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NONCOMMUTATIVE EXTENSIONS OF CLASSICAL AND MULTIPLE RECURRENCE THEOREMS

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Dedicated to the memory of our colleague Ioana Ciorănescu

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Abstract. The aim of this paper is to extend the classical recurrence theorem of A.Y. Khintchine, as well as certain multiple recurrence results of H. Furstenberg concerning weakly mixing and almost periodic measure preserving transformations, to the framework of C^*-algebras $\mathfrak{A}$ and positive linear maps $\Phi : \mathfrak{A} \to \mathfrak{A}$ preserving a state φ on $\mathfrak{A}$. For the proof of the multiple weak mixing results we use a slight extension of a convergence result of Furstenberg in Hilbert spaces, which is derived from a non-commutative generalization of Van der Corput's "Fundamental Inequality" in Theory of uniform distribution modulo 1, proved in Appendix A.

Keywords: *Poincaré recurrence, C^*-dynamical system, almost periodicity, weak mixing, multiple weak mixing.*

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1. INTRODUCTION

Recurrence was introduced by Poincaré ([30], Chapter XXVI) in connection with its study on Celestial Mechanics and refers to the property of an orbit to come arbitrarily close to positions already occupied. Poincaré noticed that almost all orbits of any bounded Hamiltonian system are recurrent. His result, stated in a measure-theoretical form, is known as Poincaré's recurrence theorem:

Let T be a measure-preserving transformation of a probability space (Ω, Σ, μ). Then

$$\mu\Big(A \cap \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} T^{-k} A\Big) = \mu(A)$$

for every set $A \in \Sigma$ i.e., the orbit of almost every $x \in A$ returns infinitely often in A.

The literature concerning this theorem is very large but all the basic facts can be found in the monographs of U. Krengel ([23], Section 1.3) and K. Petersen ([29], Section 2.3).

Let us recall that (for T a measurable transformation of a measure space (Ω, Σ, μ) such that inverse images of sets of zero measure have zero measure) the statement of Poincaré's recurrence theorem is equivalent with the nonexistence of wandering subsets of non-zero measure:

(NW) $\qquad A \in \Sigma,\ \mu(A) > 0$ *implies* $\mu(A \cap T^{-n}A) > 0$ *for some* $n \geqslant 1$.

See [23], Section 1.3, Theorem 3.1 for details.

Khintchine's recurrence theorem ([21]; see also [29], Chapter 2, Theorem 3.3) provides a quantitative version of the validity of (NW), hence of Poincaré's recurrence theorem:

If T is a measure-preserving transformation of a probability space (Ω, Σ, μ) *then, for every $A \in \Sigma$ and every $\varepsilon > 0$, there exists a relatively dense subset $\mathcal{N}$ of* $\mathbb{N} = \{0, 1, 2, \ldots\}$ *such that*

$$\mu(A \cap T^{-n}A) \geqslant \mu(A)^2 - \varepsilon \quad \textit{for all } n \in \mathcal{N}.$$

Recall that a subset $\mathcal{N}$ of $\mathbb{N}$ is called *relatively dense* provided that there exists an $L > 0$ such that in every interval of natural numbers having length $\geqslant L$ one can find a number $n \in \mathcal{N}$. If T is a mixing mapping then $\mu(A \cap T^{-n}A) \to \mu(A)^2$, so that the above estimate is the best possible.

Much later, in order to produce an ergodic-theoretical proof for a famous combinatorial theorem of E. Szemerédi ([36]), H. Furstenberg gave another kind of extension of Poincaré's recurrence theorem by proving the following multiple recurrence theorem:

For T any measure-preserving transformation of a probability space (Ω, Σ, μ),

$$\begin{aligned}
&A \in \Sigma,\ \mu(A) > 0, j \geqslant 1 \\
&\quad \Rightarrow \liminf_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \mu(A \cap T^{-k}A \cap T^{-2k}A \cap \cdots \cap T^{-jk}A) > 0 \\
&\quad \Rightarrow \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-jn}A) > 0 \textit{ for some } n \geqslant 1.
\end{aligned}$$

See [14], [16] and [15] for details.

Let us reformulate the above recurrence theorems in the framework of operator algebras. For the theory of von Neumann algebras we refer the reader to [34], and for the general theory of C^*-algebras to [35].

If T is a measure-preserving transformation of a probability space (Ω, Σ, μ) then the formula $\Phi(f) = f \circ T$ defines a $*$-endomorphism Φ of the commutative von Neumann algebra $L^\infty(\mu)$, which leaves invariant the faithful normal state $\varphi : L^\infty(\mu) \ni f \mapsto \int f \, \mathrm{d}\mu$.

Then Poincaré's recurrence theorem can be stated as

$$p \wedge \bigwedge_{n=0}^{\infty} \bigvee_{k=n}^{\infty} \Phi^k(p) = p \qquad \textit{for every projection } p \in L^\infty(\mu),$$

where $\bigwedge$ and $\bigvee$ denote the lattice operations on the projection lattice of the von Neumann algebra $L^\infty(\mu)$.

Similarly, Khintchine's recurrence theorem can be reformulated in this setting as follows:

For every projection $p \in L^\infty(\mu)$ and every $\varepsilon > 0$, there exists a relatively dense subset $\mathcal{N}$ of $\mathbb{N}$ such that

$$\varphi(p\Phi^n(p)) \geqslant \varphi(p)^2 - \varepsilon \quad \textit{for all } n \in \mathcal{N}.$$

Finally, Furstenberg's multiple recurrence theorem states that

$$\begin{aligned} & p \in L^\infty(\mu) \textit{ projection, } j \geqslant 1 \\ & \quad \Rightarrow \liminf_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left|\varphi(p\Phi^k(p)\Phi^{2k}(p)\cdots\Phi^{jk}(p))\right| > 0. \end{aligned}$$

It is easy to get an extension of Poincaré's recurrence theorem to the setting of arbitrary von Neumann algebras:

THEOREM 1.1. *Let $\mathcal{M}$ be a von Neumann algebra, φ a faithful normal state on $\mathcal{M}$, and $\Phi : \mathcal{M} \to \mathcal{M}$ a $*$-homomorphism such that $\varphi \circ \Phi = \varphi$. Then*

$$\bigvee_{k=n}^{\infty} \Phi^k(p) = \bigvee_{k=0}^{\infty} \Phi^k(p) \geqslant p \quad \textit{for every projection } p \in \mathcal{M} \textit{ and every } n \geqslant 0,$$

so

$$p \wedge \bigwedge_{n=0}^{\infty} \bigvee_{k=n}^{\infty} \Phi^k(p) = p \quad \textit{for every projection } p \in \mathcal{M}.$$

Proof. First we show that Φ is unital and normal. Indeed, since the image $\Phi(1_\mathcal{M})$ of the unit $1_\mathcal{M}$ of $\mathcal{M}$ is a projection, we have $1_\mathcal{M} - \Phi(1_\mathcal{M}) \geqslant 0$. Since $\varphi(1_\mathcal{M} - \Phi(1_\mathcal{M})) = 0$, by the faithfulness of φ we infer that $1_\mathcal{M} - \Phi(1_\mathcal{M}) = 0$. Further, if $0 \leqslant a_\iota \nearrow a$ in $\mathcal{M}$ then $(\Phi(a_\iota))_\iota$ is upward directed and bounded above by $\Phi(a)$, so there exists $b \leqslant \Phi(a)$ in $\mathcal{M}$ such that $\Phi(a_\iota) \nearrow b$ (see [34], Proposition 2.16). But then

$$\varphi(b) = \lim_\iota \varphi(\Phi(a_\iota)) = \lim_\iota \varphi(a_\iota) = \varphi(a) = \varphi(\Phi(a))$$

and the faithfulness of φ yields $b = \Phi(a)$.

Next we prove that, for any $n \geqslant 0$,

$$(*) \qquad \Phi\Big(\bigvee_{k=n}^{\infty} \Phi^k(p)\Big) = \bigvee_{k=n+1}^{\infty} \Phi^k(p).$$

For we notice that Φ is w-continuous (see [34], E.5.17), so $\Phi(\mathcal{M})$ is a von Neumann subalgebra of $\mathcal{M}$ (see [34], Corollary 3.12) and $\mathrm{Ker}(\Phi)$ is a w-closed two-sided ideal of $\mathcal{M}$, hence there exists a central projection q in $\mathcal{M}$ such that $\mathrm{Ker}(\Phi) = \mathcal{M}(1_\mathcal{M} - q)$ (see [34], 3.20). Then the restriction of Φ to q is a $*$-isomorphism onto $\Phi(\mathcal{M})$, so

$$\begin{aligned} \Phi\Big(\bigvee_{k=n}^{\infty} \Phi^k(p)\Big) &= \Phi\Big(\Big(\bigvee_{k=n}^{\infty} \Phi^k(p)\Big)q\Big) = \Phi\Big(\bigvee_{k=n}^{\infty} (\Phi^k(p)q)\Big) \\ &= \bigvee_{k=n}^{\infty} \Phi(\Phi^k(p)q) = \bigvee_{k=n+1}^{\infty} \Phi^k(p). \end{aligned}$$

According to (*) and $\varphi \circ \Phi = \varphi$, the sequence $\varphi\Big(\bigvee_{k=n}^{\infty} \Phi^k(p)\Big)$, $n \geqslant 0$, is constant and by the faithfulness of φ it follows that also the sequence $\bigvee_{k=n}^{\infty} \Phi^k(p)$, $n \geqslant 0$, is constant. ∎

However, in the case of non-commutative $\mathcal{M}$ the strength of the statement of Theorem 1.1 can be quite reduced, because the more non-commutative $\mathcal{M}$ is, the less the lattice operations in the projection lattice of $\mathcal{M}$ reflect the nearness of the projections. For example, it does not matter how close two non-equal, one-dimensional projections p, q are in the algebra $M_2(\mathbb{C})$ of all complex 2×2-matrices, $p \wedge q$ is always zero, while $p \vee q$ is equal to the unit element of $M_2(\mathbb{C})$.

In contrast with Poincaré's recurrence theorem, Khintchine's recurrence theorem has a right substantial extension to the setting of non-commutative operator algebras. Let us recall that, for $\mathfrak{A}, \mathfrak{B}$ two C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ a linear map, if Φ is positive then the following Schwarz type inequalities of Kadison hold:

$$\Phi(a)^*\Phi(a) \leqslant \|\Phi\|\Phi(a^*a) \quad \text{for every normal } a \in \mathfrak{A}$$

(see e.g. [35], 5.8). If Φ is positive and $x \in \mathfrak{A}$ is arbitrary then, summing up Kadison's inequality for $a = x + x^*$ and for $a = \mathrm{i}(x - x^*)$, we have the following inequality of Störmer:

$$\Phi(x)^*\Phi(x) + \Phi(x)\Phi(x)^* \leqslant \|\Phi\|\Phi(x^*x + xx^*)$$

(see [33]). In particular, if Φ is positive and τ is a trace on $\mathfrak{B}$ such that also $\tau \circ \Phi$ is a trace then

$$\tau(\Phi(x)^*\Phi(x)) \leqslant \|\Phi\|(\tau \circ \Phi)(x^*x) \quad \text{for every } x \in \mathfrak{A}.$$

We also recall that Φ is called *Schwarz map* if

$$\Phi(x)^*\Phi(x) \leqslant \Phi(x^*x) \quad \text{for every } x \in \mathfrak{A},$$

in which case Φ is positive and of norm $\leqslant 1$ (see e.g. [35], 5.10). If Φ is a $*$-homomorphism or, more generally, Φ is 2-positive and of norm $\leqslant 1$, then it is a Schwarz map.

The following non-commutative generalization of Khintchine's recurrence theorem, already proven in [28] if $\mathfrak{A}$ is unital and Φ is a unital $*$-homomorphism, will be proved in the next section:

THEOREM 1.2. *Let $\mathfrak{A}$ be a C^*-algebra, φ a state on $\mathfrak{A}$ and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\varphi \circ \Phi = \varphi$. Let us assume that*

$$\text{(C)} \qquad \varphi(\Phi(x)^*\Phi(x)) \leqslant \varphi(x^*x) \quad \textit{for every } x \in \mathfrak{A},$$

which happens whenever Φ is a Schwarz map or φ is tracial and $\|\Phi\| \leqslant 1$. Then, for every $x \in \mathfrak{A}$ and $\varepsilon > 0$, there exists a relatively dense subset $\mathcal{N}$ of $\mathbb{N}$ such that

$$\operatorname{Re}\varphi(x^*\Phi^n(x)) \geqslant |\varphi(x)|^2 - \varepsilon \quad \textit{for all } n \in \mathcal{N}.$$

It is easily seen that, for $\mathfrak{A}, \varphi$ and Φ as in the above theorem, we have also the upper estimate

$$\operatorname{Re}\varphi(x^*\Phi^n(x)) \leqslant \varphi(x^*x) - \frac{1}{2}\varphi((\Phi^n(x) - x)^*(\Phi^n(x) - x)), \quad x \in \mathfrak{A},\ n \in \mathbb{N}.$$

On the other hand, the above theorem and Lemma 9.4 (see Appendix B below) imply that

$$\liminf_{n\to\infty} \frac{1}{n+1} \operatorname{Re} \sum_{k=0}^{n} \varphi(x^* \Phi^n(x)) > 0 \quad \text{whenever } x \in \mathfrak{A},\ \varphi(x) \neq 0.$$

Partial extensions of Furstenberg's multiple recurrence theorem to the non-commutative setting will be proved in two particular cases: for weakly mixing C^*-dynamical systems and for almost periodic C^*-dynamical systems.

Let $\mathfrak{A}$ be a C^*-algebra, φ a state on $\mathfrak{A}$ and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\varphi \circ \Phi = \varphi$ and $\varphi(\Phi(x)^*\Phi(x)) \leqslant \varphi(x^*x)$ for every $x \in \mathfrak{A}$. We say that:

— Φ is *ergodic* with respect to φ if

$$\text{(E)} \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \big(\varphi(y\Phi^k(x)) - \varphi(y)\varphi(x)\big) = 0 \quad \text{for all } x, y \in \mathfrak{A};$$

— Φ is *weakly mixing* with respect to φ if

$$\text{(WM)} \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \big|\varphi(y\Phi^k(x)) - \varphi(y)\varphi(x)\big| = 0 \quad \text{for all } x, y \in \mathfrak{A};$$

— Φ is *almost periodic* with respect to φ if, denoting by $\pi_\varphi : \mathfrak{A} \to \mathcal{L}(H_\varphi)$ the GNS representation associated to φ and by ξ_φ its canonical cyclic vector,

$$\text{(AP)} \qquad \{\pi_\varphi(\Phi^n(x))\xi_\varphi : n \in \mathbb{N}\} \text{ is relatively compact in } H_\varphi \text{ for all } x \in \mathfrak{A}.$$

We shall call a pair $(\mathfrak{A}, \Phi)$ consisting of a C^*-algebra $\mathfrak{A}$ and a $*$-homomorphism $\Phi : \mathfrak{A} \to \mathfrak{A}$, a *$C^*$-dynamical system.*

Furthermore, we shall call any triplet $(\mathfrak{A}, \varphi, \Phi)$ consisting of a C^*-algebra $\mathfrak{A}$, a state φ on $\mathfrak{A}$ and a $*$-homomorphism $\Phi : \mathfrak{A} \to \mathfrak{A}$ with $\varphi \circ \Phi = \varphi$, that is a C^*-dynamical system leaving invariant a state, a *state preserving C^*-dynamical system.* In other words, a state preserving C^*-dynamical system is a non-commutative C^*-probability space $(\mathfrak{A}, \varphi)$ (see [39], [7]) together with a $*$-endomorphism Φ of $\mathfrak{A}$ preserving the non-commutative probability φ. We say that the state preserving C^*-dynamical system $(\mathfrak{A}, \varphi, \Phi)$ is *ergodic* (respectively *weakly mixing*, *almost periodic*) if Φ is ergodic (respectively weakly mixing, almost periodic) with respect to φ. Furthermore, for any integers $p \geqslant 1$ and $m_1, \ldots, m_p \geqslant 1$, $m_j \neq m_{j'}$ for $j \neq j'$, we say that:

($\mathrm{WM}_{m_1,\ldots,m_p}$) $(\mathfrak{A}, \varphi, \Phi)$ is *weakly mixing of order* $(m_1, \ldots, m_p)$ whenever

$$\lim_{n\to\infty} \frac{1}{n+1} \cdot \sum_{k=0}^{n} \big|\varphi\big(x_0 \Phi^{m_1\cdot k}(x_1) \cdots \Phi^{m_p\cdot k}(x_p)\big) - \varphi(x_0)\varphi(x_1)\cdots\varphi(x_p)\big| = 0$$

for all $x_0, x_1, \ldots, x_p \in \mathfrak{A}$.

($\mathrm{UWM}_{m_1,\ldots,m_p}$) $(\mathfrak{A}, \varphi, \Phi)$ is *uniformly weakly mixing of order* $(m_1, \ldots, m_p)$ whenever

$$\lim_{n\to\infty} \sup_{\substack{x\in\mathfrak{A} \\ \varphi(xx^*)\leqslant 1}} \frac{1}{n+1} \cdot \sum_{k=0}^{n} \big|\varphi\big(x \Phi^{m_1\cdot k}(x_1) \cdots \Phi^{m_p\cdot k}(x_p)\big) - \varphi(x)\varphi(x_1)\cdots\varphi(x_p)\big| = 0$$

for all $x_1, \ldots, x_p \in \mathfrak{A}$.

Denoting by $\pi_\varphi : \mathfrak{A} \to \mathcal{L}(H_\varphi)$ the GNS representation associated to φ and by ξ_φ its canonical cyclic vector, $(\mathrm{WM}_{m_1,\ldots,m_p})$ means that the bounded sequence

$$\xi_n = \pi_\varphi\big(\Phi^{m_1 \cdot n}(x_1) \cdots \Phi^{m_p \cdot n}(x_p)\big)\xi_\varphi - \varphi(x_1) \cdots \varphi(x_p)\xi_\varphi \in H_\varphi, \quad n \geqslant 1$$

is weakly mixing to zero as defined in Appendix B (before Theorem 9.5), while $(\mathrm{WM}_{m_1,\ldots,m_p})$ means that the above sequence is uniformly mixing to zero. Weak mixing (uniformly weak mixing) of order $(1, \ldots, p)$ will be called simply *weak mixing (uniformly weak mixing)* of order p.

H. Furstenberg has proved that if $(\mathfrak{A}, \varphi, \Phi)$ is a weakly mixing state preserving C^*-dynamical system with commutative $\mathfrak{A}$ then it is weakly mixing of all orders (see [14], Theorem 4.11). The following example shows that there are weakly mixing C^*-dynamical systems which are not weakly mixing of order 2.

Let H be a Hilbert space with orthonormal basis $\{\xi_0\} \cup \{\eta_j : j \in \mathbb{Z}\}$ and let U denote the unitary operator on H defined by

$$U\xi_0 = \xi_0, \quad U\eta_j = \eta_{j+1}, \quad j \in \mathbb{Z}.$$

Then $\mathrm{Ad}(U) : x \mapsto UxU^*$ is a $*$-automorphism of the C^*-algebra $\mathcal{L}(H)$ of all bounded linear operators on H, leaving invariant the state $\omega_{\xi_0} : x \mapsto (x\xi_0 \mid \xi_0)$. For any linear combinations

$$\xi = \lambda_0\xi_0 + \sum_{j=-k}^{k} \lambda_j \eta_j, \quad \eta = \mu_0\xi_0 + \sum_{j=-k}^{k} \mu_j \eta_j$$

we have

$$(U^n\xi \mid \eta) = \lambda_0\overline{\mu_0} = (\xi \mid \xi_0) \cdot (\xi_0 \mid \eta) = 0 \quad \text{for } n > k$$

and it follows that

$$(U^n\xi \mid \eta) \to (\xi \mid \xi_0) \cdot (\xi_0 \mid \eta) \quad \text{for all } \xi, \eta \in H.$$

Consequently, for every $x, y \in \mathcal{L}(H)$,

$$\omega_{\xi_0}(y\mathrm{Ad}(U)^n(x)) = (U^n x\xi_0 \mid y^*\xi_0) \to (x\xi_0 \mid \xi_0) \cdot (\xi_0 \mid y^*\xi_0) = \omega_{\xi_0}(y)\omega_{\xi_0}(x).$$

In particular, the state preserving C^*-dynamical system $(\mathcal{L}(H), \omega_{\xi_0}, \mathrm{Ad}(U))$ is weakly mixing. However it is not weakly mixing of order 2. Indeed, if x_2 denotes the partial isometry which carries ξ_0 in η_0 and vanishes on the orthogonal complement of ξ_0, $x_0 = {x_2}^*$ and x_1 stands for the unitary on H defined by

$$x_1\xi_0 = \xi_0, \quad x_1\eta_j = \eta_{-j}, \quad j \in \mathbb{Z},$$

then we have

$$\begin{aligned}\omega_{\xi_0}(x_0\mathrm{Ad}(U)^k(x_1)\mathrm{Ad}(U)^{2k}(x_2)) &= (U^k x_1 U^k x_2\xi_0 \mid x_2\xi_0) = (U^k x_1 U^k \eta_0 \mid \eta_0) \\ &= (U^k x_1 \eta_k \mid \eta_0) = (U^k \eta_{-k} \mid \eta_0) = (\eta_0 \mid \eta_0) = 1,\end{aligned}$$

and

$$\frac{1}{n+1}\sum_{k=0}^{n} \omega_{\xi_0}\big(x_0\mathrm{Ad}(U)^k(x_1)\mathrm{Ad}(U)^{2k}(x_2)\big) = 1, \quad n \geqslant 0,$$

while $\omega_{\xi_0}(x_0)\omega_{\xi_0}(x_1)\omega_{\xi_0}(x_2) = 0$ (as $\omega_{\xi_0}(x_2) = 0$).

Nevertheless, the implication (WM) $\Rightarrow$ $(\mathrm{WM}_{1,2})$ holds under a mild commutativity assumption concerning the support of the invariant state. More precisely, it will be proved in Section 7:

Theorem 1.3. *Let* $(\mathfrak{A}, \varphi, \Phi)$ *be a weakly mixing state preserving* C^**-dynamical system, such that the support projection of* φ *in the second dual* $\mathfrak{A}^{**}$ *is central. Then* $(\mathfrak{A}, \varphi, \Phi)$ *is weakly mixing of order* (m_1, m_2) *for any integers* $1 \leqslant m_1 < m_2$.

We recall that if φ is a state on a C^*-algebra $\mathfrak{A}$ and the support $s(\varphi)$ of φ in $\mathfrak{A}^{**}$ is central, then there exists a unique one-parameter group $\sigma^\varphi : \mathbb{R} \ni t \mapsto \sigma_t^\varphi$ of $*$-automorphisms of $\mathfrak{A}^{**}$, called the *modular automorphism group* of φ, such that

(1) φ satisfies the KMS-condition with respect to the group σ^φ and
(2) every σ_t^φ acts identically on $\mathfrak{A}^{**}\big(1_{\mathfrak{A}^{**}} - s(\varphi)\big)$;

(see e.g. [34], 10.17 or [4], Theorem 5.3.10). Every $*$-automorphism α of $\mathfrak{A}^{**}$ satisfying $\varphi \circ \alpha = \varphi$ commutes with the modular automorphism group of φ. Also, the support projection in $\mathfrak{A}^{**}$ of any tracial state τ on $\mathfrak{A}$ is central and the modular automorphism group of τ is the identity group, which commutes with any map on $\mathfrak{A}^{**}$. Also the following theorem will be proved in Section 7:

Theorem 1.4. *Let* $(\mathfrak{A}, \varphi, \Phi)$ *be a weakly mixing state preserving* C^**-dynamical system, such that the support projection* $s(\varphi)$ *of* φ *in the second dual* $\mathfrak{A}^{**}$ *is central and* $s(\varphi)\Phi^{**}$ *commutes with the modular automorphism group of* φ*. Then* $(\mathfrak{A}, \varphi, \Phi)$ *is uniformly weakly mixing of order* (m_1, m_2) *for any integers* $1 \leqslant m_1 < m_2$.

We do not know, whether every weakly mixing state preserving C^*-dynamical system $(\mathfrak{A}, \varphi, \Phi)$ with invertible Φ, for which the support projection of φ in $\mathfrak{A}^{**}$ is central, is weakly mixing of order 3. Implication (WM) $\Rightarrow$ ($\mathrm{UWM}_{m_1,\ldots,m_p}$) for any integers $1 \leqslant m_1 < \cdots < m_p$ will be proved only assuming that $(\mathfrak{A}, \Phi)$ is *norm-asymptotically abelian in density*, that is it verifies

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \|[\Phi^k(x), y]\| = 0 \quad \text{for all } x, y \in \mathfrak{A},$$

where $[x, y] = xy - yx$ stands for the commutator:

Theorem 1.5. *Let* $(\mathfrak{A}, \varphi, \Phi)$ *be a weakly mixing state preserving* C^**-dynamical system such that* $(\mathfrak{A}, \Phi)$ *is norm-asymptotically abelian in density. Then* $(\mathfrak{A}, \varphi, \Phi)$ *is uniformly weakly mixing of order* $(m_1, \ldots, m_p)$ *for any integers* $p \geqslant 1$ *and* $1 \leqslant m_1 < \cdots < m_p$.

In Section 4 we shall prove a splitting result for state preserving C^*-dynamical systems $(\mathfrak{A}, \varphi, \Phi)$, with central $s(\varphi)$, similar to the classical splitting theorem of K. Jacobs, K. de Leeuw and I. Glicksberg (Theorem 4.2; see also Proposition 5.5). It will follow that such a C^*-dynamical system is weakly mixing if and only if it has no almost periodic "non-scalar subsystem" (Proposition 5.4).

For almost periodic state preserving C^*-dynamical systems, the following multiple recurrence theorem will follow from Corollary 4.3:

Theorem 1.6. *Let $(\mathfrak{A}, \varphi, \Phi)$ be an almost periodic state preserving C^*-dynamical system, such that the support projection of φ in the second dual $\mathfrak{A}^{**}$ is central. Then, for every integers $m_0, m_1, \ldots, m_p \geqslant 0$, $x_0, x_1, \ldots, x_p \in \mathfrak{A}$ and $\varepsilon > 0$, there exists a relatively dense subset $\mathcal{N}$ of $\mathbb{N}$ such that*

$$\left|\varphi\big(\Phi^{m_0 \cdot n}(x_0)\Phi^{m_1 \cdot n}(x_1)\cdots\Phi^{m_p \cdot n}(x_p)\big) - \varphi(x_0 x_1 \cdots x_p)\right| \leqslant \varepsilon \quad \textit{for all } n \in \mathcal{N}.$$

By Theorems 1.3, 1.5, 1.6 and by Lemma 9.4, if $(\mathfrak{A}, \varphi, \Phi)$ is a state preserving C^*-dynamical system and $1 \leqslant m_1 < \cdots < m_p$ are integers, then

$$\liminf_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left|\varphi\big(a\Phi^{m_1 \cdot k}(a)\cdots\Phi^{m_p \cdot k}(a)\big)\right| > 0 \quad \text{if } 0 \leqslant a \in \mathfrak{A}, \varphi(a) > 0$$

in each one of the following situations:

(1) $s(\varphi)$ is central, Φ is weakly mixing with respect to φ and $p \leqslant 2$,
(2) $(\mathfrak{A}, \Phi)$ is asymptotically abelian and Φ is weakly mixing with respect to φ,
(3) $s(\varphi)$ is central and Φ is almost periodic with respect to φ.

Frequently used notations. (1) If H is a Hilbert space then $\mathcal{L}(H)$ denotes the C^*-algebra of all bounded linear operators on H and, for any $\xi \in H$, ω_ξ stands for the linear functional $\mathcal{L}(H) \ni x \mapsto (x\xi \mid \xi)$. Furthermore, wo stands for the weak operator topology on $\mathcal{L}(H)$, so for the strong operator topology, and w for the weak topology defined on $\mathcal{L}(H)$ by the norm-closed linear span of $\{\omega_\xi : \xi \in H\}$ (see e.g. [34], Chapter 1).

