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SINGULAR INTEGRAL OPERATORS ASSOCIATED WITH MEASURES OF VARYING DENSITY

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Abstract. Let $1 < p < \infty$ and let μ be a compactly supported regular Borel measure on $\mathbb{R}^n$ which has the property that there exists a $t > 1/(p-1)$ such that

$$\sup_{0<r\leqslant 1} \int_{\mathbb{R}^n} \Big(\frac{\mu(B(x,r))}{r^p}\Big)^t \, \mathrm{d}\mu(x) < \infty.$$

We show that, for such a μ, any singular integral operator on $L^2(\mathbb{R}^n,\mu)$ with a smooth, homogeneous kernel of degree -1 belongs to the norm ideal $\mathcal{C}^+_{p/(p-1)}$.

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1. INTRODUCTION

In this paper we study singular integral operators of the type

$$(T_{K,\mu}f)(x) = \int K(x-y)f(y)\,\mathrm{d}\mu(y),$$

where K is a homogeneous function of degree -1. We are interested in the question when $T_{K,\mu}$ belongs to the Lorentz-like ideal $\mathcal{C}^+_{p/(p-1)}$, $1 < p < \infty$. In other words, we would like to know under what condition on the measure μ are the s-numbers of $T_{K,\mu}$ such that $\sup\limits_{\nu>1} \nu^{-1/p} \sum\limits_{j=1}^{\nu} s_j(T_{K,\mu}) < \infty$? This question is closely related to the problem of diagonalizing commuting tuples of self-adjoint operators modulo the ideal $\mathcal{C}^-_p$, which is the pre-dual of $\mathcal{C}^+_{p/(p-1)}$. Besides this operator-theoretical connection, the reader will see the important role that geometric measure theory plays in this investigation. More precisely, the question we investigate is about an interplay between operator theory and a certain kind of measure density. To explain our results, some background information is necessary.

Recall that, for a compact operator A on a separable, infinite dimensional Hilbert space H, its s-numbers $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A) \geqslant \cdots$ are defined as follows: If $\operatorname{rank}(A) < \infty$, then this sequence is simply a descending arrangement of the eigenvalues of $|A|$ counting multiplicity. If $\operatorname{rank}(A) = \infty$, then this sequence is a descending arrangement of the *nonzero* eigenvalues of $|A|$ counting multiplicity. See [5].

Given a Hilbert space H and a $p \in [1, \infty)$, the classes $\mathcal{C}_p^-$, $\mathcal{C}_p$ and $\mathcal{C}_p^+$ respectively consist of all the compact operators A on H such that $\|A\|_p^- < \infty$, $\|A\|_p < \infty$ and $\|A\|_p^+ < \infty$. Here, $\|A\|_p = \{\operatorname{tr}(|A|^p)\}^{1/p}$ and

$$\|A\|_p^- = \sum_{j=1}^{\infty} j^{(1-p)/p} s_j(A), \quad \|A\|_p^+ = \sup_{\nu \geqslant 1} \frac{\sum_{j=1}^{\nu} s_j(A)}{\sum_{j=1}^{\nu} j^{-1/p}}.$$

$\mathcal{C}_p$ is usually refereed to as the Schatten p-class and we have $\mathcal{C}_{p'}^- \subset \mathcal{C}_{p'} \subset \mathcal{C}_{p'}^+ \subset \mathcal{C}_p^- \subset \mathcal{C}_p \subset \mathcal{C}_p^+$ for all $1 \leqslant p' < p < \infty$. $\mathcal{C}_p^-$, $\mathcal{C}_p$ and $\mathcal{C}_p^+$ are examples of a much larger class of ideals called *norm ideals* of compact operators ([8]). (In some literature they are known as *symmetrically normed ideals*. See, e.g., [5].)

Throughout the paper, K denotes a homogeneous function of degree -1 which is C^∞ on $\mathbb{R}^n \setminus \{0\}$ for some $n \in \mathbb{N}$. That is, the function $x \mapsto K(x)$ is smooth on $\mathbb{R}^n \setminus \{0\}$ and

$$\lambda K(\lambda x) = K(x) \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\} \text{ and } \lambda > 0.$$

Suppose that μ is a regular Borel measure on $\mathbb{R}^n$. We let $T_{K,\mu}$ denote the singular integral operator

$$(T_{K,\mu} f)(x) = \int K(x-y) f(y) \, \mathrm{d}\mu(y), \quad f \in L^2(\mathbb{R}^n, \mu),$$

whenever it makes sense.

In this paper we consider the question, when does $T_{K,\mu}$ belong to $\mathcal{C}_{p/(p-1)}^+$? The investigation of this problem started with Voiculescu's works ([10]–[12]) and the joint work ([3]) by David and Voiculescu. Besides the intrinsic interest associated with $T_{K,\mu}$, there is a full explanation in the Introduction of [3] as to why such a problem is important. Rather than repeating the entire explanation here, we simply remind the reader the following: Suppose that $1 < p < \infty$ and that the measure μ on $\mathbb{R}^n$ has the property that $T_{K,\mu} \in \mathcal{C}_{p/(p-1)}^+$ for $K(x) = x_j/|x|^2$, $j = 1, \ldots, n$, where we write $x = (x_1, \ldots, x_n)$. Then the n-tuple $(M_1^\mu, \ldots, M_n^\mu)$ of self-adjoint operators on $L^2(\mathbb{R}^n, \mu)$ cannot be simultaneously diagonalized modulo $\mathcal{C}_p^-$ ([3]; see also [12], Proposition 2.1). Here and in what follows, M_j^μ denotes the multiplication operator on $L^2(\mathbb{R}^n, \mu)$ defined by the formula

$$(M_j^\mu f)(x_1, \ldots, x_n) = x_j f(x_1, \ldots, x_n).$$

As usual, in a metric space (X, d), we denote the ball $\{y \in X : d(x, y) < r\}$ by $B(x, r)$. To motivate our investigation, let us recall a well-known result.

Theorem 1.1. ([3]) *Suppose that $1 < p < \infty$. Suppose that μ is a compactly supported regular Borel measure on $\mathbb{R}^n$ for which there is a positive number C such that*

$$\frac{\mu(B(x,r))}{r^p} \leqslant C \quad \text{for all } x \in \mathbb{R}^n \text{ and } 0 < r \leqslant 1. \tag{1.1}$$

Then $T_{K,\mu} \in \mathcal{C}^+_{p/(p-1)}$ for every C^∞ homogeneous function K of degree -1 on $\mathbb{R}^n \setminus \{0\}$. Consequently, for such a measure μ, $(M_1^\mu, \dots, M_n^\mu)$ cannot be simultaneously diagonalized modulo $\mathcal{C}^-_p$.

