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COMPRESSIONS AND PINCHINGS

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Abstract. There exist operators A such that for any sequence of contractions $\{A_n\}$, there is a total sequence of mutually orthogonal projections $\{E_n\}$ such that $\sum E_n A E_n = \bigoplus A_n$.

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INTRODUCTION

By an operator, we mean an element in the algebra $L(\mathcal{H})$ of all bounded linear operators acting on the usual (complex, separable, infinite dimensional) Hilbert space $\mathcal{H}$. We denote by the same letter a projection and the corresponding subspace. If F is a projection and A is an operator, we denote by A_F the compression of A by F, that is the restriction of FAF to the subspace F. Given a total sequence of nonzero mutually orthogonal projections $\{E_n\}$, we consider the pinching

$$\mathcal{P}(A) = \sum_{n=1}^{\infty} E_n A E_n = \bigoplus_{n=1}^{\infty} A_{E_n}.$$

If $\{A_n\}$ is a sequence of operators acting on separable Hilbert spaces with A_n unitarily equivalent to A_{E_n} for all n, we also naturally write $\mathcal{P}(A) = \bigoplus\limits_{n=1}^{\infty} A_n$. The main result of this paper can then be stated as:

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of operators acting on separable Hilbert spaces. Assume that $\sup\limits_n \|A_n\| < 1$. Then, we have a pinching

$$\mathcal{P}(A) = \bigoplus_{n=1}^{\infty} A_n$$

for any operator A whose essential numerical range contains the unit disc.

This result is proved in the second section of the paper. We have included a first section concerning some well-known properties of the essential numerical range.

1. PROPERTIES OF THE ESSENTIAL NUMERICAL RANGE

We denote by $\langle \cdot, \cdot \rangle$ the inner product (linear in the second variable), by $\mathrm{co}S$ the convex hull of a subset S of the complex plane $\mathbb{C}$. $W(A) = \{\langle h, Ah\rangle : \|h\| = 1\}$ is the numerical range of the operator A and $\overline{W}(A)$ is the closure of $W(A)$. The celebrated Hausdorff-Toeplitz theorem (cf. [6], Chapter 1) states that $W(A)$ is convex. A corollary is Parker' s theorem ([6], p. 20): Given an n by n matrix A, there is a matrix B unitarily equivalent to A and with all its diagonal elements equal to $\mathrm{Tr}\, A/n$.

Here are three equivalent definitions of the *essential numerical range of* A, denoted by $W_{\mathrm{e}}(A)$:

(1) $W_{\mathrm{e}}(A) = \bigcap \overline{W}(A + K)$ where the intersection runs over the compact operators K;

(2) Let $\{E_n\}$ be any sequence of finite rank projections converging strongly to the identity and denote by B_n the compression of A to the subspace $E_n^{\perp}$. Then $W_{\mathrm{e}}(A) = \bigcap_{n\geqslant 1} \overline{W}(B_n)$;

(3) $W_{\mathrm{e}}(A) = \{\lambda :$ there is an orthonormal system $\{e_n\}_{n=1}^{\infty}$ with $\lim\langle e_n, Ae_n\rangle = \lambda\}$.

It follows that $W_{\mathrm{e}}(A)$ is a compact convex set containing the essential spectrum of A, $\mathrm{Sp}_{\mathrm{e}}(A)$. The equivalence between these definitions has been known since the early seventies if not sooner (see for instance [1]). The very first definition of $W_{\mathrm{e}}(A) =$ is (1); however (3) is also a natural notion and easily entails convexity and compactness of the essential numerical range. We mention the following result of Chui-Smith-Smith-Ward ([4]):

PROPOSITION 1.1. *Every operator A admits some compact perturbation $A + K$ for which $W_{\mathrm{e}}(A) = \overline{W}(A + K)$.*

Another characterization of the essential numerical range of A is

$$W_{\mathrm{e}}(A) = \{\lambda : \text{there is a basis } \{e_n\}_{n=1}^{\infty} \text{ with } \lim\langle e_n, Ae_n\rangle = \lambda\}.$$