(2) For any $U \in \mathcal{L}(H)$, $\mathrm{Ad}(U)$ stands for the completely positive linear map $\mathcal{L}(H) \ni x \mapsto UxU^*$. If U is isometrical, then $\mathrm{Ad}(U)$ will be a $*$-homomorphism, while if U is unitary, then $\mathrm{Ad}(U)$ is a $*$-automorphism.

(3) If φ is a positive linear functional on a C^*-algebra $\mathfrak{A}$, then $\pi_\varphi : \mathfrak{A} \to \mathcal{L}(H_\varphi)$ stands for the associated GNS-representation, and ξ_φ for the canonical cyclic vector of π_φ satisfying $\varphi = \omega_{\xi_\varphi} \circ \pi_\varphi$. We notice that $\|\xi_\varphi\|^2 = \|\varphi\|$ and, for every bounded left approximate unit $(u_\iota)_\iota$ for $\mathfrak{A}$,

$$\pi_\varphi(u_\iota) \xrightarrow{\text{so}} 1_{H_\varphi}, \text{ hence } \pi_\varphi(u_\iota)\xi_\varphi \to \xi_\varphi, \text{ and } \varphi(u_\iota) \to \|\varphi\|,\ \varphi(u_\iota^* u_\iota) \to \|\varphi\|$$

(see e.g. [35], Lemma 3/4.1 and Theorem 4.5). Moreover, by the von Neumann density theorem (see e.g. [35], Theorem 7.11), the double commutant $\pi_\varphi(\mathfrak{A})''$ of $\pi_\varphi(\mathfrak{A})$ is equal to $\overline{\pi_\varphi(\mathfrak{A})}^{\text{wo}} = \overline{\pi_\varphi(\mathfrak{A})}^{\text{so}} = \overline{\pi_\varphi(\mathfrak{A})}^{\text{w}}$.

We notice also that π_φ can be uniquely extended to a normal $*$-homomorphism of $\mathfrak{A}^{**}$ onto the von Neumann algebra $\pi_\varphi(\mathfrak{A})''$, what we shall still denote by π_φ, and the kernel of this extension is $\big(1_{\mathfrak{A}^{**}} - z(\varphi)\big)\mathfrak{A}^{**}$, where $z(\varphi)$ stands for the central support of φ in $\mathfrak{A}^{**}$ (see e.g. [35], Corolary 8/8.4 and Corollary 8.7). Therefore $\pi_\varphi | z(\varphi)\mathfrak{A}^{**} : z(\varphi)\mathfrak{A}^{**} \to \pi_\varphi(\mathfrak{A})''$ is a $*$-isomorphism and $\pi_\varphi\big(z(\varphi)\big) = 1_{H_\varphi}$, where 1_{H_φ} denotes the identity operator on H_φ.

(4) The notation $U_{\Phi,\varphi}$ is defined in Lemma 2.1, $\Psi_{\Phi,\varphi}$ in Proposition 3.1, $\Psi^{\sim}_{\Phi,\varphi}$ in Proposition 3.3, and $\mathcal{M}^{\text{AP}}_{\Phi,\varphi}, E^{\text{AP}}_{\Phi,\varphi}$ in Theorem 4.2.

(5) If $\mathcal{S}$ stands for a unital multiplicative semigroup and $(a_k)_{k\in F}$ is a family in $\mathcal{S}$ with finite $F \subset \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, then we denote

$$\prod_{k\in F} a_k = \begin{cases} a_{k_1} \cdots a_{k_n} & \text{if } F = \{k_1, \ldots, k_n\},\ k_1 < \cdots < k_n, \\ 1_{\mathcal{A}} & \text{if } F = \emptyset\ . \end{cases}$$

2. THE NON-COMMUTATIVE KHINTCHINE RECURRENCE THEOREM

The goal of this section is to give a proof for Theorem 1.2, a non-commutative extension of the classical Khintchine recurrence theorem.

The main idea of the proof is to take advantage of the hilbertian structure associated to a positive linear form φ on a C^*-algebra $\mathfrak{A}$, that is of the existence of the GNS-representation $\pi_\varphi : \mathfrak{A} \to \mathcal{L}(H_\varphi)$ and the canonical cyclic vector ξ_φ. In order to translate the content of Theorem 1.2 in terms of H_φ, we have to associate to Φ a certain linear operator $U_{\Phi,\varphi}$ on H_φ:

LEMMA 2.1. *Let $\mathfrak{A}$ be a C^*-algebra, φ a state on $\mathfrak{A}$, and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\varphi \circ \Phi = \varphi$.*

(i) *If $\varphi\big(\Phi(x)^*\Phi(x)\big) \leqslant \varphi(x^*x)$ for every $x \in \mathfrak{A}$, then there is a unique linear contraction $U_{\Phi,\varphi}$ on H_φ such that*

$$U_{\Phi,\varphi}\big(\pi_\varphi(x)\xi_\varphi\big) = \pi_\varphi\big(\Phi(x)\big)\xi_\varphi \quad \textit{for all } x \in \mathfrak{A}. \tag{2.1}$$

Moreover, $U_{\Phi,\varphi}\xi_\varphi = \xi_\varphi$ and, denoting by $P^{U_{\Phi,\varphi}}$ the orthogonal projection onto

$$\{\xi \in H_\varphi : U_{\Phi,\varphi}\xi = \xi\} = \{\xi \in H_\varphi : U^*_{\Phi,\varphi}\xi = \xi\},$$

we have

$$U_{\Phi,\varphi}P^{U_{\Phi,\varphi}} = P^{U_{\Phi,\varphi}}U_{\Phi,\varphi} = P^{U_{\Phi,\varphi}}, \qquad \frac{1}{n+1}\sum_{k=0}^{n} U^k_{\Phi,\varphi} \xrightarrow{\text{so}} P^{U_{\Phi,\varphi}}.$$

(ii) *If Φ is multiplicative, hence a $*$-homomorphism, then $U_{\Phi,\varphi}$ defined in* (i) *is isometrical, $U_{\Phi,\varphi}U^*_{\Phi,\varphi}$ is equal to the orthogonal projection onto $\overline{\pi_\varphi\big(\Phi(\mathfrak{A})\big)\xi_\varphi}$ and belonging thus to the commutant $\pi_\varphi\big(\Phi(\mathfrak{A})\big)'$ of $\pi_\varphi(\Phi(\mathfrak{A}))$, and*

$$U_{\Phi,\varphi}\pi_\varphi(a) = \pi_\varphi\big(\Phi(a)\big)U_{\Phi,\varphi} \quad \textit{for all } a \in \mathfrak{A}. \tag{2.2}$$

Proof. (i) $U = U_{\Phi,\varphi}$ is a well defined linear contraction because of the density of $\pi_\varphi(\mathfrak{A})\xi_\varphi$ in H_φ and

$$\|\pi_\varphi(\Phi(x))\xi_\varphi\|^2 = \varphi(\Phi(x)^*\Phi(x)) \leqslant \varphi(x^*x) = \|\pi_\varphi(x)\xi_\varphi\|^2 \quad \text{for every } x \in \mathfrak{A}.$$

Letting $(u_\iota)_{\iota\in I}$ a bounded left approximate unit for $\mathfrak{A}$, we have

$$\begin{aligned}\|\xi_\varphi - U\pi_\varphi(u_\iota)\xi_\varphi\|^2 &= \|\xi_\varphi - \pi_\varphi(\Phi(u_\iota))\xi_\varphi\|^2\\ &= \|\varphi\| + \varphi(\Phi(u_\iota)^*\Phi(u_\iota)) - 2\mathrm{Re}\,\varphi(\Phi(u_\iota))\\ &\leqslant \|\varphi\| + \varphi(u_\iota^*u_\iota) - 2\mathrm{Re}\,\varphi(u_\iota) \to 0,\end{aligned}$$

so $U\pi_\varphi(u_\iota)\xi_\varphi \to \xi_\varphi$. Taking into account that $\pi_\varphi(u_\iota)\xi_\varphi \to \xi_\varphi$, it follows that $U\xi_\varphi = \xi_\varphi$.

According to Section 144 in Chapitre X of [31] we have

$$\{\xi \in H_\varphi : U\xi = \xi\} = \{\xi \in H_\varphi : U^*\xi = \xi\}.$$

The orthogonal projection $P = P^U$ onto the above subspace clearly satisfies $UP = P$ and $U^*P = P$, which imply $UP = PU = P$. Finally,

$$\frac{1}{n+1}\sum_{k=0}^{n} U^k \xrightarrow{\text{so}} P^U$$

is exactly the statement of the mean ergodic theorem of von Neumann for U (see e.g. [23], Theorem 1.1.4 or [29], Theorem 2.1.2 or [34], E.2.25).

(ii) By the multiplicativity of Φ we have

$$\|\pi_\varphi(\Phi(x))\xi_\varphi\|^2 = \varphi(\Phi(x)^*\Phi(x)) = \varphi(\Phi(x^*x)) = \varphi(x^*x) = \|\pi_\varphi(x)\xi_\varphi\|^2, \quad x \in \mathfrak{A},$$

so $U = U_{\Phi,\varphi}$ is isometrical. It follows that UU^* is the orthogonal projection onto

$$UH_\varphi = U\overline{\pi_\varphi(\mathfrak{A})\xi_\varphi} = \overline{\pi_\varphi\big(\Phi(\mathfrak{A})\big)\xi_\varphi}.$$

For every $a \in \mathfrak{A}$ and every $x \in \mathfrak{A}$ we have

$$\begin{aligned} U\pi_\varphi(a)\pi_\varphi(x)\xi_\varphi &= U\pi_\varphi(ax)\xi_\varphi = \pi_\varphi\big(\Phi(ax)\big)\xi_\varphi = \pi_\varphi\big(\Phi(a)\big)\pi_\varphi\big(\Phi(x)\big)\xi_\varphi \\ &= \pi_\varphi\big(\Phi(a)\big)U\pi_\varphi(x)\xi_\varphi, \end{aligned}$$

hence $U\pi_\varphi(a) = \pi_\varphi\big(\Phi(a)\big)U$. ∎

We recall that the normal positive linear map $\Phi^{**} : \mathfrak{A}^{**} \to \mathfrak{A}^{**}$ is necessarily s*-continuous (see e.g. [35], 8.17 (17)). Taking into account that $\pi_\varphi(\mathfrak{A})$ is s*-dense in $\pi_\varphi(\mathfrak{A})''$ (see e.g. [35], 8.5), formula (2.1) implies by density

$$U_{\Phi,\varphi}\big(\pi_\varphi(x)\xi_\varphi\big) = \pi_\varphi\big(\Phi^{**}(x)\big)\xi_\varphi \quad \text{for all } x \in \mathfrak{A}^{**}. \tag{2.3}$$

We notice also that

$$\pi_\varphi\big((\Phi^{**})^n(1_{\mathfrak{A}^{**}})\big)\xi_\varphi = \xi_\varphi \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

Indeed, since $\big(\pi_\varphi(1_{\mathfrak{A}^{**}})\xi_\varphi \mid \pi_\varphi(x)\xi_\varphi\big) = \varphi(x^*1_{\mathfrak{A}^{**}}) = (\xi_\varphi \mid \pi_\varphi(x)\xi_\varphi)$ for all $x \in \mathfrak{A}$, by the density of $\pi_\varphi(\mathfrak{A})\xi_\varphi$ in H_φ we have $\pi_\varphi(1_{\mathfrak{A}^{**}})\xi_\varphi = \xi_\varphi$ and it follows, for every $n \in \mathbb{N}$, $\pi_\varphi\big((\Phi^{**})^n(1_{\mathfrak{A}^{**}})\big)\xi_\varphi = U^n_{\Phi,\varphi}\pi_\varphi(1_{\mathfrak{A}^{**}})\xi_\varphi = \xi_\varphi$.

If Φ is multiplicative, then (2.2) implies by passing to the adjoints

$$\pi_\varphi(a)U^*_{\Phi,\varphi} = U^*_{\Phi,\varphi}\pi_\varphi\big(\Phi(a)\big) \quad \text{for all } a \in \mathfrak{A}. \tag{2.5}$$

Taking into account that $U^*_{\Phi,\varphi}U_{\Phi,\varphi} = 1_{H_\varphi}$, it follows for every $a \in \mathfrak{A}$

$$\pi_\varphi(a) = U^*_{\Phi,\varphi}\pi_\varphi\big(\Phi(a)\big)U_{\Phi,\varphi}, \quad U_{\Phi,\varphi}\pi_\varphi(a)U^*_{\Phi,\varphi} = \pi_\varphi\big(\Phi(a)\big)U_{\Phi,\varphi}U^*_{\Phi,\varphi}. \tag{2.6}$$

If Φ is a $*$-automorphism, then $U_{\Phi,\varphi}$ is clearly unitary and therefore

$$\pi_\varphi(\Phi(a)) = U_{\Phi,\varphi}\pi_\varphi(a)U^*_{\Phi,\varphi}, \quad a \in \mathfrak{A}. \tag{2.7}$$

Proof of Theorem 1.2. Let $U = U_{\Phi,\varphi}$ and $P^U = P^{U_{\Phi,\varphi}}$ be as in Lemma 2.1. For every $x \in \mathfrak{A}$ and $\varepsilon > 0$, denoting $\xi = \pi_\varphi(x)\xi_\varphi$, there exists $n_\xi \geqslant 1$ such that

$$\Big\|\frac{1}{n_\xi + 1}\sum_{k=0}^{n_\xi} U^k\xi - P^U\xi\Big\| \leqslant \frac{\varepsilon}{\|\xi\|}.$$

For every $l \geqslant n_\xi$ we have

$$\begin{aligned} \Big\|\frac{1}{(n_\xi+1)^2}\sum_{j,k=0}^{n_\xi} U^{l+k-j}\xi - P^U\xi\Big\| &= \Big\|\frac{1}{n_\xi+1}\sum_{j=0}^{n_\xi} U^{l-j}\Big(\frac{1}{n_\xi+1}\sum_{k=0}^{n_\xi} U^k\xi - P^U\xi\Big)\Big\| \\ &\leqslant \Big\|\frac{1}{n_\xi+1}\sum_{k=0}^{n_\xi} U^k\xi - P^U\xi\Big\| \leqslant \frac{\varepsilon}{\|\xi\|}. \end{aligned}$$

Consequently

$$\frac{1}{(n_\xi+1)^2}\sum_{j,k=0}^{n_\xi}\operatorname{Re}\varphi(x^*\Phi^{l+k-j}(x)) = \operatorname{Re}\frac{1}{(n_\xi+1)^2}\sum_{j,k=0}^{n_\xi}(U^{l+k-j}\xi \mid \xi)$$

$$= \operatorname{Re}\Big(\frac{1}{(n_\xi+1)^2}\sum_{j,k=0}^{n_\xi}(U^{l+k-j}\xi - P^U\xi \mid \xi)\Big) + \operatorname{Re}(P^U\xi \mid \xi) \geqslant \big\|P^U\xi\big\|^2 - \varepsilon.$$

On the other hand,

$$\Big(\frac{1}{n+1}\sum_{j=0}^{n}U^j\xi \,\Big|\, \xi_\varphi\Big) = \frac{1}{n+1}\sum_{j=0}^{n}\varphi(\Phi^j(x)) = \varphi(x), \quad n \geqslant 1$$

implies by passing to the limit for $n\to\infty$,

$$(P^U\xi \mid \xi_\varphi) = \varphi(x),$$

hence $|\varphi(x)| \leqslant \|P^U\xi\|\,\|\varphi\| = \|P^U\xi\|$. It follows that

$$\frac{1}{(n_\xi+1)^2}\sum_{j,k=0}^{n_\xi}\operatorname{Re}\varphi(x^*\Phi^{l+k-j}(x)) \geqslant |\varphi(x)|^2 - \varepsilon,$$

so for each integer $m \geqslant 1$ there exist integers $j(m), k(m) \in [0, n_\xi]$ such that

$$\operatorname{Re}\varphi(x^*\Phi^{mn_\xi+k(m)-j(m)}(x)) \geqslant |\varphi(x)|^2 - \varepsilon.$$

Therefore the statement of Theorem 1.2 holds with

$$\mathcal{N} = \{mn_\xi + k(m) - j(m) : m \geqslant 1 \text{ integer}\}. \quad ∎$$

Let us prove for multiplicative Φ also another variant of Theorem 1.2, a non-commutative extension of [29], Chapter 4, Lemma 4.7. We recall that a subset $\mathcal{N}$ of $\mathbb{N}$ is an *IP-set* if there exists a sequence $p_1, p_2, \ldots$ in $\mathbb{N}\setminus\{0\}$ for which

$$\mathcal{N} = \{p_{j_1} + p_{j_2} + \cdots + p_{j_n} : 1 \leqslant j_1 < j_2 < \cdots < j_n, n \geqslant 1\}.$$

The terminology is motivated by the fact that 0 together with all sums $p_{j_1} + p_{j_2} + \cdots + p_{j_n}$ form an *infinite-dimensional parallelepiped*,

$$\{0, p_1\} \cup \{p_2, p_1 + p_2\} \cup \{p_3, p_1 + p_3, p_2 + p_3, p_1 + p_2 + p_3\} \cup \cdots,$$

where each set is a translate of the union of the preceding ones.

PROPOSITION 2.2. *Let $\mathfrak{A}$ be a C^*-algebra, φ a state on $\mathfrak{A}$ and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a $*$-homomorphism such that $\varphi\circ\Phi = \varphi$. Then, for every $x \in \mathfrak{A}$, every* IP*-set $\mathcal{N} \subset \mathbb{N}$ and every $\varepsilon > 0$, the set*

$$\{n \in \mathcal{N} : \operatorname{Re}\varphi(x^*\Phi^n(x)) \geqslant |\varphi(x)|^2 - \varepsilon\}$$

is infinite.

Proof. Suppose that $\mathcal{N}$ is generated by $p_1, p_2, \ldots$. Choosing a sequence $1 = j_1 < j_2 < \cdots$ of integers such that

$$\sum_{j=j_k+1}^{j_{k+1}} p_j > n_k = \sum_{j=1}^{j_k} p_j \quad \text{for every integer } k \geqslant 1,$$

the differences $n_k - n_l$ belong to $\mathcal{N}$ for all $1 \leqslant l < k$, and

$$1 \leqslant l < k,\ 1 \leqslant l' < k',\ k < k' \Rightarrow n_k - n_l < n_{k'} - n_{l'}.$$

Now let us assume that the set $\{n \in \mathcal{N} : \operatorname{Re}\varphi(x^*\Phi^n(x)) \geqslant |\varphi(x)|^2 - \varepsilon\}$ is finite. Then there is $k_0 \geqslant 1$ such that

$$\operatorname{Re}\varphi(\Phi^{n_l}(x)^*\Phi^{n_k}(x)) = \operatorname{Re}\varphi(x^*\Phi^{n_k-n_l}(x)) < |\varphi(x)|^2 - \varepsilon \quad \text{for } k > l \geqslant k_0.$$

Taking into account that, for all integers $n, n' \geqslant 1$,

$$\begin{aligned}
&\big(\pi_\varphi(\Phi^n(x))\xi_\varphi - \varphi(x)\xi_\varphi \,\big|\, \pi_\varphi(\Phi^{n'}(x))\xi_\varphi - \varphi(x)\xi_\varphi\big)\\
&\qquad = \varphi\big(\Phi^{n'}(x)^*\Phi^n(x)\big) - \overline{\varphi(x)}\varphi(\Phi^n(x)) - \varphi(x)\overline{\varphi(\Phi^{n'}(x))} + |\varphi(x)|^2 =\\
&\qquad = \varphi\big(\Phi^{n'}(x)^*\Phi^n(x)\big) - |\varphi(x)|^2,
\end{aligned}$$

it follows, for every integer $m > k_0$,

$$\begin{aligned}
&\Big\| \sum_{k=k_0}^{k_0+m-1} \big(\pi_\varphi(\Phi^{n_k}(x))\xi_\varphi - \varphi(x)\xi_\varphi\big)\Big\|^2\\
&\qquad = \sum_{k,l=k_0}^{k_0+m-1} \big(\pi_\varphi(\Phi^{n_k}(x))\xi_\varphi - \varphi(x)\xi_\varphi \,\big|\, \pi_\varphi(\Phi^{n_l}(x))\xi_\varphi - \varphi(x)\xi_\varphi\big)\\
&\qquad = \sum_{k,l=k_0}^{k_0+m-1} \Big(\varphi(\Phi^{n_l}(x)^*\Phi^{n_k}(x)) - |\varphi(x)|^2\Big)\\
&\qquad = m\cdot\varphi(x^*x) + 2\cdot \sum_{\substack{l,k=k_0\\ l<k}}^{k_0+m-1} \operatorname{Re}\varphi(\Phi^{n_l}(x)^*\Phi^{n_k}(x)) - m^2\cdot|\varphi(x)|^2\\
&\qquad < m\cdot\varphi(x^*x) + (m^2-m)\cdot(|\varphi(x)|^2 - \varepsilon) - m^2\cdot|\varphi(x)|^2\\
&\qquad = m\cdot(\varphi(x^*x) - |\varphi(x)|^2) - (m^2-m)\cdot\varepsilon,
\end{aligned}$$

hence $(m-1)\cdot\varepsilon < \varphi(x^*x) - |\varphi(x)|^2$. But this inequality is not true for sufficiently large m. ∎

3. SPATIAL REPRESENTATION OF STATE PRESERVING POSITIVE LINEAR MAPS

Given a C^*-algebra $\mathfrak{A}$, a state φ on $\mathfrak{A}$, and a positive linear map $\Phi : \mathfrak{A} \to \mathfrak{A}$ such that $\varphi \circ \Phi = \varphi$, we are looking for a w-continuous map $\Psi : \overline{\pi_\varphi(\mathfrak{A})}^{\mathrm{w}} \to \overline{\pi_\varphi(\mathfrak{A})}^{\mathrm{w}}$ satisfying $\Psi\big(\pi_\varphi(a)\big) = \pi_\varphi\big(\Phi(a)\big)$ for all $a \in \mathfrak{A}$. If such a map Ψ exists, it is uniquely determined and positive.

Assuming that Φ is a $*$-automorphism, (2.7) yields that

$$\Psi = \operatorname{Ad}(U_{\Phi,\varphi}) \mid \overline{\pi_\varphi(\mathfrak{A})}^{\mathrm{w}}$$

is a map as required above. However, assuming only that Φ is a $*$-homomorphism, it is in general not even true that $\pi_\varphi\big(\Phi(a)\big)$ is uniquely determined by $\pi_\varphi(a)$. Indeed, if we denote

— by H a Hilbert space with orthonormal basis $\{\xi_j : j \geqslant 0\} \cup \{\eta_j : j \geqslant 0\}$,
— by p the orthogonal projection onto the closed linear span of $\{\xi_j : j \geqslant 0\}$,
— $\mathfrak{A} = \{a \in \mathcal{L}(H) : ap = pa\}$,
— $\varphi = \omega_{\xi_0}|\mathfrak{A}$,
— by U the linear isometry on H defined by $U\xi_j = \xi_{2j}, U\eta_j = \xi_{2j+1}$
— and by Φ the $*$-homomorphism $\mathfrak{A} \ni x \mapsto UxU^* \in \mathfrak{A}$,

then π_φ can be identified with $\mathfrak{A} \ni a \mapsto a|pH \in \mathcal{L}(H)$ and we have for all $a \in \mathfrak{A}$: $\|\Phi(a)|pH\| = \|UaU^*p\| = \|a\|$. Consequently, if $0 \neq a \in \mathfrak{A}$ and $ap = 0$, then $\pi_\varphi(a) = 0$ but $\pi_\varphi\big(\Phi(a)\big) \neq 0$.

The situation changes if we assume that the support projection $s(\varphi)$ of φ in the second dual $\mathfrak{A}^{**}$ is central. We notice that this happens if and only if ξ_φ is cyclic also for the commutant $\pi_\varphi(\mathfrak{A})'$, or equivalently, it is separating for $\pi_\varphi(\mathfrak{A})''$. Indeed, since $\pi_\varphi|z(\varphi)\mathfrak{A}^{**} : z(\varphi)\mathfrak{A}^{**} \to \pi_\varphi(\mathfrak{A})''$ is a $*$-isomorphism, it carries $z(\varphi)$ in 1_{H_φ}, and $s(\varphi)$ in the support of $\omega_{\xi_\varphi}|\pi_\varphi(\mathfrak{A})''$, that is in the orthogonal projection onto $\overline{\pi_\varphi(\mathfrak{A})'\xi_\varphi}$. Consequently, $s(\varphi) = z(\varphi) \Leftrightarrow \overline{\pi_\varphi(\mathfrak{A})'\xi_\varphi} = H_\varphi$.

The following result answers the question raised at the beginning of the section if $s(\varphi)$ is central and the contraction condition (C) (as stated in Theorem 1.2) is satisfied:

PROPOSITION 3.1. *Let $\mathfrak{A}$ be a C^*-algebra, φ a state on $\mathfrak{A}$ such that the support $s(\varphi)$ of φ in $\mathfrak{A}^{**}$ is central, and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\varphi \circ \Phi = \varphi$.*

(i) *If $\varphi(\Phi(x)^*\Phi(x)) \leqslant \varphi(x^*x)$ for every $x \in \mathfrak{A}$, then there exists, for every $T \in \pi_\varphi(\mathfrak{A})''$, a unique $\Psi_{\Phi,\varphi}(T) \in \pi_\varphi(\mathfrak{A})''$ such that $\Psi_{\Phi,\varphi}(T)\xi_\varphi = U_{\Phi,\varphi}T\xi_\varphi$ and $\Psi_{\Phi,\varphi} : \pi_\varphi(\mathfrak{A})'' \to \pi_\varphi(\mathfrak{A})''$ is a normal positive linear map, carrying 1_{H_φ} in 1_{H_φ} (hence of norm $\|\Psi_{\Phi,\varphi}\| \leqslant 1$), preserving $\omega_{\xi_\varphi}|\pi_\varphi(\mathfrak{A})''$ and satisfying $\Psi_{\Phi,\varphi}\big(\pi_\varphi(a)\big) = \pi_\varphi\big(\Phi^{**}(a)\big)$ for all $a \in \mathfrak{A}^{**}$.*

(ii) *If Φ is multiplicative, then $\pi_\varphi\big(\Phi(\mathfrak{A})\big)'' = \overline{\pi_\varphi\big(\Phi(\mathfrak{A})\big)}^{\mathrm{wo}} = \overline{\pi_\varphi\big(\Phi(\mathfrak{A})\big)}^{\mathrm{so}} = \overline{\pi_\varphi\big(\Phi(\mathfrak{A})\big)}^{\mathrm{w}}$ is a von Neumann subalgebra of $\pi_\varphi(\mathfrak{A})''$ and the central support of the projection $U_{\Phi,\varphi}U_{\Phi,\varphi}^* \in \pi_\varphi\big(\Phi(\mathfrak{A})\big)'$ in $\pi_\varphi\big(\Phi(\mathfrak{A})\big)''$ is 1_{H_φ}. Therefore, for every $T \in \pi_\varphi(\mathfrak{A})''$, $\Psi_{\Phi,\varphi}(T)$ is the unique element of $\pi_\varphi(\Phi(\mathfrak{A}))''$ satisfying*

$$\Psi_{\Phi,\varphi}(T)U_{\Phi,\varphi}U_{\Phi,\varphi}^* = U_{\Phi,\varphi}TU_{\Phi,\varphi}^*.$$

In particular, $\pi_\varphi(\mathfrak{A})'' \ni T \mapsto \Psi_{\Phi,\varphi}(T) \in \pi_\varphi\big(\Phi(\mathfrak{A})\big)''$ is a $*$-isomorphism.

Proof. We shall omit for convenience the subscript Φ, φ, writing simply $U = U_{\Phi,\varphi}$.