The results of [3] also cover other norm ideals and measures with growth rates other than r^p. Nevertheless every measure in [3] was assumed to have a uniform upper bound for growth rate,

$$\mu(B(x,r)) \leqslant C_1 h(r) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0.$$

And it was the function h that was matched with the norm ideal in question ([3], Theorem 3.1). The obvious question here is what happens if one drops such a uniform upper bound? As it turns out, we can prove $T_{K,\mu} \in \mathcal{C}^+_{p/(p-1)}$ under a condition weaker than (1.1).

Theorem 1.2. *Suppose that $1 < p < \infty$. Let μ be a compactly supported regular Borel measure on $\mathbb{R}^n$. Suppose that there is a $t > 1/(p-1)$ such that*

$$\sup_{0<r\leqslant 1} \int_{\mathbb{R}^n} \Big(\frac{\mu(B(x,r))}{r^p}\Big)^t \,\mathrm{d}\mu(x) < \infty. \tag{1.2}$$

Then $T_{K,\mu} \in \mathcal{C}^+_{p/(p-1)}$ for every C^∞ homogeneous function K of degree -1 on $\mathbb{R}^n \setminus \{0\}$. Consequently, for such a measure μ, $(M_1^\mu, \dots, M_n^\mu)$ cannot be simultaneously diagonalized modulo $\mathcal{C}^-_p$.

Since we are only dealing with finite measures, any μ which satisfies (1.1) automatically satisfies (1.2) for every $t > 1/(p-1)$ (indeed for every $t > 0$). Compared with Theorem 1.1, our main improvement lies in the fact that we allow $x \mapsto \mu(B(x,r))$ to vary over a wide range for each fixed r. In other words, under our assumption the function $r \mapsto \mu(B(x,r))/r^p$ need not be bounded for any x; to ensure $T_{K,\mu} \in \mathcal{C}^+_{p/(p-1)}$, the boundedness of a certain kind of "average" of $\mu(B(x,r))/r^p$ will suffice.

Our improvement over Theorem 1.1 is not vacuous. Indeed, for each integer $N \geqslant 2$ and each $t > 1/(N-1)$ there exists a compactly supported probability measure $\omega = \omega_{N,t}$ on $\mathbb{R}^{N+1}$ such that

$$\sup_{0<r\leqslant 1} \int_{\mathbb{R}^{N+1}} \Big(\frac{\omega(B(x,r))}{r^N}\Big)^t \,\mathrm{d}\omega(x) < \infty$$

and such that

$$\limsup_{r\downarrow 0} \frac{\omega(B(x,r))}{r^N} = \infty \quad \text{for every } x \in \Delta_\omega, \tag{1.3}$$

where Δ_ω denotes the support of ω. By the usual Covering Lemma ([4], [7], [9]) (1.3) implies that the N-dimensional Hausdorff measure of Δ_ω is zero.

As it turns out, even (1.2) is not a necessary condition for $T_{K,\mu} \in \mathcal{C}^+_{p/(p-1)}$. One can also produce a measure ω_0 on $\mathbb{R}^{N+1}$ which has the properties

$$\sup_{0<r\leqslant 1} \int_{\mathbb{R}^{N+1}} \Big(\frac{\omega_0(B(x,r))}{r^N}\Big)^t \, \mathrm{d}\omega_0(x) = \infty \quad \text{for every } t > \frac{1}{N-1}$$

and

$$\sup_{0<r\leqslant 1} \int_{\mathbb{R}^{N+1}} \Big(\frac{\omega_0(B(x,r))}{r^N}\Big)^{1/(N-1)} \, \mathrm{d}\omega_0(x) < \infty,$$

but $T_{K,\omega_0} \in \mathcal{C}^+_{N/(N-1)}$ for every C^∞-homogeneous function K of degree -1 on $\mathbb{R}^{N+1} \setminus \{0\}$.

Theorem 1.2 and ω_0 might lead to the speculation that the natural condition for $T_{K,\mu} \in \mathcal{C}^+_{p/(p-1)}$, $1 < p < \infty$, is that

$$\sup_{0<r\leqslant 1} \int_{\mathbb{R}^n} \Big(\frac{\mu(B(x,r))}{r^p}\Big)^{1/(p-1)} \, \mathrm{d}\mu(x) < \infty. \tag{1.4}$$

But this turns out to be false. One can show by an example that, in general, (1.4) alone does not even guarantee the *compactness* of $T_{K,\mu}$, much less membership in $\mathcal{C}^+_{p/(p-1)}$. The reason that (1.4) might lead to undesirable situation is that this condition by itself does not rule out the possibility that $\mu(B(x,r)) \approx r$ for a set of x whose measure is on the order of r. In other words, such a μ might be too singular for $T_{K,\mu}$ to be compact.

Due to the technicalities involved, the construction of the measures mentioned above will be omitted. The focus of the paper will be on the proof of Theorem 1.2.

2. PROOF OF THEOREM 1.2

Throughout the section, we fix an $n \in \mathbb{N}$ and we let

$$Q = [0,1)^n = [0,1) \times \cdots \times [0,1),$$

the unit cube in $\mathbb{R}^n$. For each $\ell \in \mathbb{N}$, we let W_ℓ be the set of words of length ℓ with $\{1,2,3,\ldots,2^n\}$ being the set of alphabet. That is,

$$W_\ell = \{w_1 \cdots w_\ell : w_1, \ldots, w_\ell \in \{1,2,3,\ldots,2^n\}\}.$$

Let $\Gamma = \{(s_1,\ldots,s_n) : s_1,\ldots,s_n \in \{0,1\}\}$ and let $\gamma_1,\ldots,\gamma_{2^n}$ be an enumeration of the elements in Γ. Given $w = w_1 \cdots w_\ell \in W_\ell$, we define

$$Q_w = Q_{w_1\cdots w_\ell} = [0,2^{-\ell})^n + 2^{-1}\gamma_{w_1} + \cdots + 2^{-\ell}\gamma_{w_\ell}.$$

