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A TENSOR PRODUCT APPROACH TO THE OPERATOR CORONA PROBLEM

PASCALE VITSE

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ABSTRACT. Let F be a bounded analytic function on the unit disc $\mathbb{D}$ having values in the space $L(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. The Operator Corona Problem is to decide whether the existence of a uniformly bounded family of left inverses of $F(z)$, $z \in \mathbb{D}$, guarantees the existence of a bounded analytic left inverse of F. When $\mathcal{H}$ is infinite dimensional, in general, the answer is known to be negative. Some sufficient conditions (on values and/or functional properties of F) are given for the answer to be positive. The technique uses the tensor product slicing method and the Grothendieck Approximation Property.

KEYWORDS: *Operator Corona problem, Bézout equations, tensor product, slicing, Grothendieck Approximation Property, Nevanlinna type meromorphic pseudocontinuation, Sz.-Nagy-Foiaş model spaces.*

MSC (2000): 30D50, 47A56, 39B42, 46B28, 30D55.

NOTATION

X, Y are Banach spaces. $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \ldots$ are separable Hilbert spaces. $L(X, Y)$ denotes the Banach space of bounded linear operators from X to Y, and $L(X) = L(X, X)$. On $L(\mathcal{H}_1, \mathcal{H}_2)$, we write SOT for the Strong Operator Topology, and WOT for the Weak Operator Topology. For any $\mathcal{H}$, we denote by id the identity operator on $\mathcal{H}$. $\mathbb{D}$ is the open unit disc of the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathbb{T}$ the unit circle, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We denote by μ the Lebesgue measure on $\mathbb{T}$. $\mathrm{Hol}(\mathbb{D}, X)$ is the space of all X-valued analytic functions on $\mathbb{D}$. $H^\infty(X)$ is the Banach space of X-valued bounded analytic functions on $\mathbb{D}$ equipped with the supremum norm, and $H^\infty = H^\infty(\mathbb{C})$. $\mathcal{C}(K, X)$ is the Banach space of X-valued continuous functions on K, where K is a compact subset of $\mathbb{C}$, equipped with the supremum norm.

1. INTRODUCTION

The operator corona problem is a tentative to generalize the famous Carleson corona theorem ([6]) to operator valued functions. It was raised by Sz.-Nagy in 1978 in the following form ([25]).

Let $F \in H^\infty(L(\mathcal{H}_1,\mathcal{H}_2))$ satisfying $\|F(z)x\| \geqslant \delta\|x\|$ for every $x \in \mathcal{H}_1$ and every $z \in \mathbb{D}$, where $\delta > 0$ is a constant. Does there exist $G \in H^\infty(L(\mathcal{H}_2,\mathcal{H}_1))$ such that $G(z)F(z) = \mathrm{id}$ for every $z \in \mathbb{D}$?

This problem is of great interest in control theory, as well as in the operator model theory and in the study of the invariant subspace problem. It is also related to the study of submodules of H^∞ and to many other subjects of analysis. Obviously, the condition required on F, usually called the *corona assumption*, is necessary. It means the existence of a uniformly bounded family of left inverses of $F(z)$, $z \in \mathbb{D}$. The question is whether this condition is sufficient for the existence of a bounded analytic left inverse of F. If it is, we say that the corona theorem is true for F. In general, the answer to Sz.-Nagy's question is known to be negative (see in [27]). But in some specific cases, it is positive. In particular, Carleson's theorem saying that any scalar Bezout equation $\sum\limits_{i=1}^{n} g_i f_i = 1$ is solvable with $\{g_i\}_{i=1}^n \subset H^\infty$ as soon as $\{f_i\}_{i=1}^n \subset H^\infty$ satisfies $\sum\limits_{i=1}^{n} |f_i(z)|^2 > \delta^2$ for every $z \in \mathbb{D}$, with $\delta > 0$, means that the answer is positive when $\dim\mathcal{H}_1 = 1$, $\dim\mathcal{H}_2 = n < \infty$. This theorem was called the corona theorem because of an equivalent formulation in the maximal ideal theory; see Section 2. More generally, the answer to the Sz.-Nagy question is positive as soon as $\dim\mathcal{H}_1 < \infty$ (see [26] and [22] for $\dim\mathcal{H}_1 = 1$; see [12] for $\dim\mathcal{H}_1, \dim\mathcal{H}_2 < \infty$; see Vasyunin's theorem in [20] for the general case).

When $F \in H^\infty(L(\mathcal{H}))$ with $\dim\mathcal{H} = \infty$, we need to make some additional assumptions for the corona theorem to be true for F. In this paper, some sufficient conditions are given in terms of approximation by functions with finite dimensional ranges. This approach is based on the Bochner-Phillips-Allan-Markus-Sementsul theory of central projections which generalizes the maximal ideal theory (see Section 2). We deal with the problem in the following more general form.

Let X be a unital Banach algebra with unit 1, and $F \in H^\infty(X)$ such that there exists a left inverse G_z of $F(z)$ for every $z \in \mathbb{D}$, satisfying $\sup\limits_{z\in\mathbb{D}} \|G_z\| < \infty$. Does there exist $G \in H^\infty(X)$ such that $G(z)F(z) = 1$ for every $z \in \mathbb{D}$?

We start deriving from the theory of central projections that the corona theorem holds for functions in a subalgebra of $H^\infty(X)$ defined as the following tensor product,

$$H^\infty \otimes X = \mathrm{span}\{f(\,\cdot\,)x : f \in H^\infty,\ x \in X\},$$

where *span* stands for the closed linear span in the uniform norm topology; see Theorem 2.2. Next, we want to recognize functions in $H^\infty \otimes X$. First we notice that every $F \in H^\infty \otimes X$ has a relatively compact range in X. Then we focus on the problem whether $H^\infty \otimes X$ coincides with $H^\infty_{\mathrm{comp}}(X)$,

$$H^\infty_{\mathrm{comp}}(X) = \{F \in H^\infty(X) : F(\mathbb{D}) \text{ is a relatively compact set in } X\}.$$

If X satisfies the Grothendieck approximation property (AP), then $H^\infty \otimes X = H^\infty_{\text{comp}}(X)$. More generally, if $F(\mathbb{D})$ is relatively compact and satisfies (AP) in X, then $F \in H^\infty \otimes X$.

Using a scalarisation (*left slicing*) method we prove the two following basic properties. First, we show that $H^\infty \otimes X$ and $H^\infty_{\text{comp}}(X)$ are equal for every dual Banach space X if and only if H^∞ satisfies (AP). The latter property represents a delicate question which seems to remain open. Secondly, if $X = (X_*)^*$ and the scalarisation map of F, $\phi_F \in L(X_*, H^\infty)$, defined by $(\phi_F(u))(z) = \langle F(z), u\rangle$, $u \in X_*$, $z \in \mathbb{D}$, maps the unit ball of X_* into a relatively compact set satisfying (AP) in H^∞, then $F \in H^\infty \otimes X$.

From these properties we deduce for example that the corona theorem is true for functions having a relatively compact range and, either being $*$-weakly (or WOT) Nevanlinna pseudocontinuable with sparsed singularities in $\mathbb{C} \setminus \overline{\mathbb{D}}$ (see Theorem 5.14), or having the "same operator structure" for all values $F(z)$, $z \in \mathbb{D}$ (e.g. Toeplitz valued functions (Theorem 6.3), or functions subordinated to a fixed structure (Theorem 6.11)).

Speaking on other possible approaches to the Operator Corona Problem, we first mention that in the Hilbert space setting the *left* Bezout equation $G(z)F(z) = \text{id}$, $z \in \mathbb{D}$, is equivalent to the *right* one $F_*(z)G_*(z) = \text{id}$, where $F_*(z) = F(\overline{z})^*$, $z \in \mathbb{D}$. For right Bezout equations, there is a general Arveson-Sz.-Nagy-Foias-Schubert theorem saying that it is equivalent to the left invertibility of the corresponding (operator valued) Toeplitz operator $T_{(F_*)^*}$ on the Hardy space $H^2(\mathcal{H}_1)$, that is to the condition

$$\|T_{(F_*)^*} f\|_{H^2(\mathcal{H}_2)} = \|P_+(F_*)^* f\|_{H^2(\mathcal{H}_2)} \geqslant \delta \|f\|_{H^2(\mathcal{H}_1)}$$

for every $f \in H^2(\mathcal{H}_1)$. In this language, the assumption of the corona theorem, $\|F(z)x\| \geqslant \delta\|x\|$, $x \in \mathcal{H}_1$, see (2.2) below, is now equivalent to say that $\|T_{(F_*)^*} k_z x\|_{H^2(\mathcal{H}_2)} \geqslant \delta \|k_z x\|_{H^2(\mathcal{H}_1)}$ for every $z \in \mathbb{D}$ and every $x \in \mathcal{H}_1$, where k_z is the reproducing kernel of the Hardy space H^2, $k_z(\zeta) = \frac{1}{1-\zeta\overline{z}}$. We refer to [20], [23], [28], [3] for further details, references and results obtained by this approach (mostly related to estimates of the left inverses for *matrix* valued functions F).

The paper is organized as follows. In Section 2, we explain that the corona theorem is true for $H^\infty \otimes X$. In Section 3, we study the spaces $H^\infty \otimes X$ and $H^\infty_{\text{comp}}(X)$, in particular their relations with the (AP) in X. In Section 4, we use the scalarisation map to examine functions from $H^\infty \otimes X$ and $H^\infty_{\text{comp}}(X)$ from the point of view of their boundary behavior. Sections 5 and 6 are devoted to general examples of functions satisfying the corona theorem.

2. CORONA THEOREM, BEZOUT EQUATIONS AND THE TENSOR PRODUCT $H^\infty \otimes X$

Let $\mathcal{A}$ be a unital commutative Banach algebra, and suppose that $\mathcal{A}$ is a function algebra on a set M. This means that the elements $f \in \mathcal{A}$ are functions on M and the functionals $f \mapsto f(m)$, $m \in M$, are bounded homomorphisms of $\mathcal{A}$. It is well-known that M is dense in $\mathfrak{M}(\mathcal{A})$, the maximal ideal space of $\mathcal{A}$, if and only if any Bezout equation $\sum_{i=1}^{n} g_i f_i = 1$ (where 1 denotes the unit of $\mathcal{A}$) is solvable with $g_1, g_2, \ldots, g_n \in \mathcal{A}$ whenever $f_1, f_2, \ldots, f_n \in \mathcal{A}$ are such that $\sum_{i=1}^{n} f_i(z)^2 \geqslant \delta^2 > 0$ for every $z \in M$. The latter property, when it is true, is usually called the *corona theorem for* $\mathcal{A}$ *and* M. It is of interest to know whether a similar "corona theorem" holds for matrix valued, or even operator valued functions $f_1, f_2, \ldots, f_n$ with matrix entries in the algebra $\mathcal{A}$. This tensoring passage (from scalar functions to matrix or operator valued functions) can often be justified by using the following theorem (in fact, a special case of the theory referred to S. Bochner, R. Phillips, G. Allan, A. Markus and A. Sementsul; see [2] and [19] for the history and more details).

Let $\mathfrak{A}$ be a unital Banach algebra with unit 1, and Z its center. Let $\mathcal{A}$ be a closed subalgebra of Z, and $\mathcal{B}$ a closed subalgebra of $\mathfrak{A}$. We denote by $\mathcal{A} \otimes \mathcal{B}$ the following subalgebra of $\mathfrak{A}$,

$$\mathcal{A} \otimes \mathcal{B} = \operatorname{clos}\Big\{ \sum_{k=1}^{n} f_k b_k : n \geqslant 1,\ f_k \in \mathcal{A},\ b_k \in \mathcal{B} \Big\}.$$

Let $\varphi \in \mathfrak{M}(\mathcal{A})$. Suppose that there exists a constant C such that

$$\Big\| \sum_{k=1}^{n} \varphi(f_k) b_k \Big\| \leqslant C \Big\| \sum_{k=1}^{n} f_k b_k \Big\|,$$

for every $n \geqslant 1$, every $f_1, f_2, \ldots \in \mathcal{A}$ and every $b_1, b_2, \ldots \in \mathcal{B}$. Then we can define a bounded multiplicative projection P_φ from $\mathcal{A} \otimes \mathcal{B}$ onto $\mathcal{B}$ by the following formula,

$$P_\varphi \Big(\sum_{k=1}^{n} f_k b_k \Big) = \sum_{k=1}^{n} \varphi(f_k) b_k.$$

THEOREM 2.1. *In the above notation, suppose that M is a dense subset of $\mathfrak{M}(\mathcal{A})$ such that P_φ is defined for every $\varphi \in M$ and $\sup_{\varphi \in M} \|P_\varphi\| < \infty$. For $F \in \mathcal{A} \otimes \mathcal{B}$, the following are equivalent:*

(i) *F is left invertible in $\mathcal{A} \otimes \mathcal{B}$;*

(ii) *the family $\{P_\varphi(F), \varphi \in M\}$ is uniformly left invertible in $\mathcal{B}$, that is,*

$$\forall\, \varphi \in M, \exists\, b_\varphi \in \mathcal{B} \text{ such that } b_\varphi P_\varphi(F) = 1 \text{ and } \sup_{\varphi \in M} \|b_\varphi\| < \infty.$$

For example, let X be a unital Banach algebra with unit 1 and let $\mathfrak{A} = \mathcal{C}_{\mathrm{A}}(X) = \operatorname{Hol}(\mathbb{D}, X) \cap \mathcal{C}(\overline{\mathbb{D}}, X)$ be the vector valued disc algebra. Set $\mathcal{A} = \mathcal{C}_{\mathrm{A}} := \mathcal{C}_{\mathrm{A}}(\mathbb{C})$ (where $f \in \mathcal{C}_{\mathrm{A}}$ is identified with $f(\,\cdot\,)1 \in \mathcal{C}_{\mathrm{A}}(X)$) and $\mathcal{B} = X$ (constant functions). It is well-known that $\mathfrak{M}(\mathcal{C}_{\mathrm{A}}) = \overline{\mathbb{D}}$, and it is easy to see that $\mathcal{C}_{\mathrm{A}}(X) =$

$\mathcal{C}_{\mathrm{A}} \otimes X$ (the Fejer polynomials of F tend to F in $\mathcal{C}_{\mathrm{A}}(X)$). As a corollary to Theorem 2.1 we get (following G. Allan) that the corona theorem is true in $\mathcal{C}_{\mathrm{A}}(X)$, that is, $F \in \mathcal{C}_{\mathrm{A}}(X)$ is left invertible in $\mathcal{C}_{\mathrm{A}}(X)$ if and only if

$$\forall z \in \mathbb{D},\ \exists\, G_z \in X \text{ such that } G_z F(z) = 1 \text{ and } \sup_{z\in\mathbb{D}} \|G_z\| < \infty. \tag{2.1}$$

As already mentioned in the introduction, in the case when $X = L(\mathcal{H})$, property (2.1) can be written in the following way,

$$\exists\, \delta > 0 \text{ such that } \|F(z)x\| \geqslant \delta \|x\|, \quad \forall z \in \mathbb{D}, \forall x \in \mathcal{H}. \tag{2.2}$$

From Carleson's corona theorem we also know that $\mathbb{D}$ is dense in $\mathfrak{M}(H^\infty)$. Applying Theorem 2.1 to $\mathfrak{A} = H^\infty(X)$, $\mathcal{A} = H^\infty$ and $\mathcal{B} = X$ we get the following theorem.

THEOREM 2.2. *Let X be a unital Banach algebra and $F \in H^\infty \otimes X$. Then F is left invertible in $H^\infty(X)$ if and only if F satisfies property* (2.1).

Theorem 2.2 is an improvement of Allan's result on $\mathcal{C}_{\mathrm{A}}(X)$ mentionned above, as $H^\infty \otimes X$ is a larger subalgebra of $H^\infty(X)$ in which the corona theorem is true (more precisely, if $\dim X = \infty$ then $\mathcal{C}_{\mathrm{A}}(X) \subsetneqq H^\infty \otimes X \subsetneqq H^\infty(X)$). In what follows, we study under which conditions a function $F \in H^\infty(X)$ belongs to $H^\infty \otimes X$. We will examine two types of such conditions: properties of the values of F (see Sections 3 and 6), and properties of its boundary behavior (see Sections 4 and 5).