(i) By the Kaplansky density theorem there is a net $(a_\iota)_\iota$ in $\mathfrak{A}$ satisfying $\|a_\iota\| \leqslant \|T\|$ and $\pi_\varphi(a_\iota) \xrightarrow{\mathrm{so}} T$, hence $\pi_\varphi(a_\iota)\xi_\varphi \to T\xi_\varphi$ and then

$$\pi_\varphi(\Phi(a_\iota))\xi_\varphi = U\pi_\varphi(a_\iota)\xi_\varphi \to UT\xi_\varphi.$$

If T_0 is any weak operator limit point of the bounded net $\big(\pi_\varphi(\Phi(a_\iota))\big)_\iota$ then $T_0\xi_\varphi$ will be a weak limit point of the convergent net $\big(\pi_\varphi(\Phi(a_\iota))\xi_\varphi\big)_\iota$, hence $T_0\xi_\varphi = UT\xi_\varphi$. It follows that the net $\big(\pi_\varphi(\Phi(a_\iota))\big)_\iota$ is weak operator convergent to some $\Psi(T) \in \pi_\varphi(\mathfrak{A})''$, which is uniquely determined by its value $\Psi(T)\xi_\varphi = UT\xi_\varphi$.

The map $\pi_\varphi(\mathfrak{A})'' \ni T \mapsto \Psi(T)$ is clearly linear, of norm $\leqslant \|\Phi\|$, carrying 1_{H_φ} in 1_{H_φ} and preserving $\omega_{\xi_\varphi}|\pi_\varphi(\mathfrak{A})''$. For every $\xi_j = T'_j\xi_\varphi$, $T'_j \in \pi_\varphi(\mathfrak{A})'$, $j = 1, 2$, the linear functional

$$\pi_\varphi(\mathfrak{A})'' \ni T \mapsto (\Psi(T)\xi_1 \mid \xi_2) = (\Psi(T)\xi_\varphi \mid (T'_1)^*T'_2\xi_\varphi) = (T\xi_\varphi \mid U^*(T'_1)^*T'_2\xi_\varphi)$$

is wo-continuous and, using $\overline{\pi_\varphi(\mathfrak{A})'\xi_\varphi} = H_\varphi$, standard arguments (see e.g. [34], Theorem 1.10) yield the w-continuity of Ψ. Since Ψ is an extension of $\pi_\varphi(\mathfrak{A}) \ni \pi_\varphi(a) \mapsto \pi_\varphi\big(\Phi(a)\big)$, it follows by density the positivity of Ψ and the equality $\Psi\big(\pi_\varphi(a)\big) = \pi_\varphi\big(\Phi^{**}(a)\big)$ for all $a \in \mathfrak{A}^{**}$.

(ii) Since Ψ is w-continuous and $\pi_\varphi(\mathfrak{A})$ is w-dense in $\pi_\varphi(\mathfrak{A})''$, $1_{H_\varphi} = \Psi(1_{H_\varphi})$ belongs to the w-closure of the $*$-subalgebra $\pi_\varphi\big(\Phi(\mathfrak{A})\big)$ of $\pi_\varphi(\mathfrak{A})''$. It follows that $\pi_\varphi\big(\Phi(\mathfrak{A})\big)'' = \overline{\pi_\varphi\big(\Phi(\mathfrak{A})\big)}^{\mathrm{wo}} = \overline{\pi_\varphi\big(\Phi(\mathfrak{A})\big)}^{\mathrm{so}} = \overline{\pi_\varphi\big(\Phi(\mathfrak{A})\big)}^{\mathrm{w}}$ is a von Neumann subalgebra of $\pi_\varphi(\mathfrak{A})''$.

Furthermore, since the central support of the projection $UU^* \in \pi_\varphi\big(\Phi(\mathfrak{A})\big)'$ in $\pi_\varphi\big(\Phi(\mathfrak{A})\big)''$ is an element $P \in \pi_\varphi(\mathfrak{A})''$ leaving ξ_φ fixed and the vector ξ_φ is separating for $\pi_\varphi(\mathfrak{A})''$, we have $P = 1_{H_\varphi}$. ∎

We notice that, for multiplicative Φ, (2.2) and (2.5) imply

$$U_{\Phi,\varphi}T = \Psi_{\Phi,\varphi}(T)U_{\Phi,\varphi}, TU^*_{\Phi,\varphi} = U^*_{\Phi,\varphi}\Psi_{\Phi,\varphi}(T) \quad \text{for all } T \in \pi_\varphi(\mathfrak{A})''. \tag{3.1}$$

Let φ be a state on a C^*-algebra such that the support of φ in $\mathfrak{A}^{**}$ is central, that is the vector ξ_φ is separating for $\pi_\varphi(\mathfrak{A})''$. Then we have by the Tomita-Takesaki theory of standard von Neumann algebras (see e.g. [34], Chapter 10 or [3], Section 2.5):

(1) the densely defined antilinear operator $\pi_\varphi(\mathfrak{A})''\xi_\varphi \ni T\xi_\varphi \mapsto T^*\xi_\varphi$ has closure S_φ and $\overline{S_\varphi|\pi_\varphi(\mathfrak{A})\xi_\varphi} = S_\varphi$;

(2) the adjoint S^*_φ is the closure of $\pi_\varphi(\mathfrak{A})'\xi_\varphi \ni T'\xi_\varphi \mapsto T'^*\xi_\varphi$;

(3) $\Delta_\varphi = S^*_\varphi S_\varphi$ is a non-singular, positive, self-adjoint linear operator in H_φ (called the *modular operator* of φ) satisfying

$$\Delta^{\mathrm{i}t}_\varphi \pi_\varphi(\mathfrak{A})''\Delta^{-\mathrm{i}t}_\varphi = \pi_\varphi(\mathfrak{A})'' \quad \text{for all } t \in \mathbb{R};$$

(4) if $S_\varphi = J_\varphi\Delta^{1/2}_\varphi = \Delta^{-1/2}_\varphi J_\varphi$ is the polar decomposition of S_φ then J_φ is an involutive antilinear isometry (called the *modular conjugation* of φ) and

$$J_\varphi\pi_\varphi(\mathfrak{A})''J_\varphi = \pi_\varphi(\mathfrak{A})'. \tag{3.2}$$

If σ^φ denotes the modular automorphism group of φ then

$$\pi_\varphi\big(\sigma^\varphi_t(a)\big) = \Delta^{\mathrm{i}t}_\varphi\pi_\varphi(a)\Delta^{-\mathrm{i}t}_\varphi, \quad t \in \mathbb{R},\, a \in \mathfrak{A}^{**}, \tag{3.3}$$

where π_φ stands also for the normal extension on $\mathfrak{A}^{**}$ of the GNS representation of φ, that is for the GNS representation of φ considered as state on $\mathfrak{A}^{**}$. We recall that Δ_φ and J_φ leave ξ_φ fixed and σ^φ acts identically on the center of $\mathfrak{A}^{**}$.

Now let A be a non-singular, positive, self-adjoint linear operator in a Hilbert space H and put $\alpha_t = \mathrm{Ad}(A^{\mathrm{i}t})$, $t \in \mathbb{R}$. For every $z \in \mathbb{C}$ a linear operator α_z can be defined in $\mathcal{L}(H)$ as follows: $(T, T_z) \in \mathrm{Graph}(\alpha_z)$ means that $\mathbb{R} \ni t \mapsto \alpha_t(T) \in$

$\mathcal{L}(H)$ has a w-continuous extension on the closed strip $\mathbb{S}_z = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| \leqslant |\operatorname{Im} z|, \operatorname{Im}\zeta \cdot \operatorname{Im} z \geqslant 0\}$ which is analytic in the interior, and whose value at z is T_z.

Then (see [6], Theorem 2.4 and Theorem 6.2, as well as [40], Theorem 1.6)

(3.4) $\quad \alpha_{-z} = \alpha_z{}^{-1}$ and $\alpha_{z_1}\alpha_{z_2} = \alpha_{z_1+z_2}$ if $\operatorname{Im} z_1 \cdot \operatorname{Im} z_2 \geqslant 0$,

(3.5) $\quad$ for $T, T_z \in \mathcal{L}(H)$ we have $(T, T_z) \in \operatorname{Graph}(\alpha_z) \Leftrightarrow TA^{-\mathrm{i}z} \subset A^{-\mathrm{i}z}T_z$,

(3.6) $\quad$ for T in the domain of α_z we have $\alpha_z(T)^* = \alpha_{\overline{z}}(T^*)$.

In particular, if T is in the domain of α_z then $\mathbb{S}_z \ni \zeta \mapsto \alpha_\zeta(T)$ is bounded, so the maximum principle yields that $\sup\{\|\alpha_\zeta(T)\| : \zeta \in \mathbb{S}_z\} = \max\{\|T\|, \|\alpha_z(T)\|\}$. We notice also that, for $T \in \mathcal{L}(H)$,

$$\alpha_{\mathrm{i}}(T) = T \Leftrightarrow \alpha_t(T) = T \quad \text{for all } t \in \mathbb{R}. \tag{3.7}$$

Indeed, if $\alpha_{\mathrm{i}}(T) = T$ then $\alpha_{t+\mathrm{i}}(T) = \alpha_t(T)$ for all $t \in \mathbb{R}$, so $\mathbb{R} \ni t \mapsto \alpha_t(T)$ has a bounded entire extension.

The next result on the eigenspaces of $U_{\Phi,\varphi}$ and $\Psi_{\Phi,\varphi}$ extends Lemma 4.3 of [26]:

PROPOSITION 3.2. *Let $\mathfrak{A}$ be a C^*-algebra, φ a state on $\mathfrak{A}$ whose support in $\mathfrak{A}^{**}$ is central, and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\varphi \circ \Phi = \varphi$ and*

$$\varphi(\Phi(x)^*\Phi(x)) \leqslant \varphi(x^*x) \quad \textit{for all } x \in \mathfrak{A}.$$

Let further $\lambda \in \mathbb{C}$, $|\lambda| = 1$, be arbitrary and let P_λ denote the orthogonal projection onto $\big\{\xi \in H_\varphi : U_{\Phi,\varphi}\xi = \lambda\xi\big\}$. Then

$$\overline{\big\{T \in \pi_\varphi(\mathfrak{A})'' : \Psi_{\Phi,\varphi}(T) = \lambda T\big\}\xi_\varphi} = \big\{\xi \in H_\varphi : U_{\Phi,\varphi}\xi = \lambda\xi\big\} \tag{3.8}$$

and there exists, for every $T \in \pi_\varphi(\mathfrak{A})''$, an $E_\lambda(T) \in \pi_\varphi(\mathfrak{A})''$ such that

$$\frac{1}{n+1}\sum_{k=0}^{n} \overline{\lambda}^k \Psi^k_{\Phi,\varphi}(T) \xrightarrow{\text{so}} E_\lambda(T), \quad E_\lambda(T)\xi_\varphi = P_\lambda T\xi_\varphi.$$

Moreover, E_λ is a w*-continuous linear projection of norm $\leqslant 1$ from $\pi_\varphi(\mathfrak{A})''$ onto $\big\{T \in \pi_\varphi(\mathfrak{A})'' : \Psi_{\Phi,\varphi}(T) = \lambda T\big\}$, which commutes with the modular automorphism group $\mathbb{R} \ni t \mapsto \operatorname{Ad}(\Delta_\varphi^{\mathrm{i}t})|\pi_\varphi(\mathfrak{A})''$ of $\omega_{\xi_\varphi}|\pi_\varphi(\mathfrak{A})''$.*

Proof. We shall again omit the subscripts Φ, φ, writing $U = U_{\Phi,\varphi}$ and $\Psi = \Psi_{\Phi,\varphi}$.

First we prove the existence of E_λ. For let $T \in \pi_\varphi(\mathfrak{A})''$ be arbitrary. If T_0 is any wo-limit point of $\Big(\frac{1}{n+1}\sum\limits_{k=0}^{n} \overline{\lambda}^k \Psi^k(T)\Big)_{n\geqslant 0}$ then $T_0 \in \pi_\varphi(\mathfrak{A})''$, $\|T_0\| \leqslant \|T\|$ and $T_0\xi_\varphi = P_\lambda T\xi_\varphi$, as

$$\frac{1}{n+1}\sum_{k=0}^{n} \overline{\lambda}^k \Psi^k(T)\xi_\varphi = \frac{1}{n+1}\sum_{k=0}^{n} \overline{\lambda}^k U^k T\xi_\varphi \xrightarrow{\|\cdot\|} P_\lambda T\xi_\varphi \tag{3.9}$$

by the mean ergodic theorem. Therefore the sequence $\Big(\frac{1}{n+1}\sum\limits_{k=0}^{n} \overline{\lambda}^k \Psi^k(T)\Big)_{n\geqslant 0}$ is weak operator convergent to some $E_\lambda(T) \in \pi_\varphi(\mathfrak{A})''$ with $\|E_\lambda(T)\| \leqslant \|T\|$

and $E_\lambda(T)\xi_\varphi = P_\lambda T\xi_\varphi$. Actually $\frac{1}{n+1}\sum_{k=0}^{n}\overline{\lambda}^k\Psi^k(T) \xrightarrow{\text{so}} E_\lambda(T)$; (3.9) yields that $\frac{1}{n+1}\sum_{k=0}^{n}\overline{\lambda}^k\Psi^k(T)\xi \xrightarrow{\|\cdot\|} E_\lambda(T)\xi$ for all $\xi\in\pi_\varphi(\mathfrak{A})'\xi\varphi$ and by $\overline{\pi_\varphi(\mathfrak{A})'\xi_\varphi} = H_\varphi$, this convergence holds for all $\xi\in H_\varphi$.

Since $\Psi\big(E_\lambda(T)\big)\xi_\varphi = UE_\lambda(T)\xi_\varphi = \lambda E_\lambda(T)\xi_\varphi$, we have $\Psi\big(E_\lambda(T)\big) = \lambda E_\lambda(T)$. On the other hand, if $\Psi(T)=\lambda T$ then

$$UT\xi_\varphi = \Psi(T)\xi_\varphi = \lambda T\xi_\varphi \Rightarrow E_\lambda(T)\xi_\varphi = P_\lambda T\xi_\varphi = T\xi_\varphi \Rightarrow E_\lambda(T) = T.$$

Thus E_λ is a linear projection of $\pi_\varphi(\mathfrak{A})''$ onto $\big\{T\in\pi_\varphi(\mathfrak{A})'' : \Psi(T)=\lambda T\big\}$. Since $\|E_\lambda(T)\|\leqslant\|T\|$, we have $\|E_\lambda\|\leqslant 1$. The proof of the w-continuity of E_λ is similar to that one of Ψ in Proposition 3.1.

First, for every $\xi_j = T_j'\xi_\varphi$, $T_j'\in\pi_\varphi(\mathfrak{A})'$, $j=1,2$, the linear functional

$$\pi_\varphi(\mathfrak{A})''\ni T\mapsto (E_\lambda(T)\xi_1\mid\xi_2) = (E_\lambda(T)\xi_\varphi\mid(T_1')^*T_2'\xi_\varphi) = (T\xi_\varphi\mid P_\lambda(T_1')^*T_2'\xi_\varphi)$$

is wo-continuous. Using $\overline{\pi_\varphi(\mathfrak{A})'\xi_\varphi} = H_\varphi$, it follows that $T\mapsto(E_\lambda(T)\xi_1\mid\xi_2)$ is w-continuous for all $\xi_1,\xi_2\in H_\varphi$, which yields the w-continuity of E_λ.

Now we prove (3.8). Since the inclusion

$$\overline{\big\{T\in\pi_\varphi(\mathfrak{A})'' : \Psi(T)=\lambda T\big\}\xi_\varphi} \subset \big\{\xi\in H_\varphi : U\xi=\lambda\xi\big\}$$

is trivial, we have to verify only the converse inclusion.

For let $\xi\in H_\varphi$, $U\xi=\lambda\xi$ and $\varepsilon>0$ be arbitrary. Choosing $T\in\pi_\varphi(\mathfrak{A})''$ with $\|T\xi_\varphi-\xi\|\leqslant\varepsilon$, we get

$$\|\overline{\lambda}^k\Psi^k(T)\xi_\varphi-\xi\| = \|U^k(T\xi_\varphi)-\lambda^k\xi\| = \|U^k(T\xi_\varphi-\xi)\|\leqslant\varepsilon,\qquad k\geqslant 0,$$

$$\Big\|\tfrac{1}{n+1}\sum_{k=0}^{n}\overline{\lambda}^k\Psi^k(T)\xi_\varphi-\xi\Big\| = \Big\|\tfrac{1}{n+1}\sum_{k=0}^{n}\Big(\overline{\lambda}^k\Psi^k(T)\xi_\varphi-\xi\Big)\Big\|\leqslant\varepsilon,\quad n\geqslant 0.$$

Passing to limit for $n\to\infty$, it follows that $\|E_\lambda(T)\xi_\varphi-\xi\|\leqslant\varepsilon$.

Finally, the commutation of E_λ with $\mathbb{R}\ni t\mapsto \mathrm{Ad}(\Delta_\varphi^{\mathrm{i}t})|\pi_\varphi(\mathfrak{A})''$ follows once we show that P_λ is left fixed by every $\sigma_t = \mathrm{Ad}(\Delta_\varphi^{\mathrm{i}t}) : \mathcal{L}(H_\varphi)\to\mathcal{L}(H_\varphi)$. Indeed, then we have for every $t\in\mathbb{R}$ and $T\in\pi_\varphi(\mathfrak{A})''$:

$$E_\lambda(\Delta_\varphi^{\mathrm{i}t}T\Delta_\varphi^{-\mathrm{i}t})\xi_\varphi = P_\lambda\Delta_\varphi^{\mathrm{i}t}T\xi_\varphi = \Delta_\varphi^{\mathrm{i}t}P_\lambda T\xi_\varphi = \Delta_\varphi^{\mathrm{i}t}E_\lambda(T)\Delta_\varphi^{-\mathrm{i}t}\xi_\varphi.$$

For we notice that, for every $T\in\pi_\varphi(\mathfrak{A})''$,

$$\begin{aligned} E_\lambda(T^*) &= \text{wo-}\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\overline{\lambda}^k\Psi^k(T^*)\\ &= \text{wo-}\lim_{n\to\infty}\frac{1}{n+1}\Big(\sum_{k=0}^{n}\lambda\Psi^k(T)\Big)^* = E_{\overline{\lambda}}(T)^*, \end{aligned}$$

so $P_\lambda S_\varphi T\xi_\varphi = P_\lambda T^*\xi_\varphi = E_\lambda(T^*)\xi_\varphi = E_{\overline{\lambda}}(T)^*\xi_\varphi = S_\varphi E_{\overline{\lambda}}(T)\xi_\varphi = S_\varphi P_{\overline{\lambda}}T\xi_\varphi$. It follows that

$$P_\lambda\Delta_\varphi^{-1/2}J_\varphi = P_\lambda S_\varphi \subset S_\varphi P_{\overline{\lambda}} = \Delta_\varphi^{-1/2}J_\varphi P_{\overline{\lambda}},\text{ that is } P_\lambda\Delta_\varphi^{-1/2}\subset\Delta_\varphi^{-1/2}J_\varphi P_{\overline{\lambda}}J_\varphi.$$

Now (3.5) implies that P_λ belongs to the domain of $\sigma_{-\mathrm{i}/2}$ and $\sigma_{-\mathrm{i}/2}(P_\lambda) = J_\varphi P_{\overline{\lambda}}J_\varphi$. Using (3.6) and (3.4), we get successively:

$$\sigma_{\mathrm{i}/2}(P_\lambda) = \sigma_{-\mathrm{i}/2}(P_\lambda)^* = J_\varphi P_{\overline{\lambda}}J_\varphi = \sigma_{-\mathrm{i}/2}(P_\lambda),\quad \sigma_{\mathrm{i}}(P_\lambda) = P_\lambda.$$

Thus, according to (3.7), P_λ is left fixed by σ. ∎

Let us also discuss the commutation of σ^φ with φ-preserving positive linear maps.

PROPOSITION 3.3. *Let $\mathfrak{A}$ be a C^*-algebra, φ a state on $\mathfrak{A}$ such that the support $s(\varphi)$ of φ in $\mathfrak{A}^{**}$ is central, and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\varphi \circ \Phi = \varphi$.*

(i) *If $\varphi(\Phi(x)^*\Phi(x)) \leqslant \varphi(x^*x)$ for every $x \in \mathfrak{A}$, then the following conditions are equivalent:*

(a) *$s(\varphi)\Phi^{**}$ commutes with σ^φ;*

(b) *$U_{\Phi,\varphi}\Delta_\varphi^{\mathrm{it}} = \Delta_\varphi^{\mathrm{it}} U_{\Phi,\varphi}$ for all $t \in \mathbb{R}$;*

(c) *$U_{\Phi,\varphi}$ commutes with J_φ;*

(d) *for every $T' \in \pi_\varphi(\mathfrak{A})'$ there is a (necessarily unique) $\Psi'_{\Phi,\varphi}(T') \in \pi_\varphi(\mathfrak{A})'$ such that $\Psi'_{\Phi,\varphi}(T')\xi_\varphi = U_{\Phi,\varphi}T'\xi_\varphi$ and $\Psi'_{\Phi,\varphi}(T')^*\xi_\varphi = U_{\Phi,\varphi}T'^*\xi_\varphi$;*

(e) *for every $T \in \pi_\varphi(\mathfrak{A})''$ there is a (necessarily unique) $\Psi^\sim_{\Phi,\varphi}(T) \in \pi_\varphi(\mathfrak{A})''$ such that $\Psi^\sim_{\Phi,\varphi}(T)\xi_\varphi = U^*_{\Phi,\varphi}T\xi_\varphi$.*

Moreover, $\Psi'_{\Phi,\varphi} : \pi_\varphi(\mathfrak{A})' \to \pi_\varphi(\mathfrak{A})'$ in (d) *and $\Psi^\sim_{\Phi,\varphi} : \pi_\varphi(\mathfrak{A})'' \to \pi_\varphi(\mathfrak{A})''$ in* (e) *are normal positive linear maps preserving $\omega_{\xi_\varphi}|\pi_\varphi(\mathfrak{A})''$, whenever they exist.*

(ii) *If Φ is multiplicative, then the conditions in* (i) *are equivalent also with each one of the following:*

(f) *$U^*_{\Phi,\varphi}TU_{\Phi,\varphi} \in \pi_\varphi(\mathfrak{A})''$ for all $T \in \pi_\varphi(\mathfrak{A})''$;*

(g) *$\Delta_\varphi^{\mathrm{it}}\pi_\varphi\big(\Phi(\mathfrak{A})\big)''\Delta_\varphi^{-\mathrm{it}} \subset \pi_\varphi\big(\Phi(\mathfrak{A})\big)''$ for all $t \in \mathbb{R}$.*

In this case $\Psi^\sim_{\Phi,\varphi} = \mathrm{Ad}(U^*_{\Phi,\varphi})|\pi_\varphi(\mathfrak{A})''$ is a left inverse of $\Psi_{\Phi,\varphi}$ and $\Psi_{\Phi,\varphi} \circ \Psi^\sim_{\Phi,\varphi}$ is a normal conditional expectation from $\pi_\varphi(\mathfrak{A})''$ onto $\pi_\varphi\big(\Phi(\mathfrak{A})\big)''$.

Proof. We shall omit the subscripts Φ, φ, writing $U = U_{\Phi,\varphi}$, $\Psi = \Psi_{\Phi,\varphi}$, and further, $\Psi' = \Psi'_{\Phi,\varphi}$, $\Psi^\sim = \Psi^\sim_{\Phi;\varphi}$

(i) For (a) $\Leftrightarrow$ (b). We notice that (a) means

$$\pi_\varphi\big(\Phi^{**}\big(\sigma_t^\varphi(x)\big)\big) = \pi_\varphi\big(\sigma_t^\varphi\big(\Phi^{**}(x)\big)\big) \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathfrak{A}^{**}.$$

Since, according to (2.3) and (3.3),

$$\begin{aligned}
\pi_\varphi\big(\Phi^{**}\big(\sigma_t^\varphi(x)\big)\big)\xi_\varphi &= U\pi_\varphi\big(\sigma_t^\varphi(x)\big)\xi_\varphi = U\Delta_\varphi^{\mathrm{it}}\pi_\varphi(x)\xi_\varphi,\\
\pi_\varphi\big(\sigma_t^\varphi\big(\Phi^{**}(x)\big)\big)\xi_\varphi &= \Delta_\varphi^{\mathrm{it}}\pi_\varphi\big(\Phi^{**}(x)\big)\xi_\varphi = \Delta_\varphi^{\mathrm{it}}U\pi_\varphi(x)\xi_\varphi
\end{aligned}$$

and the vector ξ_φ is cyclic and separating for $\pi_\varphi(\mathfrak{A})''$, (a) is equivalent with (b).

For (b) $\Rightarrow$ (c). Condition (b) yields by standard arguments (see e.g. [34], E.9.23) that $U\Delta_\varphi^{1/2} \subset \Delta_\varphi^{1/2}U$. Using now (2.1), it follows for every $x \in \mathfrak{A}$

$$\begin{aligned}
UJ_\varphi\pi_\varphi(x)\xi_\varphi &= UJ_\varphi S_\varphi\pi_\varphi(x^*)\xi_\varphi = U\Delta_\varphi^{1/2}\pi_\varphi(x^*)\xi_\varphi\\
&= \Delta_\varphi^{1/2}U\pi_\varphi(x^*)\xi_\varphi = \Delta_\varphi^{1/2}\pi_\varphi\big(\Phi(x^*)\big)\xi_\varphi\\
&= J_\varphi S_\varphi\pi_\varphi\big(\Phi(x^*)\big)\xi_\varphi = J_\varphi\pi_\varphi\big(\Phi(x)\big)\xi_\varphi = J_\varphi U\pi_\varphi(x)\xi_\varphi
\end{aligned}$$

and we conclude that $UJ_\varphi = J_\varphi U$.

For (c) $\Rightarrow$ (d) and the properties of Ψ': By (3.2) we have

$$\Psi'(T') = J_\varphi\Psi(J_\varphi T'J_\varphi)J_\varphi \in \pi_\varphi(\mathfrak{A})', \quad T' \in \pi_\varphi(\mathfrak{A})',\ J_\varphi T'J_\varphi \in \pi_\varphi(\mathfrak{A})''$$

and using (c) we get also $\Psi'(T')\xi_\varphi = J_\varphi U(J_\varphi T' J_\varphi)\xi_\varphi = UT'\xi_\varphi$. Clearly,

$$\Psi' : \pi_\varphi(\mathfrak{A})' \ni T' \mapsto J_\varphi \Psi(J_\varphi T' J_\varphi) J_\varphi \in \pi_\varphi(\mathfrak{A})'$$

is a normal positive linear map.

For (d) $\Rightarrow$ (b). By (2.1) and by (d), we have for every $x \in \mathfrak{A}$ and $T' \in \pi_\varphi(\mathfrak{A})'$

$$\begin{aligned} US_\varphi \pi_\varphi(x)\xi_\varphi &= \pi_\varphi\big(\Phi(x^*)\big)\xi_\varphi = S_\varphi \pi_\varphi\big(\Phi(x)\big)\xi_\varphi = S_\varphi U \pi_\varphi(x)\xi_\varphi, \\ US_\varphi^* T'\xi_\varphi &= U{T'}^*\xi_\varphi = \Psi'(T')^*\xi_\varphi = S_\varphi^* \Psi'(T')\xi_\varphi = S_\varphi^* U T'\xi_\varphi. \end{aligned} \tag{3.10}$$

so

$$US_\varphi \subset S_\varphi U, \quad US_\varphi^* \subset S_\varphi^* U. \tag{3.11}$$

Now (3.11) implies $U\Delta_\varphi \subset \Delta_\varphi U$, which is equivalent to (b) (see again [34], E.9.23).