It is clear that $\bigcup\limits_{w\in W_\ell} Q_w = Q$ and that $Q_w \cap Q_{w'} = \emptyset$ for $w \neq w'$ in W_ℓ. Define

$$\mathcal{W} = \bigcup_{\ell=1}^{\infty} W_\ell.$$

The homogeneity of the kernel $K(x-y)$ is such that, for the proof of Theorem 1.2, we only need to consider measures which are concentrated on Q. Therefore, for the rest of the section let μ be a regular Borel measure on $\mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus Q) = 0$. Furthermore, we assume that $\dim(L^2(\mathbb{R}^n, \mu)) = \infty$. For each $w \in \mathcal{W}$, we define the element $e_w \in L^2(\mathbb{R}^n, \mu)$ by the formula

$$e_w = \begin{cases} (\mu(Q_w))^{-1/2}\chi_{Q_w} & \text{if } \mu(Q_w) > 0, \\ 0 & \text{if } \mu(Q_w) = 0. \end{cases}$$

Let $\Lambda = \{(s_1, \ldots, s_n) : s_1, \ldots, s_n \in \{-1, 0, 1\}\}$. Given $w \in W_\ell$ and $\lambda \in \Lambda$, we have either $Q_w + 2^{-\ell}\lambda = Q_{w'}$ for some $w' \in W_\ell$ or $Q_w + 2^{-\ell}\lambda \subset \mathbb{R}^n \setminus Q$. Thus for $w \in W_\ell$ and $\lambda \in \Lambda$, we define the element $e(w, \lambda) \in L^2(\mathbb{R}^n, \mu) = L^2(Q, \mu)$ as follows:

$$e(w,\lambda) = \begin{cases} e_{w'} & \text{if } Q_w + 2^{-\ell}\lambda = Q_{w'},\ w' \in W_\ell, \\ 0 & \text{if } Q_w + 2^{-\ell}\lambda \subset \mathbb{R}^n \setminus Q. \end{cases}$$

Similarly, for $w \in W_\ell$ and $\lambda \in \Lambda$, we define

$$\mu(w,\lambda) = \begin{cases} \mu(Q_{w'}) & \text{if } Q_w + 2^{-\ell}\lambda = Q_{w'},\ w' \in W_\ell, \\ 0 & \text{if } Q_w + 2^{-\ell}\lambda \subset \mathbb{R}^n \setminus Q. \end{cases}$$

For the rest of the section we let K denote a C^∞-homogeneous function of degree -1 on $\mathbb{R}^n \setminus \{0\}$. Let $0 \leqslant \widetilde{\eta} \leqslant 1$ be a C^∞-function on $[0, \infty)$ such that $\widetilde{\eta} = 1$ on $[0, 1/2]$ and $\widetilde{\eta} = 0$ on $[5/8, \infty)$. Define $\eta(r) = \widetilde{\eta}(r) - \widetilde{\eta}(2r)$, $r \in [0, \infty)$. It is easy to see that

$$\eta = 0 \text{ on } \Big[0, \frac{1}{4}\Big] \cup \Big[\frac{5}{8}, \infty\Big) \quad \text{and} \quad \eta = 1 \text{ on } \Big[\frac{1}{3}, \frac{1}{2}\Big]. \tag{2.1}$$

Let $\ell_0 \in \mathbb{N}$ be such that $2^{\ell_0 - 1} > \sqrt{n}$. Now $\sum_{\ell=-k}^{k'} \eta(2^\ell r) = \widetilde{\eta}(2^{-k}r) - \widetilde{\eta}(2^{k'+1}r)$. Since $|u| \leqslant \sqrt{n} < 2^{\ell_0 - 1}$ for every $u \in [-1, 1]^n$, we have

$$\sum_{\ell=-\ell_0}^{\infty} \eta(2^\ell |u|) = 1 \quad \text{if } u \in [-1,1]^n \text{ and } u \neq 0. \tag{2.2}$$

(2.1) implies that $K(u)\eta(|u|) = 0$ if $0 < |u| \leqslant 1/4$. Hence there is a periodic C^∞-function φ on $\mathbb{R}^n$ with $(2^{\ell_0+2}\mathbb{Z})^n$ as its period lattice such that

$$\varphi(u) = K(u)\eta(|u|) \quad \text{if } u \in [-2^{\ell_0}, 2^{\ell_0}] \text{ and } u \neq 0. \tag{2.3}$$

Such a φ has a Fourier expansion

$$\varphi(u) = \sum_{z \in \mathbb{Z}^n} c_z \exp(2^{-\ell_0 - 1}\mathrm{i}\pi\langle u, z\rangle) \quad \text{with} \quad \sum_{z \in \mathbb{Z}^n} |c_z| < \infty. \tag{2.4}$$

For $-\ell_0 \leqslant k < k'$, we set

$$K_k(u) = \sum_{\ell=-\ell_0}^{k} K(u)\eta(2^\ell|u|), \quad K_{k,k'}(u) = \sum_{\ell=k+1}^{k'} K(u)\eta(2^\ell|u|).$$

Accordingly, for such k and k', we define the operators

$$(T_k f)(x) = \int K_k(x-y)f(y)\,\mathrm{d}\mu(y), \quad (T_{k,k'} f)(x) = \int K_{k,k'}(x-y)f(y)\,\mathrm{d}\mu(y)$$

on $L^2(\mathbb{R}^n,\mu)$. By (2.3), (2.4) and the fact that $K(u) = 2^\ell K(2^\ell u)$, it is clear that $T_1 \in \mathcal{C}_1$.

For $w \in W_\ell$ and $z \in \mathbb{Z}^n$, we set $f_w^z(x) = \exp(2^{\ell-\ell_0-1}\mathrm{i}\pi\langle x,z\rangle)$. For $1 \leqslant k < k'$, $z \in \mathbb{Z}^n$ and $\lambda \in \Lambda$, we define the operator

$$(2.5)\qquad A_{k,k',z,\lambda} = \sum_{\ell=k+1}^{k'} \sum_{w\in W_\ell} 2^\ell\{\mu(w,\lambda)\mu(Q_w)\}^{1/2}(f_w^z e(w,\lambda)) \otimes (f_w^z e_w)$$

on $L^2(\mathbb{R}^n,\mu)$. The proof of Theorem 1.2 relies on the following decomposition of $T_{k,k'}$.

