0$ to be determined in the sequel and $\nu \in \Delta(\theta;\omega)$.

Alternatively, it follows from (2.19) and a chain argument (cf. also Lemma 2.2 in [18]) that there exist $C, \eta > 0$ such that

$$(\rho(g)\rho(km))^{-1/2} \leqslant C\rho(g)^{-1/2}\mathrm{e}^{\eta|k|}\mathrm{e}^{\eta|m|}$$

for all $m \in M$ and $g, k \in G$, and, moreover, it follows from (2.20) and a chain argument (cf. also (87) in [18]) that

$$|S(\dot{k}, gm^{-1}k^{-1})| \leqslant C'\mathrm{e}^{\eta'(|g|+|m|+|k|)}$$

for all $m \in M$ and $g, k \in G$. Next, there exist $C, \eta > 0$ such that

$$\Delta_G(m^{-1}k^{-1}) \leqslant C\mathrm{e}^{\eta|k|}\mathrm{e}^{\eta|m|}$$

for all $m \in M$ and $k \in G$.

Finally, let $g, k \in G$ be such that $\dot{g} \neq \dot{k}$. Then there exists a $\widetilde{C} > 0$ such that $|gm^{-1}k^{-1}| > \widetilde{C}$ for all $m \in M$. Indeed, if there is a sequence $m_1, m_2, \ldots$ in M such that $|gm_n^{-1}k^{-1}| \leqslant 1/n$ for all $n \in \mathbb{N}$ then $m_n \in \overline{B_{|g|+|k|+1}}$ for all $n \in \mathbb{N}$. Since $\overline{B_{|g|+|k|+1}}$ is compact and M is closed, one can pass to a convergent subsequence in M and there is an $m \in M$ with $|gm^{-1}k^{-1}| = 0$. But then $\dot{g} = \dot{k}$.

Hence, there exists an $\omega_0 \geqslant \omega$, depending on $\delta > 0$, such that the integral is absolutely convergent, uniformly for all $\nu \in \Delta(\theta;\omega)$, with $\operatorname{Re}\nu > \omega_0$, and the lemma follows. ∎

Let $a_1, \ldots, a_d$ be a basis for the Lie algebra $\mathfrak{g}$ of G. In Proposition 2.10 of [18] we proved the existence of the continuous heat kernel $\kappa : X \times X \to \mathbb{C}$ of the holomorphic semigroup generated by $\overline{H}$. Next, consider the contragredient representation of U in each $L_p(X;\mathrm{d}x)$ defined by

$$(\check{U}(g)\varphi)(x) = S(x, g^{-1})R(x, g^{-1})^{1/2}\varphi(g^{-1}x), \quad \text{a.e. } x \in X$$

for all $\varphi \in L_p(X;\mathrm{d}x)$. For all $i \in \{1, \ldots, d\}$, let R_i denote the infinitesimal generator in the direction a_i affiliated to $\check{U}$. Let V be the representation of G in $L_\infty(X;\mathrm{d}x)$ defined by

$$(V(g)\varphi)(x) = \varphi(g^{-1}x), \quad \text{a.e. } x \in X$$

for all $\varphi \in L_\infty(X;\mathrm{d}x)$ and $g \in G$. Let $B_i = \mathrm{d}V(a_i)$ denote the infinitesimal generator of the one parameter group $t \mapsto V(\exp(-ta_i))$ in $L_\infty(X;\mathrm{d}x)$. Consider the metric $d : X \times X \to [0, \infty)$ on X defined by

$$d(x;y) = \sup\Big\{|\psi(x) - \psi(y)| : \psi \in C_{\mathrm{b};\infty}(X) \text{ real and } \sum_{i=1}^{d} |B_i\psi|^2 \leqslant 1\Big\} \tag{3.2}$$

for all $x, y \in X$, where $C_{\mathrm{b};\infty}(X)$ denotes the space of all infinitely differentiable functions on X with uniformly bounded derivatives. Introduce the balls $B(x;r)$ by

$$B(x;r) = \{y \in X : d(x;y) < r\}$$

for all $x \in X$ and $r > 0$. Let $R_1, \ldots, R_d$ denote the infinitesimal generators in the directions $a_1, \ldots, a_d$ associated to the representation $\check{U}$. Then in Proposition 2.11 of [18] it was shown that κ_t is pointwise C^∞ in the second variable with respect to $R_1, \dots, R_d$, and, moreover, if $\beta \in J(d)$ and R^β denotes the pointwise (multi-)-derivative, with respect to the second variable, then $R^\beta \kappa_t$ is pointwise C^∞ in the first variable, with respect to the infinitesimal generators $A_1, \dots, A_d$. Further, if $\alpha, \beta \in J(d)$, A^α denotes the pointwise (multi-)derivative, with respect to the first variable, and R^β denotes, again, the pointwise (multi-)derivative, with respect to the second variable, then $A^\alpha R^\beta \kappa_t$ is given by the reduction formula