For (d) and the properties of $\Psi' \Rightarrow$(e) and the properties of $\Psi^\sim$:

First we show that for every $0 \leqslant T \in \pi_\varphi(\mathfrak{A})''$ there is some $0 \leqslant \Psi^\sim(T) \in \pi_\varphi(\mathfrak{A})''$ such that $\Psi^\sim(T)\xi_\varphi = U^*T\xi_\varphi$. Indeed, since we have

$$\begin{aligned} (T'U^*T\xi_\varphi \mid T'\xi_\varphi) &= (T\xi_\varphi \mid U{T'}^*T'\xi_\varphi) = (T\xi_\varphi \mid \Psi'({T'}^*T')\xi_\varphi) \\ &= (\Psi'({T'}^*T')^{1/2} T \Psi'({T'}^*T')^{1/2}\xi_\varphi \mid \xi_\varphi) \end{aligned} \tag{3.12}$$

for all $T' \in \pi_\varphi(\mathfrak{A})'$, the linear operator $L^0_{U^*T\xi_\varphi} : \pi_\varphi(\mathfrak{A})'\xi_\varphi \ni T'\xi_\varphi \mapsto T'U^*T\xi_\varphi$ is positive and it follows that its Friedrichs extension $\Psi^\sim(T)$ is affiliated to $\pi_\varphi(\mathfrak{A})''$ (see [34], 10.8 and 10.9). But, again by (3.12),

$$\big(L^0_{U^*T\xi_\varphi}(T'\xi_\varphi) \mid T'\xi_\varphi\big) \leqslant \|T\| \cdot \|T'\xi_\varphi\|^2, \quad T' \in \pi_\varphi(\mathfrak{A})',$$

so, for every $\xi \in H_\varphi$, if $(T'_n)_{n\geqslant 1}$ is a sequence in $\pi_\varphi(\mathfrak{A})'$ with $T'_n\xi_\varphi \to \xi$ then we have automatically $\lim\limits_{n,k\to\infty} \big(L^0_{U^*T\xi_\varphi}(T'_n\xi_\varphi - T'_k\xi_\varphi) \mid T'_n\xi_\varphi - T'_k\xi_\varphi\big) = 0$. By the Friedrichs extension theorem, as formulated in [34], 9.6, it follows that the domain of $\Psi^\sim(T)$ is H_φ, that is $\Psi^\sim(T) \in \pi_\varphi(\mathfrak{A})''$. Clearly, $\Psi^\sim(T)\xi_\varphi = L^0_{U^*T\xi_\varphi}\xi_\varphi = U^*T\xi_\varphi$.

Thus we have a positive linear map $\pi_\varphi(\mathfrak{A})'' \ni T \mapsto \Psi^\sim(T) \in \pi_\varphi(\mathfrak{A})''$ satisfying $\Psi^\sim(T)\xi_\varphi = U^*T\xi_\varphi$. The proof of the normality of $\Psi^\sim$ is completely similar to that one of Ψ in Proposition 3.1 and of E_λ in Proposition 3.2.

(e) $\Rightarrow$ (b) is similar to (d) $\Rightarrow$ (b). By (2.1) we have (3.10) for all $x \in \mathfrak{A}$ and it follows that $US_\varphi \subset S_\varphi U$. On the other hand, by (e) we have for every $T \in \pi_\varphi(\mathfrak{A})''$

$$U^*S_\varphi T\xi_\varphi = U^*T^*\xi_\varphi = \Psi^\sim(T^*)\xi_\varphi = S_\varphi \Psi^\sim(T)\xi_\varphi = S_\varphi U^*T\xi_\varphi,$$

so $U^*S_\varphi \subset S_\varphi U^*$ and it follows that $US_\varphi^* \subset (S_\varphi U^*)^* \subset (U^*S_\varphi)^* = S_\varphi^* U$ (see e.g. [34], 9.2). We conclude that (3.11) holds, which implies (b) as in the proof of (d) $\Rightarrow$ (b).

(ii) Let us now assume that Φ is multiplicative.

For (c) $\Rightarrow$ (f). By (2.2) and (2.5), we have $U^*\pi_\varphi(\mathfrak{A})'U \subset U^*\pi_\varphi\big(\Phi(\mathfrak{A})\big)'U \subset \pi_\varphi(\mathfrak{A})'$ and, taking into account (c) and (3.2), it follows that

$$U^*TU = J_\varphi U^* J_\varphi T J_\varphi U J_\varphi \in \pi_\varphi(\mathfrak{A})'' \quad \text{for all } T \in \pi_\varphi(\mathfrak{A})'',$$

and taking into account that $U^*J_\varphi T J_\varphi U$, we get $J_\varphi T J_\varphi \in \pi_\varphi(\mathfrak{A})'$.

On the other hand, if (f) holds then (e) is satisfied with $\Psi^{\sim}(T) = \mathrm{Ad}(U^*)(T)$ and (3.1) implies that $\Psi^{\sim}$ is a left inverse of Ψ. Furthermore, using (3.1) we get for every $T_{-1}, T_0, T_1 \in \pi_\varphi(\mathfrak{A})''$

$$\begin{aligned}\Psi^{\sim}\big(\Psi(T_{-1})T_0\Psi(T_1)\big) &= U^*\Psi(T_{-1})T_0\Psi(T_1)U \\ &= T_{-1}U^*T_0UT_1 = T_{-1}\Psi^{\sim}(T_0)T_1,\end{aligned} \tag{3.13}$$

which implies that $(\Psi\circ\Psi^{\sim})\big(\Psi(T_{-1})T_0\Psi(T_1)\big) = \Psi(T_{-1})(\Psi\circ\Psi^{\sim})(T_0)\Psi(T_1)$. Therefore $\Psi\circ\Psi^{\sim}$ is a conditional expectation onto $\Psi\big(\pi_\varphi(\mathfrak{A})''\big) = \pi_\varphi\big(\Phi(\mathfrak{A})\big)''$.

For (b) $\Rightarrow$ (g). We have for every $T \in \pi_\varphi(\mathfrak{A})''$ and $t \in \mathbb{R}$

$$\Delta_\varphi^{\mathrm{i}t}\Psi(T)\Delta_\varphi^{-\mathrm{i}t}\xi\varphi = \Delta_\varphi^{\mathrm{i}t}UT\xi\varphi = U\Delta_\varphi^{\mathrm{i}t}T\xi\varphi = \Psi(\Delta_\varphi^{\mathrm{i}t}T\Delta_\varphi^{-\mathrm{i}t})\xi_\varphi,$$

so $\Delta_\varphi^{\mathrm{i}t}\Psi(T)\Delta_\varphi^{-\mathrm{i}t} = \Psi(\Delta_\varphi^{\mathrm{i}t}T\Delta_\varphi^{-\mathrm{i}t}) \in \Psi\big(\pi_\varphi(\mathfrak{A})''\big) = \pi_\varphi\big(\Phi(\mathfrak{A})\big)''$.

For (g) $\Rightarrow$ (a). Let σ and σ^Ψ denote the modular automorphism groups of $\omega_{\xi_\varphi}|\pi_\varphi(\mathfrak{A})''$ respectively $\omega_{\xi_\varphi}|\pi_\varphi\big(\Phi(\mathfrak{A})\big)''$. Since $\omega_{\xi_\varphi}|\pi_\varphi\big(\Phi(\mathfrak{A})\big)''$ satisfies the KMS condition with respect to $\mathbb{R}\ni t\mapsto \sigma_t|\pi_\varphi\big(\Phi(\mathfrak{A})\big)'' = \mathrm{Ad}(\Delta_\varphi^{\mathrm{i}t})|\pi_\varphi\big(\Phi(\mathfrak{A})\big)''$, we have $\sigma_t^\Psi = \sigma_t|\pi_\varphi\big(\Phi(\mathfrak{A})\big)''$ for all $t\in\mathbb{R}$. Now, $\Psi : \pi_\varphi(\mathfrak{A})'' \to \pi_\varphi\big(\Phi(\mathfrak{A})\big)''$ being a $*$-isomorphism which leaves ω_{ξ_φ} invariant, it follows that $\Psi\circ\sigma_t = \sigma_t^\Psi\circ\Psi = \sigma_t\circ\Psi$ for all $t\in\mathbb{R}$. ∎

We notice that (3.13) implies by induction over $k \geqslant 1$

$$(\Psi^{\sim})^k\big(\Psi^k(T_{-1})T_0\Psi^k(T_1)\big) = T_{-1}(\Psi^{\sim})^k(T_0)T_1, \quad T_{-1}, T_0, T_1 \in \pi_\varphi(\mathfrak{A})''$$

and it follows that

$$\varphi\big(\Phi^k(y_1)x\Phi^k(y_2)\big) = \omega_{\xi_\varphi}\Big(\pi_\varphi(y_1)(\Psi^{\sim})^k\big(\pi_\varphi(x)\big)\pi_\varphi(y_2)\Big), \quad x, y_1, y_2 \in \mathfrak{A}. \tag{3.14}$$

4. ALMOST PERIODICITY

If U is a linear contraction on a Hilbert space H then

$$H_{\mathrm{AP}}^U = \big\{\xi\in H : \{U^n(\xi) : n\in\mathbb{N}\} \text{ is relatively norm-compact}\big\}$$

is clearly a closed linear subspace of H, left invariant by U. Moreover, if U is isometrical then Corollaries 9.10 and 9.6 imply that

$$\begin{aligned}H_{\mathrm{AP}}^U &= \big\{\xi\in H : \{n\in\mathbb{N} : \|U^n(\xi)-\xi\|\leqslant\varepsilon\} \text{ is relatively dense } \forall\varepsilon>0\big\} \\ &= \text{ the closed linear span of all eigenvectors of } U,\end{aligned} \tag{4.1}$$

so in particular we have $UH_{\mathrm{AP}}^U = H_{\mathrm{AP}}^U$. We notice that the decomposition $H = H_{\mathrm{AP}}^U \oplus \big(H\ominus H_{\mathrm{AP}}^U\big)$ is a particular case of the general splitting theorem of K. Jacobs, K. de Leeuw and I. Glicksberg (see e.g. [23], Theorems 2.4.4 and 2.4.5).

Given a state preserving C^*-dynamical system $(\mathfrak{A}, \varphi, \Phi)$ with central $s(\varphi)$, in this section we investigate the lifting of $(H_\varphi)_{\mathrm{AP}}^{U_{\Phi,\varphi}}$ in $\pi_\varphi(\mathfrak{A})''$, that is the set of all those $T\in\pi_\varphi(\mathfrak{A})''$ for which $T\xi_\varphi \in (H_\varphi)_{\mathrm{AP}}^{U_{\Phi,\varphi}}$. As application we shall prove a multiple recurrence result for the operators in the above set.

The following general result is of interest in itself.

LEMMA 4.1. *Let H be a Hilbert space, $\mathcal{M} \subset \mathcal{L}(H)$ a* w*-closed linear subspace, $\xi_0 \in H$ a vector with $\overline{\mathcal{M}'\xi_0} = H$ and $U : H \mapsto H$ a linear isometry such that $\mathcal{M}\xi_0 \subset H^U_{\mathrm{AP}}$. Then the set $\mathcal{G}$ of all linear contractions $\Theta : \mathcal{M} \to \mathcal{M}$ satisfying*

$$\Theta(T)\xi_0 \in \overline{\{U^n T\xi_0 : n \in \mathbb{N}\}} \quad \text{for all } T \in \mathcal{M},$$

endowed with the topology of the pointwise so*-convergence, is a compact topological group with respect to composition, having the identical map of $\mathcal{M}$ as neutral element. Moreover, every $\Theta \in \mathcal{G}$ is* w*-continuous and $\mathcal{G}$ has the following recurrence property:*

For every integer $p \geqslant 1$, $\Theta_1, \ldots, \Theta_p \in \mathcal{G}$, $T_1, \ldots, T_p \in \mathcal{M}$, $\xi_1, \ldots, \xi_p \in H$ and $\varepsilon > 0$, there exists a relatively dense $\mathcal{N} \subset \mathbb{N}$ with

$$\|\Theta_j^n(T_j)\xi_j - T_j\xi_j\| \leqslant \varepsilon \quad \text{for all } 1 \leqslant j \leqslant p \text{ and } n \in \mathcal{N}.$$

Proof. Let $\Theta \in \mathcal{G}$ be arbitrary. Then we have, for every $T \in \mathcal{M}$,

$$\Theta(T)\xi_0 \in \overline{\{U^n T\xi_0 : n \in \mathbb{N}\}} \subset \{\xi \in H : \|\xi\| = \|T\xi_0\|\},$$

hence $\|\Theta(T)\xi_0\| = \|T\xi_0\|$. Consequently there exists a well defined linear isometry $U_\Theta : \overline{\mathcal{M}\xi_0} \to \overline{\mathcal{M}\xi_0}$ such that $U_\Theta(T\xi_0) = \Theta(T)\xi_0$ for all $T \in \mathcal{M}$.

If $\Theta_1, \Theta_2 \in \mathcal{G}$ then also $\Theta_1 \circ \Theta_2 : \mathcal{M} \to \mathcal{M}$ belongs to $\mathcal{G}$:

$$(\Theta_1 \circ \Theta_2)(T)\xi_0 \in \overline{\{U^n\Theta_2(T)\xi_0 : n \in \mathbb{N}\}} \subset \overline{\{U^{n+k}T\xi_0 : n, k \in \mathbb{N}\}}, \quad T \in \mathcal{M}.$$

Since the identical map on $\mathcal{M}$ clearly belongs to $\mathcal{G}$, it follows that $\mathcal{G}$ is a semigroup with respect to composition, having the identical map of $\mathcal{M}$ as neutral element. Moreover, $U_{\Theta_1 \circ \Theta_2} = U_{\Theta_1} U_{\Theta_2}$ for all $\Theta_1, \Theta_2 \in \mathcal{G}$. In particular, if $\Theta \in \mathcal{G}$ then we have $\Theta^n \in \mathcal{G}$ and $U_{\Theta^n} = U_\Theta^n$ for all $n \geqslant 1$, hence

$$\{U_\Theta^n(T\xi_0) : n \in \mathbb{N}\} = \{U_{\Theta^n}(T\xi_0) : n \in \mathbb{N}\} = \{\Theta^n(T)\xi_0 : n \in \mathbb{N}\}$$

is contained in the norm-compact set $\overline{\{U^n T\xi_0 : n \in \mathbb{N}\}}$. Consequently,

$$\mathcal{M}\xi_0 \subset \left(\overline{\mathcal{M}\xi_0}\right)^{U_\Theta}_{\mathrm{AP}}, \quad \Theta \in \mathcal{G}. \tag{4.2}$$

The proof of the w-continuity of every $\Theta \in \mathcal{G}$ is similar to that one of $\Psi_{\Phi,\varphi}$ in Proposition 3.1: for every $\xi_j = T_j'\xi_0, T_j' \in \mathcal{M}', j = 1, 2$, the linear functional

$$\mathcal{M} \ni T \mapsto (\Theta(T)\xi_1 \mid \xi_2) = (\Theta(T)\xi_0 \mid (T_1')^* T_2'\xi_0) = (T\xi_0 \mid (U_\Theta)^*(T_1')^* T_2'\xi_0),$$

where $(U_\Theta)^*$ denotes the adjoint of U_Θ considered an application $\overline{\mathcal{M}\xi_0} \to H$, is wo-continuous and, using $\overline{\mathcal{M}'\xi_0} = H$, standard arguments (see e.g. [34], Theorem 1.10) yield the desired w-continuity.

For the recurrence property of $\mathcal{G}$, let $p \geqslant 1$, $\Theta_1, \ldots, \Theta_p \in \mathcal{G}$, $T_1, \ldots, T_p \in \mathcal{M}$, $\xi_1, \ldots, \xi_p \in H$ and $\varepsilon > 0$ be arbitrary. Choose first $T_1', \ldots, T_p' \in \mathcal{M}'$ such that

$$\|\xi_j - T_j'\xi_0\| \leqslant \frac{\varepsilon}{3\|T_j\|}, \quad 1 \leqslant j \leqslant p. \tag{4.3}$$

Taking into account (4.2), by (the remark after) Corollary 9.10 we get a relatively dense set $\mathcal{N} \subset \mathbb{N}$ such that

$$\|\Theta_j^n(T_j)\xi_0 - T_j\xi_0\| = \|U_{\Theta_j}^n(T_j\xi_0) - T_j\xi_0\| \leqslant \frac{\varepsilon}{3\|T_j'\|}, \quad 1 \leqslant j \leqslant p, \, n \in \mathcal{N}. \tag{4.4}$$

Now (4.3) and (4.4) yield for all $1 \leqslant j \leqslant p$ and $n \in \mathcal{N}$:

$$\begin{aligned}
&\|\Theta_j^n(T_j)\xi_j - T_j\xi_j\| \\
&\quad \leqslant \big\|\Theta_j^n(T_j)\big(\xi_j - T_j'\xi_0\big)\big\| + \big\|T_j'\big(\Theta_j^n(T_j)\xi_0 - T_j\xi_0\big)\big\| + \big\|T_j\big(T_j'\xi_0 - \xi_j\big)\big\| \\
&\quad \leqslant \|T_j\| \cdot \frac{\varepsilon}{3\|T_j\|} + \|T_j'\| \cdot \frac{\varepsilon}{3\|T_j'\|} + \|T_j\| \cdot \frac{\varepsilon}{3\|T_j\|} = \varepsilon.
\end{aligned}$$

In order to prove the pointwise so-compactness of $\mathcal{G}$, we notice that the topology of the pointwise so-convergence on $\mathcal{G}$ is defined by the pseudo-metrics

$$\mathcal{G} \times \mathcal{G} \ni (\Theta_1, \Theta_2) \mapsto \|\Theta_1(T)\xi_0 - \Theta_2(T)\xi_0\|, \quad T \in \mathcal{M}:$$

the proof is standard by using $\overline{\mathcal{M}'\xi_0} = H$ and the inequality

$$\begin{aligned}
&\|\Theta_1(T)\xi - \Theta_2(T)\xi\| \\
&\quad \leqslant \big\|\Theta_1(T)\big(\xi - T'\xi_0\big)\big\| + \big\|T'\big(\Theta_1(T)\xi_0 - \Theta_2(T)\xi_0\big)\big\| + \big\|\Theta_2(T)\big(T'\xi_0 - \xi\big)\big\| \\
&\quad \leqslant 2 \cdot \|T\| \cdot \|\xi - T'\xi_0\| + \|T'\| \cdot \|\Theta_1(T)\xi_0 - \Theta_2(T)\xi_0\|,
\end{aligned}$$

valid for all $T \in \mathcal{G}$, $\xi \in H$ and $T' \in \mathcal{M}'$. It follows that the map $\mathcal{G} \ni \Theta \mapsto \big(\Theta(T)\xi_0\big)_T \in \prod\limits_{T \in \mathcal{M}} \overline{\{U^n T\xi_0 : n \in \mathbb{N}\}}$ is a homeomorphism onto its range, where on the range space the product topology of the compact norm-topologies is considered. Thus the compactness of $\mathcal{G}$ will follow once we show that the range of the above map is closed.

For let $\big(\xi(T)\big)_T$ be an element of the closure of the range. Then there is a net $(\Theta_\iota)_\iota$ in $\mathcal{G}$ such that, for every $T \in \mathcal{M}$, $\Theta_\iota(T)\xi_0 \to \xi(T)$ in the norm-topology. It follows that $\Theta_\iota(T)T'\xi_0 = T'\Theta_\iota(T)\xi_0 \to T'\xi(T)$ for all $T \in \mathcal{M}, T' \in \mathcal{M}'$ and, using the inequality

$$|\Theta_{\iota_1}(T)\xi - \Theta_{\iota_2}(T)\xi\| \leqslant 2 \cdot \|T\| \cdot \|\xi - T'\xi_0\| + \|T'\| \cdot \|\Theta_{\iota_1}(T)\xi_0 - \Theta_{\iota_2}(T)\xi_0\|,$$

it is easy to verify that $\big(\Theta_\iota(T)\xi\big)_\iota$ is a Cauchy sequence for all $T \in \mathcal{M}$ and $\xi \in H$. For every $T \in \mathcal{M}$, $H \ni \xi \mapsto \lim\limits_\iota \Theta_\iota(T)\xi \in H$ is a bounded linear operator $\Theta(T)$ with $\|\Theta(T)\| \leqslant \|T\|$. Since $\mathcal{M} \ni \Theta_\iota(T) \xrightarrow{\text{so}} \Theta(T)$ and the closed balls in $\mathcal{M}$ are so-closed, we have $\Theta(T) \in \mathcal{M}$ for all $t \in \mathcal{M}$. Now an easy verification shows that the obtained map $\Theta : \mathcal{M} \to \mathcal{M}$ belongs to $\mathcal{G}$. Clearly, $\Theta(T)\xi_0 = \lim\limits_\iota \Theta_\iota(T)\xi_0 = \xi(T)$ for all $T \in \mathcal{M}$.

Next we prove that the semigroup operation on $\mathcal{G}$ is continuous. Indeed, we have for any $\Theta_1, \Theta_2, \Theta_1', \Theta_2' \in \mathcal{G}$ and $T \in \mathcal{M}$

$$\begin{aligned}
&\big\|\big(\Theta_1 \circ \Theta_1'\big)(T)\xi_0 - \big(\Theta_2 \circ \Theta_2'\big)(T)\xi_0\big\| \\
&\quad \leqslant \big\|\Theta_1\big(\Theta_1'(T)\big)\xi_0 - \Theta_1\big(\Theta_2'(T)\big)\xi_0\big\| + \big\|\Theta_1\big(\Theta_2'(T)\big)\xi_0 - \Theta_2\big(\Theta_2'(T)\big)\xi_0\big\| \\
&\quad = \big\|U_{\Theta_1}\big(\Theta_1'(T)\xi_0 - \Theta_2'(T)\xi_0\big)\big\| + \big\|\Theta_1\big(\Theta_2'(T)\big)\xi_0 - \Theta_2\big(\Theta_2'(T)\big)\xi_0\big\| \\
&\quad = \big\|\Theta_1'(T)\xi_0 - \Theta_2'(T)\xi_0\big\| + \big\|\Theta_1\big(\Theta_2'(T)\big)\xi_0 - \Theta_2\big(\Theta_2'(T)\big)\xi_0\big\|.
\end{aligned}$$

Finally we prove that every $\Theta \in \mathcal{G}$ is invertible and Θ^{-1} belongs to $\mathcal{G}$. Indeed, according to the above proved recurrence property of $\mathcal{G}$, for every finite $\mathcal{F} \subset \mathcal{M}$ and every $\varepsilon > 0$ there exists some integer $n(\mathcal{F}, \varepsilon) \geqslant 1$ such that

$$\|\Theta^{n(\mathcal{F},\varepsilon)}(T)\xi_0 - T\xi_0\| \leqslant \varepsilon \quad \text{for all } T \in \mathcal{F}.$$

Then the net $\left(\Theta^{n(\mathcal{F},\varepsilon)}\right)_{(\mathcal{F},\varepsilon)}$ (where $(\mathcal{F},\varepsilon) \leqslant (\mathcal{F}',\varepsilon')$ means that $\mathcal{F} \subset \mathcal{F}'$ and $\varepsilon \geqslant \varepsilon'$) converges in $\mathcal{G}$ to the identity map on $\mathcal{M}$, that is to the neutral element of the semigroup $\mathcal{G}$. It follows that, for any limit point Θ' of the net $\left(\Theta^{n(\mathcal{F},\varepsilon)-1}\right)_{(\mathcal{F},\varepsilon)}$ in $\mathcal{G}$, we have $\Theta \circ \Theta' = \Theta' \circ \Theta =$ the identity map on $\mathcal{M}$.

Now, since $\mathcal{G}$ is a group with respect to composition and the composition is continuous with respect to the compact topology of $\mathcal{G}$, according to a theorem of R. Ellis ([10], see also [11], [27], Section 3 and [5], Section 4), we infer that $\mathcal{G}$ is actually a compact topological group. ∎

Now we prove the main result of this section.

Theorem 4.2. *Let $(\mathfrak{A},\varphi,\Phi)$ be a state preserving C^*-dynamical system such that the support $s(\varphi)$ of φ in $\mathfrak{A}^{**}$ is central and let us denote*

$$\begin{aligned} P^{\mathrm{AP}}_{\Phi,\varphi} &= \textit{the orthogonal projection of } H_\varphi \textit{ onto } (H_\varphi)^{U_{\Phi,\varphi}}_{\mathrm{AP}}, \\ \mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi} &= \left\{T \in \pi_\varphi(\mathfrak{A})'' : T\xi_\varphi \in (H_\varphi)^{U_{\Phi,\varphi}}_{\mathrm{AP}}\right\}, \\ \mathcal{A}^{\mathrm{AP}}_{\Phi,\varphi} &= \textit{the linear span of } \bigcup_{\substack{\lambda\in\mathbb{C}\\|\lambda|=1}} \left\{T \in \pi_\varphi(\mathfrak{A})'' : \Psi_{\Phi,\varphi}(T) = \lambda T\right\}. \end{aligned}$$

Then:

(i) $\overline{\mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi}\xi_\varphi} = (H_\varphi)^{U_{\Phi,\varphi}}_{\mathrm{AP}}$*;*

(ii) $\mathcal{A}^{\mathrm{AP}}_{\Phi,\varphi}$ *is a $*$-subalgebra of $\pi_\varphi(\mathfrak{A})''$ containing 1_{H_φ}, whose* w*-closure is* $\mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi}$*;*

(iii) $\mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi}$ *is a von Neumann algebra, which is left invariant by the modular automorphism group $\mathbb{R} \ni t \mapsto \mathrm{Ad}(\Delta^{\mathrm{it}}_\varphi)|\pi_\varphi(\mathfrak{A})''$ of $\omega_{\xi_\varphi}|\pi_\varphi(\mathfrak{A})''$;*

(iv) $\Psi_{\Phi,\varphi}|\mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi}$ *is a $*$-automorphism of $\mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi}$;*

(v) *there is a normal conditional expectation $E^{\mathrm{AP}}_{\Phi,\varphi}$ of $\pi_\varphi(\mathfrak{A})''$ onto $\mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi}$, which preserves $\omega_{\xi_\varphi}|\pi_\varphi(\mathfrak{A})''$ and it is uniquely defined by*

$$E^{\mathrm{AP}}_{\Phi,\varphi}(T)\xi_\varphi = P^{\mathrm{AP}}_{\Phi,\varphi}T\xi_\varphi, \quad T \in \pi_\varphi(\mathfrak{A})'' \tag{4.5}$$

and commutes with $\Psi_{\Phi,\varphi}$.

Moreover:

(UR) *For any integers $p \geqslant 1$ and $m_1,\dots,m_p \geqslant 1$, $T_1,\dots,T_p \in \mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi}$, $\xi_1,\dots,\xi_p \in H_\varphi$ and $\varepsilon > 0$, there exists a relatively dense $\mathcal{N} \subset \mathbb{N}$ with*

$$\|\Psi^{m_j\cdot n}_{\Phi,\varphi}(T_j)\xi_j - T_j\xi_j\| \leqslant \varepsilon \quad \textit{for all } 1 \leqslant j \leqslant p \textit{ and } n \in \mathcal{N}.$$

Proof. We shall write for convenience $U = U_{\Phi,\varphi}$, $\Psi = \Psi_{\Phi,\varphi}$, $P^{\mathrm{AP}} = P^{\mathrm{AP}}_{\Phi,\varphi}$, $\mathcal{M}^{\mathrm{AP}} = \mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi}$, $\mathcal{A}^{\mathrm{AP}} = \mathcal{A}^{\mathrm{AP}}_{\Phi,\varphi}$ and $E^{\mathrm{AP}} = E^{\mathrm{AP}}_{\Phi,\varphi}$.

Clearly, $\mathcal{A}^{\mathrm{AP}}$ is a $*$-subalgebra of $\pi_\varphi(\mathfrak{A})''$ containing 1_{H_φ} and contained in $\mathcal{M}^{\mathrm{AP}}$. By Proposition 3.1 and by (4.1) we have

$$\overline{\mathcal{A}^{\mathrm{AP}}\xi_\varphi} = \text{ the closed linear span of all eigenvectors of } U = (H_\varphi)^U_{\mathrm{AP}},$$

so $\overline{\mathcal{M}^{\mathrm{AP}}\xi_\varphi} = (H_\varphi)^U_{\mathrm{AP}}$.

$\mathcal{M}^{\mathrm{AP}}$ is plainly a wo-closed, hence w-closed linear subspace of $\pi_\varphi(\mathfrak{A})''$, left invariant by Ψ. In the sequel we prove the w-density of $\mathcal{A}^{\mathrm{AP}}$ in $\mathcal{M}^{\mathrm{AP}}$. In particular, it will follow that $\mathcal{M}^{\mathrm{AP}}$ is a von Neumann algebra.