LEMMA 2.1. *For any $1 \leqslant k < k'$, $T_{k,k'} = \sum_{\lambda\in\Lambda}\sum_{z\in\mathbb{Z}^n} c_z A_{k,k',z,\lambda}$. Therefore, by (2.4), there is a constant $C_{2.1}(n,K) > 0$ which depends only on n and on K such that for any norm ideal $\mathcal{C}$ of compact operators on $L^2(\mathbb{R}^n,\mu)$ and for any such k and k', we have*

$$\|T_{k,k'}\|_{\mathcal{C}} \leqslant C_{2.1}(n,K)\sup\{\|A_{k,k',z,\lambda}\|_{\mathcal{C}} : z \in \mathbb{Z}^n, \lambda \in \Lambda\}.$$

Proof. By (2.1), for each $\ell \in \mathbb{N}$, $\eta(2^\ell|x-y|) \neq 0$ only if $2^\ell(x-y) \in (-1,1)^n$, i.e., only if $x \in y + (-2^{-\ell}, 2^{-\ell})^n$. Hence if $y \in Q_w$, $w \in W_\ell$, then $\eta(2^\ell|x-y|) \neq 0$ only if $x \in \bigcup_{\lambda\in\Lambda}(Q_w + 2^{-\ell}\lambda)$. On the other hand, if $y \in Q_w$, $w \in W_\ell$, and $x \in \bigcup_{\lambda\in\Lambda}(Q_w + 2^{-\ell}\lambda)$, then $2^\ell(x-y) \in [-2,2]^n$ and, therefore, $K(2^\ell(x-y))\eta(2^\ell|x-y|) = \varphi(2^\ell(x-y))$. By this observation and (2.4), for $x \neq y$ in Q,

$$\begin{aligned} K(2^\ell(x-y))\eta(2^\ell|x-y|) &= \sum_{\lambda\in\Lambda}\sum_{w\in W_\ell} \chi_{Q_w+2^{-\ell}\lambda}(x)K(2^\ell(x-y))\eta(2^\ell|x-y|)\chi_{Q_w}(y) \\ &= \sum_{\lambda\in\Lambda}\sum_{w\in W_\ell} \chi_{Q_w+2^{-\ell}\lambda}(x)\varphi(2^\ell(x-y))\chi_{Q_w}(y) \\ &= \sum_{z\in\mathbb{Z}^n} c_z \sum_{\lambda\in\Lambda}\sum_{w\in W_\ell} \exp(2^{-\ell_0-1}\mathrm{i}\pi\langle 2^\ell(x-y),z\rangle)\chi_{Q_w+2^{-\ell}\lambda}(x)\chi_{Q_w}(y). \end{aligned}$$

From this and the identity $K_{k,k'}(u) = \sum_{\ell=k+1}^{k'} 2^\ell K(2^\ell u)\eta(2^\ell|u|)$ for $u \in [-1,1]^n \setminus \{0\}$ we obtain the decomposition of $T_{k,k'}$. ∎

This lemma reduces the proof of Theorem 1.2 to estimate of $\|A_{k,k',z,\lambda}\|^+_{p/(p-1)}$, which we will take up next. Let us first record three elementary lemmas.

LEMMA 2.2. *Suppose that $0 < \tau < \infty$. Then there is a positive number C which depends only on n and τ such that, for any $\ell \in \mathbb{N}$,*

$$(2.6)\quad \frac{1}{C}\int(\mu(B(x,2^{-\ell})))^\tau\,\mathrm{d}\mu(x) \leqslant \sum_{w\in W_\ell}(\mu(Q_w))^{1+\tau} \leqslant \int(\mu(B(x,2^{-\ell+\ell_0})))^\tau\,\mathrm{d}\mu(x).$$

Proof. Let $B = B(\tau,n) > 0$ be such that $(a_1+\cdots+a_{3^n})^\tau \leqslant B(a_1^\tau+\cdots+a_{3^n}^\tau)$ whenever $a_1,\ldots,a_{3^n}$ are non-negative numbers. Suppose that $w \in W_\ell$ and $u \in$

Q_w. Then it is obvious that $B(u, 2^{-\ell}) \subset \bigcup_{\lambda\in\Lambda} (Q_w + 2^{-\ell}\lambda)$. Hence for every $x \in Q$ we have

$$(2.7) \qquad \begin{aligned}(\mu(B(x, 2^{-\ell})))^\tau &= \sum_{w\in W_\ell} (\mu(B(x, 2^{-\ell})))^\tau \chi_{Q_w}(x) \\ &\leqslant \sum_{w\in W_\ell} \Big(\mu\Big(\bigcup_{\lambda\in\Lambda} (Q_w + 2^{-\ell}\lambda)\Big)\Big)^\tau \chi_{Q_w}(x).\end{aligned}$$

Since there are exactly 3^n elements in Λ, we have

$$\int (\mu(B(x, 2^{-\ell})))^\tau \,\mathrm{d}\mu(x) \leqslant B \sum_{w\in W_\ell} \sum_{\lambda\in\Lambda} (\mu(Q_w + 2^{-\ell}\lambda))^\tau \mu(Q_w).$$

By Hölder's inequality, $\sum_{w\in W_\ell} (\mu(Q_w + 2^{-\ell}\lambda))^\tau \mu(Q_w) \leqslant \sum_{w\in W_\ell} (\mu(Q_w))^{1+\tau}$ for every $\lambda \in \Lambda$. Therefore $C = 3^n B$ will do for the first half of (2.6).

To prove the second half of (2.6), we observe that, since $\sqrt{n} < 2^{\ell_0-1}$, $B(u, 2^{-\ell+\ell_0}) \supset Q_w$ if $u \in Q_w$ and $w \in W_\ell$. Thus $\sum_{w\in W_\ell} (\mu(Q_w))^\tau \chi_{Q_w}(x) \leqslant (\mu(B(x, 2^{-\ell+\ell_0})))^\tau$ for every $x \in Q$. The second half of (2.6) is now obtained by integrating this inequality. ∎

Let $\{v_1, \ldots, v_k, \ldots\}$ be an orthonormal set in a Hilbet space H. Define the orthogonal projections $P_k = v_1 \otimes v_1 + \cdots + v_k \otimes v_k$, $k \in \mathbb{N}$.