3. THE SPACES $H^\infty \otimes X$ AND $H^\infty_{comp}(X)$

We start fixing the notation and basic facts on vector valued functions.

SPACES OF MEASURABLE FUNCTIONS. For p, $1 \leqslant p \leqslant \infty$, the following spaces are defined:

$$\begin{aligned}
L^p_{\mathrm{U}}(X) &= \{F : \mathbb{T} \to X : F \text{ uniformly (norm) measurable and } \|F(\,\cdot\,)\|_X \in L^p\},\\
L^p_{\mathrm{w}}(X) &= \{F : \mathbb{T} \to X : F \text{ weakly measurable and } \|F(\,\cdot\,)\|_X \in L^p\},\\
L^p_{\mathrm{w}^*}(X^*) &= \{F : \mathbb{T} \to X^* : F\ *\text{-weakly measurable and } \|F(\,\cdot\,)\|_X \in L^p\},\\
L^p(L(\mathcal{H})) &= \{F : \mathbb{T} \to L(\mathcal{H}) : F \text{ WOT-measurable and } \|F(\,\cdot\,)\|_X \in L^p\}.
\end{aligned}$$

For measurability definitions, see Chapter 3 of [16]. Recall that $F \in L^1_{\mathrm{U}}(X)$ is always almost separably valued. Besides, it is well-known that if X is separable, then $L^p_{\mathrm{w}}(X) = L^p_{\mathrm{U}}(X)$ (the Gelfand-Pettis theorem; [16]). On the other hand $L^p(L(\mathcal{H})) \neq L^p_{\mathrm{U}}(L(\mathcal{H}))$ as soon as $\mathcal{H}$ is infinite dimensional. For $F \in L^p_{\mathrm{U}}(X)$, $L^p_{\mathrm{w}}(X)$, $L^p_{\mathrm{w}^*}(X^*)$ or $L^p(L(\mathcal{H}))$, the L^p-norm of F is $\|F\|_p = \Big(\int_{\mathbb{T}} \|F(\xi)\|^p \,\mathrm{d}\mu(\xi)\Big)^{\frac{1}{p}}$ if $1 \leqslant p < \infty$, and $\|F\|_\infty = \operatorname{ess\,sup}_{\mathbb{T}} \|F(\xi)\|$ if $p = \infty$. All the spaces $L^p_{\mathrm{U}}(X)$, $L^p_{\mathrm{w}}(X)$, $L^p_{\mathrm{w}^*}(X)$ and $L^p(L(\mathcal{H}))$, $p \geqslant 1$, are Banach spaces.

$P(z,\xi)$ will denote the Poisson kernel,

$$P(z,\xi) = \frac{1 - |z|^2}{|1 - \bar{\xi} z|^2}, \quad z \in \mathbb{D},\ \xi \in \mathbb{T}. \tag{3.1}$$

If $F \in L^1_{\mathrm{U}}(X)$, $L^1_{\mathrm{w}}(X)$, $L^1_{\mathrm{w}^*}(X^*)$ or $L^1(L(\mathcal{H}))$, the Fourier coefficients $\widehat{F}(k)$, $k \in \mathbb{Z}$, and the harmonic Poisson extension $\widetilde{F}$ of F are given by the usual formulas. For instance,

$$\widetilde{F}(z) = \int_{\mathbb{T}} P(z,\xi)F(\xi)\,\mathrm{d}\mu(\xi), \quad z \in \mathbb{D}, \tag{3.2}$$

where the integral is defined in the norm, weak, weak-$*$ or WOT sense, depending on the case. It is well-known that $\widetilde{F}(z)$ tends to $F(\xi)$ nontangentially for almost every $\xi \in \mathbb{T}$ in the norm topology if $F \in L^1_{\mathrm{U}}(X)$ (the converse is also true when X is the dual of a separable Banach space); it tends to $F(\xi)$ in the SOT if $X = L(\mathcal{H})$. On the other hand, for functions from $L^1_{\mathrm{w}}(X)$, $\widetilde{F}(z)$ may have no boundary limits in any sense.

SOME SUBSPACES OF $L^\infty_U(X)$. If $f \in L^\infty$ and $x \in X$, then it is clear that $F = f(\,\cdot\,)x \in L^\infty_{\mathrm{U}}(X)$. Now we consider the space generated by all such functions,

$$L^\infty \otimes X := \mathrm{span}_{\mathrm{U}}\{f(\,\cdot\,)x : f \in L^\infty, x \in X, n \geqslant 1\} = \mathrm{clos}_{\mathrm{U}} \bigcup_{n\geqslant 1} \Big(L^\infty \bigotimes_n X\Big),$$

where $L^\infty \bigotimes_n X := \Big\{ \sum_{i=1}^n f_i x_i : f_i \in L^\infty, x_i \in X \Big\} \subset L^\infty_{\mathrm{U}}(X)$, and $\mathrm{span}_{\mathrm{U}}$ and $\mathrm{clos}_{\mathrm{U}}$ stand respectively for the closed linear span and the closure in the uniform norm topology. In particular, $L^\infty \otimes X \subset L^\infty_{\mathrm{U}}(X)$.

Note now that if $F \in L^\infty_{\mathrm{w}}(X)$ and $F(\mathbb{T})$ is almost relatively compact in X, which means that there exists $\sigma \subset \mathbb{T}$, $\mu(\sigma) = 0$, such that $F(\mathbb{T} \setminus \sigma)$ is relatively compact in X, then $F \in L^\infty_{\mathrm{U}}(X)$. Let

$$L^\infty_{\mathrm{comp}}(X) := \{F \in L^\infty_{\mathrm{w}}(X) : F(\mathbb{T}) \text{ almost relatively compact in } X\}.$$

So, $L^\infty_{\mathrm{comp}}(X) \subset L^\infty_{\mathrm{U}}(X)$. The proof of the following property is standard.

PROPOSITION 3.1. $L^\infty_{\mathrm{comp}}(X) = L^\infty \otimes X$. *In particular*, $L^\infty_{\mathrm{comp}}(X)$ *is a closed subspace (subalgebra if X is a Banach algebra) of* $L^\infty_{\mathrm{U}}(X)$. *Moreover*, $L^\infty_{\mathrm{comp}}(X) \neq L^\infty_{\mathrm{U}}(X)$ *if* $\dim X = \infty$.

Now, we pass to $H^\infty(X)$. If X, Y are two Banach spaces in duality, the duality product from $X \times Y$ to $\mathbb{C}$ will be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle_{X,Y}$.

RIGHT SLICING. The *right slicing* mapping comes from the tensor product theory (see e.g. [17]). Here it is adapted and simplified for our purposes. For $n \geqslant 1$, let

$$H^\infty \bigotimes_n X = \Big\{F \in H^\infty(X) : F = \sum_{k=1}^n f_k x_k,\ (f_k)_{k=1}^n \subset H^\infty,\ (x_k)_{k=1}^n \subset X\Big\},$$

and let $H^\infty \bigotimes_{\mathrm{a}} X$ be the algebraic tensor product of H^∞ and X, $H^\infty \bigotimes_{\mathrm{a}} X = \bigcup_{n\geqslant 1} \Big(H^\infty \bigotimes_n X\Big)$. τ will denote the bounded pointwise convergence, that is, for $(F_n)_{n\geqslant 1} \subset H^\infty(X)$,

$$\tau\text{-}\lim_{n\to\infty} F_n = F \Leftrightarrow \begin{cases} \lim_{n\to\infty} \|F_n(z) - F(z)\| = 0, & \forall z \in \mathbb{D}; \\ \sup_{n\geqslant 1} \|F_n\|_\infty < \infty. \end{cases} \tag{3.3}$$

Clearly the τ-limit F must be in $H^\infty(X)$. Furthermore, $H^\infty \bigotimes_{\mathrm{a}} X$ is τ-dense in $H^\infty(X)$ because the Fejer polynomials of $F \in H^\infty(X)$ are τ-convergent to F. Recall also that in the scalar case, the τ-convergence coincides on H^∞ with the weak-$*$ convergence induced by the duality $L^\infty - L^1$. Denote by $H^\infty_* = L^1/H^1_-$ the predual of H^∞ for this duality, where $H^1_- = \overline{zH^1} = \{f \in L^1 : \widehat{f}(n) = 0, \forall n \geqslant 0\}$. Functions in L^1 will be identified with their class in $H^\infty_* = L^1/H^1_-$. The duality form can be written as $\langle f, g\rangle_{H^\infty, H^\infty_*} = \int_{\mathbb{T}} f(\xi)\overline{g(\xi)}\,\mathrm{d}\mu(\xi)$ for $f \in H^\infty$, $g \in H^\infty_*$.

LEMMA 3.2. *Let $\varphi \in H^\infty_*$. The map*

$$\bar{l}_\varphi : H^\infty \bigotimes\nolimits_{\mathrm{a}} X \longrightarrow X, \quad \sum_{k=1}^n f_k x_k \longmapsto \sum_{k=1}^n \langle f_k, \varphi\rangle_{H^\infty, H^\infty_*}\, x_k$$

is well defined, linear and bounded from $H^\infty \bigotimes_{\mathrm{a}} X$ to X. Furthermore, it is τ-continuous and can be continuously extended up to a bounded linear map l_φ from $(H^\infty(X), \tau)$ to X. For every $F \in H^\infty(X)$ and every $\varphi \in H^\infty_*$, we have $\|l_\varphi(F)\| \leqslant \|\varphi\|_{H^\infty_*} \|F\|_\infty$.

Proof. First, we can see that $\|\bar{l}_\varphi(F)\| \leqslant \|\varphi\|_{H^\infty_*}\|F\|_\infty$ for every $F \in H^\infty \bigotimes_{\mathrm{a}} X$ and every $\varphi \in H^\infty_*$. Indeed, let $(f_i)_{i=1}^n \subset H^\infty$, $(x_i)_{i=1}^n \subset X$. Then, for every $h \in H^1_-$,

$$\begin{aligned}\bar{l}_\varphi\Big(\sum_{k=1}^n f_k x_k\Big) &= \sum_{k=1}^n \langle f_k, \varphi\rangle x_k = \sum_{k=1}^n \int_{\mathbb{T}} f_k(\xi)\overline{(\varphi + h)(\xi)} x_k \,\mathrm{d}\mu(\xi)\\ &= \int_{\mathbb{T}} \Big(\sum_{k=1}^n f_k(\xi) x_k\Big)\overline{(\varphi + h)(\xi)}\,\mathrm{d}\mu(\xi).\end{aligned}$$

Therefore $\|\bar{l}_\varphi(F)\| \leqslant \|F\|_\infty \|\varphi + h\|_1$ holds for every $F \in H^\infty \bigotimes_{\mathrm{a}} X$. Hence, $\bar{l}_\varphi$ is well defined, linear and bounded from $H^\infty \bigotimes_{\mathrm{a}} X$ to X. Now we show that $\bar{l}_\varphi$ is τ-continuous. Let $(F_n)_{n\geqslant 1} \subset H^\infty \bigotimes_{\mathrm{a}} X$ be a sequence which τ-converges to a function $F \in H^\infty \bigotimes_{\mathrm{a}} X$. For $n \geqslant 1$ and $\varphi \in H^\infty_*$, set $T_n(\varphi) = \bar{l}_\varphi(F_n)$. Then T_n is a continuous linear map from H^∞_* to X, and $\|T_n\| \leqslant \|F_n\|_\infty \leqslant C$. If $\varphi \in H^\infty_*$, then $\lim_{n\to\infty} T_n(\varphi) = T(\varphi)$, where $T(\varphi) = \bar{l}_\varphi(F)$. Indeed, this is the case for $\varphi = P(z, \cdot\,)$ for a fixed $z \in \mathbb{D}$, where $P(z, \xi)$ stands for the Poisson kernel (see formula (3.1)), because $T_n(P(z,\cdot\,)) = \int_{\mathbb{T}} F_n(\xi)P(z,\xi)\,\mathrm{d}\mu(\xi) = F_n(z)$ and $T(P(z,\cdot\,)) = F(z)$. Moreover, by the Hahn-Banach theorem, $\mathrm{lin}\{P(z,\cdot\,) : z \in \mathbb{D}\}$ is dense in H^∞_*. Applying the Banach-Steinhaus theorem we get $\lim_{n\to\infty} T_n(\varphi) = T(\varphi)$ for every $\varphi \in H^\infty_*$. The continuity of $\bar{l}_\varphi : \Big(H^\infty \bigotimes_{\mathrm{a}} X, \tau\Big) \to X$ follows. The τ-density of $H^\infty \bigotimes_{\mathrm{a}} X$ in $H^\infty(X)$ completes the proof. ∎

For $\varphi \in H^\infty_*$, the map l_φ defined in Lemma 3.2 is called the *right slicing map*. When F admits boundary values almost everywhere (e.g. when $F \in H^\infty(L(\mathcal{H}))$ or $F \in H^\infty_{\mathrm{U}}(X)$, see below) then $l_\varphi(F)$ is given by the formula

$$l_\varphi(F) = \int_{\mathbb{T}} F(\xi)\overline{\varphi(\xi)}\,\mathrm{d}\mu(\xi), \tag{3.4}$$

where the integral is defined in the same sense as the boundary values. The following lemma can be proved by a standard argument. By $\sigma(X,Y)$ we denote the weak topology induced by a set $Y \subset X^*$.

LEMMA 3.3. *Let $F \in H^\infty(X)$. For every $\varphi \in H_*^\infty$ and every $u \in X_*$,*

$$\langle l_\varphi(F), u\rangle_{X,X_*} = \langle(\langle F(\,\cdot\,), u\rangle_{X,X_*}), \varphi\rangle_{H^\infty,H_*^\infty}.$$

Therefore, the map $\varphi \mapsto l_\varphi(F)$ from $(H_*^\infty, \sigma(H_*^\infty, H^\infty))$ to $(X, \sigma(X, X_*))$ is continuous. Moreover, $\operatorname{clos}_{\sigma(X,X_*)}\{l_\varphi(F) : \varphi \in H_*^\infty, \|\varphi\|_{H_*^\infty} \leqslant 1\} = \overline{\operatorname{conv}}\{F(z) : z \in \mathbb{D}\}$, where $\overline{\operatorname{conv}}$ stands for the norm closed convex hull in X.

THE SPACES $H_U^\infty(X)$, $H^\infty \otimes X$ AND $H_{comp}^\infty(X)$. First, we make some remarks on the subspace $H_{\mathrm{U}}^\infty(X)$ of $H^\infty(X)$ consisting of functions having nontangential limits almost everywhere on $\mathbb{T}$ for the norm topology of X. When X is an arbitrary Banach space, the boundary values of $F \in H^\infty(X)$ do not need to exist in any sense. Nevertheless the Fatou theorem works for $X = L(\mathcal{H})$ (existence of SOT boundary values almost everywhere) or $X = (X_*)^*$ with X_* separable (existence of weak-$*$ boundary values almost everywhere). Moreover, when boundary values of $F \in H^\infty(X)$ do exist in a particular sense, then F coincides with the harmonic Poisson extension of its boundary values defined in the same sense, see formula (3.2). For $F \in H_{\mathrm{U}}^\infty(X)$, it is clear that the boundary values define a function from $L_{\mathrm{U}}^\infty(X)$, and thus we get

$$H_{\mathrm{U}}^\infty(X) = H^\infty(X) \cap L_{\mathrm{U}}^\infty(X).$$

In particular, $H_{\mathrm{U}}^\infty(X) = H^\infty(X)$ if X is a separable dual space. In a more general situation, it may happen that $H_{\mathrm{U}}^\infty(X) \neq H^\infty(X)$. For instance, $H_{\mathrm{U}}^\infty(c_0) \neq H^\infty(c_0)$, where c_0 is the space of scalar sequences tending to zero, and $H_{\mathrm{U}}^\infty(L(\mathcal{H})) \neq H^\infty(L(\mathcal{H}))$ if $\dim \mathcal{H} = \infty$ (consider $F(z) = \operatorname{diag}(1, z, z^2, z^3, \ldots)$).