Let $\mathcal{G}$ denote the set of all linear contractions $\Theta : \mathcal{M}^{\mathrm{AP}} \to \mathcal{M}^{\mathrm{AP}}$ satisfying

$$\Theta(T)\xi_\varphi \in \overline{\{U^n T\xi_\varphi : n \in \mathbb{N}\}} \quad \text{for all } T \in \mathcal{M}^{\mathrm{AP}}.$$

By Lemma 4.1, every $\Theta \in \mathcal{G}$ is w-continuous and $\mathcal{G}$ is a compact topological group with respect to composition and the topology of the pointwise so-convergence, having the identical map on $\mathcal{M}^{\mathrm{AP}}$ as neutral element. Since $\Psi(T)\xi_\varphi = UT\xi_\varphi$, $\Psi|\mathcal{M}^{\mathrm{AP}}$ belongs to $\mathcal{G}$. Therefore $\Psi|\mathcal{M}^{\mathrm{AP}} : \mathcal{M}^{\mathrm{AP}} \to \mathcal{M}^{\mathrm{AP}}$ is invertible and $\mathbb{Z} \ni n \mapsto (\Psi|\mathcal{M}^{\mathrm{AP}})^n \in \mathcal{G}$ is a group homomorphism.

Let $B(\mathbb{Z})$ denote the Bohr compactification of the discrete additive group $\mathbb{Z}$, and $n \mapsto b(n)$ the canonical imbedding of $\mathbb{Z}$ into $B(\mathbb{Z})$ (see e.g. [32], Section 1.8). We notice that the dual map of the above imbedding is a group isomorphism of the discrete group $\widehat{B(\mathbb{Z})}$ onto the compact multiplicative group $\widehat{\mathbb{Z}} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Let $g \mapsto \langle g, \lambda\rangle$ denote the continuous character on $B(\mathbb{Z})$ corresponding to $\lambda \in \widehat{\mathbb{Z}}$, so that $\langle b(n), \lambda\rangle = \lambda^n$.

By the universality property of the Bohr compactification (see e.g. [9], Section 16.1) there exists a continuous group homomorphism $B(\mathbb{Z}) \ni g \mapsto \rho(g) \in \mathcal{G}$ such that $(\Psi|\mathcal{M}^{\mathrm{AP}})^n = \rho(b(n))$ for all $n \in \mathbb{Z}$. Denoting the normalized Haar measure of $B(\mathbb{Z})$ by m, let us consider the weak integrals

$$T_\lambda = \text{w-}\int\limits_{B(\mathbb{Z})} \overline{\langle g, \lambda\rangle}\rho(g)(T)\,\mathrm{d}m(g) \in \mathcal{M}^{\mathrm{AP}}, \quad T \in \mathcal{M}^{\mathrm{AP}},\, \lambda \in \widehat{\mathbb{Z}}.$$

It is easily seen that $\rho(g)\big(T_\lambda\big) = \langle g, \lambda\rangle T_\lambda$, hence $\Psi\big(T_\lambda\big) = \rho(b(1))\big(T_\lambda\big) = \lambda T_\lambda$, and so $T_\lambda \in \mathcal{A}^{\mathrm{AP}}$. Therefore the w-density of $\mathcal{A}^{\mathrm{AP}}$ in $\mathcal{M}^{\mathrm{AP}}$ follows if we show that every $T \in \mathcal{M}^{\mathrm{AP}}$ belongs to the w-closed linear span of $\{T_\lambda : \lambda \in \widehat{\mathbb{Z}}\}$. But this follows easily by using the Hahn-Banach theorem.

If ψ is a w-continuous linear functional on $\mathcal{M}^{\mathrm{AP}}$ vanishing on $\mathfrak{A}$, then the Fourier coefficients of the continuous function $B(\mathbb{Z}) \ni g \mapsto \psi\big(\rho(g)(T)\big)$ are

$$\int\limits_{B(\mathbb{Z})} \overline{\langle g, \lambda\rangle}\psi\big(\rho(g)(T)\big)\,\mathrm{d}m(g) = \psi\big(T_\lambda\big) = 0, \quad \lambda \in \widehat{\mathbb{Z}}$$

and the uniqueness theorem for Fourier transforms (see e.g. [32], 1.7.3 (b)) yields that it vanishes identically. In particular, $\psi(T) = \psi\big(\rho(b(0))(T)\big) = 0$.

Next we notice that, according to Proposition 3.2, the modular automorphism group of $\omega_{\xi_\varphi}|\pi_\varphi(\mathfrak{A})''$ leaves $\{T \in \pi_\varphi(\mathfrak{A})'' : \Psi(T) = \lambda T\}$ invariant for any $\lambda \in \mathbb{C}, |\lambda| = 1$, so it leaves $\mathcal{A}^{\mathrm{AP}}$, and also the w-closure $\mathcal{M}^{\mathrm{AP}}$ of this, invariant. By a well known result of M. Takesaki (see [38]) it follows the existence of a normal conditional expectation E^{AP} of $\pi_\varphi(\mathfrak{A})''$ onto $\mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi}$, which preserves $\omega_{\xi_\varphi}|\pi_\varphi(\mathfrak{A})''$.

For (4.5) we notice that, for every $T \in \pi_\varphi(\mathfrak{A})''$, $E^{\mathrm{AP}}(T)\xi\varphi \in (H_\varphi)^U_{\mathrm{AP}}$ by the definition of $\mathcal{M}^{\mathrm{AP}}$, and $T\xi\varphi - E^{\mathrm{AP}}(T)\xi\varphi$ is orthogonal to $(H_\varphi)^U_{\mathrm{AP}}$ because

$$\big(T\xi_\varphi - E^{\mathrm{AP}}(T)\xi_\varphi \,\big|\, T_0\xi_\varphi\big) = \big(T_0^*T\xi_\varphi - E^{\mathrm{AP}}(T_0^*T)\xi_\varphi \,\big|\, \xi_\varphi\big) = 0$$

for all $T_0 \in \mathcal{M}^{\mathrm{AP}}$ and $\overline{\mathcal{M}^{\mathrm{AP}}\xi_\varphi} = (H_\varphi)^U_{\mathrm{AP}}$. The commutation of E^{AP} with Ψ follows by using the commutation of P^{AP} with U:

$$E^{\mathrm{AP}}\big(\Psi(T)\big)\xi_\varphi = P^{\mathrm{AP}}UT\xi_\varphi = UP^{\mathrm{AP}}\xi_\varphi = \Psi\big(E^{\mathrm{AP}}(T)\big)\xi_\varphi.$$

On the other hand, since the injective normal $*$-endomorphism Ψ of $\pi_\varphi(\mathfrak{A})''$ maps $\mathcal{A}^{\mathrm{AP}}$ onto itself, it follows that Ψ maps $\mathcal{M}^{\mathrm{AP}} = \overline{\mathcal{A}^{\mathrm{AP}}}^{\mathrm{w}}$ bijectively onto itself.

Finally, applying to $\mathcal{G}$ the recurrence property from Lemma 4.1 with $\Theta_j = (\Psi|\mathcal{M}^{\mathrm{AP}})^{m_j}$, the uniform recurrence property (UR) follows. ∎

It follows that the operators in $\mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi}$ have a strong multiple recurrence property:

Corollary 4.3. *Let $(\mathfrak{A}, \varphi, \Phi)$ be a state preserving C^*-dynamical system such that $s(\varphi) \in \mathfrak{A}^{**}$ is central. Then, for any integers $p \geqslant 1$ and $m_1, \dots, m_p \geqslant 1$, $T_1, \dots, T_p \in \mathcal{M}^{\mathrm{AP}}_{\Phi,\varphi}$, $S_1, \dots, S_{p-1} \in \mathcal{L}(H_\varphi)$, $\xi \in H_\varphi$ and $\varepsilon > 0$, there exists a relatively dense subset $\mathcal{N}$ of $\mathbb{N}$ such that*

$$\big\|\Psi^{m_1\cdot n}_{\Phi,\varphi}(T_1)S_1\Psi^{m_2\cdot n}_{\Phi,\varphi}(T_2)S_2\cdots\Psi^{m_p\cdot n}_{\Phi,\varphi}(T_p)\xi - T_1S_1T_2S_2\cdots T_p\xi\big\| \leqslant \varepsilon$$

for all $n \in \mathcal{N}$.

Proof. We shall write simply $\Psi = \Psi_{\Phi,\varphi}$.

Applying (UR) from Theorem 4.2 with $\xi_j = S_jT_{j+1}S_{j+1}\cdots T_p\xi$, we get a relatively dense $\mathcal{N} \subset \mathbb{N}$ such that

$$\big\|\big(\Psi^{m_j\cdot n}(T_j) - T_j\big)S_jT_{j+1}S_{j+1}\cdots T_p\xi\big\| \leqslant \varepsilon\Big(\sum_{j=1}^p \|T_1\|\,\|S_1\|\cdots\|T_{j-1}\|\,\|S_{j-1}\|\Big)^{-1}$$

for all $1 \leqslant j \leqslant p$ and $n \in \mathcal{N}$ (where $\|T_1\|\,\|S_1\|\cdots\|T_{j-1}\|\,\|S_{j-1}\| = 1$ for $j = 1$). Since

$$\begin{aligned}&\Psi^{m_1\cdot n}(T_1)S_1\Psi^{m_2\cdot n}(T_2)S_2\cdots\Psi^{m_p\cdot n}(T_p) - T_1S_1T_2S_2\cdots T_p\\ &\quad= \sum_{j=1}^p \Psi^{m_1\cdot n}(T_1)S_1\cdots\Psi^{m_{j-1}\cdot n}(T_{j-1})S_{j-1}\big(\Psi^{m_j\cdot n}(T_j) - T_j\big)S_jT_{j+1}S_{j+1}\cdots T_p,\end{aligned}$$

it follows for every $n \in \mathcal{N}$:

$$\begin{aligned}&\big\|\Psi^{m_1\cdot n}_{\Phi,\varphi}(T_1)S_1\Psi^{m_2\cdot n}_{\Phi,\varphi}(T_2)S_2\cdots\Psi^{m_p\cdot n}_{\Phi,\varphi}(T_p)\xi - T_1S_1T_2S_2\cdots T_p\xi\big\|\\ &\quad\leqslant \sum_{j=1}^p \|T_1\|\,\|S_1\|\cdots\|T_{j-1}\|\,\|S_{j-1}\|\big\|\big(\Psi^{m_j\cdot n}(T_j) - T_j\big)S_jT_{j+1}S_{j+1}\cdots T_p\xi\big\|\\ &\quad\leqslant \varepsilon. \quad ∎\end{aligned}$$

Theorem 1.6 is an immediate consequence of Corollary 4.3.

5. ERGODICITY AND WEAK MIXING

Ergodicity of a linear contraction U on a Hilbert space H is usually defined by requiring that the fixed point space $H^U = \{\xi \in H : U\xi = \xi\}$ be one-dimensional. By the mean ergodic theorem of von Neumann this is equivalent to the existence of some $\xi_0 \in H$ with $\|\xi_0\| = 1$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} U^k \xi = (\xi \mid \xi_0) \cdot \xi_0 \quad \text{for all } \xi \in H,$$

or equivalently,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \big((U^k \xi \mid \eta) - (\xi \mid \xi_0) \cdot (\xi_0 \mid \eta)\big) = 0 \quad \text{for all } \xi, \eta \in H.$$

Since by Section 144 in Chapitre X from [31] we have $H^U = H^{U^*}$, U is ergodic if and only if U^* is ergodic.

Let us now characterize ergodicity by using invariant means on the additive semigroup $\mathbb{N}$. We notice that an invariant mean on $\mathbb{N}$ is a state M on $l^\infty(\mathbb{N})$ such that $M\big((\lambda_j)_{j\geqslant 0}\big) = M\big((\lambda_{j+1})_{j\geqslant 0}\big)$ for all $(\lambda_j)_{j\geqslant 0} \in l^\infty(\mathbb{N})$. It is well known that there exists invariant mean on $\mathbb{N}$ (see [8] or [17], Theorem 1.2.1).

LEMMA 5.1. *Let U be a linear contraction on a Hilbert space H, $\xi_0 \in H$, $\|\xi_0\| = 1$, and M an invariant mean on $\mathbb{N}$. Then $H^U = \mathbb{C} \cdot \xi_0$ if and only if*

$$M\Big(\big((U^j \xi \mid \eta) - (\xi \mid \xi_0)(\xi_0 \mid \eta)\big)_{j\geqslant 0}\Big) = 0 \quad \textit{for all } \xi, \eta \in H.$$

Proof. If $H^U = \mathbb{C} \cdot \xi_0$ then we have for every $\xi, \eta \in H$

$$\begin{aligned}
&\Big|\frac{1}{n+1} \sum_{k=0}^{n} \big((U^{j+k} \xi \mid \eta) - (\xi \mid \xi_0) \cdot (\xi_0 \mid \eta)\big)\Big| \\
&\qquad = \Big|\Big(\frac{1}{n+1} \sum_{k=0}^{n} U^k \xi - (\xi \mid \xi_0) \cdot \xi_0 \Big| (U^*)^j \eta\Big)\Big| \\
&\qquad \leqslant \Big\|\frac{1}{n+1} \sum_{k=0}^{n} U^k \xi - (\xi \mid \xi_0) \cdot \xi_0\Big\| \, \|\eta\|,
\end{aligned}$$

for $j \in \mathbb{N}$, so

$$\lim_{n\to\infty} \Big(\frac{1}{n+1} \sum_{k=0}^{n} \big((U^{j+k} \xi \mid \eta) - (\xi \mid \xi_0) \cdot (\xi_0 \mid \eta)\big)\Big)_{j\geqslant 0} = 0 \quad \text{in } l^\infty(\mathbb{N}).$$

Applying now the invariant mean M to the above equality, it follows that

$$M\Big(\big((U^j \xi \mid \eta) - (\xi \mid \xi_0)(\xi_0 \mid \eta)\big)_{j\geqslant 0}\Big) = 0. \tag{5.1}$$

Conversely, if (5.1) holds for some $\xi \in H^U$ and every $\eta \in H$, then we have $(\xi \mid \eta) - (\xi \mid \xi_0)(\xi_0 \mid \eta) = 0$ for all $\eta \in H$, so $\xi = (\xi \mid \xi_0) \cdot \xi_0 \in \mathbb{C} \cdot \xi_0$. ∎

Let $\mathfrak{A}$ be a C^*-algebra, φ a state on $\mathfrak{A}$, and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\varphi \circ \Phi = \varphi$ and the contraction condition (C) in Theorem 1.2 is satisfied. Taking into account that $\pi_\varphi(\mathfrak{A})\xi_\varphi$ is dense in H_φ, it is easy to verify that Φ is ergodic with respect to φ (that is (E) from Section 1 holds) if and only if the linear contraction $U_{\Phi,\varphi}$ is ergodic, in which case $H_\varphi^{U_{\Phi,\varphi}} = \mathbb{C} \cdot \xi_\varphi$.

Given an invariant mean M on $\mathbb{N}$, Φ is ergodic with respect to φ if and only if

$$(\mathrm{E}_M) \qquad M\Big(\big(\varphi(y\Phi^j(x)) - \varphi(y)\varphi(x)\big)_{j\geqslant 0}\Big) = 0 \quad \text{for all } x, y \in \mathfrak{A}.$$

(cf. [3], end of Section 4.3.3). Indeed, the above condition means that

$$M\Big(\big((U^j_{\Phi,\varphi}\xi \mid \eta) - (\xi \mid \xi_\varphi)(\xi_\varphi \mid \eta)\big)_{j\geqslant 0}\Big) = 0 \tag{5.2}$$

holds for all $\xi, \eta \in \pi_\varphi(\mathfrak{A})\xi_\varphi$, so by $\overline{\pi_\varphi(\mathfrak{A})\xi_\varphi} = H_\varphi$ it is equivalent to the validity of (5.2) for all $\xi, \eta \in H_\varphi$. But, according to Lemma 5.1, this is equivalent to the ergodicity of $U_{\Phi,\varphi}$.

Let $\mathfrak{A}$ be again a C^*-algebra, φ a state on $\mathfrak{A}$ such that $s(\varphi) \in \mathfrak{A}^{**}$ is central, and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\varphi \circ \Phi = \varphi$ and the condition (C) is satisfied. Then, according to (3.8) in Proposition 3.2, Φ is ergodic with respect to φ if and only if $\big\{T \in \pi_\varphi(\mathfrak{A})'' : \Psi_{\Phi,\varphi}(T) = T\big\} = \mathbb{C} \cdot 1_{H_\varphi}$. Furthermore, the ergodicity of Φ with respect to φ is equivalent to each one of the conditions

$$\text{so-}\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \pi_\varphi\big(\Phi^k(x)\big) = \varphi(x)1_{H_\varphi} \qquad \text{for all } x \in \mathfrak{A}, \tag{5.3}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \big(\varphi(y_1\Phi^k(x)y_2) - \varphi(y_1y_2)\varphi(x)\big) = 0 \quad \text{for all } x, y_1, y_2 \in \mathfrak{A}. \tag{5.4}$$

Indeed, if Φ is ergodic with respect to φ and $x \in \mathfrak{A}$, then we have

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \pi_\varphi\big(\Phi^k(x)\big)T'\xi_\varphi = T'\Big(\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} U^k_{\Phi,\varphi}\pi_\varphi(x)\xi_\varphi\Big) = \varphi(x)T'\xi_\varphi$$

for all $T' \in \pi_\varphi(\mathfrak{A})'$. Since $\overline{\pi_\varphi(\mathfrak{A})'\xi_\varphi} = H_\varphi$ and, according to Proposition 3.1 (i), $\big\|\pi_\varphi\big(\Phi^k(x)\big)\big\| = \big\|\Psi^k_{\Phi,\varphi}\big(\pi_\varphi(x)\big)\big\| \leqslant \|\pi_\varphi(x)\|$ for all $k \geqslant 0$, the equality in (5.3) follows. Now (5.3) clearly implies (5.4) and it is easy to verify that (5.4) implies $\text{wo-}\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \pi_\varphi\big(\Phi^k(x)\big) = \varphi(x)1_{H_\varphi}$ for all $x \in \mathfrak{A}$, hence also (E).

If, additionally, Φ is a $*$-homomorphism such that $s(\varphi)\Phi^{**}$ commutes with σ^φ, then Φ is ergodic with respect to φ if and only if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \big(\varphi(\Phi^k(y_1)x\Phi^k(y_2)) - \varphi(y_1y_2)\varphi(x)\big) = 0, \quad x, y_1, y_2 \in \mathfrak{A}. \tag{5.5}$$

Indeed, Φ is ergodic with respect to φ if and only if $U_{\Phi,\varphi}$, hence $U^*_{\Phi,\varphi}$ is ergodic, so if and only if the map $\widetilde{\Psi_{\Phi,\varphi}} : \pi_\varphi(\mathfrak{A})'' \ni T \mapsto U^*_{\Phi,\varphi}TU_{\Phi,\varphi} \in \pi_\varphi(\mathfrak{A})''$ considered in Proposition 3.3, is ergodic with respect to ω_{ξ_φ}. But, according to (3.14), (5.4) written for $\widetilde{\Psi_{\Phi,\varphi}}$ and ω_{ξ_φ} means (5.5).

We summarize the aboves in the following proposition:

PROPOSITION 5.2. *Let $\mathfrak{A}$ be a C^*-algebra, φ a state on $\mathfrak{A}$, and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\varphi \circ \Phi = \varphi$ and the contraction condition* (C) *is satisfied. Given an invariant mean M on $\mathbb{N}$, the following statements are equivalent:*

(a) *Φ is ergodic with respect to φ, that is*

$$\text{(E)} \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \big(\varphi(y\Phi^k(x)) - \varphi(y)\varphi(x)\big) = 0, \quad x, y \in \mathfrak{A};$$

(b) *the linear contraction $U_{\Phi,\varphi}$ is ergodic;*

(c) *we have*

$$(\mathrm{E}_M) \qquad M\Big(\big(\varphi(y\Phi^j(x)) - \varphi(y)\varphi(x)\big)_{j\geqslant 0}\Big) = 0, \quad x, y \in \mathfrak{A}.$$

For central $s(\varphi) \in \mathfrak{A}^{**}$, the above statements are equivalent also with each one of the following:

(d) $\big\{T \in \pi_\varphi(\mathfrak{A})'' : \Psi_{\Phi,\varphi}(T) = T\big\} = \mathbb{C} \cdot 1_{H_\varphi}$*;*

(e) $\lim\limits_{n\to\infty} \frac{1}{n+1} \sum\limits_{k=0}^{n} \big(\varphi(y_1\Phi^k(x)y_2) - \varphi(y_1y_2)\varphi(x)\big) = 0$*,* $x, y_1, y_2 \in \mathfrak{A}$*,*

while if $s(\varphi) \in \mathfrak{A}^{**}$ is central, Φ is a $*$-homomorphism and $s(\varphi)\Phi^{**}$ commutes with σ^φ, then they are equivalent with

(f) $\lim\limits_{n\to\infty} \frac{1}{n+1} \sum\limits_{k=0}^{n} \big(\varphi(\Phi^k(y_1)x\Phi^k(y_2)) - \varphi(y_1y_2)\varphi(x)\big) = 0$*,* $x, y_1, y_2 \in \mathfrak{A}$*.*

For a similar treatment of the weak mixing property we need some preparation.

We call a linear contraction U on a Hilbert space H *weakly mixing* if there exists a vector $\xi_0 \in H$ with $\|\xi_0\| = 1$ such that

$$(5.6) \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \big|(U^k\xi \mid \eta) - (\xi \mid \xi_0)\cdot(\xi_0 \mid \eta)\big| = 0 \quad \text{for all } \xi, \eta \in H.$$

Then U is clearly ergodic, in particular $\xi_0 \in H^U$. It is easy to verify that if U is weakly mixing then also U^* is weakly mixing and, for any other weakly mixing linear contraction V on some Hilbert space K, the linear contraction $U \mathbin{\overline{\otimes}} V$ on the tensor product Hilbert space $H \mathbin{\overline{\otimes}} K$ is again weakly mixing. Furthermore, if U is weakly mixing then by Lemma 9.3 (see Appendix B below) any power U^n, $n \geqslant 1$ is again weakly mixing.

The next basic characterizations of the weak mixing property for a linear contraction on a Hilbert space are standard.

LEMMA 5.3. *For a linear contraction U on a Hilbert space H and an invariant mean M on $\mathbb{N}$, the following statements are equivalent:*

(a) *U is weakly mixing;*

(b) *there exists $\xi_0 \in H$, $\|\xi_0\| = 1$ such that, denoting the orthogonal projection onto $\mathbb{C}\cdot\xi_0$ by P^{ξ_0}, we have for all $\xi \in H$*

$$\text{D-}\lim_{n\to\infty} U^n(\xi) = P^{\xi_0}(\xi) \text{ with respect to the weak topology of } H;$$

(c) 1 *is the only eigenvalue of U of modulus* 1 *and it is simple;*

(d) $H^U \neq \{0\}$ *and the linear contraction $U \mathbin{\overline{\otimes}} U^*$ on the tensor product Hilbert space $H \mathbin{\overline{\otimes}} H$ is ergodic;*

(e) *there exists* $\xi_0 \in H$, $\|\xi_0\| = 1$ *such that*

$$M\Big(\big(\big|(U^j\xi|\eta) - (\xi \mid \xi_0)(\xi_0 \mid \eta)\big|\big)_{j\geqslant 0}\Big) = 0 \quad \textit{for all } \xi,\eta \in H.$$

If U *is isometrical then the above statements are equivalent also with the next one:*

(f) *there exists* $\xi_0 \in H^U$, $\|\xi_0\| = 1$, *such that the only vectors* $\xi \in H$ *having relatively norm-compact orbit* $\{U^n(\xi) : n \in \mathbb{N}\}$ *are those in* $\mathbb{C} \cdot \xi_0$.

Proof. For (a) $\Rightarrow$ (b) we notice that, for any $\xi \in H$, the vector $\xi - (\xi \mid \xi_0)\xi_0$ satisfies condition (a) in Theorem 9.5 (Appendix B), hence also condition (d) in the same theorem. The implications (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c) are trivial. Assuming now that (c) holds and denoting by P the orthogonal projection onto the one-dimensional eigenspace corresponding to the eigenvalue 1, for any $\xi \in H$ the vector $\xi - P(\xi)$ satisfies condition (b) in Theorem 9.5 and Theorem 9.5 implies that we have with respect to the weak topology D-$\lim\limits_{n\to\infty} U^n(\xi) = P(\xi)$. Hence the conditions (a), (b) and (c) are equivalent.

Now (a) $\Rightarrow$ (d) by the remarks before the statement. Conversely, let us assume that (d) holds. If $\xi_0 \in H^U$, $\|\xi_0\| = 1$ then $\xi_0 \otimes \xi_0 \in (H \overline{\otimes} H)^{U \overline{\otimes} U^*}$, $\|\xi_0 \otimes \xi_0\| = 1$ and by the ergodicity of $U \overline{\otimes} U^*$ we have $(H \overline{\otimes} H)^{U \overline{\otimes} U^*} = \mathbb{C} \cdot (\xi_0 \otimes \xi_0)$. But if ξ is any eigenvector of U corresponding to some eigenvalue λ of modulus 1, then

$$\|U^*\xi - \overline{\lambda}\xi\|^2 = \|U^*\xi\|^2 + \|\xi\|^2 - 2\,\mathrm{Re}\,\big(\overline{\lambda}(U\xi \mid \xi)\big) = \|U^*\xi\|^2 - \|\xi\|^2 \leqslant 0$$

(cf. [31], Chapitre X, Section 144) yields $U^*\xi = \overline{\lambda}\xi$. It follows that $\xi \otimes \xi$ belongs to $(H \overline{\otimes} H)^{U \overline{\otimes} U^*}$ and so ξ is a scalar multiple of ξ_0. We conclude that (c) holds.

For the proof of (e) $\Rightarrow$ (c) we first notice that, according to Lemma 5.1, (e) implies that $H^U = \mathbb{C} \cdot \xi_0$. On the other hand, if (e) holds and $U\xi = \lambda\xi$ for some $1 \neq \lambda \in \mathbb{C}$ of modulus 1 then, taking into account that

$$\lambda(\xi \mid \xi_0) = (U\xi \mid \xi_0) = (\xi \mid U^*\xi_0) = (\xi \mid \xi_0) \to (\xi \mid \xi_0) = 0,$$

hence $\big|(U^j\xi \mid \xi) - (\xi \mid \xi_0)(\xi_0 \mid \xi)\big| = \big|(\lambda^j(\xi \mid \xi)\big| = \|\xi\|^2$ for all $j \geqslant 0$, we have $\xi = 0$.

Next we show that (a) together with (d) imply (e). Indeed, if U and $U \overline{\otimes} U^*$ are ergodic then, for every $\xi, \eta \in H$, putting

$$\lambda_j = (U^j\xi \mid \eta), \quad \lambda = (\xi \mid \xi_0)(\xi_0 \mid \eta),$$

we have

$$\big((U \overline{\otimes} U^*)^j(\xi \otimes \eta)\,\big|\,\eta \otimes \xi\big) = |\lambda_j|^2, \quad (\xi \otimes \eta \mid \xi_0 \otimes \xi_0)(\xi_0 \otimes \xi_0 \mid \eta \otimes \xi) = |\lambda|^2$$

and Lemma 5.1 yields

$$M\big((\lambda_j)_{j\geqslant 0}\big) = \lambda, \quad M\big((|\lambda_j|^2)_{j\geqslant 0}\big) = |\lambda|^2. \tag{5.7}$$

Now a moment's reflection shows that, for any $(\lambda_j)_{j\geqslant 0} \in l^\infty(\mathbb{N})$,

$$\begin{aligned} M\big((\lambda_j)_{j\geqslant 0}\big) = \lambda \text{ and } M\big((|\lambda_j|^2)_{j\geqslant 0}\big) = |\lambda|^2 &\Rightarrow M\big((|\lambda_j - \lambda|^2)_{j\geqslant 0}\big) = 0 \\ &\Rightarrow M\big((|\lambda_j - \lambda|)_{j\geqslant 0}\big) = 0, \end{aligned}$$

hence (5.7) implies the equality in (e).