Lemma 2.3. *Suppose that $A_1, \ldots, A_j$ are finite-rank operators on a Hilbert space. Suppose $k_1 \leqslant \cdots \leqslant k_j$ are integers such that* $\operatorname{rank}(A_i) \leqslant k_i$, *$i = 1, 2, \ldots, j$. Then in any norm ideal $\mathcal{C}$ of compact operators, we have*

$$\|A_1 + \cdots + A_j\|_{\mathcal{C}} \leqslant \Big\|\bigoplus_{i=1}^{j} (\|A_i\| + \cdots + \|A_j\|)P_{k_i}\Big\|_{\mathcal{C}}.$$

Proof. It is known that $\|A_1+\cdots+A_j\|_{\mathcal{C}} \leqslant \|\, \|A_1\|P_{k_1} + \cdots + \|A_j\|P_{k_j}\|_{\mathcal{C}}$ ([13], Lemma 4.4). If we set $P_{k,k'} = v_{k+1} \otimes v_{k+1} + \cdots + v_{k'} \otimes v_{k'}$ for $k < k'$ and $P_{k,k} = 0$, then $\sum_{i=1}^{j} \|A_i\|P_{k_i} = \sum_{i=1}^{j} \sum_{\nu=i}^{j} \|A_\nu\|P_{k_{i-1},k_i}$, where $k_0 = 0$. Obviously there is a partial isometries U such that $\sum_{i=1}^{j} \sum_{\nu=i}^{j} \|A_\nu\|P_{k_{i-1},k_i} = U^*\Big\{\bigoplus_{i=1}^{j} \sum_{\nu=i}^{j} \|A_\nu\|P_{k_i}\Big\}U$. ∎

Suppose now that we are given a $p \in (1, \infty)$. For the rest of the section, let s and τ be such that $1 < s \leqslant p$ and $\tau > 1/(s-1)$.

Lemma 2.4. *There is a positive number $C(s)$ which depends only on s such that the following holds true: Let ρ and S be positive numbers and let $A_1, \ldots, A_J$ be trace-class operators on a Hilbert space satisfying the conditions $\|A_j\| \leqslant 2^{-sj}\rho$, $\|A_j\|_1 \leqslant S$, $j = 1, \ldots, J$. Then*

$$\Big\|\sum_{j=1}^{J} 2^j A_j\Big\|_{s/(s-1)}^{+} \leqslant C(s)(\rho^{1/s} S^{(s-1)/s} + \rho).$$

Proof. Let $j_0 = \min\{j \in \mathbb{Z} : 2^{sj}S/\rho \geqslant 1\}$ and let $J_0 = \max\{1, j_0\}$. Without loss of generality, we may assume that $J > J_0$. Define

$$k_j = \Big[\frac{2^{sj}S}{\rho}\Big], \quad J_0 \leqslant j \leqslant J. \tag{2.8}$$

(As usual, $[R]$ denotes the largest integer not exceeding R.) Let $C > 0$ be such that $C\sum_{i=1}^{\nu} i^{-(s-1)/s} \geqslant \nu^{1/s}$ for all $\nu \in \mathbb{N}$. Let the projections P_k be the same as in the preceding lemma. We first show that

$$\Big\| \sum_{j=J_0}^{J} 2^{-(s-1)j} P_{k_j} \Big\|^{+}_{s/(s-1)} \leqslant 3C(1-2^{1-s})^{-1}\Big(\frac{S}{\rho}\Big)^{(s-1)/s}. \tag{2.9}$$

Denote $T = \sum_{j=J_0}^{J} 2^{-(s-1)j}P_{k_j}$. Set $k_{J_0-2} = k_{J_0-1} = 0$. If $k_{j-1} < \ell \leqslant k_j$, $J_0 \leqslant j \leqslant J$, then $s_\ell(T) = \sum_{i=j}^{J} 2^{-(s-1)i} \leqslant (1-2^{1-s})^{-1}2^{-(s-1)j}$. Given $k_{m-1} < \nu \leqslant k_m$, $J_0 \leqslant m \leqslant J$, let us write $\nu = \nu_0 + k_{m-1}$ with $1 \leqslant \nu_0 \leqslant k_m - k_{m-1}$. We have

$$\begin{aligned}
&\frac{\sum_{i=1}^{\nu} s_i(T)}{\sum_{i=1}^{\nu} i^{-(s-1)/s}} \\
&\leqslant C(1-2^{1-s})^{-1}\nu^{-1/s}\Big(2^{-(s-1)m}\nu_0 + \sum_{j=J_0-1}^{m-1} 2^{-(s-1)j}(k_j - k_{j-1})\Big) \\
&\leqslant C(1-2^{1-s})^{-1}\Big(2^{-(s-1)m}k_m^{(s-1)/s} + \sum_{j=J_0-1}^{m-1} (2^{-(s-1)j}k_j^{(s-1)/s})\Big(\frac{k_j}{\nu}\Big)^{1/s}\Big).
\end{aligned} \tag{2.10}$$

By (2.8), $2^{-(s-1)j}k_j^{(s-1)/s} \leqslant (S/\rho)^{(s-1)/s}$, $J_0 - 1 \leqslant j \leqslant J$. The $\sum_{j=J_0-1}^{m-1} \cdots$ term above is 0 if $m = J_0$. If $m \geqslant J_0 + 1$, since $\nu \geqslant 1 + k_{m-1} \geqslant 2^{s(m-1)}S/\rho$, we have $(k_j/\nu)^{1/s} \leqslant 2^{-(m-1-j)}$ for $J_0 - 1 \leqslant j \leqslant m-1$. Substituting these into (2.10), we see that $\sum_{i=1}^{\nu} s_i(T)/\sum_{i=1}^{\nu} i^{-(s-1)/s} \leqslant 3C(1-2^{1-s})^{-1}(S/\rho)^{(s-1)/s}$, which yields (2.9).