$$
\begin{aligned}
(A^\alpha R^\beta \kappa_t)(\dot{g}; \dot{k}) = &\sum_{(\gamma,\delta) \in Lb(\beta)} d_\delta \\
&\cdot \int_M (\widetilde{A}^\alpha \widetilde{R}^\gamma K_t)(gm^{-1}k^{-1}) S(\dot{k}, gm^{-1}k^{-1}) (\rho(g)\rho(km))^{-1/2} \Delta_G(m^{-1}k^{-1}) \, \mathrm{d}m
\end{aligned}
$$

for all $g, k \in G$ and $t > 0$. Further, it follows from Theorem 2.12 in [18] that for all $\alpha, \beta \in J(d)$ there exist $a, b > 0$ and $\omega' \geqslant 0$ such that

$$
\begin{aligned}
|(A^\alpha R^\beta \kappa_t)(x; y)| \leqslant c\big(&\mathrm{Vol}_X(B(x;1)) \mathrm{Vol}_X(B(y;1))\big)^{-1/2} \\
&\cdot t^{-(|\alpha|+|\beta|+d-d_M)/n} \mathrm{e}^{\omega' t} \mathrm{e}^{-b(d(x;y)^n t^{-1})^{1/(n-1)}}
\end{aligned}
\tag{3.3}
$$

for all $x, y \in G/M$ and $t > 0$.

By Lemma 3.1 one can define, for $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu$ sufficiently large, the function $r_{\nu,\delta} : X \times X \setminus \{(x,x) : x \in X\} \to \mathbb{C}$ by

$$
\begin{aligned}
r_{\nu,\delta}(\dot{g}; \dot{k})
&= \int_M R_{\nu,\delta}(gm^{-1}k^{-1}) S(\dot{k}, gm^{-1}k^{-1}) (\rho(g)\rho(km))^{-1/2} \Delta_G(m^{-1}k^{-1}) \, \mathrm{d}m \\
&= \int_M \Gamma(\delta)^{-1} \int_0^\infty \mathrm{e}^{-\nu t} t^{\delta-1} K_t(gm^{-1}k^{-1}) \, \mathrm{d}t \qquad (3.4)\\
&\qquad \cdot S(\dot{k}, gm^{-1}k^{-1}) (\rho(g)\rho(km))^{-1/2} \Delta_G(m^{-1}k^{-1}) \, \mathrm{d}m \\
&= \Gamma(\delta)^{-1} \int_0^\infty \mathrm{e}^{-\nu t} t^{\delta-1} \kappa_t(\dot{g}; \dot{k}) \, \mathrm{d}t
\end{aligned}
$$

for all $g, k \in G$ such that $\dot{g} \neq \dot{k}$.

Remark 3.2. If $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu$ sufficiently large, then the expression for $r_{\nu,\delta}$, in terms of the integral of κ_t, is absolutely convergent, in virtue of the upper estimates for κ_t stated in (3.3). We may reverse the order of integration in the double integral, because one can show, by a similar estimation argument as used in the proof of Theorem A.1 in [8] that there exist $a, b > 0$ such that

$$
\int_0^\infty \Gamma(\delta)^{-1} \mathrm{e}^{-\nu t} t^{\delta-1} |K_t(gm^{-1}k^{-1})| \, \mathrm{d}t \leqslant a \mathrm{e}^{-b\rho(\nu;\Delta)^{1/n}|m|}
$$

for all $m \in M$ if $\dot{g} \neq \dot{k}$. Now, by a similar bounding argument as used in the proof of Lemma 3.1, the repeated integral in (3.4) is absolutely convergent. ∎

The $r_{\nu,\delta}$ are reduced resolvent kernels by the following identity.

PROPOSITION 3.3. *If $p \in [1,\infty]$ and $\varphi \in L_p(X; \mathrm{d}x)$ then*

$$((\nu I + \overline{H})^{-\delta}\varphi)(x) = \int\limits_X r_{\nu,\delta}(x;y)\varphi(y)\,\mathrm{d}y$$

for a.e. $x \in X$.