Finally, assuming that U is isometrical, (c) $\Leftrightarrow$ (f) follows from Corollary 9.6. ∎

Now we are prepared to prove the counterpart of Proposition 5.2 for weak mixing (cf. [3], Proposition 4.3.36 and [16], Theorem 5.1):

Proposition 5.4. *Let $\mathfrak{A}$ be a C^*-algebra, φ a state on $\mathfrak{A}$, and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\varphi \circ \Phi = \varphi$ and the contraction condition* (C) *is satisfied. Given an invariant mean M on $\mathbb{N}$, the following statements are equivalent:*

(a) *Φ is weakly mixing with respect to φ, that is*

$$\text{(WM)} \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left|\varphi(y\Phi^k(x)) - \varphi(y)\varphi(x)\right| = 0, \quad x, y \in \mathfrak{A};$$

(b) *the linear contraction $U_{\Phi,\varphi}$ is weakly mixing;*

(c) *we have*

$$(\mathrm{WM}_M) \qquad M\big(\big(\left|\varphi(y\Phi^j(x)) - \varphi(y)\varphi(x)\right|\big)_{j\geqslant 0}\big) = 0, \quad x, y \in \mathfrak{A}.$$

For central $s(\varphi) \in \mathfrak{A}^{**}$ the above statements are equivalent also with each one of the following:

(d) *1 is the only eigenvalue of modulus 1 of the normal positive linear map $\Psi_{\Phi,\varphi}$ and it is simple;*

(e) $\lim\limits_{n\to\infty} \frac{1}{n+1} \sum\limits_{k=0}^{n} \left|\varphi(y_1\Phi^k(x)y_2) - \varphi(y_1y_2)\varphi(x)\right| = 0$, *$x, y_1, y_2 \in \mathfrak{A}$.*

If $s(\varphi) \in \mathfrak{A}^{**}$ is central and Φ is a $*$-homomorphism, then (a)–(e) *are equivalent with*

(f) *there exists no von Neumann algebra $\mathbb{C}\cdot 1_{H_\varphi} \neq \mathcal{M} \subset \pi_\varphi(\mathfrak{A})''$, left invariant by $\Psi_{\Phi,\varphi}$ and such that $\{\Psi^n_{\Phi,\varphi}(T)\xi_\varphi : n \in \mathbb{N}\}$ is relatively compact in H_φ for all $T \in \mathcal{M}$.*

If $s(\varphi) \in \mathfrak{A}^{**}$ is central, Φ is a $*$-homomorphism and $s(\varphi)\Phi^{**}$ commutes with σ^φ, then (a)–(f) *are equivalent with*

(g) $\lim\limits_{n\to\infty} \frac{1}{n+1} \sum\limits_{k=0}^{n} \left|\varphi(\Phi^k(y_1)x\Phi^k(y_2)) - \varphi(y_1y_2)\varphi(x)\right| = 0$, *$x, y_1, y_2 \in \mathfrak{A}$.*

Proof. We shall omit again the subscript Φ, φ, writing $U = U_{\Phi,\varphi}$ and $\Psi = \Psi_{\Phi,\varphi}$.

Taking into account that U is a contraction and $\pi_\varphi(\mathfrak{A})\xi_\varphi$ is dense in H_φ, it is easy to verify that Φ is weakly mixing with respect to φ if and only if (5.6) holds for $H = H_\varphi$ and $\xi_0 = \xi_\varphi$, that is U is weakly mixing. In other words, (a) $\Leftrightarrow$ (b).

Clearly (WM_M) means that

$$(5.8) \qquad M\big(\big(\left|(U^j\xi \mid \eta) - (\xi \mid \xi_\varphi)(\xi_\varphi \mid \eta)\right|\big)_{j\geqslant 0}\big) = 0$$

holds for all $\xi, \eta \in \pi_\varphi(\mathfrak{A})\xi_\varphi$. Since $\|U\| \leqslant 1$ and $\overline{\pi_\varphi(\mathfrak{A})\xi_\varphi} = H_\varphi$, it follows that (WM_M) is equivalent to the validity of (5.8) for all $\xi, \eta \in H_\varphi$. Therefore Lemma 5.3 yields (b) $\Leftrightarrow$ (c).

In the remaining part of the proof we shall assume that the support projection of φ in $\mathfrak{A}^{**}$ is central, or equivalently, that ξ_φ is cyclic also for $\pi_\varphi(\mathfrak{A})'$. Then the equivalence (b) $\Leftrightarrow$ (d) follows immediately by using Proposition 3.2.

If (b) holds then, using the equality in (5.6) for $H = H_\varphi$, $\xi_0 = \xi_\varphi$ and with $\xi = \pi_\varphi(x)\xi_\varphi$, $\eta = T'^*\pi_\varphi(y_1^*)\xi_\varphi$, where $x, y_1 \in \mathfrak{A}$ and $T' \in \pi_\varphi(\mathfrak{A})'$, we get

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left| \left(\pi_\varphi\big(y_1\Phi^k(x)\big)\xi \,\middle|\, \xi_\varphi\right) - (\pi_\varphi(y_1)\xi \mid \xi_\varphi)\varphi(x)\right| = 0 \tag{5.9}$$

for $\xi = T'\xi_\varphi$. Since $\overline{\pi_\varphi(\mathfrak{A})'\xi_\varphi} = H_\varphi$ and Proposition 3.1 implies $\left\|\pi_\varphi\big(\Phi^k(x)\big)\right\| = \left\|\Psi^k\big(\pi_\varphi(x)\big)\right\| \leqslant \|\pi_\varphi(x)\|$ for all $k \geqslant 0$, it follows (5.9) for all $x, y_1 \in \mathfrak{A}$ and $\xi \in H_\varphi$, which yields the equality in (e) for $\xi = \pi_\varphi(y_2)\xi_0$. Conversely, it is easily seen that (e) implies (5.6) for $H = H_\varphi$ and $\xi_0 = \xi_\varphi$, hence also (WM). We conclude that the conditions (a)–(e) are all equivalent.

Assuming that Ψ is a $*$-homomorphism, (d) $\Leftrightarrow$ (f) by Theorem 4.2.

Finally, assuming that Φ is a $*$-homomorphism and $s(\varphi)\Phi^{**}$ commutes with σ^φ, we prove that (b) $\Leftrightarrow$ (g). Indeed, U is weakly mixing if and only if U^* is weakly mixing, hence, by the above part of the proof, if and only if (e) holds for $(\mathfrak{A}, \varphi, \Phi)$ replaced by $\pi_\varphi(\mathfrak{A})''$, ω_{ξ_φ} and the map $\Psi_{\widetilde{\Phi},\dot{\varphi}}\;\pi_\varphi(\mathfrak{A})'' \ni T \mapsto U^*TU \in \pi_\varphi(\mathfrak{A})''$ considered in Proposition 3.3. But by (3.14) this means (g). ∎

The next weak mixing result completes Theorem 4.2 by showing that it is a matter of a splitting result:

Proposition 5.5. *Let $(\mathfrak{A}, \varphi, \Phi)$ be a state preserving C^*-dynamical system such that $s(\varphi) \in \mathfrak{A}^{**}$ is central. Then, for every $T \in \pi_\varphi(\mathfrak{A})''$,*

$$\text{D-}\lim_{n\to\infty} \Psi^n_{\Phi,\varphi}\big(T - E^{\mathrm{AP}}_{\Phi,\varphi}(T)\big) = 0 \quad \textit{with respect to the } \text{wo-}\textit{topology.} \tag{5.10}$$

Proof. We shall write simply $U = U_{\Phi,\varphi}$, $\Psi = \Psi_{\Phi,\varphi}$, $P = P^{\mathrm{AP}}_{\Phi,\varphi}$ and $E = E^{\mathrm{AP}}_{\Phi,\varphi}$.

Putting $\xi_n = \Psi^n\big(T - E(T)\big)\xi_\varphi = U^n\big(T\xi_\varphi - PT\xi_\varphi\big)$, since $T\xi_\varphi - PT\xi_\varphi$ is orthogonal to all eigenvectors of U, Theorem 9.5 yields that D-$\lim\limits_{n\to\infty} \xi_n = 0$ with respect to the weak topology of H_φ. Let $\mathcal{E} \subset \mathbb{N}$ be a set of density zero such that $\lim\limits_{\mathcal{E}\not\ni n\to\infty} (\xi_n \mid \eta) = 0$ for all $\eta \in H_\varphi$. Then we have, for any $\eta \in H_\varphi$, first

$$\lim_{\mathcal{E}\not\ni n\to\infty} \big(\Psi^n\big(T - E(T)\big)T'\xi_\varphi \mid \eta\big) = \lim_{\mathcal{E}\not\ni n\to\infty} \big(\xi_n \mid (T')^*\eta\big) = 0 \quad \text{for all } T' \in \pi_\varphi(\mathfrak{A})',$$

and then, by $\|\Psi\| \leqslant 1$ and the density of $\pi_\varphi(\mathfrak{A})'\xi_\varphi$ in H_φ,

$$\lim_{\mathcal{E}\not\ni n\to\infty} \big(\Psi^n\big(T - E(T)\big)\xi \mid \eta\big) = 0 \quad \text{for all } \xi \in H_\varphi. \quad ∎$$

6. COVARIANT REPRESENTATIONS OF STATE PRESERVING C^*-DYNAMICAL SYSTEMS

In this section, of interest for itself, we investigate the possibility to imbed a state preserving C^*-dynamical system $(\mathfrak{A}, \varphi, \Phi)$ in a state preserving C^*-dynamical system of the form $(\mathcal{M}, \omega_\xi|\mathcal{M}, \mathrm{Ad}(U)|\mathcal{M})$, where $\mathcal{M} \subset \mathcal{L}(H)$ is a von Neumann algebra, ξ is some cyclic vector for $\mathcal{M}$, and $U : H \to H$ is a unitary with $U\xi = \xi$ (cf. [13]). If Φ is a $*$-automorphism of $\mathfrak{A}$ then the GNS representation π_φ yields such an imbedding with $\mathcal{M} = \pi_\varphi(\mathfrak{A})''$, $\xi = \xi_\varphi$ and $U = U_{\Phi,\varphi}$.

In order to get a representation of an arbitrary state preserving C^*-dynamical system $(\mathfrak{A}, \varphi, \Phi)$, in which Φ allows an implementation like (2.7), we need unitary dilations of the linear isometry $U_{\Phi,\varphi}$.

Let us recall that for any linear isometry U on a Hilbert space H there exist:

(1) a unitary operator $\widetilde{U}$ on a Hilbert space $\widetilde{H}$ and
(2) a linear isometry $V : H \to \widetilde{H}$ such that
  (a) $VU = \widetilde{U}V$ (so, $\widetilde{U}VH \subset VH$) and
  (b) $\bigcup_{k=-\infty}^{+\infty} \widetilde{U}^k VH = \bigcup_{k=-\infty}^{-1} \widetilde{U}^k VH$ is dense in $\widetilde{H}$

(see [37], Proposition I.2.3). One can put

$$\widetilde{H} = H \oplus \bigoplus_{k=-\infty}^{-1} (H \ominus UH),$$

$$\widetilde{U}\Big(\xi \oplus \bigoplus_{k=-\infty}^{-1} \xi_k\Big) = (U\xi + \xi_{-1}) \oplus \bigoplus_{k=-\infty}^{-1} \xi_{k-1},$$

$$V\xi = \xi \oplus \bigoplus_{k=-\infty}^{-1} 0.$$

The pair $(\widetilde{U}, V)$ is called the *minimal unitary dilation* of U and it is unique up to natural unitary equivalence. It is easily seen that, for any $\lambda \in \mathbb{C}, |\lambda| = 1$,

$$\{\widetilde{\xi} \in \widetilde{H} : \widetilde{U}\widetilde{\xi} = \lambda\widetilde{\xi}\} = V\big(\{\xi \in H : U\xi = \lambda\xi\}\big). \tag{6.1}$$

For any $T, S \in \mathcal{L}(H)$ with $UT = SU$ we have

$$\widetilde{U}\big(VTV^* + \widetilde{U}^*V\big(S(1_H - UU^*)\big)V^*\widetilde{U}\big)\widetilde{U}^* = V\big(UTU^* + S - SUU^*\big)V^* = VSV^*.$$

Consequently, if $T_0, T_1, T_2, \ldots \in \mathcal{L}(H)$ are such that $UT_0 = T_1U$ and the series

$$\widetilde{T}(T_0, T_1, T_2, \ldots) = VT_0V^* + \sum_{k=1}^{\infty} \widetilde{U}^{-k}V\big(T_k(1_H - UU^*)\big)V^*\widetilde{U}^k \tag{6.2}$$

converges with respect to the weak operator topology of $\mathcal{L}(\widetilde{H})$, then

$$\begin{aligned}&\widetilde{U}\Big(VT_0V^* + \sum_{k=1}^{\infty} \widetilde{U}^{-k}V\big(T_k(1_H - UU^*)\big)V^*\widetilde{U}^k\Big)\widetilde{U}^* \\ &\quad = VT_1V^* + \sum_{k=1}^{\infty} \widetilde{U}^{-k}V\big(T_{k+1}(1_H - UU^*)\big)V^*\widetilde{U}^k.\end{aligned} \tag{6.3}$$

Since the projections $\widetilde{U}^{-k}V(1_H - UU^*)V^*\widetilde{U}^k = \widetilde{U}^{-k}VV^*\widetilde{U}^k - \widetilde{U}^{-k+1}VV^*\widetilde{U}^{k-1}$ and VV^*, $k \geqslant 1$, are mutually orthogonal, if $T_0, T_1, T_2, \ldots \in \mathcal{L}(H)$ are such that

$$\sup_{k\geqslant 0}\|T_k\| < +\infty, \quad UT_0 = T_1U \quad \text{and} \quad T_kUU^* = UU^*T_k, \quad k \geqslant 1 \tag{6.4}$$

then the series (6.2) converges with respect to the strong operator topology of $\mathcal{L}(\widetilde{H})$, hence (6.3) holds. Moreover, if $S_0, S_1, S_2, \ldots \in \mathcal{L}(H)$ is another sequence with

$$\sup_{k\geqslant 0}\|S_k\| < +\infty, \quad US_0 = S_1U \quad \text{and} \quad S_kUU^* = UU^*S_k, \quad k \geqslant 1$$

then straightforward verification shows that

$$\widetilde{T}(T_0, T_1, T_2, \ldots)\widetilde{T}(S_0, S_1, S_2, \ldots) = \widetilde{T}(T_0S_0, T_1S_1, T_2S_2, \ldots). \tag{6.5}$$

We notice that if $T_0, T_1, T_2, \ldots \in \mathcal{L}(H)$ satisfy (6.4) then we have

$$\widetilde{U}^{-k}VT_kUU^*V^*\widetilde{U}^k = \widetilde{U}^{-k}VUU^*T_kUU^*V^*\widetilde{U}^k = \widetilde{U}^{-k+1}VU^*T_kUV^*\widetilde{U}^{k-1}, \quad k \geqslant 1$$

and (6.2) yields

$$\begin{aligned}&\widetilde{T}(T_0, T_1, T_2, \ldots)\\ &\quad = \text{so-}\lim_{n\to\infty}\Big(\sum_{k=2}^{n}\widetilde{U}^{-k+1}V\big(T_{k-1} - U^*T_kU\big)V^*\widetilde{U}^{k-1} + \widetilde{U}^{-n}VT_nV^*\widetilde{U}^n\Big).\end{aligned} \tag{6.6}$$

LEMMA 6.1. *Let $(\mathfrak{A}, \varphi, \Phi)$ be a state preserving C^*-dynamical system and let $U_{\Phi,\varphi}$ denote the linear isometry defined on H_φ by $U_{\Phi,\varphi}(\pi_\varphi(x)\xi_\varphi) = \pi_\varphi(\Phi(x))\xi_\varphi$, $x \in \mathfrak{A}$. Let further*

$$\begin{array}{ccc} H_\varphi & \xrightarrow{U_{\Phi,\varphi}} & H_\varphi \\ {\scriptstyle V_{\Phi,\varphi}}\Big\downarrow & & \Big\downarrow{\scriptstyle V_{\Phi,\varphi}} \\ \widetilde{H}_{\Phi,\varphi} & \xrightarrow{\widetilde{U}_{\Phi,\varphi}} & \widetilde{H}_{\Phi,\varphi} \end{array}$$

be the minimal unitary dilation of $U_{\Phi,\varphi}$. Then

$$\begin{aligned}\widetilde{\pi}_{\Phi,\varphi} : \mathfrak{A} \ni a \mapsto{}& V_{\Phi,\varphi}\pi_\varphi(a)V^*_{\Phi,\varphi}\\ &+ \sum_{k=1}^{\infty}\widetilde{U}^{-k}_{\Phi,\varphi}V_{\Phi,\varphi}\big(\pi_\varphi(\Phi^k(a))(1_{H_\varphi} - U_{\Phi,\varphi}U^*_{\Phi,\varphi})\big)V^*_{\Phi,\varphi}\widetilde{U}^k_{\Phi,\varphi}\\ ={}& \text{so-}\lim_{n\to\infty}\widetilde{U}^{-n}_{\Phi,\varphi}V_{\Phi,\varphi}\pi_\varphi(\Phi^n(a))V^*_{\Phi,\varphi}\widetilde{U}^n_{\Phi,\varphi},\end{aligned}$$

where the series is so*-convergent, is a $*$-representation $\widetilde{\pi}_{\Phi,\varphi} : \mathfrak{A} \to \mathcal{L}(\widetilde{H}_{\Phi,\varphi})$ and*

$$\widetilde{\pi}_{\Phi,\varphi}\big(\Phi(a)\big) = \widetilde{U}_{\Phi,\varphi}\widetilde{\pi}_{\Phi,\varphi}(a)\widetilde{U}^*_{\Phi,\varphi} \quad \text{for all } a \in \mathfrak{A}.$$

Furthermore, the so*-closure of the $*$-subalgebra*

$$\widetilde{\mathcal{A}}_{\Phi,\varphi} = \bigcup_{k=-\infty}^{+\infty}\widetilde{U}^k_{\Phi,\varphi}\widetilde{\pi}_{\Phi,\varphi}(\mathfrak{A})\widetilde{U}^{-k}_{\Phi,\varphi} = \bigcup_{k=-\infty}^{-1}\widetilde{U}^k_{\Phi,\varphi}\widetilde{\pi}_{\Phi,\varphi}(\mathfrak{A})\widetilde{U}^{-k}_{\Phi,\varphi}$$

of $\mathcal{L}(\widetilde{H}_{\Phi,\varphi})$ *is a von Neumann algebra* $\widetilde{\mathcal{M}}_{\Phi,\varphi}$ *and* $\widetilde{\xi}_{\Phi,\varphi} = V_{\Phi,\varphi}\xi_\varphi$ *is a cyclic vector for* $\widetilde{\mathcal{M}}_{\Phi,\varphi}$ *satisfying* $\widetilde{U}_{\Phi,\varphi}\widetilde{\xi}_{\Phi,\varphi} = \widetilde{\xi}_{\Phi,\varphi}$ *and*

$$\varphi(x) = (\widetilde{\pi}_{\Phi,\varphi}(x)\widetilde{\xi}_{\Phi,\varphi} \mid \widetilde{\xi}_{\Phi,\varphi}) \quad \text{for every } x \in \mathfrak{A}.$$

Proof. Put for convenience $U = U_{\Phi,\varphi}$, $\widetilde{H} = \widetilde{H}_{\Phi,\varphi}$, $\widetilde{U} = \widetilde{U}_{\Phi,\varphi}$ and $V = V_{\Phi,\varphi}$.

If $a \in \mathfrak{A}$ and $T_k = \pi_\varphi(\Phi^k(a))$, then by Lemma 2.1 we have $UT_{k-1} = T_kU$ for all $k \geqslant 1$, in particular the conditions in (6.4) are satisfied. Therefore the series (6.2) is so-convergent and (6.6) holds with $T_{k-1} - U^*T_kU = 0$, so we can define

$$\begin{aligned}\widetilde{\pi}(a) =& V\pi_\varphi(a)V^* + \sum_{k=1}^{\infty} \widetilde{U}^{-k}V\big(\pi_\varphi(\Phi^k(a))(1_{H_\varphi} - UU^*)\big)V^*\widetilde{U}^k = \\ =& \text{so-}\lim_{n\to\infty} \widetilde{U}^{-n}V\pi_\varphi(\Phi^n(a))V^*\widetilde{U}^n.\end{aligned}$$

Using (6.5) with $T_k = \pi_\varphi(\Phi^k(a)), a \in \mathfrak{A}$ and $S_k = \pi_\varphi(\Phi^k(b)), b \in \mathfrak{A}$, or the second expression for $\widetilde{\pi}(a)$, it is easy to verify that $\widetilde{\pi} : \mathfrak{A} \ni a \mapsto \widetilde{\pi}(a) \in \mathcal{L}(\widetilde{H})$ is a $*$-representation. On the other hand, using (6.3) with $T_k = \pi_\varphi(\Phi^k(a))$, $a \in \mathfrak{A}$, we get also $\widetilde{\pi}\big(\Phi(a)\big) = \widetilde{U}\widetilde{\pi}(a)\widetilde{U}^*$.

Denoting $\widetilde{\xi} = V\xi$, we have clearly $\widetilde{U}\widetilde{\xi} = \widetilde{U}V\xi_\varphi = VU\xi_\varphi = V\xi_\varphi = \widetilde{\xi}$. Since

$$\overline{\widetilde{U}^k\widetilde{\pi}(\mathfrak{A})\widetilde{U}^{-k}\widetilde{\xi}} = \overline{\widetilde{U}^kV\pi_\varphi(\mathfrak{A})\xi_\varphi} = \widetilde{U}^kVH_\varphi,$$

the minimality of the unitary dilation $(\widetilde{U}, V)$ of U implies that the vector $\widetilde{\xi}$ is cyclic for the $*$-subalgebra

$$\bigcup_{k=-\infty}^{+\infty} \widetilde{U}^k\widetilde{\pi}(\mathfrak{A})\widetilde{U}^{-k} = \bigcup_{k=-\infty}^{-1} \widetilde{U}^k\widetilde{\pi}(\mathfrak{A})\widetilde{U}^{-k} \subset \mathcal{L}(\widetilde{H}).$$

In particular, this $*$-subalgebra is non-degenerate, so its strong operator closure is a von Neumann algebra. Finally, the verification of $\varphi(x) = (\widetilde{\pi}(x)\widetilde{\xi} \mid \widetilde{\xi})$ for every $x \in \mathfrak{A}$ is straightforward. ∎

Let $(\mathfrak{A}, \varphi, \Phi)$ be a state preserving C^*-dynamical system and let us assume that the $*$-representation $\widetilde{\pi} : \mathfrak{A} \to \mathcal{L}(\widetilde{H})$, the unitary operator $\widetilde{U} : \widetilde{H} \to \widetilde{H}$ and the vector $\widetilde{\xi} \in \widetilde{H}$ satisfy the conditions:

(i) $\widetilde{\pi}(\Phi(a)) = \widetilde{U}\widetilde{\pi}(a)\widetilde{U}^*$for all $a \in \mathfrak{A}$;

(ii) $\widetilde{U}\widetilde{\xi} = \widetilde{\xi}$;

(iii) $\widetilde{\xi}$ is cyclic for the $*$-subalgebra $\widetilde{\mathcal{A}} = \bigcup_{k=-\infty}^{+\infty} \widetilde{U}^k\widetilde{\pi}(\mathfrak{A})\widetilde{U}^{-k} \subset \mathcal{L}(\widetilde{H})$;

(iv) $\varphi(x) = (\widetilde{\pi}(x)\widetilde{\xi} \mid \widetilde{\xi})$ for every $x \in \mathfrak{A}$.

Then $\widetilde{\mathcal{A}}\widetilde{\xi} \ni \widetilde{U}^k\widetilde{\pi}(x)\widetilde{\xi} \mapsto \widetilde{U}^k_{\Phi,\varphi}\widetilde{\pi}_{\Phi,\varphi}(x)\widetilde{\xi}_{\Phi,\varphi}$ extends to a unitary operator $W : \widetilde{H} \to \widetilde{H}_{\Phi,\varphi}$ such that

(1) $W\widetilde{\pi}(a) = \widetilde{\pi}_{\Phi,\varphi}(a)W$ for all $a \in \mathfrak{A}$,

(2) $W\widetilde{U} = \widetilde{U}_{\Phi,\varphi}W$,

(3) $W\widetilde{\xi} = \widetilde{\xi}_{\Phi,\varphi}$.

We notice that for the verification of $W\widetilde{\xi} = \widetilde{\xi}_{\Phi,\varphi}$ we need the existence of a bounded net $(v_\kappa)_{\kappa\in K} \subset \mathfrak{A}$ and a net $(n_\kappa)_{\kappa\in K} \subset \mathbb{Z}$ such that, simultaneously,

— $\big(\widetilde{U}^{n_\kappa}\widetilde{\pi}(v_\kappa)\widetilde{U}^{-n_\kappa}\big)_{\kappa\in K}$ is a left approximate unit for $\widetilde{\mathcal{A}}$ and

— $\big(\widetilde{U}^{n_\kappa}_{\Phi,\varphi}\widetilde{\pi}_{\Phi,\varphi}(v_\kappa)\widetilde{U}^{-n_\kappa}_{\Phi,\varphi}\big)_{\kappa\in K}$ is a left approximate unit for $\widetilde{\mathcal{A}}_{\Phi,\varphi}$,

because then [35], Lemma 3/4.1 implies that

$$\widetilde{U}^{n_\kappa}\widetilde{\pi}(v_\kappa)\widetilde{\xi} \to \widetilde{\xi} \quad \text{and} \quad W\big(\widetilde{U}^{n_\kappa}\widetilde{\pi}(v_\kappa)\widetilde{\xi}\big) = \widetilde{U}^{n_\kappa}_{\Phi,\varphi}\widetilde{\pi}_{\Phi,\varphi}(v_\kappa)\widetilde{\xi}_{\Phi,\varphi} \to \widetilde{\xi}_{\Phi,\varphi}.$$

For we put $K = \mathcal{F} \times \{1, 2, \ldots\}$, where $\mathcal{F}$ stands for the set of all finite subsets of $\mathfrak{A} \times \mathbb{Z}$. Then K is upward directed with respect to the order

$$(F_1, p_1) \leqslant (F_2, p_2) \Leftrightarrow F_1 \subset F_2 \quad \text{and } p_1 \leqslant p_2.$$

Let also $(u_\iota)_{\iota\in I}$ be a bounded left approximate unit for $\mathfrak{A}$. Now, for any $\kappa = (F, p) \in K$ we first choose in $\mathbb{Z}$ some $n_\kappa \leqslant \min\{k : (a, k) \in F\}$, and then some $\iota_\kappa \in I$ such that, for every $(a, k) \in F$, hold

$$\begin{aligned}\big\|\big(\widetilde{U}^{n_\kappa}\widetilde{\pi}(u_{\iota_\kappa})\widetilde{U}^{-n_\kappa}\big)&\big(\widetilde{U}^{k}\widetilde{\pi}(a)\widetilde{U}^{-k}\big) - \widetilde{U}^{k}\widetilde{\pi}(a)\widetilde{U}^{-k}\big\| \\ &= \big\|\widetilde{\pi}(u_{\iota_\kappa})\widetilde{U}^{k-n_\kappa}\widetilde{\pi}(a)\widetilde{U}^{n_\kappa-k} - \widetilde{U}^{k-n_\kappa}\widetilde{\pi}(a)\widetilde{U}^{n_\kappa-k}\big\| \\ &= \big\|\widetilde{\pi}\big(u_{\iota_\kappa}\Phi^{k-n_\kappa}(a) - \Phi^{k-n_\kappa}(a)\big)\big\| \leqslant \frac{1}{p}\end{aligned}$$

and the inequality obtained replacing in the above one $\widetilde{\pi}$ by $\widetilde{\pi}_{\Phi,\varphi}$ and $\widetilde{U}$ by $\widetilde{U}_{\Phi,\varphi}$. Then the nets $(u_{\iota_\kappa})_{\kappa\in K}$ and $(n_\kappa)_{\kappa\in K}$ satisfy the required conditions.