Write $X = \sum_{j=J_0}^{J} 2^j A_j$ and $Y = \sum_{j=J_0}^{J} 2^{1-(s-1)j}\rho P_{k_j} = 2\rho T$. Our next step is to show that

$$\|X\|^{+}_{s/(s-1)} \leqslant \|Y\|^{+}_{s/(s-1)}, \tag{2.11}$$

which, when combined with (2.9), yields

$$\Big\| \sum_{j=J_0}^{J} 2^j A_j \Big\|^{+}_{s/(s-1)} \leqslant 6C(1-2^{1-s})^{-1}S^{(s-1)/s}\rho^{1/s}. \tag{2.12}$$

To establish (2.11), obviously it suffices to show that $\sum_{i=1}^{\nu} s_i(X) \leqslant \sum_{i=1}^{\nu} s_i(Y)$ for every $\nu \in \mathbb{N}$. But it is obvious that

$$\sum_{i=1}^{\nu} s_i(Y) = \operatorname{tr}(YP_\nu) = \sum_{\substack{k_j \leqslant \nu \\ J_0 \leqslant j \leqslant J}} 2^{1-(s-1)j}\rho k_j + \nu \sum_{\substack{k_j > \nu \\ J_0 \leqslant j \leqslant J}} 2^{1-(s-1)j}\rho, \tag{2.13}$$

where the first (respectively second) term is 0 if there are no $J_0 \leqslant j \leqslant J$ which satisfy $k_j \leqslant \nu$ (respectively $k_j > \nu$). Let B be an operator such that $\|B\| \leqslant 1$ and $\operatorname{rank}(B) \leqslant \nu$. For $J_0 \leqslant j \leqslant J$ such that $k_j \leqslant \nu$, we have

$$|\operatorname{tr}(2^j A_j B)| \leqslant \|2^j A_j\|_1 \|B\| \leqslant 2^j S = (2^{-(s-1)j}\rho) \times \left(\frac{2^{sj} S}{\rho}\right) \leqslant 2^{1-(s-1)j}\rho k_j. \tag{2.14}$$

For $J_0 \leqslant j \leqslant J$ such that $k_j > \nu$, we have

$$|\operatorname{tr}(2^j A_j B)| \leqslant \|2^j A_j\|\,\|B\|_1 \leqslant 2^{-(s-1)j}\rho\nu < \nu 2^{1-(s-1)j}\rho. \tag{2.15}$$

It follows from (2.13)–(2.15) that $|\operatorname{tr}(XB)| \leqslant \sum_{i=1}^{\nu} s_i(Y)$. Since

$$\sum_{i=1}^{\nu} s_i(X) \leqslant \sup\{|\operatorname{tr}(XB)| : \|B\| \leqslant 1, \operatorname{rank}(B) \leqslant \nu\},$$

this completes the proof of (2.11).

Since (2.12) is now proven, the lemma follows if $J_0 = 1$. Suppose that $J_0 > 1$. By the definition of J_0, we have $S < \rho/2^{sj}$ for $1 \leqslant j \leqslant J_0 - 1$. Hence

$$\left\| \sum_{j=1}^{J_0-1} 2^j A_j \right\|_1 \leqslant \sum_{j=1}^{J_0-1} 2^j S \leqslant \rho \sum_{j=1}^{J_0-1} 2^{-(s-1)j} \leqslant \rho 2^{-(s-1)}(1 - 2^{-(s-1)})^{-1}.$$

This completes the proof. ∎

For the proof of Theorem 1.1, Lemmas 2.1 and 2.4 will suffice. But the proof of Theorem 1.2 requires an extra step. For integers $1 \leqslant k < k'$, define

$$M(s,\tau,k,k') = \sup\left\{ \int \left(\frac{\mu(B(x,r))}{r^s}\right)^\tau \mathrm{d}\mu(x) : 2^{-k'+\ell_0} \leqslant r \leqslant 2^{-k-1+\ell_0} \right\}.$$

Proposition 2.5. *If $\tau > 1/(s-1)$, then there is a constant $C(n,s,\tau) > 0$ which depends only on n, s and τ such that:*

$$\left\| \sum_{\ell=k+1}^{k'} \sum_{w \in W_\ell} 2^\ell \mu(Q_w)(f_w e_w) \otimes (f_w e_w) \right\|_{s/(s-1)}^{+} \\ \leqslant C(n,s,\tau)\{(M(s,\tau,k,k'))^{1/(1+\tau)} + (M(s,\tau,k,k'))^{1/s(1+\tau)}(\mu(Q))^{(s-1)/s}\}$$

for $1 \leqslant k < k'$, and $f_w \in L^\infty(\mathbb{R}^n, \mu)$ with $\|f_w\|_\infty \leqslant 1$ and $w \in \bigcup_{\ell=k+1}^{k'} W_\ell$.

Proof. We fix $1 \leqslant k < k'$ and denote $M = 2^{\tau s \ell_0} M(s, \tau, k, k')$. For $k+1 \leqslant \ell \leqslant k'$, we set

$$W_{\ell,0} = \{w \in W_\ell : \mu(Q_w) < 2^{1-s\ell} M^{1/(\tau+1)}\},$$
$$W_{\ell,m} = \{w \in W_\ell : 2^{m-s\ell} M^{1/(\tau+1)} \leqslant \mu(Q_w) < 2^{m+1-s\ell} M^{1/(\tau+1)}\}, \quad m \in \mathbb{N}.$$

For $m = 0, 1, 2, \ldots$ and $k+1 \leqslant \ell \leqslant k'$, define $A_{\ell,m} = \sum_{w \in W_{\ell,m}} \mu(Q_w)(f_w e_w) \otimes (f_w e_w)$. (Naturally, $\sum_{w \in \emptyset} \cdots$ means 0.) Let

$$B_m = \sum_{\ell=k+1}^{k'} 2^\ell A_{\ell,m}, \quad m = 0, 1, 2, \ldots. \tag{2.16}$$

Thus $\Big\| \sum_{m \geqslant 0} B_m \Big\|^+_{s/(s-1)}$ is the quantity to be estimated.

If w and w' are distinct elements in W_ν, then the supports of $f_w e_w$ and $f_{w'} e_{w'}$ are disjoint. Hence $\|A_{\nu,m}\| \leqslant M^{1/(1+\tau)} 2^{m+1-s\nu}$. Thus there is a $C_1 > 0$ which depends only on s such that for any $k+1 \leqslant \ell \leqslant k'$,

$$\sum_{\nu=\ell}^{k'} \|2^\nu A_{\nu,m}\| \leqslant C_1 M^{1/(1+\tau)} 2^{m-(s-1)\ell}. \tag{2.17}$$

Suppose that $m \geqslant 1$. Let $k_1 = k_1(m)$ be the smallest integer such that $2^{sk_1-(1+\tau)m} \geqslant 1$ and let $k_0 = \max\{k+1, k_1\}$. It follows from the definition of $W_{\ell,m}$ and (2.6) that, for $k+1 \leqslant \ell \leqslant k'$,

$$\begin{aligned} \operatorname{card}(W_{\ell,m}) \times (2^{m-s\ell})^{1+\tau} M &\leqslant \sum_{w \in W_{\ell,m}} (\mu(Q_w))^{1+\tau} \\ &\leqslant 2^{\tau s \ell_0} M(s, \tau, k, k') 2^{-\tau s \ell} = M 2^{-\tau s \ell}. \end{aligned}$$