Now consider the subspace $H^\infty \otimes X$ of $H^\infty(X)$ generated by $H^\infty \bigotimes_{\mathrm{a}} X$,

$$H^\infty \otimes X = \operatorname{clos}_{H^\infty(X)} H^\infty \bigotimes\nolimits_{\mathrm{a}} X.$$

This space was already introduced in Section 2. We have seen that the corona theorem is true for functions from $H^\infty \otimes X$, and that makes this subspace of special interest. The following subspace of $H^\infty(X)$ also turns out to be of interest,

$$H_{\mathrm{comp}}^\infty(X) = \{F \in H^\infty(X) : F(\mathbb{D}) \text{ is a relatively compact set in } X\}.$$

LEMMA 3.4. (i) *$H^\infty \otimes X$ and $H_{\mathrm{comp}}^\infty(X)$ are closed subspaces of $H^\infty(X)$, and Banach subalgebras of $H^\infty(X)$ if X is a Banach algebra. Always $H^\infty \otimes X \subset H_{\mathrm{comp}}^\infty(X)$.*

(ii) *Every $F \in H_{\mathrm{comp}}^\infty(X)$ has norm boundary values in X almost everywhere on $\mathbb{T}$; this means that $H_{\mathrm{comp}}^\infty(X) \subset H_{\mathrm{U}}^\infty(X)$.*

(iii) *$H_{\mathrm{comp}}^\infty(X) = L_{\mathrm{comp}}^\infty(X) \cap H_{\mathrm{U}}^\infty(X) = L_{\mathrm{comp}}^\infty(X) \cap H^\infty(X)$.*

(iv) *$\mathcal{C}_{\mathrm{A}}(X) \subset H^\infty \otimes X$.*

(v) *If $\dim X < \infty$, then $H_{\mathrm{comp}}^\infty(X) = H_{\mathrm{U}}^\infty(X) = H^\infty(X) = H^\infty \otimes X$. If $\dim X = \infty$, then $H_{\mathrm{comp}}^\infty(X) \neq H_{\mathrm{U}}^\infty(X)$.*

(vi) *If $X = (X_*)^*$ is separable then $H_{\mathrm{U}}^\infty(X) = H^\infty(X)$. On the other hand, if $\mathcal{H}$ is an infinite dimensional Hilbert space then $H_{\mathrm{U}}^\infty(L(\mathcal{H})) \neq H^\infty(L(\mathcal{H}))$.*

Proof. The proof of (i) and (ii) consists in a standard use of compactness, the separability of span$F(\mathbb{D})$, and the scalar Fatou theorem. For the nontrivial inclusion of (iii), take $F \in L^\infty_{\mathrm{comp}}(X) \cap H^\infty(X)$ and use the Poisson formula (3.2) to obtain that $F(\mathbb{D}) \subset \overline{\mathrm{conv}}(F(\mathbb{T} \setminus \sigma))$, where $\sigma \subset \mathbb{T}$ is chosen in such a way that $\mu(\sigma) = 0$ and $F(\mathbb{T} \setminus \sigma)$ is relatively compact. This implies that $F \in H^\infty_{\mathrm{comp}}(X)$. The less obvious statement from (v) and (vi), namely that $H^\infty_{\mathrm{comp}}(X) \neq H^\infty_{\mathrm{U}}(X)$ for $\dim X = \infty$, can be proved using the Jones-Vinogradov interpolation operator (see [20], Lecture 8). ∎

THE APPROXIMATION PROPERTY (AP). It is still unknown whether $H^\infty_{\mathrm{comp}}(X) = H^\infty \otimes X$ for every X. Below, we link this equality with the classical approximation property for X and prove its equivalence to some other finite rank approximations.

Recall that a Banach space X has the *approximation property*, (AP) in short, if every compact subset K of X satisfies the following:

$$\forall \varepsilon > 0,\ \exists T \in L(X),\ \mathrm{rank}\, T < \infty,\ \text{such that } \|Tx - x\| < \varepsilon,\ \forall x \in K. \tag{3.5}$$

A (relatively) compact subset K of a Banach space X satisfies (AP) in X if (3.5) holds. See [18] for basic facts and references on (AP).

PROPOSITION 3.5. *Let $F \in H^\infty_{\mathrm{comp}}(X)$. If $F(\mathbb{D})$ satisfies* (AP) *in X, then $F \in H^\infty \otimes X$. In particular, if X satisfies* (AP)*, then $H^\infty_{\mathrm{comp}}(X) = H^\infty \otimes X$.*

Proof. Let $\varepsilon > 0$ and let $T \in L(X)$ be a finite rank operator such that $\|T(F(z)) - F(z)\| < \varepsilon$, $\forall z \in \mathbb{D}$. Write $T = \sum_{k=1}^{N} \langle \cdot\,, x_k^* \rangle x_k$, where $\{x_k\}_{k=1}^N \subset X$ and $\{x_k^*\}_{k=1}^N \subset X^*$. Then $T(F(z)) = \sum_{k=1}^{N} \langle F(z), x_k^* \rangle x_k$ for every $z \in \mathbb{D}$. For $1 \leqslant k \leqslant N$, set $f_k = \langle F(\cdot\,), x_k^* \rangle$. Then $f_k \in H^\infty$. Therefore the function $z \mapsto T(F(z))$, $z \in \mathbb{D}$, belongs to $H^\infty \otimes X$ and so does F. ∎

Many Banach spaces satisfy (AP), e.g. the space $\mathcal{C}(K)$ of continuous functions on any compact set K, the $L^p(\Omega, \nu)$ spaces, $1 \leqslant p \leqslant \infty$, the Schatten classes $S_p(X, Y)$, $1 \leqslant p \leqslant \infty$, where X et Y are any Banach spaces with a Schauder basis. Nevertheless there do exist some (even separable) Banach spaces not satisfying (AP) ([11], [8]), and it is also known that $L(\mathcal{H})$ does not satisfy (AP) when $\mathcal{H}$ is infinite dimensional ([24]). The problem whether the space H^∞ satisfies (AP) seems to be open. In what follows, we exhibit some subspaces of $L(\mathcal{H})$ and H^∞ satisfying (AP) and relate them to the operator corona problem. Note that in the proof of Proposition 3.5, the operator T needs not to be defined on entire X, but only on span$\,(F(\mathbb{D}))$. As long as T takes values in X this is still the same property as (AP) for $F(\mathbb{D})$ in X (because of the Hahn-Banach theorem). Sometimes we will restrict ourselves to such a finite rank approximation $T : \mathrm{span}\,(F(\mathbb{D})) \to X$, also calling it (AP). But first, we mention a simple metric property of a compact set K in a Banach space X which guarantees that K satisfies (AP) in X. The following approximate numbers $\varepsilon(K, n)$ will be used,

$$\varepsilon(K, n) := \inf_{\dim L = n} \sup_{x \in K} \mathrm{dist}(x, L),$$

where L denotes a finite dimensional subspace of X. Compactness of K implies that $\varepsilon(K,n)$ decreases to 0 as n tends to ∞. The following proposition gives a rate of decrease of $\varepsilon(F(\mathbb{D}),n)$, $F \in H^\infty_{\mathrm{comp}}(X)$, which guarantees that F belongs to $H^\infty \otimes X$.

PROPOSITION 3.6. *Let $K \subset X$ be a relatively compact set. If $\varepsilon(K,n) = \mathrm{o}\Big(\frac{1}{\sqrt{n}}\Big)$, $n \to \infty$, then K satisfies* (AP).

Proof. The following result is due to D.R. Lewis (see [31], p. 116): for any Banach space X and any n-dimensional subspace Y of X, there exists a projection $P \in \mathcal{L}(X)$ such that $PX = Y$ and $\|P\| \leqslant \sqrt{n}$. Now, for $n \geqslant 1$, choose an n-dimensional subspace L_n of X such that $\sup\limits_{x\in F(\mathbb{D})} \mathrm{dist}(x, L_n) \leqslant 2\varepsilon(F(\mathbb{D}),n)$ and a projection P_n from X onto L_n such that $\|P_n\| \leqslant \sqrt{n}$. Given $x \in K$, there exists $y \in L_n$ such that $\|x-y\| \leqslant 3\,\varepsilon(K,n)$, and consequently,

$$\|x - P_n x\| \leqslant \|x-y\| + \|P_n x - y\| \leqslant 3(1+\sqrt{n})\,\varepsilon(K,n)$$

as $P_n x - y = P_n(x-y)$. Therefore, K satisfies (AP). ∎

4. SCALARISATION

In this section the subspaces $H^\infty_{\mathrm{comp}}(X)$ and $H^\infty \otimes X$ are characterized by functional properties in H^∞ by using a scalarisation process (which corresponds to the *left slicing* in the tensor product theory). In the case of operator valued functions, the membership of F in $H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$ or in $H^\infty \otimes L(\mathcal{H})$ is characterized by properties of the scalar functions $\langle F(\,\cdot\,)x, y\rangle \in H^\infty$, $x, y \in \mathcal{H}$, and in some cases even by properties of the matrix entries $\langle F(\,\cdot\,)e_j, e_i\rangle$ of F with respect to some orthonormal basis $(e_i)_{i\geqslant 1}$ of $\mathcal{H}$. Then the functional language is used to compare approximation properties in X and H^∞. Later on, these results will be used to show that some block diagonal functions from $H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$ actually belong to $H^\infty \otimes L(\mathcal{H})$ (see Section 6), and that some pseudocontinuable functions from $H^\infty_{\mathrm{comp}}(X)$ also are in $H^\infty \otimes X$ (see Section 5).

Only the case when X is a dual space, $X = (X_*)^*$, is treated here. Note that the operator case fits into this setting as $L(\mathcal{H})$ is the dual space of the space of trace class operators, $L(\mathcal{H}) = (S_1(\mathcal{H}))^*$, in the trace duality. In all this section the duality product $\langle\cdot,\cdot\rangle$ corresponds to the (X, X_*) duality. B_* stands for the closed unit ball of X_*.

Let $F \in H^\infty(X)$. We define a scalarisation map $\phi_F \in L(X_*, H^\infty)$ by $\phi_F(u) = \langle F(\,\cdot\,), u\rangle$, $u \in X_*$.

PROPOSITION 4.1. *The map $\Phi : F \mapsto \phi_F$ is an isometric linear operator from $H^\infty(X)$ onto $L(X_*, H^\infty)$.*

Proof. The linearity is obvious. The following equalities show that Φ is isometric.

$$\begin{aligned}\|\phi_F\| &= \sup_{u\in B_*} \|\phi_F(u)\| = \sup_{u\in B_*}\sup_{z\in\mathbb{D}} |\langle F(z), u\rangle| \\ &= \sup_{z\in\mathbb{D}}\sup_{u\in B_*} |\langle F(z), u\rangle| = \sup_{z\in\mathbb{D}} \|F(z)\| = \|F\|_\infty.\end{aligned}$$

To see that Φ is onto, let $\phi \in L(X_*, H^\infty)$. Then, for every $z \in \mathbb{D}$, the map $u \mapsto (\phi(u))(z)$ is a continuous linear functional on X_* and hence there exists a unique $F(z) \in X = (X_*)^*$ such that $(\phi(u))(z) = \langle F(z), u\rangle$ for every $u \in X_*$. It remains to show that the map $z \mapsto F(z)$ belongs to $H^\infty(X)$. First we see that F is bounded, because

$$\|F(z)\| = \sup_{u\in B_*} |\langle F(z), u\rangle| = \sup_{u\in B_*} |(\phi(u))(z)| \leqslant \sup_{u\in B_*} \|\phi(u)\|_\infty = \|\phi\|,$$

for every $z \in \mathbb{D}$. Furthermore, $\langle F(\,\cdot\,), u\rangle = \phi(u) \in H^\infty$ for every $u \in X_*$. In particular, F is weak-$*$ analytic, and therefore analytic. Consequently, $F \in H^\infty(X)$, and it is clear that $\phi = \phi_F = \Phi(F)$. ∎

CHARACTERIZATION OF $H^\infty_{comp}(X)$ AND $H^\infty \otimes X$. $\mathcal{F}(X,Y)$ denotes the space of bounded finite rank operators from X to Y, and $S_\infty(X,Y)$ denotes the space of compact linear operators from X to Y.

PROPOSITION 4.2. *Let $F \in H^\infty(X)$. The following are equivalent:*
(i) $F \in H^\infty_{\mathrm{comp}}(X)$;
(ii) $\phi_F \in S_\infty(X_*, H^\infty)$.

Proof. We show that (i) implies (ii). The fact that (ii) implies (i) can be proved in a similar way. Let $\varepsilon > 0$. Choose a finite ε-net $\{F(z_i)\}_{i=1}^p$ of $F(\mathbb{D})$ in X. For every i, $1 \leqslant i \leqslant p$, define $h_i : u \mapsto \langle F(z_i), u\rangle = \phi_F(u)(z_i)$, and $h : u \mapsto h(u) = (h_1(u), h_2(u), \ldots, h_p(u))$, where $u \in X_*$. Then h_i is a bounded linear functional on X_*, $\|h_i\| \leqslant \|F(z_i)\| \leqslant \|F\|_\infty$. Take the uniform norm on $\mathbb{C}^p$, $\|\lambda\|_\infty = \sup_{1\leqslant i\leqslant p} |\lambda_i|$, $\lambda \in \mathbb{C}^p$. As $h(B_*)$ is bounded in $\mathbb{C}^p$ we can take a finite ε-net $\{h(u_j)\}_{j=1}^q$ of $h(B_*)$ in $\mathbb{C}^p$. Now we show that $\{\phi_F(u_j)\}_{j=1}^q$ is a 3ε-net of $\phi_F(B_*)$ in H^∞. Let $\phi_F(u) \in \phi_F(B_*)$ and $z \in \mathbb{D}$. Let i, j, $1 \leqslant i, j \leqslant q$, be such that $\|F(z) - F(z_i)\| \leqslant \varepsilon$ and $\|h(u) - h(u_j)\|_\infty \leqslant \varepsilon$. Then

$$\begin{aligned}
&|\phi_F(u)(z) - \phi_F(u_j)(z)| \\
&\leqslant |\phi_F(u)(z) - \phi_F(u)(z_i)| + |\phi_F(u)(z_i) - \phi_F(u_j)(z_i)| + |\phi_F(u_j)(z_i) - \phi_F(u_j)(z)| \\
&= |\langle F(z) - F(z_i), u\rangle| + |h_i(u) - h_i(u_j)| + |\langle F(z_i) - F(z), u_j\rangle| \\
&\leqslant \|F(z) - F(z_i)\| + \varepsilon + \|F(z_i) - F(z)\| \leqslant 3\varepsilon.
\end{aligned}$$

Therefore $\|\phi_F(u) - \phi_F(u_j)\|_\infty \leqslant 3\varepsilon$ and $\{\phi_F(u_j)\}_{j=1}^q$ is a 3ε-net of $\phi_F(B_*)$ in H^∞. Hence, $\phi_F(B_*)$ is relatively compact in H^∞. ∎

Now we come to a characterization of $H^\infty \otimes X$. The following lemma is obvious.

LEMMA 4.3. *Let $F \in H^\infty(X)$. The following are equivalent:*
(i) $F \in H^\infty \bigotimes_n X$;
(ii) ϕ_F *has finite rank less or equal to n.*
Furthermore, if E is a closed subspace of H^∞, then $F \in E \bigotimes_n X$ if and only if ϕ_F has finite rank less or equal to n and $\operatorname{Im}\phi_F \subset E$.

THEOREM 4.4. *Let $F \in H^\infty(X)$. Then $F \in H^\infty \otimes X$ if and only if $\phi_F \in$* $\mathrm{clos}\mathcal{F}(X_*, H^\infty)$, *where* clos *stands for the norm closure in $L(X_*, H^\infty)$.*

Proof. It is straightforward from Proposition 4.1 and Lemma 4.3. ∎

If $F \in H^\infty \otimes X$ and the mapping ϕ_F takes values in a closed subspace E of H^∞, then the natural question arises whether $F \in E \otimes X$. The answer is yes if E is a complemented subspace of H^∞, since one can simply project approximating functions from $H^\infty \bigotimes_{\mathrm{a}} X$ onto $E \otimes X$. Later on, we prove that the same conclusion is true for E satisfying (AP). But for a general $E \subset H^\infty$, we do not know the answer.

The following corollary is a consequence of Theorem 4.4 and the Grotendieck characterization of (AP); see [18].

COROLLARY 4.5. *The following are equivalent:*
(i) $H^\infty \otimes X = H^\infty_{\mathrm{comp}}(X)$ *for every dual Banach space* X*;*
(ii) H^∞ *has* (AP).