We conclude that the triple $(\widetilde{\pi}, \widetilde{U}, \widetilde{\xi})$ is uniquely determined by the properties (i)–(iv) up to natural unitary equivalence. We shall call any triplet in this equivalence class the *covariant GNS representation* of $(\mathfrak{A}, \varphi, \Phi)$. Mostly we shall work with the triplet $(\widetilde{\pi}_{\Phi,\varphi}, \widetilde{U}_{\Phi,\varphi}\xi_{\Phi,\varphi})$ given in Lemma 6.1. We notice that the link between π_φ and $\widetilde{\pi}_{\Phi,\varphi}$ is supplied by $V_{\Phi,\varphi}$:

$$V_{\Phi,\varphi}\pi_\varphi(a)V^*_{\Phi,\varphi} = \widetilde{\pi}_{\Phi,\varphi}(a)V_{\Phi,\varphi}V^*_{\Phi,\varphi} = V_{\Phi,\varphi}V^*_{\Phi,\varphi}\widetilde{\pi}_{\Phi,\varphi}(a), \quad a \in \mathfrak{A}. \tag{6.7}$$

The covariant GNS representation of $(\mathfrak{A}, \varphi, \Phi)$ yields the answer to the imbedding question raised at the beginning of this section. Let us investigate additionally the case in which the support projection of φ in $\mathfrak{A}^{**}$ is central:

PROPOSITION 6.2. *Let $(\mathfrak{A}, \varphi, \Phi)$ be a state preserving C^*-dynamical system such that the support $s(\varphi)$ of φ in $\mathfrak{A}^{**}$ is central and $s(\varphi)\Phi^{**}$ commutes with σ^φ. Then, denoting by $(\widetilde{\pi}, \widetilde{U}, \widetilde{\xi})$ the covariant* GNS *representation of $(\mathfrak{A}, \varphi, \Phi)$, $\widetilde{\xi}$ is cyclic also for the commutant of $\widetilde{\mathcal{A}} = \bigcup\limits_{k=-\infty}^{+\infty} \widetilde{U}^k\widetilde{\pi}(\mathfrak{A})\widetilde{U}^{-k} = \bigcup\limits_{k=-\infty}^{-1} \widetilde{U}^k\widetilde{\pi}(\mathfrak{A})\widetilde{U}^{-k}$, or equivalently, separating for the von Neumann algebra $\widetilde{\mathcal{M}} = \widetilde{\mathcal{A}}''$.*

Proof. We shall omit for convenience the subscript Φ, φ, writing $U = U_{\Phi,\varphi}$.

We identify $(\widetilde{\pi}, \widetilde{U}, \widetilde{\xi})$ with the triple $(\widetilde{\pi}_{\Phi,\varphi}, \widetilde{U}_{\Phi,\varphi}, \widetilde{\xi}_{\Phi,\varphi})$ defined in Lemma 6.1 and denote simply $V = V_{\Phi,\varphi}$ and $\widetilde{H} = \widetilde{H}_{\Phi,\varphi}$. Then

$$\widetilde{T} \in \widetilde{\mathcal{M}} \Rightarrow V^*\widetilde{T}V \in \pi_\varphi(\mathfrak{A})''. \tag{6.8}$$

For it is enough to show that $V^*\widetilde{U}^{-k}\widetilde{\pi}(a)\widetilde{U}^k V \in \pi_\varphi(\mathfrak{A})''$ for every $a \in \mathfrak{A}$ and $k \geqslant 1$. But using $\widetilde{U}V = VU$ and (6.7), we get

$$V^*\widetilde{U}^{-k}\widetilde{\pi}(a)\widetilde{U}^k V = (U^*)^k V^*\widetilde{\pi}(a)VU^k = (U^*)^k\pi_\varphi(a)U^k$$

and by Proposition 3.3 we have $(U^*)^k\pi_\varphi(a)U^k \in \pi_\varphi(\mathfrak{A})''$.

Now let $\widetilde{T} \in \widetilde{\mathcal{M}}$ be such that $\widetilde{T}\widetilde{\xi} = 0$. Then we have for every $n \geqslant 1$

$$V^*(\widetilde{U}^n\widetilde{T}^*\widetilde{T}\widetilde{U}^{-n})V\xi_\varphi = V^*\widetilde{U}^n\widetilde{T}^*\widetilde{T}\widetilde{\xi} = 0.$$

Since $\widetilde{U}^n\widetilde{T}^*\widetilde{T}\widetilde{U}^{-n} \in \widetilde{\mathcal{M}}$, (6.8) yields $V^*(\widetilde{U}^n\widetilde{T}^*\widetilde{T}\widetilde{U}^{-n})V \in \pi_\varphi(\mathfrak{A})''$ and, taking into account that the vector ξ_φ is separating for $\pi_\varphi(\mathfrak{A})''$, it follows successively

$$V^*(\widetilde{U}^n\widetilde{T}^*\widetilde{T}\widetilde{U}^{-n})V = 0, \quad \widetilde{T}\widetilde{U}^{-n}V = 0.$$

We conclude by the density of $\bigcup_{n=1}^{+\infty} \widetilde{U}^{-n}VH_\varphi$ in $\widetilde{H}$ that $\widetilde{T} = 0$. ∎

Proposition 6.2 is a non-commutative variant of the arguments used to reduce the proof of Theorem 7.13 in [14], to the case of invertible measure preserving transformations. It can be used to reduce the study of state preserving C^*-dynamical systems $(\mathfrak{A}, \varphi, \Phi)$, for which the support projection $s(\varphi) \in \mathfrak{A}^{**}$ is central and $s(\varphi)\Phi^{**}$ commutes with σ^φ, to the study of similar state preserving C^*-dynamical systems, for which additionally Φ is a $*$-automorphism.

Using the notations of Proposition 6.2, it would be interesting to find a characterization of the situation when the vector $\widetilde{\xi}$ is separating for $\widetilde{\mathcal{M}}$. It can be easily seen that the centrality of $s(\varphi)$ is a necessary condition, while Proposition 6.2 shows that the centrality of $s(\varphi)$ together with the commutation of $s(\varphi)\Phi^{**}$ and σ^φ is a sufficient condition.

7. MULTIPLE WEAK MIXING

We begin this section by proving an abstract theorem in Hilbert spaces, which extends Van der Corput's difference theorem in the theory of uniform distribution modulo one (see e.g. [24], Chapter 1, Theorem 3.1 and Chapter 4, Theorem 2.1):

THEOREM 7.1. *Let $\xi_1, \xi_2, \ldots$ be a bounded sequence in a Hilbert space H. If*

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{d=1}^{h}\Big|\frac{1}{n}\sum_{k=1}^{n}\mathrm{Re}\,(\xi_{k+d}\,|\,\xi_k)\Big| = 0, \tag{7.1}$$

then

$$\lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{k=1}^{n}\xi_k\Big\| = 0.$$

On the other hand, if

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{d=1}^{h}\frac{1}{n}\sum_{k=1}^{n}|\mathrm{Re}\,(\xi_{k+d}\,|\,\xi_k)| = 0, \tag{7.2}$$

then the sequence $(\xi_k)_{k\geqslant 1}$ *is uniformly weakly mixing to zero, that is*

$$\lim_{n\to\infty} \sup\Big\{\frac{1}{n}\sum_{k=1}^{n}|(\xi_k \mid \eta)| : \eta\in H,\ \|\eta\|\leqslant 1\Big\} = 0.$$

In particular, in this case

$$\text{D-}\lim_{n\to\infty}\xi_n = 0 \quad \textit{with respect to the weak topology of } H.$$

Proof. Denoting $c = \sup_{k\geqslant 1}\|\xi_k\| < +\infty$, by inequality (8.7) we get for any natural numbers $n \geqslant h \geqslant 1$:

$$\begin{aligned}\Big\|\frac{1}{n}\sum_{j=1}^{n}\xi_{k_j}\Big\|^2 &\leqslant \frac{n+h}{n^2(h+1)}\sum_{j=1}^{n}\|\xi_{k_j}\|^2 + \frac{2(n+h)}{n^2(h+1)}\sum_{d=1}^{h}\frac{h-d+1}{h+1}\operatorname{Re}\sum_{j=1}^{n}(\xi_{k_j}\mid\xi_{k_{j+d}})\\ &\leqslant \frac{n+h}{n(h+1)}c + 2\Big(1+\frac{h}{n}\Big)\frac{1}{n(h+1)}\sum_{l=1}^{L\cdot h}\sum_{j=1}^{n}|\operatorname{Re}(\xi_{k_j}\mid\xi_{k_j+l})|\\ &\leqslant \Big(\frac{1}{n}+\frac{1}{h+1}\Big)c + \frac{4}{n(h+1)}\sum_{l=1}^{L\cdot h}\Big|\sum_{k=1}^{L\cdot n}\operatorname{Re}(\xi_k\mid\xi_{k+l})\Big|,\end{aligned}$$

so

$$\Big\|\frac{1}{n}\sum_{j=1}^{n}\xi_{k_j}\Big\|^2 \leqslant \Big(\frac{1}{n}+\frac{1}{h}\Big)c + \frac{4}{h}\sum_{d=1}^{h}\Big|\frac{1}{n}\sum_{k=1}^{n}\operatorname{Re}(\xi_k\mid\xi_{k+l})\Big|. \tag{7.3}$$

Let us first assume that (7.1) holds and let $\varepsilon > 0$ be arbitrary. Choosing an integer $h_\varepsilon \geqslant 1$ with $\frac{c}{h_\varepsilon}\leqslant\varepsilon$, (9.8) yields for every $h\geqslant h_\varepsilon$:

$$\lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{k=1}^{n}\xi_k\Big\|^2 \leqslant \varepsilon + 4\limsup_{n\to\infty}\frac{1}{h}\sum_{d=1}^{h}\Big|\frac{1}{n}\sum_{k=1}^{n}\operatorname{Re}(\xi_{k+d}\mid\xi_k)\Big|.$$

By (7.1) it follows that

$$\lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{k=1}^{n}\xi_k\Big\|^2 \leqslant \varepsilon + 4\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{d=1}^{h}\Big|\frac{1}{n}\sum_{k=1}^{n}\operatorname{Re}(\xi_{k+d}\mid\xi_k)\Big| = \varepsilon.$$

Let us next assume that (7.2) holds. By Theorem 9.8 (Appendix B), $(\xi_k)_{k\geqslant 1}$ follows to be uniformly weakly mixing to zero once we prove that, for every relatively dense sequence $1\leqslant k_1<k_2<\cdots$ in $\mathbb{N}$,

$$\lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{j=1}^{n}\xi_{k_j}\Big\| = 0. \tag{7.4}$$

Further, by the first part of the proof, (7.4) is implied by

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{d=1}^{h}\Big|\frac{1}{n}\sum_{j=1}^{n}\operatorname{Re}(\xi_{k_{j+d}}\mid\xi_{k_j})\Big| = 0. \tag{7.5}$$

Put $k_0 = 0$ and denote $L = \sup\limits_{j\geqslant 0}(k_{j+1} - k_j) < +\infty$. Then $d \leqslant k_{j+d} - k_j \leqslant L\cdot d$ for all $j, d \geqslant 0$, in particular $j \leqslant k_j \leqslant L \cdot j$. Therefore

$$\sum_{d=1}^{h}\Big|\sum_{j=1}^{n}\mathrm{Re}\,(\xi_{k_{j+d}} \mid \xi_{k_j})\Big| \leqslant \sum_{d=1}^{h}\sum_{j=1}^{n}|\mathrm{Re}\,(\xi_{k_{j+d}} \mid \xi_{k_j})|$$
$$\leqslant \sum_{l=1}^{L\cdot h}\sum_{j=1}^{n}|\mathrm{Re}\,(\xi_{k_j+l} \mid \xi_{k_j})| \leqslant \sum_{l=1}^{L\cdot h}\sum_{k=1}^{L\cdot n}|\mathrm{Re}\,(\xi_{k+l} \mid \xi_k)|$$

and (7.5) is thus consequence of (7.2).

The convergence $\mathrm{D}\text{-}\lim\limits_{n\to\infty} \xi_n = 0$ with respect to the weak topology of H follows now by Lemma 9.2 (see Appendix B). ∎

By formula (9.1) in Appendix B, the above theorem implies immediately the following slight extension of Lemmas 4.9 ad 7.5 in [14]:

COROLLARY 7.2. *Let $\xi_1, \xi_2, \ldots$ be a bounded sequence in a Hilbert space H such that*

$$\mathrm{D}\text{-}\lim_{d\to\infty}\big(\mathrm{D}\text{-}\limsup_{n\to\infty}|\,\mathrm{Re}(\xi_{n+d} \mid \xi_n)|\big) = 0.$$

Then the sequence $(\xi_k)_{k\geqslant 1}$ is uniformly weakly mixing to zero, in particular

$$\mathrm{D}\text{-}\lim_{n\to\infty} \xi_n = 0 \quad \textit{with respect to the weak topology of } H.$$

Let $(\mathfrak{A}, \varphi, \Phi)$ be a state preserving C^*-dynamical system. For any integer $p \geqslant 1$ and $m_0, m_1, \ldots, m_p \in \mathbb{N}$, $m_j \neq m_{j'}$ for $j \neq j'$, we say that:

$(\mathfrak{A}, \varphi, \Phi)$ is *symmetrically weakly mixing with respect to* $m_0, \ldots, m_p$ if

$$\mathrm{D}\text{-}\lim_{n\to\infty}\varphi\Big(\prod_{j=-p}^{p}\Phi^{m_{|j|}\cdot n}(x_j)\Big) = \varphi(x_0)\prod_{j=1}^{p}\varphi(x_{-j}x_j) \quad \text{for all } x_0, x_{\pm 1}, \ldots, x_{\pm p} \in \mathfrak{A}.$$

LEMMA 7.3. *Let $(\mathfrak{A}, \varphi, \Phi)$ be a state preserving C^*-dynamical system, $p \geqslant 1$, and $m_0, m_1, \ldots, m_p \in \mathbb{N}$, $m_j \neq m_{j'}$ for $j \neq j'$. Then* $(\mathrm{SWM}_{m_0,\ldots,m_p})$ *implies*

$$\begin{aligned}&\mathrm{D}\text{-}\lim_{n\to\infty}\varphi\Big((\Phi^{**})^{m_p\cdot n}(x_{-p})\Big(\prod_{j=-p+1}^{p-1}\Phi^{m_{|j|}\cdot n}(x_j)\Big)(\Phi^{**})^{m_p\cdot n}(x_p)\Big)\\ &\qquad = \varphi(x_0)\prod_{j=1}^{p}\varphi(x_{-j}x_j)\end{aligned} \tag{7.6}$$

for all $x_0, \ldots, x_{\pm(p-1)} \in \mathfrak{A}$ and $x_{\pm p} \in \mathfrak{A}^{**}$. In particular,

$$(\mathrm{SWM}_{m_0,\ldots,m_p}) \Rightarrow (\mathrm{SWM}_{m_0,\ldots,m_q}) \quad \textit{for every } 1 \leqslant q \leqslant p.$$

Proof. The first statement follows by using

$$\varphi\Big((\Phi^{**})^{m_p\cdot n}(x_{-p})\Big(\prod_{j=-p+1}^{p-1}\Phi^{m_{|j|}\cdot n}(x_j)\Big)(\Phi^{**})^{m_p\cdot n}(x_p)\Big)$$
$$= \Big(\pi_\varphi\Big(\prod_{j=-p+1}^{p-1}\Phi^{m_{|j|}\cdot n}(x_j)\Big)U_{\Phi,\varphi}^{m_p\cdot n}\pi_\varphi(x_p)\xi_\varphi \,|\, U_{\Phi,\varphi}^{m_p\cdot n}\pi_\varphi(x_{-p}^*)\xi_\varphi\Big)$$

and $\overline{\pi_\varphi(\mathfrak{A})\xi_\varphi} = H_\varphi$.

Assuming $p \geqslant 2$, $(\mathrm{SWM}_{m_0,\dots,m_p}) \Rightarrow (\mathrm{SWM}_{m_0,\dots,m_{p-1}})$ follows by applying (7.6) with $x_{-p}, x_p = 1_{\mathfrak{A}^{**}}$ and taking into account (2.4). ∎

LEMMA 7.4. *Let $(\mathfrak{A},\Phi)$ be a C^*-dynamical system which is norm-asymptotically abelian in density, $p \geqslant 1$, and $m_0, m_1, \dots, m_p \in \mathbb{N}$, $m_j \neq m_{j'}$ for $j \neq j'$. Then*

$$\text{D-}\lim_{n\to\infty}\Big\|\prod_{j=-p}^{p}\Phi^{m_{|j|}\cdot n}(x_j) - \Phi^{m_0\cdot n}(x_0)\prod_{j=1}^{p}\Phi^{m_j\cdot n}(x_{-j}x_j)\Big\| = 0$$

for all $x_0, x_{\pm 1}, \dots, x_{\pm p} \in \mathfrak{A}$.

Proof. Put $c = \max\limits_{-p\leqslant j\leqslant p}\|x_j\|$ and, for convenience, $T_j = \Phi^{m_{|j|}\cdot n}(x_j)$.

Straightforward verification shows that we have, for any $1 \leqslant q \leqslant p$,

$$\prod_{j=-q}^{q}T_j - \Big(\prod_{j=-q+1}^{q-1}T_j\Big)T_{-q}T_q = \sum_{k=-q+1}^{q-1}\Big(\prod_{j=-q+1}^{k-1}T_j\Big)[T_{-q},T_k]\Big(\prod_{j=k+1}^{q}T_j\Big)$$

and it follows:

$$\prod_{j=-p}^{p}T_j - \Big(\prod_{j=-q+1}^{q-1}T_j\Big)\Big(\prod_{j=q}^{p}(T_{-j}T_j)\Big)$$
$$= \sum_{r=q}^{p}\sum_{k=-r+1}^{r-1}\Big(\prod_{j=-r+1}^{k-1}T_j\Big)[T_{-r},T_k]\Big(\prod_{j=k+1}^{r}T_j\Big)\Big(\prod_{j=r+1}^{p}(T_{-j}T_j)\Big).$$

Using the above equality with $q = 1$, we get the inequality

$$\Big\|\prod_{j=-p}^{p}T_j - T_0\prod_{j=1}^{p}(T_{-j}T_j)\Big\| \leqslant c^{2p-1}\sum_{r=1}^{p}\sum_{|k|\leqslant r-1}\|[T_{-r},T_k]\|,$$

which yields the statement by the norm-asymptotic abelianess in density of $(\mathfrak{A},\Phi)$. ∎

Now we are ready to prove the main result of this section, which will imply Theorems 1.3, 1.4 and 1.5:

THEOREM 7.5. *Let $(\mathfrak{A},\varphi,\Phi)$ be a weakly mixing state preserving C^*-dynamical system, $p \geqslant 1$, and $1 \leqslant m_1, \dots, m_p \in \mathbb{N}$, $m_j \neq m_{j'}$ for $j \neq j'$. Then*

$$(\mathrm{SWM}_{m_1,\dots,m_p}) \Rightarrow (\mathrm{UWM}_{m_1,\dots,m_p}) \Rightarrow (\mathrm{WM}_{m_1,\dots,m_p}).$$

If $(\mathfrak{A},\Phi)$ is norm-asymptotically abelian in density then we have also

$$(\mathrm{WM}_{m_1,\dots,m_p}) \Leftrightarrow (\mathrm{SWM}_{0,m_1,\dots,m_p}).$$

Proof. For the first implication we assume $(\mathrm{SWM}_{m_1,\dots,m_p})$ and show that, for every $x_1,\dots,x_p\in\mathfrak{A}$, the sequence

$$\xi_n=\pi_\varphi\big(\Phi^{m_1\cdot n}(x_1)\cdots\Phi^{m_p\cdot n}(x_p)\big)\xi_\varphi-\varphi(x_1)\cdots\varphi(x_p)\xi_\varphi\in H_\varphi,\quad n\geqslant 1$$

is uniformly weakly mixing to zero.

Let us denote, for every $1\leqslant q\leqslant p$ and $n\geqslant 1$,

$$\begin{aligned}\xi_n^{(q)}&=\pi_\varphi\Big(\prod_{j=1}^{q}\Phi^{m_j\cdot n}(x_j)\Big)\xi_\varphi-\varphi(x_q)\pi_\varphi\Big(\prod_{j=1}^{q-1}\Phi^{m_j\cdot n}(x_j)\Big)\xi_\varphi\\&=\pi_\varphi\Big(\Big(\prod_{j=1}^{q-1}\Phi^{m_j\cdot n}(x_j)\Big)(\Phi^{**})^{m_p\cdot n}\big(\underbrace{x_q-\varphi(x_q)1_{\mathfrak{A}^{**}}}_{=\tilde{x}_q}\big)\Big)\xi_\varphi.\end{aligned}$$

According to Lemma 7.3,

$$\begin{aligned}(\xi_{n+d}^{(q)}\mid\xi_n^{(q)})&=\varphi\Big((\Phi^{**})^{m_q\cdot n}(\widetilde{x}_q^*)\Big(\prod_{j=-q+1}^{-1}\Phi^{m_{|j|}\cdot n}(x_{-j}^*)\Big)\\&\qquad\cdot\Big(\prod_{j=1}^{q-1}\Phi^{m_j\cdot(n+d)}(x_j)\Big)(\Phi^{**})^{m_q\cdot(n+d)}(\widetilde{x}_q)\Big)\\&=\varphi\Big((\Phi^{**})^{m_q\cdot n}(\widetilde{x}_q^*)\Big(\prod_{j=-q+1}^{-1}\Phi^{m_{|j|}\cdot n}(x_{-j}^*)\Big)\\&\qquad\cdot\Big(\prod_{j=1}^{q-1}\Phi^{m_j\cdot n}\big(\Phi^{m_j\cdot d}(x_j)\big)\Big)(\Phi^{**})^{m_q\cdot n}\big((\Phi^{**})^{m_q\cdot d}(\widetilde{x}_q)\big)\Big)\end{aligned}$$

converges in density for $n\to\infty$ to $\Big(\prod_{j=1}^{q-1}\varphi\big(x_j^*\Phi^{m_j\cdot d}(x_j)\big)\Big)\cdot\varphi\big(\widetilde{x}_q^*(\Phi^{**})^{m_q\cdot d}(\widetilde{x}_q)\big)$. Furthermore, since Φ, hence also Φ^{m_q} is weakly mixing with respect to φ, using (2.4) we get

$$\text{D-}\lim_{d\to\infty}\varphi\big(\widetilde{x}_q^*(\Phi^{**})^{m_q\cdot d}(\widetilde{x}_q)\big)=\text{D-}\lim_{d\to\infty}\varphi\big(x_q^*\Phi^{m_q\cdot d}(x_q)\big)-\overline{\varphi(x_q)}\varphi(x_q)=0.$$

Therefore $\text{D-}\lim\limits_{d\to\infty}\big(\text{D-}\lim\limits_{n\to\infty}(\xi_{n+d}^{(q)}\mid\xi_n^{(q)})\big)=0$ and, by Corollary 7.2, $(\xi_n^{(q)})_{n\geqslant1}$ is uniformly weakly mixing to zero for any $1\leqslant q\leqslant p$.

Now, since $\xi_n=\sum\limits_{q=1}^{p}\Big(\prod\limits_{j=q+1}^{p}\varphi(x_j)\Big)\xi_n^{(q)}$, we conclude that the sequence $(\xi_n)_{n\geqslant1}$ is indeed uniformly weakly mixing to zero.

Further, $(\mathrm{UWM}_{m_1,\dots,m_p})\Rightarrow(\mathrm{WM}_{m_1,\dots,m_p})$ is trivial and, if we assume that the C^*-dynamical system $(\mathfrak{A},\Phi)$ is norm-asymptotically abelian in density, then the equivalence $(\mathrm{WM}_{m_1,\dots,m_p})\Leftrightarrow(\mathrm{SWM}_{0,m_1,\dots,m_p})$ follows by using Lemma 7.4. ∎

Proof of Theorem 1.3. Let the integers $1\leqslant m_1<m_2$ be arbitrary. According to Proposition 5.4, $(\mathfrak{A},\varphi,\Phi)$ satisfies $(\mathrm{SWM}_{m_2-m_1,0})\Leftrightarrow(\mathrm{SWM}_{m_2,m_1})$

and by Theorem 7.5 it follows that $(\mathfrak{A},\varphi,\Phi)$ is weakly mixing of order (m_2,m_1). Consequently, if $x_1,x_2\in\mathfrak{A}$ then

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\Big|\Big(\pi_\varphi\big(\Phi^{m_2\cdot k}(x_2)\Phi^{m_1\cdot k}(x_1)\big)\xi_\varphi-\varphi(x_2)\varphi(x_1)\xi_\varphi\,|\,\xi\Big)\Big|=0 \tag{7.7}$$

for every $\xi\in\pi_\varphi(\mathfrak{A})\xi_\varphi$, hence for every $\xi\in H_\varphi$. (7.3) with $\xi\in\pi_\varphi(\mathfrak{A})'\xi_\varphi$ yields

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\Big|\Big(\pi_\varphi\big(\Phi^{m_2\cdot k}(x_2)\Phi^{m_1\cdot k}(x_1)\big)\xi-\varphi(x_2)\varphi(x_1)\xi\,|\,\xi_\varphi\Big)\Big|=0, \tag{7.8}$$

first for $\xi\in\pi_\varphi(\mathfrak{A})'\xi_\varphi$ and then for any $\xi\in H_\varphi$. Using (7.8) with $\xi\in\pi_\varphi(\mathfrak{A})\xi_\varphi$, we finally get D-$\lim\limits_{n\to\infty}\varphi\big(\Phi^{m_2\cdot k}(x_2)\Phi^{m_1\cdot k}(x_1)x_0\big)=\varphi(x_2)\varphi(x_0)\varphi(x_0)$ for all $x_0\in\mathfrak{A}$. Writing the above equality for x_j replaced by x_j^* and passing to the conjugates, we get (WM_{m_1,m_2}). ∎

Proof of Theorem 1.4. If the integers $1\leqslant m_1<m_2$ are arbitrary then, according to Proposition 5.4, $(\mathfrak{A},\varphi,\Phi)$ satisfies $(\mathrm{SWM}_{0,m_2-m_1})\Leftrightarrow(\mathrm{SWM}_{m_1,m_2})$ and by Theorem 7.5 it follows that $(\mathfrak{A},\varphi,\Phi)$ is uniformly weakly mixing of order (m_1,m_2). ∎

We notice that, by (6.1) and by Proposition 5.4, the statement of Theorem 1.4 still holds if, instead of assuming the commutation of $s(\varphi)\Phi^{**}$ with the modular group of φ, we assume that the canonical cyclic vector $\widetilde{\xi}$ of the covariant GNS representation $(\widetilde{\pi},\widetilde{U},\widetilde{\xi})$ of $(\mathfrak{A},\varphi,\Phi)$ is also separating for the von Neumann algebra $\Big(\bigcup\limits_{k=-\infty}^{+\infty}\widetilde{U}^k\widetilde{\pi}(\mathfrak{A})\widetilde{U}^{-k}\Big)''$ (cf. Proposition 6.2).

Proof of Theorem 1.5. We shall use induction on p.

For any $m_1\geqslant 1$, (SWM_{m_1}) being trivially satisfied, (UWM_{m_1}) follows by using Theorem 7.5. An alternative way to get this is by using Proposition 5.4 and Theorem 9.7 (see Appendix B).

If $(\mathrm{UWM}_{m_1,\dots,m_p})$ holds for some $p\geqslant 1$ and all $1\leqslant m_1<\dots<m_p$, then, for any integers $1\leqslant n_1<\dots<n_{p+1}$, condition $(\mathrm{UWM}_{n_2-n_1,\dots,n_{p+1}-n_1})$ is satisfied and Theorem 7.5 yields

$$\begin{aligned}(\mathrm{UWM}_{n_2-n_1,\dots,n_{p+1}-n_1})&\Rightarrow(\mathrm{WM}_{n_2-n_1,\dots,n_{p+1}-n_1})\\&\Rightarrow(\mathrm{SWM}_{0,n_2-n_1,\dots,n_{p+1}-n_1})\Leftrightarrow(\mathrm{SWM}_{n_1,\dots,n_{p+1}})\\&\Rightarrow(\mathrm{UWM}_{n_1,\dots,n_{p+1}}).\quad ∎\end{aligned}$$

8. APPENDIX A: A NON-COMMUTATIVE VAN DER CORPUT TYPE INEQUALITY

The goal of this section is to extend to the setting of arbitrary $*$-algebras an inequality of J.G. Van der Corput (see e.g. [24], Chapter 1, Lemma 3.1), used by him to prove his celebrated difference theorem for uniform distribution modulo one (see e.g. [24], Chapter 1, Theorem 3.1).