That is, $\operatorname{card}(W_{\ell,m}) \leqslant 2^{s\ell-(1+\tau)m}$. Set

$$n_{\ell,m} = [2^{s\ell-(1+\tau)m}] \text{ for } \ell \geqslant k_0 \quad \text{and} \quad n_{\ell,m} = 0 \text{ for } \ell < k_0.$$

Then

$$\operatorname{rank}(A_{\ell,m}) \leqslant \operatorname{card}(W_{\ell,m}) \leqslant n_{\ell,m}, \quad k+1 \leqslant \ell \leqslant k'. \tag{2.18}$$

Thus we have $B_m = 0$ in the case $k_0 > k'$. Let us assume $k_0 \leqslant k'$. From Lemma 2.3, (2.17) and (2.18) we obtain

$$\|B_m\|^+_{s/(s-1)} \leqslant C_1 M^{1/(1+\tau)} \|\widetilde{B}_m\|^+_{s/(s-1)}, \tag{2.19}$$

where $\widetilde{B}_m = \bigoplus_{\ell=k_0}^{k'} 2^{m-(s-1)\ell} P_{n_{\ell,m}}$.

Now the s-numbers of $\widetilde{B}_m$ are such that $s_i(\widetilde{B}_m) = 2^{m-(s-1)\ell}$ if $n_{k_0-1,m} + \cdots + n_{\ell-1,m} < i \leqslant n_{k_0-1,m} + \cdots + n_{\ell-1,m} + n_{\ell,m}$, where $k_0 \leqslant \ell \leqslant k'$.

This gives us sufficient information to estimate $\|\widetilde{B}_m\|^+_{s/(s-1)}$.

Suppose that $j = j_0 + n_{k_0-1,m} + \cdots + n_{\nu-1,m}$ with $1 \leqslant j_0 \leqslant n_{\nu,m}$, $k_0 \leqslant \nu \leqslant k'$. Then

$$\begin{aligned}\frac{\sum_{i=1}^{j} s_i(\widetilde{B}_m)}{\sum_{i=1}^{j} i^{-(s-1)/s}} &\leqslant C j^{-1/s}\Big(j_0 2^{m-(s-1)\nu} + \sum_{\ell=k_0-1}^{\nu-1} n_{\ell,m} 2^{m-(s-1)\ell}\Big) \\ &\leqslant C\Big(n_{\nu,m}^{(s-1)/s} 2^{m-(s-1)\nu} + \sum_{\ell=k_0-1}^{\nu-1}\Big(\frac{n_{\ell,m}}{j}\Big)^{1/s}\big(n_{\ell,m}^{(s-1)/s} 2^{m-(s-1)\ell}\big)\Big).\end{aligned} \tag{2.20}$$

Note that $n_{\ell,m}^{(s-1)/s} 2^{m-(s-1)\ell} \leqslant 2^{m-(s-1)(1+\tau)m/s}$ for $k_0 - 1 \leqslant \ell \leqslant k'$. Set $\delta = 2^{1-(s-1)(1+\tau)/s}$. Now we invoke the condition $\tau > 1/(s-1)$: It simply means $1 - (s-1)(1+\tau)/s < 0$. That is, $\delta < 1$. It follows from (2.20) that

$$\frac{\sum_{i=1}^{j} s_i(\widetilde{B}_m)}{\sum_{i=1}^{j} i^{-(s-1)/s}} \leqslant C\Big(1 + \sum_{\ell=k_0-1}^{\nu-1}\Big(\frac{n_{\ell,m}}{j}\Big)^{1/s}\Big)\delta^m.$$

Now $\sum_{\ell=k_0-1}^{\nu-1} (n_{\ell,m}/j)^{1/s} = 0$ if $\nu = k_0$. If $\nu > k_0$, we have $j \geqslant 1 + n_{\nu-1,m} \geqslant 2^{s(\nu-1)-(1+\tau)m}$. That is, $n_{\ell,m}/j \leqslant 2^{s(\ell-\nu+1)}$ for $k_0 - 1 \leqslant \ell \leqslant \nu - 1$. Thus the $\Big(1 + \sum_{\ell=k_0-1}^{\nu-1} (n_{\ell,m}/j)^{1/s}\Big)$ above is not greater than 3 in any case. Thus $\sum_{i=1}^{j} s_i(\widetilde{B}_m)/\sum_{i=1}^{j} i^{-(s-1)/s} \leqslant 3C\delta^m$ for all such j. This implies that $\|\widetilde{B}_m\|^{+}_{s/(s-1)} \leqslant 3C\delta^m$. Combining this with (2.19), we obtain

$$\|B_m\|^{+}_{s/(s-1)} \leqslant 3CC_1 M^{1/(1+\tau)}\delta^m. \tag{2.21}$$

This holds for every $m \geqslant 1$ and the constants C and C_1 depend only on s.

To estimate $\|B_0\|^{+}_{s/(s-1)}$, we note that $\|A_{\ell,0}\| \leqslant 2^{1-s\ell} M^{1/(\tau+1)}$ and $\|A_{\ell,0}\|_1 \leqslant \mu(Q)$. Thus, applying Lemma 2.4 with $\rho = 2M^{1/(\tau+1)}$ and $S = \mu(Q)$, we have

$$\|B_0\|^{+}_{s/(s-1)} \leqslant 2C(s)(M^{1/s(1+\tau)}(\mu(Q))^{(s-1)/s} + M^{1/(1+\tau)}).$$

Combining this with (2.21), we obtain the desired bound for $\Big\|\sum_{m\geqslant 0} B_m\Big\|^{+}_{s/(s-1)}$. ∎

In this paper, our definition of the rank-one operator $\xi\otimes\zeta$ is $(\xi\otimes\zeta)f = \langle f, \zeta\rangle\xi$.