The question whether H^∞ has (AP) has been studied by J. Bourgain ([4]), J. Bourgain and O. Reinov ([5]), G. Pisier ([21]), and others. Although this question is still open, in our setting, some conditions on $F \in H^\infty_{\mathrm{comp}}(X)$ can be given which make the compact operator ϕ_F a limit of bounded finite rank operators. For instance, the following proposition is an elementary consequence of the duality reasoning.

PROPOSITION 4.6. *Suppose* X *is a bidual,* $X = (X_*)^* = (X_{**})^{**}$*, and let* $F \in H^\infty(X)$*. If* X_{**} *has* (AP) *and* $\phi_F \in S_\infty(X_*, H^\infty)$ *is weak-$*$ continuous, then* $F \in H^\infty \otimes X$*.*

Now we consider the scalarisation in the case of operator valued functions. It is well-known that $L(\mathcal{H}) = S_1^*$ and $S_1 = S_\infty^*$ with respect to the standard trace duality,

$$\langle A, B\rangle = \mathrm{Tr}(AB) = \sum_{n \geqslant 1} (ABe_n, e_n),$$

where $(e_n)_{n\geqslant 1}$ is any orthonormal basis of $\mathcal{H}$. Using some standard properties of S_1 (see e.g. [14]), we obtain the following proposition (to compare with Theorem 4.11 below).

PROPOSITION 4.7. *Let* $F \in H^\infty(L(\mathcal{H}))$*. The following are equivalent:*
(i) $F \in H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$;
(ii) $\{\langle F(\,\cdot\,)u, v\rangle : u, v \in \mathcal{H},\ \|u\| = \|v\| = 1\}$ *is relatively compact in* H^∞*.*

SCALARISATION AND APPROXIMATION PROPERTY IN H^∞. In view of Proposition 4.2 and Theorem 4.4, in the case when X is a dual space we can use (AP) in H^∞ rather than in X; see Proposition 4.8. In the case of an operator valued function $F \in H^\infty(L(\mathcal{H}))$, we show that, in general, the approximation property for the set of matrix entries $\mathcal{M}_F = \{\langle F(\,\cdot\,)e_j, e_i\rangle\}_{i,j\geqslant 1}$ with respect to an orthonormal basis $(e_i)_{i\geqslant 1}$ of $\mathcal{H}$ does not lead to $F \in H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$.

Proposition 4.8. *Let X be a dual Banach space, $X = (X_*)^*$. Let $F \in H^\infty_{\mathrm{comp}}(X)$. If $\phi_F(B_*)$ satisfies* (AP) *in H^∞ then $F \in H^\infty \otimes X$. If $X = L(\mathcal{H})$, it is sufficient that the set $\{\langle F(\cdot)u, v\rangle : u, v \in \mathcal{H},\ \|u\| = \|v\| = 1\}$, or even the family $\{\langle F(\cdot)x_j, x_i\rangle\}_{i,j\geqslant 1}$, satisfy* (AP) *in H^∞, where $(x_k)_{k\geqslant 1}$ is a dense sequence in the unit sphere of $\mathcal{H}$.*

Proof. Let E be the closed linear span of $\phi_F(B_*)$ in H^∞ (in the case when $X = L(\mathcal{H})$, E coincides with the closed linear spans of $\{\langle F(\cdot)u, v\rangle : u, v \in \mathcal{H}, \|u\| = \|v\| = 1\}$ or $\{\langle F(\cdot)x_j, x_i\rangle\}_{i,j\geqslant 1}$). If $\phi_F(B_*)$ satisfies (AP) in H^∞, then for every $\varepsilon > 0$, there exists a bounded finite rank operator $T : E \to H^\infty$ such that $\|T\phi_F(u) - \phi_F(u)\|_\infty < \varepsilon$ for every $u \in B_*$. In particular, $T\phi_F$ belongs to $\mathcal{F}(X_*, H^\infty)$ and $\|T\phi_F - \phi_F\|_\infty = \sup\limits_{u\in B_*} \|T\phi_F(u) - \langle F(\cdot), u\rangle\| \leqslant \varepsilon$. Thus, $\phi_F \in \overline{\mathcal{F}(X_*, H^\infty)}$ and from Theorem 4.4 we get $F \in H^\infty \otimes X$. ∎

Corollary 4.9. *If $F \in H^\infty_{\mathrm{comp}}(X)$ with $X = (X_*)^*$, and $\phi_F(B_*) \subset E$, where E is a closed subspace of H^∞ satisfying* (AP)*, then $F \in E \otimes X$.*

Indeed, we can choose the approximation operator T taking values in E.

Remark 4.10. In Corollary 4.9 the assumption $F \in H^\infty_{\mathrm{comp}}(X)$ is essential. Indeed, there exists $F \in H^\infty(L(\mathcal{H}))$ such that $\phi_F(B_*) \subset \mathcal{C}_{\mathrm{A}}$ but $F \notin \mathcal{C}_{\mathrm{A}}(L(\mathcal{H})) = \mathcal{C}_{\mathrm{A}} \otimes L(\mathcal{H})$. For example, we can take $F(z) = \operatorname{diag}\big((z^n)_{n\geqslant 1}\big) \in L(\ell^2)$. Let $(e_n)_{n\geqslant 1}$ denote the canonical basis of ℓ^2. Then, for $T \in S_1$, we have $\phi_F(T) = \sum\limits_{n\geqslant 0} z^n (Te_n, e_n)$ and $\sum\limits_{n\geqslant 0} |(Te_n, e_n)| < \infty$. Thus $\phi_F(T) \in \mathcal{C}_{\mathrm{A}}$. But $F \notin \mathcal{C}_{\mathrm{A}}(L(\ell^2))$, and even $F \notin H^\infty_{\mathrm{U}}(L(\ell^2))$.

So, in the case when $X = (X_*)^*$, we have seen that $F \in H^\infty_{\mathrm{comp}}(X)$ belongs to $H^\infty \otimes X$ as soon as $F(\mathbb{D})$ satisfies (AP) in X or $\phi_F(B_*)$ satisfies (AP) in H^∞. Nevertheless, these two properties are independent of each other in the following sense (see [30] for the proof). If X does not satisfy (AP), then there exists $F \in H^\infty_{\mathrm{comp}}(X)$ such that $F(\mathbb{D})$ does not satisfy (AP) in X but $\phi_F(B_*)$ satisfies (AP) in H^∞. On the other hand, if H^∞ does not satisfy (AP), then there exists $F \in H^\infty \otimes \ell^\infty$ (and hence, $F(\mathbb{D}) \in$ (AP)) such that $\phi_F(B_*)$ does not satisfy (AP) in H^∞.

Now we consider the question on how to express the membership of F in $H^\infty \otimes L(\mathcal{H})$ or in $H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$ in terms of the set of its matrix entries $\mathcal{M}_F = \{f_{ij}\}_{i,j\geqslant 1}$ with respect to an orthonormal basis $\mathbf{e} = (e_i)_{i\geqslant 1}$ of $\mathcal{H}$, $f_{ij} = \langle F(\cdot)e_j, e_i\rangle$. We show that it may happen that $\mathcal{M}_F$ satisfies (AP) in H^∞ but $F \notin H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$. On the other hand, if $\mathcal{M}_F$ satisfies a stronger approximation property, called the *complete approximation property* (CAP), then $F \in H^\infty \otimes L(\mathcal{H})$, see Theorem 4.13 below.

Theorem 4.11. *Let $F \in H^\infty(L(\mathcal{H}))$. In the notation above, for $n \geqslant 1$, $F \in H^\infty \bigotimes\limits_n L(\mathcal{H})$ if and only if* $\operatorname{rank}\{f_{ij} : i, j \geqslant 1\} := \dim(\operatorname{lin}\{f_{ij} : i, j \geqslant 1\}) \leqslant n$. *On the other hand, if $\dim \mathcal{H} = \infty$, there exists $F \in H^\infty(L(\mathcal{H}))$ such that $\mathcal{M}_F$ is a relatively compact set satisfying* (AP) *in H^∞, but $F \notin H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$.*

Proof. $F \in H^\infty \bigotimes_n L(\mathcal{H})$ if and only if $\operatorname{rank} \phi_F(B_*) \leqslant n$ (see Lemma 4.3). Thus, if $F \in H^\infty \bigotimes_n L(\mathcal{H})$ then $\operatorname{rank}\{f_{ij} : i,j \geqslant 1\} = \operatorname{rank}\{\phi_F(e_j \otimes e_i) : i,j \geqslant 1\} \leqslant n$, where $e_j \otimes e_i$ stands for the rank one operator $(e_j \otimes e_i)x = \langle x, e_i\rangle e_j$, $x \in \mathcal{H}$. Conversely, let P_n be the orthogonal projection onto $\operatorname{lin}\{e_i : 1 \leqslant i \leqslant n\}$. For every $A \in S_1$, P_nAP_n tends to A in S_1 and $P_nAP_n \in \operatorname{lin}\{e_j \otimes e_i : 1 \leqslant i,j \leqslant n\}$. Therefore, $\phi_F(P_nAP_n)$ tends to $\phi_F(A)$ in H^∞ and $\phi_F(P_nAP_n) \in \operatorname{lin}\{\phi_F(e_j \otimes e_i) : 1 \leqslant i,j \leqslant n\} \subset \operatorname{lin}\{\phi_F(e_j \otimes e_i) : i,j \geqslant 1\}$. Thus, if $\operatorname{rank}\{\phi_F(e_j \otimes e_i) : i,j \geqslant 1\} \leqslant n$, then $\operatorname{rank} \phi_F(B_*) \leqslant n$.

Now we take $\mathbf{e}$ to be the canonical basis of ℓ^2, and $A = (a_{ij})_{i,j\geqslant 1} \in L(\ell^2)$ the Césaro operator defined by $Ax = \left(\frac{1}{k}\sum_{i=1}^{k} x_i\right)_{k\geqslant 1}$, $x = (x_k)_{k\geqslant 1} \in \ell^2$.

$$A = \begin{pmatrix} 1 & 0 & \cdots & & & & \\ \frac{1}{2} & \frac{1}{2} & 0 & & & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & & & \\ \vdots & & & \ddots & \ddots & & \\ \frac{1}{n} & \frac{1}{n} & \cdots & \cdots & \frac{1}{n} & 0 & \\ \vdots & & & & & \ddots & \ddots \\ \vdots & & & & & & \ddots \end{pmatrix}.$$

By the Schur test ([15]), A is a bounded operator on ℓ^2; indeed,

$$\begin{cases} \forall i \geqslant 1, & \sum_{j\geqslant 1} a_{ij}\frac{1}{\sqrt{j}} \leqslant \frac{M}{\sqrt{i}}, \\ \forall j \geqslant 1, & \sum_{i\geqslant 1} a_{ij}\frac{1}{\sqrt{i}} \leqslant \frac{M}{\sqrt{j}}; \end{cases}$$

where M is a constant. On the other hand, A is not compact. Indeed, take $x_n = \frac{1}{\sqrt{n}}(1,1,\ldots,1,0,0,\ldots)$, $n \geqslant 1$; then $\|x_n\| = 1$ and $(x_n)_{n\geqslant 1}$ tends weakly to 0 whereas $\|Ax_n\| \geqslant 1$. Let $P_n \in L(\ell^2)$, $n \geqslant 1$, be as above, and $Q_n = \mathrm{id} - P_n$. Define $A_n = Q_{n-1}A$. Then we choose a Carleson sequence $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$, and $(\varphi_n)_{n\geqslant 1} \subset H^\infty$ a sequence of interpolating functions ($\varphi_n(\lambda_k) = \delta_{nk}$, $\forall n,k \geqslant 1$) such that $\sum_{n\geqslant 1} |\varphi_n(z)| \leqslant C$ for every $z \in \mathbb{D}$ (see Section 5 below for definitions and references). Set $F(z) = \sum_{n\geqslant 1} \varphi_n(z)A_n$, $z \in \mathbb{D}$. Then $F \in H^\infty(L(\ell^2))$. Consider the family of matrix entries $\{f_{ij}\}_{i,j\geqslant 1}$ of F. We have $f_{ij}(z) = \sum_{n\geqslant 1} \varphi_n(z)(A_n)_{ij}$ for every $i,j \geqslant 1$ and every $z \in \mathbb{D}$. Let $N \geqslant 1$. If $i+j \geqslant 2N$, then $0 \leqslant (A_n)_{ij} \leqslant \frac{1}{N}$ for every $n \geqslant 1$ and $\|f_{ij}\|_\infty \leqslant \frac{C}{N}$. The set $\{f_{ij} : i+j < 2N\}$ being finite, we deduce that $\{f_{ij} : i,j \geqslant 1\}$ is a relatively compact family in H^∞.

Now suppose that $F(\mathbb{D})$ is relatively compact. As $F(\lambda_k) = A_k$, the sequence $(A_k)_{k\geqslant 1}$ must have a norm convergent subsequence. But $(A_k)_{k\geqslant 1}$ tends SOT to 0 because $\lim_{n\to\infty} \|Q_nx\| = 0$ for every $x \in \ell^2$. Then $(A_k)_{k\geqslant 1}$ must be norm convergent to 0. As $\|A - P_kA\| = \|A_k\|$, we deduce that the sequence of finite rank operators $(P_kA)_{k\geqslant 1}$ is norm convergent to A, which is a contradiction to the non-compactness of A.

Moreover, the set $\{f_{ij} : i,j \geqslant 1\}$ satisfies (AP) as it is contained in $T(\ell^\infty)$ which satisfies (AP); here $T \in L(\ell^\infty, H^\infty)$ is the interpolation operator associated with $(\varphi_k)_{k\geqslant 1}$, it is an isomorphic embedding of ℓ^∞ into H^∞ (see Lemma 5.7). ∎

Now we define a stronger approximation property for the family of matrix entries of $F \in H^\infty(L(\mathcal{H}))$, which guarantees that $F \in H^\infty \otimes L(\mathcal{H})$. $\mathcal{M}_n$ denotes the space of $n\times n$ matrices with scalar entries. $\mathcal{M}_n(E)$ stands for the space of $n\times n$ matrices with entries in E, where E is a closed subspace of H^∞. The space $\mathcal{M}_n(E)$ embeds canonically into $H^\infty(\mathcal{M}_n)$. We endow $\mathcal{M}_n(E)$ with the norm $\|\cdot\|_\infty$ induced by $H^\infty(\mathcal{M}_n)$.

DEFINITION 4.12. Let K be a relatively compact subset of H^∞, and $E = \operatorname{span}_{H^\infty} K$. The set K satisfies the complete approximation property (CAP) in H^∞ if for every $\varepsilon > 0$ and every $C > 0$, there exists $T \in L(E, H^\infty)$ such that

$$\operatorname{rank} T < \infty \quad \text{and} \quad \|(T\otimes I_n)(\varphi_{ij})_{i,j=1}^n - (\varphi_{ij})_{i,j=1}^n\|_\infty \leqslant \varepsilon \tag{4.1}$$

for every matrix $(\varphi_{ij})_{i,j=1}^n$ with $\{\varphi_{ij} : 1\leqslant i,j\leqslant N\} \subset K$ and $\|(\varphi_{ij})_{i,j=1}^n\|_\infty \leqslant C$, $n \geqslant 1$, where the operator $T\otimes I_n \in L(\mathcal{M}_n(E), \mathcal{M}_n(H^\infty))$ is defined by $(T\otimes I_n)(\varphi_{ij})_{i,j=1}^n = (T\varphi_{ij})_{i,j=1}^n$.