Proof. The proposition follows directly from Laplace transformation, and the observation that κ_t is the reduced heat kernel of $S = S_t$ generated by $\overline{H}$. ∎

Next, the left derivative in the direction a_i on the Lie group G is denoted by $\widetilde{A}_i$ and the right derivative by $\widetilde{R}_i$. If $\beta \in J(d)$, then by $Lb(\beta)$ we denote the set of all $(\gamma,\delta) \in J(d)^2$ such that γ is a multi-index obtained from β by omission of some indices and δ is the multi-index formed by the omitted indices (cf. [7], p. 747). Moreover, if $\delta = (j_1,\ldots,j_l) \in J(d)$ then we set $d_\delta = (\widetilde{R}_{j_1}\Delta_G)(e)\cdots(\widetilde{R}_{j_l}\Delta_G)(e)$. Then, by similar arguments as used in the proof of Proposition 2.11 in [18], and using the upper bounds in (2.2), one can show that for all $\alpha,\beta \in J(d)$ one has

$$\begin{aligned}(A^\alpha R^\beta r_{\nu,\delta})(\dot{g};\dot{k}) &= \sum_{(\gamma,\delta)\in Lb(\beta)} d_\delta \int\limits_M (\widetilde{A}^\alpha \widetilde{R}^\gamma R_{\nu,\delta})(gm^{-1}k^{-1}) \\ &\qquad\qquad \cdot S(\dot{k}, gm^{-1}k^{-1})(\rho(g)\rho(km))^{-1/2}\Delta_G(m^{-1}k^{-1})\,\mathrm{d}m \\ &= \Gamma(\delta)^{-1}\int\limits_0^\infty \mathrm{e}^{-\nu t}t^{\delta-1}(A^\alpha R^\beta \kappa_t)(\dot{g};\dot{k})\,\mathrm{d}t\end{aligned}$$

for all $g,k \in G$, with $\dot{g} \neq \dot{k}$.

In the following lemma we state bounds for $r_{\nu,\delta}$ and its derivatives.

LEMMA 3.4. *For all $\alpha,\beta \in J(d)$ and $\theta \in (0,\theta_C)$ there exist $a,b > 0$ and $\omega \geqslant 0$ such that*

$$\begin{aligned}|(A^\alpha R^\beta r_{\nu,\delta})(x;y)| &\leqslant a(\mathrm{Vol}_X(B(x;1))\mathrm{Vol}_X(B(y;1)))^{-1/2}\rho^{(d-d_M+|\alpha|+|\beta|-n\delta)/n} \\ &\qquad \cdot F_{|\alpha|+|\beta|,\delta}(\rho^{1/n}d(x;y))\mathrm{e}^{-b\rho^{1/n}d(x;y)}\end{aligned}$$

for all $x,y \in X$ with $x \neq y$ and $\nu \in \Delta(\theta;\omega)$, where $\rho = \rho(\nu;\Delta)$ and

$$F_{k,\delta}(x) = \begin{cases} x^{-(d-d_M+k-n\delta)} & \text{if } d-d_M+k > n\delta, \\ 1+\log^+ x^{-1} & \text{if } d-d_M+k = n\delta, \\ 1 & \text{if } d-d_M+k < n\delta, \end{cases}$$

with $\log^+ y = \log y$ if $y \geqslant 1$ and $\log^+ y = 0$ if $y \leqslant 1$.

Proof. The proof is analogous to the proof of Theorem A.1 in [8] or Theorem III.6.7 in [16], since the estimates depend on the upper bounds for κ_t and all its derivatives, stated in (3.3). ∎

By similar arguments as used in the proof of Proposition 2.11 in [18], again, but now using the upper bounds from (2.4), one can show that, if $\mathrm{Re}\, c_0$ is sufficiently large for the coefficient c_0 of H, the operator $f(\overline{H})$ has a reduced kernel κ_f such that for all $\alpha, \beta \in J(d)$

$$(A^\alpha R^\beta \kappa_f)(\dot{g};\dot{k}) = \sum_{(\gamma,\delta)\in Lb(\beta)} d_\delta$$
$$\cdot \int_M (\widetilde{A}^\alpha \widetilde{R}^\gamma K_f)(gm^{-1}k^{-1}) S(\dot{k}, gm^{-1}k^{-1})(\rho(g)\rho(km))^{-1/2}\Delta_G((km)^{-1})\, \mathrm{d}m$$

for all $g, k \in G$ such that $\dot{g} \neq \dot{k}$.

In the following lemma, we state upper bounds for κ_f and its derivatives.