Formula 8.1. If $1 \leqslant n \geqslant h \geqslant d \geqslant 0$ are natural numbers and $a_{1-h}, \dots$ $\dots, a_{n+h-d}$ are elements of an additive semigroup then

$$\sum_{k=1}^{n+h} \sum_{j=k-h}^{k-d} a_j = (h-d+1) \sum_{j=1-d}^{n} a_j + \sum_{j=1}^{h-d} (h-d+1-j)\big(a_{1-d-j} + a_{n+j}\big).$$

Proof.

$$\sum_{k=1}^{n+h} \sum_{j=k-h}^{k-d} a_j = \sum_{j=1-h}^{-d} \sum_{k=1}^{j+h} a_j + \sum_{j=1-d}^{n} \sum_{k=j+d}^{j+h} a_j + \sum_{j=n+1}^{n+h-d} \sum_{k=j+d}^{n+h} a_j$$

$$= \sum_{j=1-h}^{-d} (j+h) a_j + \sum_{j=1-d}^{n} (h-d+1) a_j + \sum_{j=n+1}^{n+h-d} (n+h-j-d+1) a_j$$

$$= (h-d+1) \sum_{j=1-d}^{n} a_j + \sum_{j=1}^{h-d} (h-d+1-j)\big(a_{1-d-j} + a_{n+j}\big). \qquad \blacksquare$$

Formula 8.1 yields immediately:

Formula 8.2. If $1 \leqslant n \geqslant h \geqslant d \geqslant 0$ are natural numbers and $a_1, \dots, a_n$ are elements of an additive semigroup with neutral element then, putting $a_j = 0$ for $j \leqslant 0$ and for $j \geqslant n+1$, we have

$$\sum_{k=1}^{n+h} \sum_{j=k-h}^{k-d} a_j = (h-d+1) \sum_{j=1}^{n} a_j.$$

Now we prove the counterpart of Formula 8.1 for double sums:

Formula 8.3. If $1 \leqslant n \geqslant h \geqslant 0$ are natural numbers and $a_{j,j'}, 1-h \leqslant$ $j, j' \leqslant n+h$ are elements of an additive semigroup then

$$\sum_{k=1}^{n+h} \sum_{j,j'=k-h}^{k} a_{j,j'} = (h+1) \sum_{j=1}^{n} a_{j,j} + \sum_{d=1}^{h} (h-d+1) \sum_{j=1-d}^{n} \big(a_{j,j+d} + a_{j+d,j}\big)$$

$$+ \sum_{d=1}^{h} (h-d+1)\big(a_{1-d,1-d} + a_{n+d,n+d}\big)$$

$$+ \sum_{d=1}^{h} \sum_{j=1}^{h-d} (h-d+1-j)\big(a_{1-d-j,1-j} + a_{1-j,1-d-j} + a_{n+j,n+d+j} + a_{n+d+j,n+j}\big).$$

Proof. Straightforward computation yields

$$\begin{aligned}\sum_{k=1}^{n+h}\sum_{j,j'=k-h}^{k} a_{j,j'} &= \sum_{k=1}^{n+h}\sum_{j=k-h}^{k} a_{j,j} + \sum_{k=1}^{n+h}\sum_{\substack{j,j'=k-h\\ j<j'}}^{k} \big(a_{j,j'} + a_{j',j}\big)\\ &= \sum_{k=1}^{n+h}\sum_{j=k-h}^{k} a_{j,j} + \sum_{k=1}^{n+h}\sum_{d=1}^{h}\sum_{j=k-h}^{k-d} \big(a_{j,j+d} + a_{j+d,j}\big)\\ &= \sum_{k=1}^{n+h}\sum_{j=k-h}^{k} a_{j,j} + \sum_{d=1}^{h}\sum_{k=1}^{n+h}\sum_{j=k-h}^{k-d} \big(a_{j,j+d} + a_{j+d,j}\big).\end{aligned}$$

Taking now in account that, according to Formula 8.1,

$$\sum_{k=1}^{n+h}\sum_{j=k-h}^{k} a_{j,j} = (h+1)\sum_{j=1}^{n} a_{j,j} + \sum_{j=1}^{h}(h+1-j)\big(a_{1-j,1-j} + a_{n+j,n+j}\big)$$

and

$$\begin{aligned}\sum_{k=1}^{n+h}\sum_{j=1}^{k-d}(a_{j,j+d} + a_{j+d,j}) &= (h-d+1)\sum_{j=1-d}^{n}(a_{j,j+d} + a_{j+d,j})\\ &\quad + \sum_{j=1}^{h-d}(h-d-j+1)\big(a_{1-d-j,1-j} + a_{1-j,1-d-j} + a_{n+j,n+d+j} + a_{n+d+j,n+j}\big),\end{aligned}$$

the equality from the statement follows. ∎

Similarly to the implication Formula 8.1 ⇒ Formula 8.2, Formula 8.3 implies

Formula 8.4. If $1 \leqslant n \geqslant h \geqslant 0$ are natural numbers and $a_{j,j'}$, $1 \leqslant j, j' \leqslant n$ are elements of an additive semigroup with neutral element then, putting $a_{j,j'} = 0$ for j or $j' \leqslant 0$ and for j or $j' \geqslant n+1$, we have

$$\sum_{k=1}^{n+h}\sum_{j,j'=k-h}^{k} a_{j,j'} = (h+1)\sum_{j=1}^{n} a_{j,j} + \sum_{d=1}^{h}(h-d+1)\sum_{j=1}^{n}\big(a_{j,j+d} + a_{j+d,j}\big). \quad ∎$$

If $n \geqslant 1$ is a natural number and $\xi_1, \ldots, \xi_n$ are elements of a normed vector space, then we have by the Cauchy inequality

$$\Big\|\sum_{k=1}^{n}\xi_k\Big\|^2 \leqslant \Big(\sum_{k=1}^{n}\|\xi_k\|\Big)^2 \leqslant n\sum_{k=1}^{n}\|\xi_k\|^2. \tag{8.1}$$

The following analogue of (8.1) for $*$-algebras can be found in Section 2.8 (3) of [35], (cf. [18], proof of Lemma 1):

Inequality 8.5. If $n \geqslant 1$ is a natural number and $a_1, \ldots, a_n$ are elements of a $*$-algebra then

$$\Big(\sum_{k=1}^{n} a_k\Big)^*\Big(\sum_{k=1}^{n} a_k\Big) \leqslant n\sum_{k=1}^{n} a_k^* a_k.$$

Proof. The inequality from the statement holds trivially for $n = 1$.

Let us now assume that it holds for some $n \geqslant 1$ and let $a_1, \dots, a_n, a_{n+1}$ be elements of a $*$-algebra. Then

$$\begin{aligned}\Big(\sum_{k=1}^{n+1} a_k\Big)^* \Big(\sum_{k=1}^{n+1} a_k\Big) &= \Big(\sum_{k=1}^{n} a_k + a_{n+1}\Big)^* \Big(\sum_{k=1}^{n} a_k + a_{n+1}\Big) \\ &= \Big(\sum_{k=1}^{n} a_k\Big)^* \Big(\sum_{k=1}^{n} a_k\Big) + \sum_{k=1}^{n} \big(a_k^* a_{n+1} + a_{n+1}^* a_k\big) + a_{n+1}^* a_{n+1} \\ &\leqslant n \sum_{k=1}^{n} a_k^* a_k + \sum_{k=1}^{n} \big(a_k^* a_k + a_{n+1}^* a_{n+1}\big) + a_{n+1}^* a_{n+1} \\ &= (n+1) \sum_{k=1}^{n+1} a_k^* a_k,\end{aligned}$$

where the inequality follows by using the induction assumption and

$$\begin{aligned}a_k^* a_{n+1} + a_{n+1}^* a_k &= a_k^* a_k + a_{n+1}^* a_{n+1} - \big(a_k - a_{n+1}\big)^* \big(a_k - a_{n+1}\big) \\ &\leqslant a_k^* a_k + a_{n+1}^* a_{n+1}. \quad \blacksquare\end{aligned}$$

Now we are prepared to prove our Van der Corput type inequality for $*$-algebras (cf. [24], Chapter 4, Lemma 2.1):

INEQUALITY 8.6. If $1 \leqslant n \geqslant h \geqslant 0$ are natural numbers and $a_1, \dots, a_n$ are elements of a $*$-algebra then

$$\begin{aligned}&(h+1)^2 \Big(\sum_{j=1}^{n} a_j\Big)^* \Big(\sum_{j=1}^{n} a_j\Big) \\ &\qquad \leqslant (n+h)(h+1) \sum_{j=1}^{n} a_j^* a_j + 2(n+h) \sum_{d=1}^{h} (h-d+1) \operatorname{Re} \sum_{j=1}^{n} a_j^* a_{j+d}.\end{aligned}$$

Proof. Put $a_j = 0$ for $j \leqslant 0$ and for $j \geqslant n+1$. Using successively Formula 8.2, Inequality 8.5 and Formula 8.4, we get

$$\begin{aligned}(h+1)^2 \Big(\sum_{j=1}^{n} a_j\Big)^* \Big(\sum_{j=1}^{n} a_j\Big) &= \Big(\sum_{k=1}^{n+h} \sum_{j=k-h}^{k} a_j\Big)^* \Big(\sum_{k=1}^{n+h} \sum_{j=k-h}^{k} a_j\Big) \\ \leqslant (n+h)^2 \sum_{k=1}^{n+h} \Big(\sum_{j=k-h}^{k} a_j\Big)^* \Big(\sum_{j=k-h}^{k} a_j\Big) &= (n+h)^2 \sum_{k=1}^{n+h} \sum_{j,j'=k-h}^{k} a_j^* a_{j'} \\ = (n+h)\Big((h+1) \sum_{j=1}^{n} a_j^* a_j + \sum_{d=1}^{h} (h-d+1) &\sum_{j=1}^{n} \big(a_j^* a_{j+d} + a_{j+d}^* a_j\big)\Big). \quad \blacksquare\end{aligned}$$

It follows immediately a Van der Corput type inequality in Hilbert spaces:

INEQUALITY 8.7. If $1 \leqslant n \geqslant h \geqslant 0$ are natural numbers and $\xi_1, \ldots, \xi_n$ are vectors in a Hilbert space then

$$(h+1)^2 \Big\| \sum_{j=1}^{n} \xi_j \Big\|^2 \\ \leqslant (n+h)(h+1) \sum_{j=1}^{n} \|\xi_j\|^2 + 2(n+h) \sum_{d=1}^{h} (h-d+1) \operatorname{Re} \sum_{j=1}^{n} (\xi_j \mid \xi_{j+d}).$$

Proof. Choose bounded linear operators a_j with $\xi_j = a_j \xi_1$ and apply Inequality 8.6 to these $a_1, \ldots, a_n$. Then the value of ω_{ξ_1} in the left-hand side will be less or equal than the value of ω_{ξ_1} in the right-hand side. ∎

9. APPENDIX B: USED INGREDIENTS FROM THE ERGODIC THEORY

We recall that the *upper density* $D^*(\mathcal{E})$ and the *lower density* $D_*(\mathcal{E})$ of some $\mathcal{E} \subset \mathbb{N} = \{0, 1, \ldots\}$ are defined by

$$D^*(\mathcal{E}) := \limsup_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \chi_{\mathcal{E}}(k) = \limsup_{n\to\infty} \frac{1}{n+1} \operatorname{card}(\mathcal{E} \cap [0,n]),$$

$$D_*(\mathcal{E}) := \liminf_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \chi_{\mathcal{E}}(k) = \liminf_{n\to\infty} \frac{1}{n+1} \operatorname{card}(\mathcal{E} \cap [0,n]),$$

where $\chi_{\mathcal{E}}$ stands for the characteristic function of $\mathcal{E} \subset \mathbb{N}$ (see e.g. [14], Chapter 3, Section 5 or [23], Section 2.3). If upper and lower densities coincide then $\mathcal{E}$ is called having *density* $D(\mathcal{E}) := D^*(\mathcal{E}) = D_*(\mathcal{E})$.

The upper (respectively lower) density of a sequence $(k_j)_{j\geqslant 1}$ in $\mathbb{N}$ means the upper (respectively lower) density of the subset $\{k_j : j \geqslant 1\}$ of $\mathbb{N}$.

We also recall that a subset $\mathcal{N}$ of $\mathbb{N}$ is called *relatively dense* if there exists $L > 0$ such that every interval of natural numbers of lenght $\geqslant L$ contains some element of $\mathcal{N}$. In this case holds clearly $D_*(\mathcal{N}) \geqslant \frac{1}{L}$, so relatively dense sets are of lower density > 0.

A sequence $(k_j)_{j\geqslant 1}$ in $\mathbb{N}$ is called *relatively dense* if the subset $\{k_j : j \geqslant 1\}$ of $\mathbb{N}$ is relatively dense. It is easy to see that a strictly increasing sequence $(k_j)_{j\geqslant 1}$ is relatively dense if and only if $\sup\limits_{j\geqslant 1}(k_{j+1} - k_j) < +\infty$.

For the subsets $\mathcal{E} \subset \mathbb{N}$ with $D(\mathcal{E}) = 0$, called of *zero density*, the following useful fact was noticed by S. Kakutani and L.K. Jones (see [19] or [29], Remark 2.6.3):

LEMMA 9.1. *Let $\mathcal{E}_1, \mathcal{E}_2, \ldots \subset \mathbb{N}$ be a sequence of zero density subsets. Then there exists a zero density subset $\mathcal{E} \subset \mathbb{N}$ such that $\mathcal{E}_j \setminus \mathcal{E}$ is finite for every $j \geqslant 1$.*

Proof. Choose by induction a sequence $1 \leqslant n_1 < n_2 < \cdots$ of integers such that

$$\frac{1}{n+1} \sum_{k=0}^{n} \chi_{\mathcal{E}_j}(k) \leqslant \frac{1}{p^2} \quad \text{for } n \geqslant n_p \text{ and } j = 1, 2, \ldots, p$$

and put $\mathcal{E} = \bigcup\limits_{j=1}^{\infty} \big(\mathcal{E}_j \cap (n_j, +\infty)\big)$. For $n_p < n \leqslant n_{p+1}$ we have

$$\frac{1}{n+1}\sum_{k=0}^{n}\chi_{\mathcal{E}}(k) \leqslant \frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=1}^{p}\chi_{\mathcal{E}_j}(k) = \sum_{j=1}^{p}\frac{1}{n+1}\sum_{k=0}^{n}\chi_{\mathcal{E}_j}(k) \leqslant p\frac{1}{p^2} = \frac{1}{p},$$

so $\mathcal{E}$ has density zero. ∎

A sequence $(\omega_n)_{n\geqslant 0}$ in a topological space Ω is said to *converge in density* to $\omega \in \Omega$ if there exists a set $\mathcal{E} \subset \mathbb{N}$ of density zero such that $\lim\limits_{\mathcal{E}\not\ni n\to\infty} \omega_n = \omega$. We then write $\text{D-}\lim\limits_{n\to\infty} \omega_n = \omega$. Lemma 9.1 implies that, for any countable family $(\omega_n^{(p)})_{n\geqslant 0}$, $p \geqslant 1$ of convergent sequences in Ω, denoting $\text{D-}\lim\limits_{n\to\infty} \omega_n^{(p)}$ by $\omega^{(p)}$, there exists a set $\mathcal{E} \subset \mathbb{N}$ of density zero such that

$$\lim_{\mathcal{E}\not\ni n\to\infty} \omega_n^{(p)} = \omega^{(p)} \quad \text{for all } p \geqslant 1.$$

Lemma 9.2. *If $(\xi_n)_{n\geqslant 0}$ is a bounded sequence in a Hilbert space H then the following statements are equivalent:*

(a) $\text{D-}\lim\limits_{n\to\infty} (\xi_n \mid \eta) = 0$ *for all* $\eta \in H$;

(b) $\text{D-}\lim\limits_{n\to\infty} \xi_n = 0$ *with respect to the weak topology of* H.

Proof. We prove only (a) ⇒ (b), the converse implication being trivial.

Let H_0 denote the closed linear span of $\{\xi_n : n \geqslant 0\}$. By (a) there are subsets $\mathcal{E}_j \subset \mathbb{N}$ of density zero such that $\lim\limits_{\mathcal{E}_j\not\ni n\to\infty} (\xi_n \mid \xi_j) = 0$ for all $j \geqslant 0$ and by Lemma 9.1 there exists a subset $\mathcal{E} \subset \mathbb{N}$ of density zero with $\mathcal{E}_j \setminus \mathcal{E}$ finite for all $j \geqslant 0$. Then $\lim\limits_{\mathcal{E}\not\ni n\to\infty} (\xi_n \mid \xi_j) = 0$ for all $j \geqslant 0$, so the closed linear subspace $\{\eta \in H : \lim\limits_{\mathcal{E}\not\ni n\to\infty} (\xi_n \mid \eta) = 0\} \subset H$ contains the sequence $(\xi_j)_{j\geqslant 0}$, hence all H_0. Since it trivially contains $H \ominus H_0$, it is equal to H. In other words, (b) holds. ∎

Using Lemma 9.1 it is easy to verify also the next classical result of B.O. Koopman and J. von Neumann ([22], see also [29], Lemma 2.6.2):

Lemma 9.3. *Let $(a_n)_{n\geqslant 0}$ be a bounded sequence in* $[0, +\infty)$. *Then* $\text{D-}\lim\limits_{n\to\infty} a_n = 0 \Leftrightarrow \{k \in \mathbb{N} : a_k > \varepsilon\}$ *has density zero for all* $\varepsilon > 0 \Leftrightarrow \lim\limits_{n\to\infty} \frac{1}{n+1}\sum\limits_{k=0}^{n} a_k = 0$.

In particular, given any real number $p > 0$, a bounded sequence $(\omega_n)_{n\geqslant 0}$ in a metric space (Ω, d) is convergent in density to $\omega \in \Omega$ if and only if

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} d(\omega_k, \omega)^p = 0.$$

The *superior limit in density* of a bounded sequence $(a_n)_{n\geqslant 0}$ in $\mathbb{R}$ is defined by

$$\text{D-}\limsup_{n\to\infty} a_n = \inf\big\{\lambda \in \mathbb{R} : \{k \in \mathbb{N} : a_k > \lambda\} \text{ has density zero}\big\}$$

(see [14], Section 7.2). Clearly,

$$\limsup_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}a_k \leqslant \text{D-}\limsup_{n\to\infty}a_n, \tag{9.1}$$

where we don't always have equality, as the example $a_n=(-1)^n$ shows. However, by Lemma 9.3,

$$\text{D-}\limsup_{n\to\infty}|a_n|=0 \Leftrightarrow \text{D-}\lim_{n\to\infty}|a_n|=0 \Leftrightarrow \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}|a_k|=0.$$

We recall also the following easily verifiable counterpart of Lemma 9.3:

Lemma 9.4. *Let $(a_n)_{n\geqslant 0}$ be a bounded sequence in $[0,+\infty)$. Then*

$$D_*\big(\{k\in\mathbb{N}: a_k>\varepsilon\}\big)=0 \text{ for all } \varepsilon>0 \Leftrightarrow \liminf_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}a_k=0.$$

Let X be a Banach space with dual space X^*. We shall say that a bounded sequence $(x_k)_{k\geqslant 1}$ in X is *weakly mixing to zero* if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|\langle x^*,x_k\rangle|=0 \Leftrightarrow \text{D-}\lim_{n\to\infty}\langle x^*,x_n\rangle=0 \quad \text{for all } x^*\in X^*,$$

and we shall say that it is *uniformly weakly mixing to zero* if

$$\lim_{n\to\infty}\sup\Big\{\frac{1}{n}\sum_{k=1}^{n}|\langle x^*,x_k\rangle| : x^*\in X^*,\ \|x^*\|\leqslant 1\Big\}=0.$$

The following theorem is again due to B.O. Koopman and J. von Neumann ([22]).

Theorem 9.5. *For a linear contraction U on a Hilbert space H and $\xi\in H$ the following statements are equivalent:*

(a) *the sequence $(U^n\xi)_{n\geqslant 1}$ is weakly mixing to zero;*

(b) *ξ is orthogonal to all eigenvectors of U corresponding to an eigenvalue of modulus* 1*;*

(c) D-$\lim_{n\to\infty}(U^n\xi \mid \xi)=0$*;*

(d) D-$\lim_{n\to\infty}U^n\xi=0$ *with respect to the weak topology of H.*

Proof. For the proof of the equivalence of (a), (b) and (c) we refer the reader to Theorem 2.3.4 in [23]. On the other hand, (a) $\Leftrightarrow$ (d) follows from Lemma 9.2. ∎

Corollary 9.6. *If U is a linear isometry on a Hilbert space H then*

$$H^U_{\mathrm{AP}}=\big\{\xi\in H: \{U^n\xi : n\in\mathbb{N}\} \text{ is relatively norm-compact}\big\}$$

is the closed linear span of all eigenvectors of U.

Proof. It is easy to see that H^U_{AP} is a closed linear subspace of H containing all eigenvectors of U.

Now let $\xi \in H_{\mathrm{AP}}^U$ be orthogonal to all eigenvectors of U. By Theorem 9.5 there exists some $\mathcal{E} \subset \mathbb{N}$ of density zero satisfying $\underset{\mathcal{E}\not\ni n\to\infty}{\text{weak-lim}}\, U^n\xi = 0$ and then, by the relative norm-compactness of the orbit of ξ, there exist $n_1 < n_2 < \cdots$ in $\mathbb{N}\setminus\mathcal{E}$ such that the sequence $\big(U^{n_j}\xi\big)_{j\geqslant 1}$ is norm-convergent. But then, taking into account that U is isometrical, we have

$$\|\xi\| = \|\lim_{j\to\infty} U^{n_j}\xi\| = \|\underset{\mathcal{E}\not\ni n\to\infty}{\text{weak-lim}}\, U^n\xi\| = 0 \Rightarrow \xi = 0. \quad ∎$$

In [20] the following results was proved.

Theorem 9.7. *Let U be a power bounded linear operator on a Banach space X, $x \in X$, and $x_k = U^k(x)$, $k \geqslant 1$. Then the following conditions are equivalent:*

(i) *the sequence $(x_k)_{k\geqslant 1}$ is weakly mixing to zero;*

(ii) *the sequence $(x_k)_{k\geqslant 1}$ is uniformly weakly mixing to zero.*

The next characterization of bounded sequences which are uniformly weakly mixing to zero was proved in [20] for the case when the sequence is an orbit of a power bounded linear operator, and in [41] in the general case (for a related result in Hilbert spaces see [1]):

Theorem 9.8. *For a bounded sequence $(x_k)_{k\geqslant 1}$ in a Banach space X, the following conditions are equivalent:*

(i) *$(x_k)_{k\geqslant 1}$ is uniformly weakly mixing to zero;*

(ii) *For every sequence $1 \leqslant k_1 < k_2 < \cdots$ in $\mathbb{N}$ of lower density > 0,*

$$\lim_{n\to\infty} \Big\| \frac{1}{n}\sum_{j=1}^{n} x_{k_j} \Big\| = 0.$$

(iii) *For every relatively dense sequence $1 \leqslant k_1 < k_2 < \cdots$ in $\mathbb{N}$,*

$$\lim_{n\to\infty} \Big\| \frac{1}{n}\sum_{j=1}^{n} x_{k_j} \Big\| = 0.$$

For the following known property of totally bounded metric spaces (used e.g. in the proof of [16], Theorem 4.1) we could not find a reference.

Proposition 9.9. *If (Ω, d) is a totally bounded metric space and $\varepsilon > 0$ then*

$$\big\{ n \in \mathbb{N} : \textit{there are } \omega_1, \ldots, \omega_n \in \Omega \textit{ such that } d(\omega_j, \omega_k) > \varepsilon \textit{ for } j \neq k \big\}$$

is bounded.

Proof. Assuming the contrary, for every integer $n \geqslant 1$ there exist $\omega_1^{(n)}, \ldots$ $\ldots, \omega_n^{(n)} \in \Omega$ with $d(\omega_j^{(n)}, \omega_k^{(n)}) > \varepsilon$ for $j \neq k$. Since the completion $\widetilde{\Omega}$ of Ω is a compact metric space, we get by induction infinite sets $\mathbb{N} \supset \mathcal{N}_1 \supset \mathcal{N}_2 \supset \cdots$, $\mathcal{N}_j \subset \{n : n \geqslant j\}$, and $\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots \in \widetilde{\Omega}$ for which $\lim\limits_{\mathcal{N}_j\ni n\to\infty} \omega_j^{(n)} = \widetilde{\omega}_j$. But then $d(\widetilde{\omega}_j, \widetilde{\omega}_k) = \lim\limits_{\mathcal{N}_k\ni n\to\infty} d(\omega_j^{(n)}, \omega_k^{(n)}) \geqslant \varepsilon$ whenever $j < k$, so the sequence $\big(\widetilde{\omega}_j\big)_{j\geqslant 1}$ has no Cauchy subsequence, in contradiction with the compactness of $\widetilde{\Omega}$. ∎

The next recurrence result is done in the proof of [16], Theorem 4.1, but it goes back to [25] (see also [12], Exercise 4.3.D):

COROLLARY 9.10. *Let* (Ω, d) *be a metric space,* $T : \Omega \to \Omega$ *an isometrical map, and* $\omega \in \Omega$. *Then the following statements are equivalent:*

(a) *the orbit* $\{T^n(\omega) : n \in \mathbb{N}\}$ *is totally bounded;*

(b) *for every* $\varepsilon > 0$ *there exists a relatively dense* $\mathcal{N} \subset \mathbb{N}$ *such that* $d(T^n(\omega), \omega) \leqslant \varepsilon$ *for all* $n \in \mathcal{N}$.

Proof. The implication (b) $\Rightarrow$ (a) is straightforward, it holds even if T is assumed only contractive. Let us now assume that (a) holds and let $\varepsilon > 0$ be arbitrary.

By Proposition 9.9 there exists a greatest integer $p \geqslant 1$ such that there are $n_1 < \cdots < n_p$ in $\mathbb{N}$ with $d\big(T^{n_j}(\omega), T^{n_k}(\omega)\big) > \varepsilon$ for $j \neq k$. Since T is isometrical, we have for every $n \in \mathbb{N}$

$$d\big(T^{n+n_j}(\omega), T^{n+n_k}(\omega)\big) = d\big(T^{n_j}(\omega), T^{n_k}(\omega)\big) > \varepsilon \quad \text{for } j \neq k$$

and by the maximality of p it follows that $d\big(T^{n+n_j}(\omega), \omega\big) \leqslant \varepsilon$ for at least one $1 \leqslant j \leqslant p$, that is

$$d\big(T^m(\omega), \omega\big) \leqslant \varepsilon \text{ for some } n \leqslant m \leqslant n + n_p. \quad \blacksquare$$

We notice that if $\mathcal{F}$ is a finite set and, for every $\iota \in \mathcal{F}$, (Ω_ι, d_ι) is a metric space, $T_\iota : \Omega_\iota \to \Omega_\iota$ is an isometrical map and $\omega_\iota \in \Omega_\iota$, then all orbits $\{T_\iota^n(\omega_\iota) : n \in \mathbb{N}\}$, $\iota \in \mathcal{F}$, are totally bounded if and only if for every $\varepsilon > 0$ there exists a relatively dense $\mathcal{N} \subset \mathbb{N}$ such that $d_\iota(T_\iota^n(\omega_\iota), \omega_\iota) \leqslant \varepsilon$ for all $\iota \in \mathcal{F}$ and $n \in \mathcal{N}$.

For it is enough to apply Corollary 9.10 to

$$\Omega = \prod_{\iota \in \mathcal{F}} \Omega_\iota, \quad d\big((\omega_\iota)_\iota, (\rho_\iota)_\iota\big) = \max_\iota d_\iota(\omega_\iota, \rho_\iota), \quad T = \prod_{\iota \in \mathcal{F}} T_\iota.$$

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