THEOREM 4.13. *Let $F \in H^\infty(L(\mathcal{H}))$ and $\mathcal{M}_F = (f_{ij})_{i,j\geqslant 1}$ its matrix in an orthonormal basis $\mathbf{e} = (e_n)_{n\geqslant 1}$. If $\{f_{ij} : i,j \geqslant 1\}$ is a relatively compact family satisfying* (CAP) *in H^∞, then $F \in H^\infty \otimes L(\mathcal{H})$.*

Proof. Set $E = \operatorname{span}_{H^\infty}\{f_{ij} : i,j\geqslant 1\}$ and $C = \|F\|_\infty$. Then $\|(f_{ij})_{i,j=1}^n\|_\infty \leqslant C$ for every $n \geqslant 1$. Let $\varepsilon > 0$ and $T \in L(E, H^\infty)$ satisfying (4.1). Let $x, y \in \mathcal{H}$, $x = \sum\limits_{j=1}^n x_je_j$, $y = \sum\limits_{i=1}^n y_ie_i$. Then $\langle F(\cdot)x, y\rangle = \sum\limits_{i,j=1}^n x_j\overline{y_i}f_{ij}$ belongs to E. Moreover, $T(\langle F(\cdot)x,y\rangle) - \langle F(\cdot)x,y\rangle = \sum\limits_{i,j=1}^n x_j\overline{y_i}(Tf_{ij} - f_{ij})$ and

$$\begin{aligned}\Big\|\sum_{i,j=1}^n x_j\overline{y_i}(Tf_{ij}-f_{ij})\Big\|_\infty &= \|\langle[(T\otimes I_n)(f_{ij})_{i,j=1}^n - (f_{ij})_{i,j=1}^n](x_j)_{j=1}^n, (y_i)_{i=1}^n\rangle\|_\infty\\ &\leqslant \|(T\otimes I_n)(f_{ij})_{i,j=1}^n - (f_{ij})_{i,j=1}^n\|_\infty \|x\|\,\|y\| \leqslant \varepsilon\|x\|\,\|y\|.\end{aligned}$$

Then $\|T(\langle F(\cdot)x,y\rangle) - \langle F(\cdot)x,y\rangle\|_\infty \leqslant \varepsilon\|x\|\,\|y\|$ for every $x,y\in\mathcal{H}$. Therefore the set $\{\langle F(\cdot)x,y\rangle : x,y\in\mathcal{H},\ \|x\| = \|y\| = 1\} = \phi_F(S_{1,*})$ satisfies (AP) in H^∞, and from Proposition 4.8 we deduce that $F \in H^\infty \otimes L(\mathcal{H})$. ∎

REMARK 4.14. The (CAP) is a restricted version of the *operator approximation property (OAP)* which was introduced by Effros and Ruan ([10]). The (OAP) is formulated in terms of tensor products $L(\mathcal{H}_1)\otimes L(\mathcal{H}_2)$. As H^∞ embeds isometrically in the space $L(H^2)$ of bounded operators on the Hardy space H^2, the (CAP) coincides with the restriction to $H^\infty \otimes L(\ell^2)$ of the (OAP) for $L(H^2)\otimes L(\ell^2)$.

5. THE CORONA THEOREM FOR WEAKLY PSEUDOCONTINUABLE FUNCTIONS

In this section, we use the scalarisation results of Section 4 to obtain some explicit sufficient conditions for the left invertibility of a function $F \in H^\infty(L(\mathcal{H}))$. To this end, we exhibit some subspaces of H^∞ satisfying (AP) and defined in terms of pseudocontinuable functions appearing in the theory of Hardy spaces H^p, $1 \leqslant p \leqslant \infty$ (see [20], Lecture 2). The proofs are based on the free interpolation techniques (see [13], [20]). We first give some general information.

SUBSPACES OF $H^\infty(X)$ DEFINED BY MINIMAL SEQUENCES. For $(f_k)_{k\geqslant 1} \subset H^\infty$, we define the following property (P) and write $(f_k)_{k\geqslant 1} \in$ (P) if

(P) $\quad (f_k)_{k\geqslant 1}$ is a uniformly $*$-minimal sequence, and $\sup_{z\in\mathbb{D}} \sum_{k\geqslant 1} |f_k(z)| < \infty$.

Recall that a sequence $(f_k)_{k\geqslant 1} \subset H^\infty$ is uniformly $*$-minimal if there exists a bounded sequence $(\varphi_k)_{k\geqslant 1} \subset L^1$ such that $\langle f_i, \varphi_j\rangle_{H^\infty, H^\infty_*} = \delta_{ij}$ for every $i, j \geqslant 1$. We refer to Section 3 for definitions of duality, τ-topology (see formula (3.3)) and $*$-convergence in H^∞.

For $(f_k)_{k\geqslant 1} \in$ (P), we define

$$\mathcal{L}((f_k)_{k\geqslant 1}, X) = \Big\{ \sum_{k\geqslant 1} f_k x_k : (x_k)_{k\geqslant 1} \subset X,\ \sup_{k\geqslant 1} \|x_k\| < \infty \Big\}$$

and $\mathcal{L}(X) = \bigcup\{\mathcal{L}((f_k)_{k\geqslant 1}, X) : (f_k)_{k\geqslant 1} \in \text{(P)}\}$.

LEMMA 5.1. *Let $(f_k)_{k\geqslant 1} \in$ (P). Then $\mathcal{L}((f_k)_{k\geqslant 1}, X)$ is a τ-closed subspace of $H^\infty(X)$.*

Proof. Let $F = \sum_{k\geqslant 1} f_k x_k \in \mathcal{L}((f_k)_{k\geqslant 1}, X)$. As $\sup_{z\in\mathbb{D}} \sum_{k\geqslant 1} |f_k(z)|\,\|x_k\| < \infty$, the series defining F is τ-convergent. Therefore, $\mathcal{L}((f_k)_{k\geqslant 1}, X) \subset H^\infty(X)$. To see that $\mathcal{L}((f_k)_{k\geqslant 1}, X)$ is τ-closed, we take a sequence $(F_n)_{n\geqslant 1} \subset \mathcal{L}((f_k)_{k\geqslant 1}, X)$, τ-convergent to some $F \in H^\infty(X)$. Let $(\varphi_k)_{k\geqslant 1} \subset L^1$, $\sup_{k\geqslant 1} \|\varphi_k\|_1 < \infty$, such that $\langle f_i, \varphi_j\rangle_{H^\infty, H^\infty_*} = \delta_{ij}$ for every $i, j \geqslant 1$.

From Lemma 3.2, if $G = \sum_{k\geqslant 1} f_k y_k$, $\sup_{k\geqslant 1} \|y_k\| < \infty$, we have $l_{\varphi_k}(G) = \sum_{j\geqslant 1} \langle f_j(z), \varphi_k\rangle y_j = y_k$ for every $k \geqslant 1$. On the other hand, we know that for fixed k, $(l_{\varphi_k}(F_n))_{n\geqslant 1}$ converges in X due to the continuity of l_{φ_k}. Set $x_k = \lim_{n\to\infty} l_{\varphi_k}(F_n)$. Using Lemma 3.2 again, we obtain $\|x_k\| \leqslant \|\varphi_k\|_{H^\infty_*} \sup_{n\geqslant 1} \|F_n\|$, and therefore $\sup_{k\geqslant 1} \|x_k\| < \infty$. Thus, $G = \sum_{k\geqslant 1} f_k x_k$ belongs to $\mathcal{L}((f_k)_{k\geqslant 1}, X)$ and it is clear that $(F_n)_{n\geqslant 1}$ is τ-convergent to G. Therefore, $F = G \in \mathcal{L}((f_k)_{k\geqslant 1}, X)$ and $\mathcal{L}((f_k)_{k\geqslant 1}, X)$ is τ-closed. ∎

EXAMPLE 5.2. For every Carleson sequence $(\lambda_k)_{k\geqslant 1} \subset \mathbb{D}$ (see below for the definition), there exist interpolating functions $(f_k)_{k\geqslant 1} \subset H^\infty$, $f_k(\lambda_j) = \delta_{kj}$, $\forall\, j, k \geqslant 1$, such that $\sup_{z\in\mathbb{D}} \sum_{k\geqslant 1} |f_k(z)| < \infty$ (see [13]). In particular, $(f_k)_{k\geqslant 1} \in$ (P). ∎

For $(f_k)_{k\geqslant 1} \in$ (P), we denote by $\mathcal{L}_{\mathrm{comp}}((f_k)_{k\geqslant 1}, X)$ the following subspace of $\mathcal{L}((f_k)_{k\geqslant 1}, X)$,

$$\mathcal{L}_{\mathrm{comp}}((f_k)_{k\geqslant 1}, X) = \Big\{ \sum_{k\geqslant 1} f_k x_k : (x_k)_{k\geqslant 1} \subset X \text{ is a relatively compact sequence in } X \Big\},$$

and further we set $\mathcal{L}_{\mathrm{comp}}(X) = \bigcup \{\mathcal{L}_{\mathrm{comp}}((f_k)_{k\geqslant 1}, X) : (f_k)_{k\geqslant 1} \in \text{(P)}\}$.

LEMMA 5.3. *$\mathcal{L}_{\mathrm{comp}}(X) \subset H^\infty \otimes X$.*

Proof. Let $F = \sum_{k\geqslant 1} f_k x_k \in \mathcal{L}_{\mathrm{comp}}((f_k)_{k\geqslant 1}, X)$. Let $\varepsilon > 0$ and let u be a map in $\mathbb{N}^*$ with a finite set of values, say $k_1, k_2, \ldots, k_p$, and such that $\|x_k - x_{u(k)}\| \leqslant \varepsilon$ for every $k \geqslant 1$. We set

$$G = \sum_{k\geqslant 1} f_k x_{u(k)} = \sum_{j=1}^{p} \Big(\sum_{k:u(k)=k_j} f_k \Big) x_{u_j}.$$

According to property (P), $\Big(\sum_{k:u(k)=k_j} f_k \Big) \in H^\infty$ for every j, $1 \leqslant j \leqslant p$. Thus, $G \in H^\infty \bigotimes_p X$. On the other hand, $\|F - G\|_\infty \leqslant \sup_{z\in\mathbb{D}} \sum_{k\geqslant 1} |f_k(z)| \, \|x_k - x_{u(k)}\| \leqslant C\varepsilon$, where $C = \sup_{z\in\mathbb{D}} \sum_{k\geqslant 1} |f_k(z)|$. Therefore, $F \in H^\infty \otimes X$. ∎

In fact, $\mathcal{L}_{\mathrm{comp}}(X)$ is a dense subset of $H^\infty \otimes X$ because finite sums $\sum_{k=1}^{n} f_k x_k$ with $\{f_k\}_{k=1}^n \subset H^\infty$ and $\{x_k\}_{k=1}^n \subset X$, $n \geqslant 1$, belong to $\mathcal{L}_{\mathrm{comp}}(X)$ and are dense in $H^\infty \otimes X$.

PROPOSITION 5.4. *Let $F \in \mathcal{L}(X)$. The following are equivalent:*

(i) $F \in H^\infty \otimes X$;

(ii) $F \in H^\infty_{\mathrm{comp}}(X)$;

(iii) $F \in \mathcal{L}_{\mathrm{comp}}(X)$.

Proof. Clearly, (i) implies (ii). Lemma 5.3 means that (iii) implies (i). It remains to prove that (ii) implies (iii). Since $F \in \mathcal{L}(X)$, there exists a sequence $(f_k)_{k\geqslant 1}$ satisfying (P) such that $F \in \mathcal{L}((f_k)_{k\geqslant 1}, X)$. Let $(\varphi_k)_{k\geqslant 1} \subset L^1$, $\sup_{k\geqslant 1} \|\varphi_k\|_1 < \infty$, satisfying $\langle f_i, \varphi_j \rangle_{H^\infty, H^\infty_*} = \delta_{ij}$ for every $i, j \geqslant 1$. As in the proof of Lemma 5.1 we have $l_{\varphi_k}(F) = x_k$ for every $k \geqslant 1$. According to Lemma 3.3, $x_k \in C\, \overline{\mathrm{conv}}\{F(z) : z \in \mathbb{D}\}$, where $C = \sup\{\|\varphi_k\|_{H^\infty_*} : k \geqslant 1\}$. Since $\overline{\mathrm{conv}}\{F(z) : z \in \mathbb{D}\}$ is compact, we get $F \in \mathcal{L}_{\mathrm{comp}}((f_k)_{k\geqslant 1}, X)$. ∎

PSEUDOCONTINUABLE FUNCTIONS. Recall that a function $f \in H^p$, $1 \leqslant p \leqslant \infty$, is Nevanlinna pseudocontinuable (p.c. in short) if there exists a function $f_* \in \mathrm{Nev}(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}) = \{g/h : g, h \in H^\infty(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})\}$ such that $f(\xi) = f_*(\xi)$ for almost all $\xi \in \mathbb{T}$ (nontangential limits). Here, $H^\infty(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$ denotes the space of bounded analytic functions in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. We refer to [20], Lecture 2, for the theory of pseudocontinuable functions. If a pseudocontinuation exists, then it is unique, and we can classify p.c. functions in terms of their pseudocontinuation. Indeed, every inner function $\theta \in H^\infty$ is p.c., and has a pseudocontinuation θ_* given by the Schwarz reflection, $\theta_*(z) = \frac{1}{\overline{\theta(\frac{1}{\bar z})}}$, $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$. In particular, $(\theta_*)^{-1} \in H^\infty(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$. Let f be p.c., and $f_* = \frac{g}{h}$ its pseudocontinuation, $g, h \in H^\infty(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$. We can suppose that $\lim_{|z|\to\infty} g(z) = 0$. Let $h = \theta_1 h_1$ be the inner-outer factorization of h, and set $\theta(z) = \overline{\theta_1(\frac{1}{\bar z})}$, $z \in \mathbb{D}$. Then $\theta \in H^\infty$ is inner and $\theta_* = \frac{1}{\theta_1}$. Thus $f_* = \frac{g}{h_1}\theta_*$, where $g, h_1 \in H^\infty(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$, h_1 outer. In this case, we say that f has a pseudocontinuation with singularities subordinated to θ. If moreover $f \in H^\infty$, then $f_* = g_1\theta_*$, where $g_1 \in H^\infty(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$, $\lim_{|z|\to\infty} g_1(z) = 0$. We say that f has a purely meromorphic pseudocontinuation if $\theta = B$ is a Blaschke product.

For every inner function $\theta \in H^\infty$, we denote by $\mathcal{PC}(\theta)$ the vector space of p.c. H^∞ functions with singularities subordinated to θ,

$$\mathcal{PC}(\theta) := \Big\{ f \in H^\infty : f_* = g\theta_*,\ g \in H^\infty(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}),\ \lim_{|z|\to\infty} g(z) = 0 \Big\}.$$

Note that the meromorphic pseudocontinuation of $f \in \mathcal{PC}(B)$ can have poles only at points $\frac{1}{\bar\lambda}$, where λ is a zero of B, with at most the same order of multiplicity. For an inner function $\theta \in H^\infty$, we set

$$K_\theta^\infty := H^\infty \cap \theta H_-^\infty = H^\infty \cap K_\theta,$$

where $H_-^\infty = \bar z \overline{H^\infty}$ and $K_\theta = H^2 \ominus \theta H^2 = H^2 \cap \theta H_-^2$ is a coinvariant subspace for the shift operator on H^2.

LEMMA 5.5. *Let $\theta \in H^\infty$ be an inner function. Then:*

(i) $\mathcal{PC}(\theta) = K_\theta^\infty$;

(ii) *K_θ^∞ is a $*$-closed subspace of H^∞;*

(iii) *if B is a Blaschke product, $B = \prod_{n\geqslant 1} b_{\lambda_n}^{k(\lambda_n)}$, $\sum_{n\geqslant 1} k(\lambda_n)(1-|\lambda_n|) < \infty$, where $b_\lambda(z) = \frac{|\lambda|}{\lambda}(\lambda - z)(1-\bar\lambda z)^{-1}$, then*

$$K_B^\infty = \operatorname{span}_* \bigg\{ \frac{1}{\big(z - \frac{1}{\bar\lambda_n}\big)^k} : n \geqslant 1,\ 1 \leqslant k \leqslant k(\lambda_n) \bigg\},$$

where span_* *stands for the $*$-closed linear span in H^∞.*

Proof. (i) is clear because a function $g \in H^\infty(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$ satisfies $\lim_{|z|\to\infty} g(z) = 0$ if and only if $g \in H_-^\infty$. Property (ii) comes from the formula $K_B^\infty = \bigcap_{n\geqslant 0} \operatorname{Ker}\varphi_n$, where $\varphi_n(f) = \int_{\mathbb{T}} f(\xi)\overline{B(\xi)}\xi^n \, d\mu(\xi)$, $f \in H^\infty$, $n \geqslant 0$. For the proof of (iii) see [20], Lecture 2. ∎

REMARK 5.6. If $f \in K_\theta^\infty$ has a pseudocontinuation f_*, then f and f_* not only have the same nontangential limits on $\mathbb{T}$, but also they extend to an analytic function on $\mathbb{C} \setminus \left\{ \frac{1}{\lambda} : \lambda \in \sigma(\theta) \right\}$, where $\sigma(\theta)$ denotes the spectrum of θ (see [20]). In particular, if $\theta = B$ is a Blaschke product, then $f \in K_B^\infty$ can be extended to an analytic function on $\mathbb{C} \setminus \operatorname{clos} \left\{ \frac{1}{\lambda} : B(\lambda) = 0 \right\}$.