Lemma 3.5. *Let $\alpha, \beta \in J(d)$, $\theta \in (0, \theta_C)$ and $\varphi \in (\pi/2, \pi]$. If the real part of the zero-order coefficient c_0 of H is sufficiently large then there exist $a, b > 0$, independent of c_0, and $c > 0$, linearly dependent on $\mathrm{Re}\, c_0$, such that*

$$|(A^\alpha R^\beta \kappa_f)(x;y)|$$
$$\leqslant a(\mathrm{Vol}_X(B(x;1))\mathrm{Vol}_X(B(y;1)))^{-1/2}\|f\|_\infty (d(x;y))^{-(d-d_M+|\alpha|+|\beta|)}\mathrm{e}^{-bc^{1/n}d(x;y)}$$

for all $f \in F_\varphi$ and $x, y \in X$ with $x \neq y$.

Proof. The proof is similar to the proof of Corollary A.2 in [8], if we use the kernel bounds for κ_t and all its derivatives, stated in (3.3). ∎

Acknowledgements. The author wishes to thank A.F.M. ter Elst for the many fruitful discussions, comments and suggestions.

REFERENCES

1. J.-P. Anker, Sharp estimates for some functions of the Laplacian on noncompact symmetric spaces, *Duke Math. J.* **65**(1992), 257–297.
2. R.J. Burns, A.F.M. ter Elst, D.W. Robinson, L_p-regularity of subelliptic operators on Lie groups, *J. Operator Theory* **31**(1994), 165–187.
3. R.R. Coifman, G. Weiss, *Transference Methods in Analysis*, CBMS Regional Conf. Ser. in Math., vol. 31. Amer. Math. Soc., Providence, RI, 1977.
4. M. Cowling, I. Doust, A. McIntosh, A. Yagi, Banach space operators with a bounded H^∞ functional calculus, *J. Austral. Math. Soc. Ser. A* **60**(1996), 51–89.
5. X.T. Duong, D.W. Robinson, Semigroup kernels, Poisson bounds, and holomorphic functional calculus, *J. Funct. Anal.* **142**(1996), 89–129.
6. A.F.M. ter Elst, H. Prado, Subelliptic operators on nilpotent Lie groups, *Math. Scand.* **90**(2000), 251–266.
7. A.F.M. ter Elst, D.W. Robinson, Subcoercive and subelliptic operators on Lie groups: variable coefficients, *Publ. Res. Inst. Math. Sci.* **29**(1993), 745–801.
8. A.F.M. ter Elst, D.W. Robinson, Functional analysis of subelliptic operators on Lie groups, *J. Operator Theory* **31**(1994), 277–301.
9. A.F.M. ter Elst, D.W. Robinson, Subcoercivity and subelliptic operators on Lie groups. I: Free nilpotent groups, *Potential Anal.* **3**(1994), 283–337.

10. A.F.M. TER ELST, D.W. ROBINSON, Subcoercivity and subelliptic operators on Lie groups. II: The general case. *Potential Anal.* **4**(1995), 205–243.
11. A.F.M. TER ELST, D.W. ROBINSON, Weighted subcoercive operators on Lie groups, *J. Funct. Anal.* **157**(1998), 88–163.
12. K. DE LEEUW, , H. MIRKEL, A priori estimates for differential operators in L_∞ norm, *Illinois J. Math.* **8**(1964), 112–124.
13. A. MCINTOSH, Operators which have an H_∞ functional calculus, in *Miniconference on Operator Theory and Partial Differential Equations*, B. Jefferies, A. McIntosh, W.J. Ricker (eds.), vol. 14, Proc. Centre Math. Anal. Austral. Nat. Univ., Austral. Nat. Univ., Canberra, Australia, 1986, pp. 210–231.
14. A. MCINTOSH, A. YAGI, Operators of type ω without a bounded H_∞ functional calculus, in *Miniconference on Operators in Analysis*, I. Doust, B. Jefferies, C. Li, A. McIntosh, (eds.), vol. 24, Proc. Centre Math. Anal. Austral. Nat. Univ., Austral. Nat. Univ., Canberra, Australia, 1989, pp. 159–172.
15. D. ORNSTEIN, A non-inequality for differential operators in the L_1 norm, *Arch. Rational Mech. Anal.* **11**(1962), 40–49.
16. D.W. ROBINSON, *Elliptic Operators and Lie Groups*, Oxford Math. Monogr., Oxford Univ. Press, Oxford 1991.
17. L. SALOFF-COSTE, D.W. STROOCK, Operateurs uniformement sous-elliptiques sur les groupes de Lie, *J. Funct. Anal.* **98**(1991), 97–121.
18. C.M.P.A. SMULDERS, Reduced heat kernels on homogeneous spaces with quasi-invariant measures, Research Report RANA 99–20, Eindhoven University of Technology, Eindhoven 1999.
19. E.M. STEIN, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton 1993.

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Received November 8, 2000.