Corollary 2.6. *Let s, τ and $C(n, s, \tau)$ be the same as in Proposition 2.5. For any $1 \leqslant k < k'$, $z \in \mathbb{Z}^n$ and $\lambda \in \Lambda$, we have*

$$\begin{aligned}\|A_{k,k',z,\lambda}\|^{+}_{s/(s-1)} \leqslant C(n,s,\tau)\big((M(s,\tau,k,k'))^{1/(1+\tau)} \\ + (M(s,\tau,k,k'))^{1/s(1+\tau)}(\mu(Q))^{(s-1)/s}\big).\end{aligned} \tag{2.22}$$

Proof. Recall that $A_{k,k',z,\lambda}$ is defined by (2.5). Let $\{\xi_w : w \in \mathcal{W}\}$ be an orthonormal set in $L^2(\mathbb{R},\mu)$. For a fixed set of $1 \leqslant k < k'$, $z \in \mathbb{Z}^n$ and $\lambda \in \Lambda$, we define

$$A = \sum_{\ell=k+1}^{k'} \sum_{w \in W_\ell} \sqrt{2^\ell \mu(Q_w)}\xi_w \otimes (f_w^z e_w),$$

$$B = \sum_{\ell=k+1}^{k'} \sum_{w \in W_\ell} \sqrt{2^\ell \mu(w,\lambda)}\xi_w \otimes (f_w^z e(w,\lambda)).$$

For $w' \in W_\ell$, set $f_{w',\lambda}^z = f_w^z$ if there is a $w \in W_\ell$ (which is necessarily unique) such that $Q_{w'} = Q_w + 2^{-\ell}\lambda$ and set $f_{w',\lambda}^z = 0$ if no such w exists. Hence

$$B^*B = \sum_{\ell=k+1}^{k'} \sum_{w' \in W_\ell} 2^\ell \mu(Q_{w'})(f_{w',\lambda}^z e_{w'}) \otimes (f_{w',\lambda}^z e_{w'}),$$

$$A^*A = \sum_{\ell=k+1}^{k'} \sum_{w \in W_\ell} 2^\ell \mu(Q_w)(f_w^z e_w) \otimes (f_w^z e_w).$$

Thus, if we write R for the right-hand side of (2.22), then it follows from Proposition 2.5 that $\|A^*A\|_{s/(s-1)}^+ \leqslant R$ and that $\|B^*B\|_{s/(s-1)}^+ \leqslant R$. Now for any $X \in \mathcal{C}_s^-$, we can write $X = Y^*Z$ with $\|Y^*Y\|_s^- = \|X\|_s^- = \|Z^*Z\|_s^-$. Since $A_{k,k',z,\lambda} = B^*A$, we have

$$\begin{aligned}|\mathrm{tr}(A_{k,k',z,\lambda}X)| &= |\mathrm{tr}(ZB^*AY^*)| \leqslant \{\mathrm{tr}(ZB^*BZ^*)\mathrm{tr}(YA^*AY^*)\}^{1/2} \\ &\leqslant \{\|Z^*Z\|_s^- \|B^*B\|_{s/(s-1)}^+ \|Y^*Y\|_s^- \|A^*A\|_{s/(s-1)}^+\}^{1/2},\end{aligned}$$

where the second inequality uses the duality between $\mathcal{C}_s^-$ and $\mathcal{C}_{s/(s-1)}^+$ ([5]). Thus $|\mathrm{tr}(A_{k,k',z,\lambda}X)| \leqslant \|X\|_s^- R$. Another application of the duality completes the proof. ∎

Proof of Theorem 1.2. As we mentioned before, we may assume $\mu(\mathbb{R}^n \backslash Q) = 0$ without loss of generality. Thus we may apply all the propositions above. By (1.2), there is a positive number B such that

$$M(p,t,k,k') \leqslant B \quad \text{for all } 1 \leqslant k < k'. \tag{2.23}$$

Since $t > 1/(p-1)$, there is an $s \in (1,p)$ such that $t > 1/(s-1)$. We have

$$M(s,t,k,k') \leqslant 2^{t(p-s)(-k-1+\ell_0)} M(p,t,k,k') \leqslant 2^{t(p-s)(-k-1+\ell_0)} B.$$

That is, $\lim_{k\to\infty} \sup_{k'>k} M(s,t,k,k') = 0$. Since $T_{k,k'} = T_{k'} - T_k$, it follows from Lemma 2.1, Corollary 2.6 and this limit that $\lim_{k\to\infty} \sup_{k'>k} \|T_{k'} - T_k\|_{s/(s-1)}^+ = 0$. Hence $\{T_k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathcal{C}_{s/(s-1)}^+$.

It is easy to deduce from (1.2) that

$$\lim_{k\to\infty} 2^k \int \mu(B(x,2^{-k}))\,\mathrm{d}\mu(x) = 0.$$

Using this limit, the definition of η and the homogeneity of K, it is easy to show that

$$\lim_{\varepsilon\downarrow 0}(\langle T_{K,\mu,\varepsilon}f,g\rangle - \langle T_{k(\varepsilon)}f,g\rangle) = 0 \quad \text{for any } f,g \in L^{\infty}(\mathbb{R}^n,\mu),$$

where $k(\varepsilon) \in \mathbf{N}$ is such that $2^{-k(\varepsilon)-2} < \varepsilon \leqslant 2^{-k(\varepsilon)-1}$ and

$$(T_{K,\mu,\varepsilon}f)(x) = \int\limits_{|x-y|\geqslant\varepsilon} K(x-y)f(y)\,\mathrm{d}\mu(y).$$

Since $\{T_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{C}^{+}_{s/(s-1)}$, this shows that the singular integral operator $T_{K,\mu}$ is well defined on $L^2(\mathbb{R}^n,\mu)$. Indeed it is the $\|\cdot\|^{+}_{s/(s-1)}$-limit of $\{T_k\}_{k=1}^{\infty}$.

Thus to prove that $T_{K,\mu} \in \mathcal{C}^{+}_{p/(p-1)}$, we only need to show that the numerical sequence $\{\|T_k\|^{+}_{p/(p-1)}\}_{k=1}^{\infty}$ is bounded. By Lemma 2.1, Corollary 2.6 and (2.23), we have

$$\|T_{k,k'}\|^{+}_{p/(p-1)} \leqslant C_{2.1}(n,K)C(n,p,t)\{B^{1/(1+t)} + B^{1/p(1+t)}(\mu(Q))^{(p-1)/p}\} \tag{2.24}$$

for all $1 \leqslant k < k'$. Since $T_k = T_{1,k} + T_1$ and $T_1 \in \mathcal{C}_1$, (2.24) implies that $\{\|T_k\|^{+}_{p/(p-1)}\}_{k=1}^{\infty}$ is bounded. ∎

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