LINEAR FREE INTERPOLATION. Our main source of subspaces of H^∞ satisfying (AP) is provided by linear free interpolation operators in H^∞. The following lemma recalls a well-known result on this subject (see [13], Chapter VIII, Section 2). Recall that a sequence $\Lambda = (\lambda_n)_{n \geqslant 1}$ of distinct points in $\mathbb{D}$ is a Carleson (interpolating) sequence ($\Lambda \in$ (C)) if

$$\text{(C)} \qquad \inf\{|B_{\lambda_n}(\lambda_n)| : n \geqslant 1\} = \delta > 0,$$

where $B_{\lambda_n} = \frac{B}{b_{\lambda_n}}$ and $B = \prod_{n \geqslant 1} b_{\lambda_n}$ is the Blaschke product associated with Λ.

LEMMA 5.7. *Let $\Lambda = (\lambda_n)_{n \geqslant 1} \subset \mathbb{D}$. Suppose that there exists a bounded linear operator $T : \ell^\infty(\Lambda) \to H^\infty$ realizing interpolation on Λ, that is $(Ta)|\Lambda = a$ for every $a \in \ell^\infty(\Lambda)$. Then $\Lambda \in$ (C) and the range $T(\ell^\infty(\Lambda))$ of T is a closed subspace of H^∞ isomorphic to $\ell^\infty(\Lambda)$, thus satisfying* (AP) *in H^∞.*

COROLLARY 5.8. *Let $X = (X_*)^*$ be a dual Banach space, and $F \in H^\infty(X)$ such that $\phi_F(B_*) \subset T(\ell^\infty(\Lambda))$, where $\Lambda \subset \mathbb{D}$, $\Lambda \in$ (C), and T is an interpolation operator for Λ. Then $F \in H^\infty \otimes X$ if and only if $F \in H^\infty_{\mathrm{comp}}(X)$.*

Indeed, this is a direct consequence of Lemma 5.7 and Proposition 4.8.

It is not easy to distinguish subspaces of the form $T(\ell^\infty(\Lambda))$ with some T and some $\Lambda \in$ (C) among all subspaces of H^∞. Several constructions exist for interpolation operators; see [13], [20], [29]. But only one of these operators fulfills the properties we need. The following lemma is inspired by a result of G.M. Airapetyan ([1]).

LEMMA 5.9. *Let $\Lambda = (\lambda_n)_{n \geqslant 1} \subset \mathbb{D}$. The following are equivalent:*

(i) *there exists $T \in L(\ell^\infty(\Lambda), H^\infty)$ such that $(Ta)|\Lambda = a$ for every $a \in \ell^\infty(\Lambda)$, and $T(\ell^\infty(\Lambda)) = K_B^\infty$, where $B = \prod_{n \geqslant 1} b_{\lambda_n}$ is the Blaschke product with the zero set Λ;*

(ii) *$\Lambda \in$ (C) and (F), where (F) stands for the uniform Frostman condition,*

$$\text{(F)} \qquad \sup_{\xi \in \mathbb{T}} \sum_{n \geqslant 1} \frac{1 - |\lambda_n|^2}{|\xi - \lambda_n|} < \infty.$$

REMARK 5.10. It is clear that the uniform Frostman condition (F) is stable with respect to finite union. Notice also that a sequence $\Lambda \in$ (F) is always "tangent to $\mathbb{T}$", more precisely there is at most a finite number of points of Λ in every Stolz angle. It is also known that a Frostman sequence is a finite union of Carleson sequences. On the other hand, if $(\theta_n)_{n \geqslant 1}$ is a sequence of distinct real numbers tending to 0, then there exists a sequence $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$, with $\lambda_n = r_n e^{i\theta_n}$, such that $\Lambda \in$ (C) and (F) (where $(r_n)_{n\geqslant 1}$ tends to 1 fast enough).

Proof of Lemma 5.9. First we show that (i) implies (ii). We know that $H^\infty|\Lambda = \ell^\infty(\Lambda)$ implies $\Lambda \in (C)$. Notice that the interpolation operator taking values in K_B^∞ is necessarily unique, because if $f_1, f_2 \in K_B^\infty$, and $f_1|\Lambda = f_2|\Lambda$, then $(f_1 - f_2) \in BH^\infty \cap K_B^\infty = \{0\}$, thus $f_1 = f_2$. Moreover, for every finitely supported sequence $a = (a(\lambda_n))_{n\geqslant 1} \in \ell^\infty(\Lambda)$ we have

$$Ta(z) = \sum_{n\geqslant 1} a(\lambda_n) \frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}z} \frac{B_{\lambda_n}(z)}{B_{\lambda_n}(\lambda_n)}, \quad z \in \mathbb{D},$$

where $B_{\lambda_n} = \frac{B}{b_{\lambda_n}}$. Indeed, $\frac{B_{\lambda_n}}{1-\overline{\lambda_n}z} = \frac{\lambda_n}{|\lambda_n|}\frac{B}{z-\lambda_n} \in K_B^\infty$ because $\frac{1}{z-\lambda_n} \in H_-^\infty$, thus $Ta \in K_B^\infty$. Now, as T is bounded, we have

$$\sup\{|Ta(z)| : \|a\|_\infty \leqslant 1\} = \sum_{n\geqslant 1} \frac{1-|\lambda_n|^2}{|1-\overline{\lambda_n}z|} \frac{|B_{\lambda_n}(z)|}{|B_{\lambda_n}(\lambda_n)|} \leqslant C,$$

for some constant $C > 0$. It is known (see [20]) that for $\Lambda \in$ (C), there exists $\delta_1 > 0$ such that

$$|B(z)| \geqslant \delta_1 \inf\{|b_{\lambda_n}(z)| : n \geqslant 1\},$$

for every $z \in \mathbb{D}$ (generalized Carleson condition). Let $0 < \varepsilon < 1$. We set

$$\Omega_\varepsilon := \mathbb{D} \setminus \Big(\bigcup_{n\geqslant 1} \{z \in \mathbb{D} : |b_{\lambda_n}(z)| < \varepsilon\}\Big).$$

Choosing ε small enough, we can suppose that the hyperbolic discs $\{z \in \mathbb{D} : |b_{\lambda_n}(z)| < \varepsilon\}$, $n \geqslant 1$, are mutually disjoint. Then $|B_{\lambda_n}(z)| \geqslant |B(z)| \geqslant \delta_1\varepsilon$ for every $z \in \Omega_\varepsilon$ and every $n \geqslant 1$. Therefore,

$$\sum_{n\geqslant 1} \frac{1-|\lambda_n|^2}{|1-\overline{\lambda_n}z|} \frac{|B_{\lambda_n}(z)|}{|B_{\lambda_n}(\lambda_n)|} \geqslant \delta_1\varepsilon \sum_{n\geqslant 1} \frac{1-|\lambda_n|^2}{|1-\overline{\lambda_n}z|}$$

for every $z \in \Omega_\varepsilon$, and we have

$$\sum_{n\geqslant 1} \frac{1-|\lambda_n|^2}{|1-\overline{\lambda_n}z|} \leqslant \frac{C}{\delta_1\varepsilon}, \quad \forall z \in \Omega_\varepsilon.$$

For every finitely supported unimodular sequence $(\varepsilon_n)_{n\geqslant 1} \subset \mathbb{C}$ ($|\varepsilon_n| = 1$ for a finite set of n, and $\varepsilon_n = 0$ elsewhere), we have

$$\Big|\sum_{n\geqslant 1} \varepsilon_n \frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}z}\Big| \leqslant \frac{C}{\delta_1\varepsilon}, \tag{5.1}$$

for every $z \in \Omega_\varepsilon$. It is clear that, by continuity, (5.1) holds for every $z \in \mathbb{T}$. Taking the supremum over $(\varepsilon_n)_{n\geqslant 1}$, we get that $\sup\limits_{z\in\mathbb{T}} \sum\limits_{n\geqslant 1} \frac{1-|\lambda_n|^2}{|z-\lambda_n|} \leqslant \frac{C}{\delta_1\varepsilon}$ for every $z \in \mathbb{T}$.

Now we show that (ii) implies (i). We check that the series

$$\sum_{n\geqslant 1} a(\lambda_n) \frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}z} \frac{B_{\lambda_n}(z)}{B_{\lambda_n}(\lambda_n)}$$

is $*$-convergent in H^∞ for every $a \in \ell^\infty(\Lambda)$. Indeed, taking $z \in \mathbb{D}$ and a finitely supported sequence $b = (b_n)_{n\geqslant 1}$ and applying the maximum principle, we obtain

$$\Big|\sum_{n\geqslant 1} b_n \frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}z}\frac{B_{\lambda_n}(z)}{B_{\lambda_n}(\lambda_n)}\Big| \leqslant \Big(\sup_{n\geqslant 1}\frac{1}{|B_{\lambda_n}(\lambda_n)|}\Big)\Big(\sup_{|\zeta|=1}\sum_{n\geqslant 1}\frac{1-|\lambda_n|^2}{|1-\overline{\lambda_n}\zeta|}\Big)\|b\|_\infty.$$

Taking the supremum over such b, $\|b\|_\infty \leqslant 1$, we get the absolute convergence of the series and a uniform upper bound for the partial sums, and hence the weak-$*$ convergence in H^∞. Set

$$Ta(z) = \sum_{n\geqslant 1} a(\lambda_n)\frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}z}\frac{B_{\lambda_n}(z)}{B_{\lambda_n}(\lambda_n)}, \quad a \in \ell^\infty(\Lambda).$$

Then T is an interpolation operator acting from $\ell^\infty(\Lambda)$ to K^∞_B (see Lemma 5.5). Moreover, T is bounded according to the previous uniform bound. Finally, if $f \in K^\infty_B$, then $T((f(\lambda_n)_{n\geqslant 1}) \in K^\infty_B$ and $T((f(\lambda_n)_{n\geqslant 1})$ coincides with f on Λ.

From the uniqueness mentioned above, we get that $f = T((f(\lambda_n)_{n\geqslant 1})$. Thus $T(\ell^\infty(\Lambda)) = K^\infty_B$. ∎

COROLLARY 5.11. *Let $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ be such that $\Lambda \in$ (C) and $\Lambda \in$ (F). Then the map P_B defined on H^∞ by*

$$P_B f = \sum_{n\geqslant 1} f(\lambda_n)\frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}z}\frac{B_{\lambda_n}}{B_{\lambda_n}(\lambda_n)}, \quad f \in H^\infty$$

is a bounded linear operator. Moreover, P_B coincides with the "Riesz projection" onto K^∞_B, that is $P_B f = f$ for every $f \in K^\infty_B$ and $P_B f = 0$ for every $f \in BH^\infty$ (equivalently, $P_B = BP_-\overline{B}$ where $P_- : L^2 \to H^2_-$ is the orthogonal projection). Consequently, $H^\infty = K^\infty_B \dot{+} BH^\infty$ (topological direct sum). ∎

COROLLARY 5.12. *Let $B = B_1\widetilde{B}$, where $B, B_1, \widetilde{B}$ are Blaschke products with $B_1 = B_\Lambda$, $\Lambda \in$ (C) and $\Lambda \in$ (F). Then $K^\infty_B = K^\infty_{B_1} \dot{+} B_1 K^\infty_{\widetilde{B}}$.*

Indeed, let $f \in K^\infty_B$. According to Corollary 5.11, we can write $f = f_1 + f_2$, where $f_1 = P_{B_1} f \in K^\infty_{B_1}$ and, since $K^\infty_{B_1} \subset K^\infty_B$, $f_2 \in B_1 H^\infty \cap (H^\infty \cap B H^\infty_-) = B_1 K^\infty_{\widetilde{B}}$. Moreover such a decomposition is unique.

We say that the divisor $d(\lambda)$, $\lambda \in \mathbb{D}$, of a Blaschke product $B = \prod_{\lambda\in\mathbb{D}} b_\lambda^{d(\lambda)}$ is a *Frostman divisor* if $\sup_{\lambda\in\mathbb{D}} d(\lambda) < \infty$ and the support of d satisfies the uniform Frostman condition (F).

COROLLARY 5.13. *Let $B = \prod_{\lambda\in\mathbb{D}} b_\lambda^{d(\lambda)}$ be a Blaschke product having a Frostman divisor d. Then $B = \prod_{k=1}^{N} B_k$, where $B_k = B_{\sigma_k}$, $\sigma_k \in$ (C) and (F), and*

$$K^\infty_B = K^\infty_{B_1} \dot{+} B_1 K^\infty_{B_2} \dot{+} \cdots \dot{+} B_1 B_2 \cdots B_{N-1} K^\infty_{B_N}. \tag{5.2}$$

Hence K^∞_B satisfies (AP).

Indeed, according to Vinogradov-Goluzina lemma (see [20]) the support of d is a finite union of sets, say $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \subset \mathbb{D}$ such that $\Lambda_k \in$ (C) and (F). Let $M = \sup_{\lambda\in\mathbb{D}} d(\lambda)$. Then there exists $(d_k)_{k=1}^N$ such that $d = \sum_{k=1}^N d_k$, where $N \leqslant nM$ and d_k is the indicator function of a sequence $\sigma_k \subset \mathbb{D}$ satisfying $\sigma_k \in$ (C) and (F). Setting $B_k = B_{\sigma_k}$, we prove the first assertion. Now we show (5.2) by induction on N. Set $\widetilde{B} = B_2 \cdots B_N$ and let $f \in K_B^\infty$. According to Corollary 5.12 we have $K_B^\infty = K_{B_1}^\infty \dot{+} B_1 K_{\widetilde{B}}^\infty$. We get the result by applying (5.2) to $\widetilde{B}$. ∎

THEOREM 5.14. *Let $B = \prod_{\lambda\in\mathbb{D}} b_\lambda^{d(\lambda)}$ be a Blaschke product having a Frostman divisor d. Assume that one of the following two conditions is satisfied:*

(i) *$F \in H^\infty(X)$, with $X = (X_*)^*$ a dual Banach space, and $\phi_F(B_*) \subset K_B^\infty$, where ϕ_F stands for the scalarisation map of F (see Section* 4*).*

(ii) *$F \in H^\infty(L(\mathcal{H}))$ and $\mathcal{M}_F \subset K_B^\infty$, where $\mathcal{M}_F = \{\langle F(\,\cdot\,)e_j, e_i\rangle : i, j \geqslant 1\}$ is the family of matrix entries of F with respect to an orthonormal basis $(e_i)_{i\geqslant 1}$ of $\mathcal{H}$.*

Then $F \in H^\infty_{\mathrm{comp}}(X)$ if and only if $F \in H^\infty \otimes X$.

Proof. Under condition (i), the result follows from Corollary 5.13 and Proposition 4.8. Condition (ii) implies condition (i). Indeed, we know that K_B^∞ is $*$-closed, see Lemma 5.5. It remains to prove that $\phi_F(A) \in \mathrm{span}_*\{\langle F(\,\cdot\,)e_j, e_i\rangle : i, j \geqslant 1\}$ for every $A \in S_1$. Let $x \in \mathcal{H}$, and let P_n be the orthogonal projection onto $\mathrm{lin}\{e_i : 1 \leqslant i \leqslant n\}$. The sequence $(h_{j,n})_{n\geqslant 1}$ defined by $h_{j,n} = \langle F(\,\cdot\,)P_n x, e_j\rangle$ tends to $\langle F(\,\cdot\,)x, e_j\rangle$ $*$-weakly for every $j \geqslant 1$. Thus $\langle F(\,\cdot\,)x, e_j\rangle \in K_B^\infty$. Applying the same process to y, we obtain that $\langle F(\,\cdot\,)x, y\rangle \in K_B^\infty$. If $A = \sum_{k\geqslant 1} \langle\,\cdot\,, x_k\rangle y_k$, $\sum_{k\geqslant 1} \|x_k\| \cdot \|y_k\| < \infty$, then the series $\phi_F(A) = \sum_{k\geqslant 1} \langle F(\,\cdot\,)x_k, y_k\rangle$ is norm convergent, and hence $\phi_F(A) \in K_B^\infty$. ∎

COROLLARY 5.15. *Let $F \in H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$ satisfying condition* (ii) *of Theorem* 5.14. *Then there exists $G \in H^\infty(L(\mathcal{H}))$ such that $GF = \mathrm{id}$ if and only if $\|F(z)x\| \geqslant \delta\|x\|$ for every $x \in \mathcal{H}$ and every $z \in \mathbb{D}$, where $\delta > 0$ is a constant.*

6. FUNCTIONS SUBORDINATED TO A FIXED STRUCTURE

In this section, we are mainly concerned with the case $X = L(\mathcal{H})$, $\dim\mathcal{H} = \infty$. The case when $\dim\mathcal{H} < \infty$ is clear; see Lemma 3.4. On the contrary, if $\dim\mathcal{H} = \infty$, $L(\mathcal{H})$ does not satisfy (AP) and we do not know whether $H^\infty \otimes L(\mathcal{H})$ and $H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$ coincide. Nevertheless, some subspaces of $L(\mathcal{H})$ do satisfy (AP), and we obtain that the corona theorem is true for functions in $H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$ with values in some special subspaces $\mathcal{E}$ of $L(\mathcal{H})$. Note that even if $\mathcal{E}$ is a subalgebra, the left inverse will not necessarily take values in $\mathcal{E}$. However, this is the case if $\mathcal{E}$ is a unital C^*-algebra ($A \in \mathcal{E} \Rightarrow A^* \in \mathcal{E}$). Indeed, under assumption (2.2) we get that $G(z) = (F(z)^*F(z))^{-1}F(z)^*$ is a bounded (nonanalytic) left inverse of $F(z)$, and if $\mathcal{E}$ is a C^*-algebra, then $G(z) \in \mathcal{E}$. Consequently, if $\mathcal{E}$ is a C^*-algebra satisfying (AP), every $F \in H^\infty_{\mathrm{comp}}(\mathcal{E}) = H^\infty \otimes \mathcal{E}$ satisfying condition (2.2) has a left inverse in $H^\infty \otimes \mathcal{E}$.

COMPACT VALUED FUNCTIONS. Recall that $S_\infty = S_\infty(\mathcal{H}_1, \mathcal{H}_2)$, the space of compact operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, satisfies (AP). In particular, $H^\infty_{\mathrm{comp}}(S_\infty) = H^\infty \otimes S_\infty$.

From this equality we deduce the following two results. The first one refers to the setting of Fuhrmann-Vasyunin's theorem (see [20], Appendix 3, and references mentionned therein).

PROPOSITION 6.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces with $\dim \mathcal{H}_1 < \infty$. Then every $F \in H^\infty_{\mathrm{comp}}(L(\mathcal{H}_1, \mathcal{H}_2))$ satisfying condition (2.2) has a left inverse in $H^\infty \otimes L(\mathcal{H}_2, \mathcal{H}_1)$. In particular, if $\dim \mathcal{H}_2 < \infty$, then every $F \in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$ satisfying (2.2) has a left inverse in $H^\infty(L(\mathcal{H}_2, \mathcal{H}_1))$.*

Proof. The second assertion is an obvious consequence of the first one, because then $F(\mathbb{D})$ is a bounded set in a finite dimensional space. When $\dim \mathcal{H}_2 = \infty$ we can reduce the problem to the case where $F \in H^\infty(L(\mathcal{H}))$. If $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$, we can identify $\mathcal{H}_1$ and $\mathcal{H}_2$. In the general case, let $F \in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$ satisfying (2.2). Condition (2.2) implies that $\dim \mathcal{H}_1 \leqslant \dim \mathcal{H}_2$, and we can assume that $\mathcal{H}_1 \subset \mathcal{H}_2$. We denote by $\mathcal{H}_1^\perp$ the orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}_2$. Let $\mathcal{H}_3$ be another infinite dimensional Hilbert space. We set $\mathcal{H} = \mathcal{H}_2 \oplus \mathcal{H}_3$ (hilbertian sum) and we define F_0 on $\mathbb{D}$ by

$$F_0(z)|\mathcal{H}_1 = F(z), \quad F_0(z)|\mathcal{H}_1^\perp \oplus \mathcal{H}_3 = U$$

for every $z \in \mathbb{D}$, where U is a constant unitary operator from $\mathcal{H}_1^\perp \oplus \mathcal{H}_3$ onto $\mathcal{H}_3$. Then $F_0 \in H^\infty(L(\mathcal{H}))$ and

$$\begin{aligned}\|F_0(z)(u \oplus v \oplus w)\|^2 &= \|F(z)u + U(v+w)\|^2 = \|F(z)u\|^2 + \|U(v+w)\|^2 \\ &\geqslant \delta^2(\|u\|^2 + \|v \oplus w\|^2) = \delta^2\|u \oplus v \oplus w\|^2\end{aligned}$$

for every $u \in \mathcal{H}_1$, $v \in \mathcal{H}_1^\perp$, $w \in \mathcal{H}_3$ (we assume that $\delta \leqslant 1$).

First we show that F_0 is left invertible if and only if F is left invertible. Suppose that F_0 has a left inverse $G_0 \in H^\infty(L(\mathcal{H}))$. Then $G_0(z)F_0(z)(x \oplus 0) = x \oplus 0$ for every $z \in \mathbb{D}$ and every $x \in \mathcal{H}_1$, thus $G(z) = P_{\mathcal{H}_1}G_0(z)|\mathcal{H}_2$ is a left inverse of $F(z)$ for every $z \in \mathbb{D}$, and $G \in H^\infty(L(\mathcal{H}_2, \mathcal{H}_1))$. For the converse, assume that F has a left inverse $G \in H^\infty(L(\mathcal{H}_2, \mathcal{H}_1))$. We define G_0 on $\mathbb{D}$ by

$$G_0(z)|\mathcal{H}_2 = G(z), \quad G_0(z)|\mathcal{H}_3 = U^{-1},$$

for every $z \in \mathbb{D}$. Then G_0 is a left inverse of F_0 in $H^\infty(L(\mathcal{H}))$.

Further, if $F \in H^\infty_{\mathrm{comp}}(L(\mathcal{H}_1, \mathcal{H}_2))$ and $\dim \mathcal{H}_1 < \infty$, then F has compact values, that is $F \in H^\infty_{\mathrm{comp}}(S_\infty) = H^\infty \otimes S_\infty$. Consequently, $F_0 \in H^\infty \otimes L(\mathcal{H})$, and we deduce from Theorem 2.2 that F_0 has a left inverse $G_0 \in H^\infty \otimes L(\mathcal{H})$. Then $G = P_{\mathcal{H}_1}G_0|\mathcal{H}_2$ is a left inverse of F, and $G \in H^\infty \otimes L(\mathcal{H}_2, \mathcal{H}_1)$. ∎

PROPOSITION 6.2. *Let $\mathcal{E}$ be a subspace of $L(\mathcal{H})$ generated by the identity operator and the space S_∞ of compact operators. If $F \in H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$ and if F takes values in $\mathcal{E}$, then $F \in H^\infty \otimes \mathcal{E}$. If moreover, F satisfies condition* (2.2) *then F is left invertible in $H^\infty \otimes \mathcal{E}$.*

Proof. This is an immediate consequence of the (AP) for S_∞ and the fact that $\mathcal{E}$ is a C^*-algebra. ∎

It would be interesting to generalize the above result to those functions in $H^\infty(L(\mathcal{H}))$ with any Fredholm values (necessarily of constant negative index in view of (2.2)).

TOEPLITZ VALUED FUNCTIONS. For general facts about Toeplitz operators, we refer to [20], Appendix 4. We denote by P_+ the Riesz (orthogonal) projection from $L^2 = L^2(\mathbb{T})$ onto the Hardy space H^2,

$$P_+\Big(\sum_{k\in\mathbb{Z}} \widehat{f}(k)z^k\Big) = \sum_{k\geqslant 0} \widehat{f}(k)z^k.$$

For $\varphi \in L^\infty = L^\infty(\mathbb{T})$, the Toeplitz operator T_φ with symbol φ is the bounded linear operator defined on H^2 by $T_\varphi u = P_+(\varphi u)$, $u \in H^2$. We will use the following well-known properties of Toeplitz operators:

(a) $\|T_\varphi\| = \|\varphi\|_\infty$ for every $\varphi \in L^\infty$.

(b) $\Big\|\sum_i \prod_j \varphi_{ij}\Big\|_\infty \leqslant \Big\|K + \sum_i \prod_j T_{\varphi_{ij}}\Big\|$ for every finite family $\{\varphi_{ij}\} \subset L^\infty$ and every compact operator K. In particular, $T_\varphi \in S_\infty$ if and only if $\varphi \equiv 0$.

We denote by T_{L^∞} the closed subspace of $L(H^2)$ consisting of all Toeplitz operators.

THEOREM 6.3. *Let $F \in H^\infty_{\mathrm{comp}}(L(H^2))$ be such that $F(z) \in T_{L^\infty} + S_\infty$ for every $z \in \mathbb{D}$. Then $F \in H^\infty \otimes L(H^2)$. In particular, if F satisfies condition* (2.2) *(with $\mathcal{H} = H^2$), then F is left invertible in $H^\infty \otimes L(H^2)$.*

Proof. The map $\varphi \mapsto T_\varphi$ is an isometric isomorphism from L^∞ onto T_{L^∞}. In particular, T_{L^∞} satisfies (AP) because L^∞ does. From the properties of Toeplitz operators mentionned above, we deduce that $T_{L^\infty} \cap S_\infty = \{0\}$ and that the map $T_\varphi + K \mapsto T_\varphi$ is a norm 1 projection from $T_{L^\infty} + S_\infty$ onto T_{L^∞}. As both T_{L^∞} and S_∞ satisfy (AP), the same holds for $T_{L^\infty} + S_\infty$. Therefore,

$$H^\infty_{\mathrm{comp}}(T_{L^\infty} + S_\infty) = H^\infty \otimes (T_{L^\infty} + S_\infty) \subset H^\infty \otimes L(H^2),$$

and we can apply Theorem 2.2 to F. ∎

REMARK 6.4. If $A \in T_{L^\infty} + S_\infty$ satisfies $\|Af\| \geqslant \delta\|f\|$ for every $f \in H^2$, $\delta > 0$, A is not necessarily a Fredholm operator with index 0, even if A is Fredholm. Moreover, $T_{L^\infty} + S_\infty$ is not an algebra. In particular, if a left inverse of F exists, it does not necessarily take values in $T_{L^\infty} + S_\infty$.

Using a theorem by L. Coburn ([7]) it is easy to deduce the following corollary.

Corollary 6.5. *Let $F \in H^\infty_{\mathrm{comp}}(L(H^2))$ be such that, for every $z \in \mathbb{D}$, $F(z) \in \operatorname{Alg} T_{\mathcal{C}(\mathbb{T})}$, the norm closed algebra generated by $T_{\mathcal{C}(\mathbb{T})}$ (Toeplitz operators with continuous symbols). Then $F \in H^\infty \otimes \operatorname{Alg} T_{\mathcal{C}(\mathbb{T})}$. In particular, if F satisfies condition* (2.2)*, then F has a left inverse in $H^\infty \otimes \operatorname{Alg} T_{\mathcal{C}(\mathbb{T})}$.*

The above results can be generalized to Toeplitz operators with matrix symbols; see [9] for the generalization of the required properties. It is also worth mentioning that a function $F \in H^\infty(T_{H^\infty})$ has a left inverse in $H^\infty(T_{L^\infty})$, but not necessarily in $H^\infty(T_{H^\infty})$, as soon as it has one in $H^\infty(L(H^2))$. This follows from the easy to check fact that the standard averaging projection P from $L(H^2)$ onto T_{L^∞} is semi-multiplicative in the sense that $P(AT_\varphi) = P(A)T_\varphi$ for every $A \in L(H^2)$ and every $\varphi \in H^\infty$; see for example [20]. However, the above property is no longer true for $F \in H^\infty(T_{L^\infty})$.

Functions subordinated to a fixed matrix structure. Here we are interested in functions in $H^\infty(L(\mathcal{H}))$ having a matrix block decomposition associated to a fixed map in the following sense. Let $\mathcal{H} = \bigoplus_{n\geqslant 1} \mathcal{H}_n$ be a hilbertian decomposition of $\mathcal{H}$, and $A \in L(\ell^2)$ with the matrix $A = (a_{ij})_{i,j\geqslant 1}$. Then $F \in H^\infty(L(\mathcal{H}))$ is subordinated to A if F has the following block structure, $F = (a_{ij}F_{ij})_{i,j\geqslant 1}$, that is $F(z)x = \bigoplus_{n\geqslant 1} \sum_{k\geqslant 1} a_{nk}F_{nk}(z)x_k$ for every $x = \Big(\bigoplus_{n\geqslant 1} x_n\Big) \in \mathcal{H}$ and every $z \in \mathbb{D}$, where $F_{ij} \in H^\infty(L(\mathcal{H}_j, \mathcal{H}_i))$ for every $i, j \geqslant 1$, and $\sup_{i,j\geqslant 1} \|F_{ij}\|_\infty < \infty$. Note that the condition $\sup_{i,j\geqslant 1} \|F_{ij}\|_\infty < \infty$ does not guarantee that $F \in H^\infty(\ell^2)$ if no assumption is made on A; see Lemma 6.10 for a sufficient condition.

First, we treat the case of block diagonal functions ($(a_{ij})_{i,j\geqslant 1} = (\delta_{ij})_{i,j\geqslant 1} = \mathrm{id}$), then we analyze a more general case. The following technical lemma is easy to prove.

Lemma 6.6. *Assume that $\mathcal{H} = \bigoplus_{n\geqslant 1} \mathcal{H}_n$, and let $F \in H^\infty(L(\mathcal{H}))$ be a block diagonal function, $F = \operatorname{diag}(F_1, F_2, \ldots)$, where $F_n \in H^\infty(L(\mathcal{H}_n))$ for every $n \geqslant 1$. Then the scalarisation map $\phi_F \in L(S_1, H^\infty)$ can be identified with $\widetilde{\phi}_F \in L(\ell^1((S_1(\mathcal{H}_n))_{n\geqslant 1}), H^\infty)$ defined by $\widetilde{\phi}_F((A_n)_{n\geqslant 1}) = \sum_{n\geqslant 1} \phi_{F_n}(A_n)$ for every sequence $(A_n)_{n\geqslant 1} \in \ell^1(S_1(\mathcal{H}_n)_{n\geqslant 1})$, that is, such that $A_n \in S_1(\mathcal{H}_n)$ for every $n \geqslant 1$, and $\sum_{n\geqslant 1} \|A\|_1 < \infty$.*

Proposition 6.7. *Let X_n, $n \geqslant 1$, and Y be Banach spaces, and let $\phi \in L(\ell^1((X_n)_{n\geqslant 1}), Y)$ be defined by $\phi((A_n)_{n\geqslant 1}) = \sum_{n\geqslant 1} \phi_n(A_n)$, where $A_n \in L(X_n, Y)$ for every $n \geqslant 1$. Then:*

(i) *ϕ is a compact operator if and only if the operators ϕ_n, $n \geqslant 1$, are simultaneously compact, that is, if $\bigcup_{n\geqslant 1} \phi_n(B_*)$ is relatively compact in Y, where B_* stands for the closed unit ball of X_n for adequate n.*

(ii) *ϕ is in the closure of finite rank operators if and only if the operators ϕ_n, $n \geqslant 1$, are uniformly norm approximable by simultaneously compact operators with uniformly finite rank, that is, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n \geqslant 1$, there exists $\psi_n \in L(X_n, Y)$ satisfying*

$$\begin{cases} \operatorname{rank} \psi_n \leqslant N, \\ \|\phi_n - \psi_n\| \leqslant \varepsilon, \\ \bigcup\limits_{n \geqslant 1} \psi_n(B_*) \text{ is relatively compact in } Y. \end{cases}$$

Proof. The proof of the first part is standard. Let us focus on the second one. We denote by $\mathcal{F} := \mathcal{F}(\ell^1((X_n)_{n\geqslant 1}), Y)$ the space of bounded finite rank operators from $\ell^1((X_n)_{n\geqslant 1})$ into Y. Let $\phi \in \overline{\mathcal{F}}$ and $\varepsilon > 0$. Let $\psi \in \mathcal{F}$ be such that $\|\phi - \psi\| \leqslant \varepsilon$. We set $\psi_n = \psi|X_n$ for every $n \geqslant 1$. Then $\|\phi_n - \psi_n\| \leqslant \varepsilon$ and $\operatorname{rank} \psi_n \leqslant \operatorname{rank} \psi < \infty$ for every $n \geqslant 1$. Clearly, we have $\bigcup\limits_{n\geqslant 1} \psi_n(B_*) \subset \psi(B_*)$ which is relatively compact because $\psi \in \mathcal{F}$.

For the converse, fix $\varepsilon > 0$. Let $N \in \mathbb{N}$ and $\psi_n \in L(X_n, Y)$, $n \geqslant 1$, satisfy the three conditions mentioned in part (ii). Every ψ_n can be written in the form

$$\psi_n = \sum_{k=1}^{N} \langle \,\cdot\, , B'_{n,k} \rangle y'_{n,k},$$

where $\{y'_{n,k}\}_{k=1}^N \subset Y$, $\{B'_{n,k}\}_{k=1}^N \subset X_n^*$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{X,X^*}$. As $E := \operatorname{Lin}\{B'_{n,k} : 1 \leqslant k \leqslant N\}$ is a finite dimensional subspace of $(X_n)^*$, we have $E = (E_*)^*$, where E_* can be identified with $X_n/E_\perp$, with $E_\perp = \{A \in X_n : \langle A, B\rangle = 0, \forall B \in E\}$. According to Auerbach's lemma (see [31]), we can choose a normalized basis of E whose biorthogonal family is a normalized basis of E^*. Thus, up to replace N by $\dim E$, we can suppose that $\|B'_{n,k}\| = 1$ for every $k \in \{1, \dots, N\}$, and that there exists $\{A'_{n,k}\}_{k=1}^N \subset E^*$ satisfying $\|A'_{n,k}\|_{E_*} = 1$ and $\langle A'_{n,j}, B'_{n,k}\rangle = \delta_{jk}$ for every $j, k \in \{1, \dots, N\}$. E_* is finite dimensional, and therefore reflexive. In particular, E^* can be identified with E_*, and $\|A'_{n,k}\|_{E_*} = \inf\{\|A'_{n,k} + A\|_1 : A \in E_\perp\}$ for every $k \in \{1, \dots, N\}$. Consequently, there exists $\{A''_{n,k} : 1 \leqslant k \leqslant N\} \subset X_n$ such that $\|A''_{n,k}\|_1 \leqslant 2$ and $\langle A''_{n,j}, B'_{n,k}\rangle = \delta_{jk}$ for every $j, k \in \{1, \dots, N\}$. Set $B_{n,k} = \|A''_{n,k}\|_1 B'_{n,k}$, $A_{n,k} = \frac{A''_{n,k}}{\|A''_{n,k}\|_1}$, and $y_{n,k} = \frac{y'_{n,k}}{\|A''_{n,k}\|_1}$. Then

$$\psi_n = \sum_{k=1}^{N} \langle \,\cdot\, , B_{n,k} \rangle y_{n,k}.$$

For every $n \geqslant 1$, $\{y_{n,k} : 1 \leqslant k \leqslant N\} \subset \psi_n(B_*)$. Indeed, $A_{n,k} \in B_*$ and $\psi_n(A_{n,k}) = \langle A_{n,k}, B_{n,k}\rangle y_{n,k} = y_{n,k}$ for every $k \in \{1, \dots, N\}$. As $\bigcup\limits_{n\geqslant 1} \psi_n(B_*)$ is relatively compact in Y, we can take in $\{y_{n,k} : n \geqslant 1,\ 1 \leqslant k \leqslant N\}$ a finite $\frac{\varepsilon}{N}$-net $\{\widetilde{y}_j\}_{j=1}^p$. For $n, k \geqslant 1$, we denote by $j(n,k) \in \{1, \dots, p\}$ an integer such that $\|y_{n,k} - \widetilde{y}_{j(n,k)}\|_\infty \leqslant \frac{\varepsilon}{N}$. We set

$$\widetilde{\psi}_n = \sum_{k=1}^{N} \langle \,\cdot\, , B_{n,k} \rangle \widetilde{y}_{j(n,k)} = \sum_{j=1}^{p} \Big\langle \cdot, \Big(\sum_{k : j(n,k) = j} B_{n,k} \Big) \Big\rangle \widetilde{y}_j.$$

Clearly, $\widetilde{\psi}_n \in L(X_n, Y)$ for every $n \geqslant 1$. Moreover, $\|\psi_n - \widetilde{\psi}_n\| \leqslant \varepsilon$, thus $\|\phi_n - \widetilde{\psi}_n\| \leqslant 2\varepsilon$. Let $\widetilde{\psi} \in L(\ell^1((X_n)_{n\geqslant 1}), Y)$ be defined by $\widetilde{\psi}((A_n)_{n\geqslant 1}) = \sum\limits_{n\geqslant 1} \widetilde{\psi}_n(A_n)$, $(A_n)_{n\geqslant 1} \in \ell^1((X_n)_{n\geqslant 1})$. Then $\|\phi - \widetilde{\psi}\| \leqslant 2\varepsilon$ and it remains to show that $\widetilde{\psi} \in \mathcal{F}$. But $\widetilde{\psi}_n(X_n) \subset \mathrm{Lin}\{\widetilde{y}_j : 1 \leqslant j \leqslant p\}$ for every $n \geqslant 1$, thus $\widetilde{\psi}(\ell^1((X_n)_{n\geqslant 1})) \subset \mathrm{Lin}\{\widetilde{y}_j : 1 \leqslant j \leqslant p\}$ and $\widetilde{\psi}$ is of finite rank. Thus $\phi \in \overline{\mathcal{F}}$. ∎

THEOREM 6.8. *Let* $\mathcal{H} = \bigoplus\limits_{n\geqslant 1} \mathcal{H}_n$, *and let* $F \in H^\infty(L(\mathcal{H}))$ *be a block diagonal function,* $F = \mathrm{diag}(F_1, F_2, \ldots)$, *where* $F_n \in H^\infty(L(\mathcal{H}_n))$ *for every* $n \geqslant 1$.

(1) *The following are equivalent:*

(i) $F \in H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$;

(ii) $\bigcup\limits_{n\geqslant 1} \{\langle F_n(\,\cdot\,)u, v\rangle : u, v \in \mathcal{H}_n, \|u\| = \|v\| = 1\}$ *is relatively compact in* H^∞;

(iii) $\bigcup\limits_{n\geqslant 1} \phi_{F_n}(B_*)$ *is relatively compact in* H^∞.

(2) *The following are equivalent:*

(i) $F \in H^\infty \otimes L(\mathcal{H})$;

(ii) $\inf\limits_{\dim M<\infty} \sup\limits_{n\geqslant 1} \mathrm{dist}(F_n, M \otimes L(\mathcal{H}_n)) = 0$, *where* M *denotes any finite dimensional subspace of* H^∞;

(iii) *for every* $\varepsilon > 0$ *and every* $n \geqslant 1$ *there exists* $\psi_n \in L(S_1(\mathcal{H}_n), H^\infty)$ *satisfying*

$$\begin{cases} \sup\limits_{n\geqslant 1} \mathrm{rank}\, \psi_n < \infty, \\ \|\phi_{F_n} - \psi_n\| \leqslant \varepsilon, \\ \bigcup\limits_{n\geqslant 1} \psi_n(B_*) \text{ is relatively compact in } H^\infty. \end{cases}$$

Proof. The first assertion follows from Lemma 6.6, Proposition 6.7 and Proposition 4.7. For the second one, the fact that (i) and (iii) are equivalent comes from Lemma 6.6 and Proposition 6.7. The proof of the equivalence (i) $\Leftrightarrow$ (ii) is similar to that of Proposition 6.7. ∎

COROLLARY 6.9. *Let* $F \in H^\infty(L(\mathcal{H}))$ *be a block diagonal function,* $F = \mathrm{diag}(f_1A_1, f_2A_2, \ldots)$, *for some Hilbertian decomposition of* $\mathcal{H}$, $\mathcal{H} = \bigoplus\limits_{n\geqslant 1} \mathcal{H}_n$, *with* $(f_n)_{n\geqslant 1} \subset H^\infty$, $\sup\limits_{n\geqslant 1} \|f_n\|_\infty < \infty$, *and* $A_n \in L(\mathcal{H}_n)$, $\|A_n\| = 1$, $\forall n \geqslant 1$. *The following are equivalent:*

(i) $F \in H^\infty \otimes L(\mathcal{H})$;

(ii) $F \in H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$;

(iii) $(f_n)_{n\geqslant 1}$ *is relatively compact in* H^∞.

Indeed, we apply Theorem 6.8 with $F_n = f_nA_n$. If $F \in H^\infty_{\mathrm{comp}}(L(\mathcal{H}))$ then

$$\bigcup_{n\geqslant 1} \{\langle F_n(\,\cdot\,)u, v\rangle : \|u\| = \|v\| = 1\} = \bigcup_{n\geqslant 1} \{f_n\langle A_nu, v\rangle : \|u\| = \|v\| = 1\}$$

is relatively compact in H^∞. Therefore $(f_n)_{n\geqslant1}$ is relatively compact in H^∞ and

$$\varepsilon((f_n)_{n\geqslant1},N)=\inf_{\dim M=N}\sup_{n\geqslant1}\operatorname{dist}(f_n,M)\to0$$

as $N\to\infty$ (where M is a subspace of H^∞). We have $\inf\limits_{\dim M<\infty}\sup\limits_{n\geqslant1}\operatorname{dist}(F_n,M\otimes L(\mathcal{H}_n))=0$ and $F\in H^\infty\otimes L(\mathcal{H})$.

It is worth mentioning that for the block diagonal functions of Corollary 6.9 the corona theorem is always true, that is, condition (2.2) implies the existence of a bounded analytic left inverse, even for functions not in $H^\infty\otimes L(\mathcal{H})$.

Now we are interested in a more general case, namely functions subordinated to a bounded matrix with positive entries. We start with a lemma justifying such a definition. Then we treat the case of scalar (i.e. one dimensional) blocks. In the case of the operator blocks, a similar generalization as for block diagonal matrices can be given.

LEMMA 6.10. *Let $\mathcal{H}=\bigoplus\limits_{n\geqslant1}\mathcal{H}_n$ and $F_{ij}\in H^\infty(L(\mathcal{H}_j,\mathcal{H}_i))$, $i,j\geqslant1$, $\sup\limits_{i,j\geqslant1}\|F_{ij}\|_\infty=C<\infty$. If $A=(a_{ij})_{i,j\geqslant1}\in L(\ell^2)$ satisfies $a_{ij}\geqslant0$ for every $i,j\geqslant1$, then the function F subordinated to A, $F=(a_{ij}F_{ij})_{i,j\geqslant1}$, belongs to $H^\infty(L(\mathcal{H}))$.*

Proof. Let $x=\Big(\bigoplus\limits_{n\geqslant1}x_n\Big)\in\mathcal{H}$ and $z\in\mathbb{D}$. Then $F(z)x=\bigoplus\limits_{i\geqslant1}\sum\limits_{j\geqslant1}a_{ij}F_{ij}(z)x_j$, thus

$$\begin{aligned}\|F(z)x\|^2&=\sum_{i\geqslant1}\Big\|\sum_{j\geqslant1}a_{ij}F_{ij}(z)x_j\Big\|^2\leqslant C^2\sum_{i\geqslant1}\Big(\sum_{j\geqslant1}a_{ij}\|x_j\|\Big)^2=C^2\|A(\|x_j\|)_{j\geqslant1}\|^2\\&\leqslant C^2\|A\|^2\sum_{j\geqslant1}\|x_j\|^2=C^2\|A\|^2\|x\|^2,\end{aligned}$$

and therefore, $F\in H^\infty(\ell^2)$. ∎

THEOREM 6.11. *Let $A=(a_{ij})_{i,j\geqslant1}\in L(\ell^2)$ satisfying $a_{ij}\geqslant0$ for every $i,j\geqslant1$, and let $(f_{ij})_{i,j\geqslant1}\subset H^\infty$ be a relatively compact family. Then the function F subordinated to A, $F=(a_{ij}f_{ij})_{i,j\geqslant1}$, belongs to $H^\infty\otimes L(\ell^2)$. In particular, if such a function F satisfies condition* (2.2), *then F is left invertible in $H^\infty(L(\ell^2))$.*

Proof. Let $\varepsilon>0$, and let $(\varphi_k)_{k=1}^N$ be a finite ε-net for $(f_{ij})_{i,j\geqslant1}$ in H^∞. We define u to be a map from $\{i\geqslant1\}\times\{j\geqslant1\}$ to $\{1,\dots,N\}$ such that $\|f_{ij}-\varphi_{u(i,j)}\|_\infty<\varepsilon$ for every $i,j\geqslant1$. Then $|a_{ij}f_{ij}(z)-a_{ij}\varphi_{u(i,j)}(z)|<\varepsilon a_{ij}$ for every $i,j\geqslant1$ and every $z\in\mathbb{D}$. Now we define G by the following matrix representation, $G=(a_{ij}\varphi_{u(i,j)})_{i,j\geqslant1}$, then $G\in H^\infty(L(\ell^2))$, $\|F-G\|_\infty<\varepsilon\|A\|$ and $G=\sum\limits_{k=1}^N\varphi_kA_k$, where A_k has the matrix representation $A_k=(a_{ij}^{(k)})_{i,j\geqslant1}$ with $a_{ij}^{(k)}=a_{ij}$ if $k(i,j)=k$ and $a_{ij}^{(k)}=0$ otherwise. In particular, we have $0\leqslant a_{ij}^{(k)}\leqslant a_{ij}$ for every $i,j\geqslant1$ and every $k\in\{1,\dots,N\}$, thus $A_k\in L(\ell^2)$, $G\in H^\infty\bigotimes\limits_n L(\ell^2)$ and therefore $F\in H^\infty\otimes L(\ell^2)$. ∎

EXAMPLES 6.12. (i) Every $F \in H^\infty(L(\ell^2))$, $F(z) = (f_{ij}(z))_{i,j\geqslant 1}$, having a finite number of nonzero diagonals and satisfying condition (2.2) is left invertible in $H^\infty(L(\ell^2))$. Indeed, this case corresponds to $A = (a_{ij})_{i,j\geqslant 1}$ with $a_{ij} = 1$ if $|i-j| \leqslant N$ and $a_{ij} = 0$ otherwise.

(ii) More generally, Theorem 6.11 applies to matrices $A = (a_{ij})_{i,j\geqslant 1}$ with positive entries such that $\sum\limits_{j\geqslant 1} a_{ij} \leqslant C_1$ for every $i \geqslant 1$ and $\sum\limits_{i\geqslant 1} a_{ij} \leqslant C_2$ for every $j \geqslant 1$.

In the setting of Example 6.12 (i), the corona theorem is always true for diagonal functions, even for F not in $H^\infty \otimes L(\ell^2)$. But for F having at least two nonzero diagonals, we do not know a better result than that of Theorem 6.11. On the other hand, this example can be considered as a particular case of functions subordinated to a fixed Toeplitz operator with a polynomial symbol $\varphi = \sum\limits_{k=-N}^{N} a_k \xi^k$, with $a_k \neq 0$ for every k, $-N \leqslant k \leqslant N$. A slightly more general example occurs when we consider a general Toeplitz matrix $A = T_\varphi = (a_{i-j})_{i,j\geqslant 0}$ with non-negative entries (T_φ is bounded if and only if $\sum\limits_{k\in\mathbb{Z}} a_k < \infty$). It would be interesting to know whether the result is still true for functions "subordinated", in a sense, to an arbitrary Toeplitz operator.

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PASCALE VITSE
Département de Mathématiques
et Statistiques
Université Laval
Québec G1K 7P4
QC CANADA

E-mail: pvitse@mat.ulaval.ca

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