Let us check the equivalence between our definition (3) with orthonormal system and the above identity which seems to be due to Q.F. Stout ([11]). Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal system such that $\lim_{n\to\infty}\langle x_n, Ax_n\rangle = \lambda$. If $\mathrm{span}\{x_n\}_{n=1}^{\infty}$ is of finite codimension p we immediately get a basis $e_1, \ldots, e_p$; $e_{p+1} = x_1, \ldots$; $e_{p+n} = x_n, \ldots$ such that $\lim_{n\to\infty}\langle e_n, Ae_n\rangle = \lambda$. If $\mathrm{span}\{x_n\}_{n=1}^{\infty}$ is of infinite codimension, we may complete this system with $\{y_n\}_{n=1}^{\infty}$ in order to obtain a basis. Let P_j be the subspace spanned by y_j and $\{x_n : 2^{j-1} \leqslant n < 2^j\}$. By Parker's theorem, there is a basis of P_j, say $\{e_l^j\}_{l\in\Lambda_j}$, with

$$\langle e_l^j, Ae_l^j\rangle = \frac{1}{\dim P_j}\mathrm{Tr}\, AP_j.$$

Since

$$\frac{1}{\dim P_j}\operatorname{Tr} AP_j \to \lambda \quad \text{as } j \to \infty,$$

we may index $\{e_l^j\}_{j\in \mathrm{N}; l\in\Lambda_j}$ in order to obtain a basis $\{f_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \langle f_n, Af_n\rangle = \lambda.$$

The essential numerical range appears closely related to the diagonal set of A which we define by

$$\Delta(A) = \{\ \lambda : \text{there is a basis } \{e_n\}_{n=1}^{\infty} \text{ with } \langle e_n, Ae_n\rangle = \lambda\}.$$

The next result is a straightforward consequence of a lemma of Peng Fan ([5]). A real operator means an operator acting on a real Hilbert space and int$X$ denotes the interior of $X \subset \mathbb{C}$.

PROPOSITION 1.2. *Let A be an operator. Then* $\operatorname{int} W_{\mathrm{e}}(A) \subset \Delta(A) \subset W_{\mathrm{e}}(A)$. *Consequently, an open set $\mathcal{U}$ is contained in $\Delta(A)$ if and only if there is a basis $\{e_n\}_{n=1}^{\infty}$ such that $\mathcal{U} \subset \operatorname{co}\{\langle e_k, Ae_k\rangle : k \geqslant n\}$ for all n. Finally, the diagonal set of a real operator is symetric about the real axis. (For A self-adjoint, the result holds with* int *denoting the interior of subsets of* $\mathbb{R}$.*)*

Curiously enough, it seems difficult to answer the following questions: Is the diagonal set always a (possibly vacuous) convex set? Is there an operator of the form self-adjoint + compact with a disconnected diagonal set?

An elementary, but very important property of $W(\,\cdot\,)$ is the so named projection property $\operatorname{Re} W(A) = W(\operatorname{Re} A)$ (see [6], p. 9), where Re stands for real part. $W_{\mathrm{e}}(\,\cdot\,)$ has also this property. This result and the Hausdorff-Toeplitz theorem are the keys to prove the following fact:

PROPOSITION 1.3. *Let A be an operator.*

(i) *If $W_{\mathrm{e}}(A) \subset W(A)$ then $W(A)$ is closed.*

(ii) *There exist normal finite rank operators R of arbitrarily small norm such that $W(A+R)$ is closed.*

Proof. Assertion (i) is due to J.S. Lancaster ([8]). We prove the second assertion and implicitly prove Lancaster's result. We may find an orthonormal system $\{f_n\}$ such that the closure of the sequence $\{\langle f_n, Af_n\rangle\}$ contains the boundary $\partial W_{\mathrm{e}}(A)$. Fix $\varepsilon > 0$. It is possible to find an integer p and scalars z_j, $1 < j < p$, with $|z_j| < \varepsilon$, such that:

$$\operatorname{co}\{\langle f_j, Af_j\rangle + z_j : 1 < j < p\} \supset \partial W_{\mathrm{e}}(A).$$

Thus, the finite rank operator $R = \sum_{1<j<p} z_j f_j \otimes f_j$ has the property that W$(A+R)$ contains $W_{\mathrm{e}}(A)$.

We need this operator R. Indeed, setting $X = A + R$, we also have W$(X) \supset W_{\mathrm{e}}(X)$. We then claim that W$(X)$ is closed (this claim implies assertion (i)). By the contrary, there would exist $z \in \partial\overline{W}(X) \setminus W_{\mathrm{e}}(X)$. Furthermore, since $\overline{W}(X)$ is the convex hull of its extreme points, we could assume that such a z is an extreme of $\overline{W}(X)$. By suitable rotation and translation, we could assume that $z = 0$ and that the imaginary axis is a line of support of $\overline{W}(X)$. The projection property for $W(\,\cdot\,)$ would imply that $W(\operatorname{Re} X) = (x, 0[$ for a certain negative number x, so that $0 \in W_{\mathrm{e}}(\operatorname{Re} X)$. Thus we would deduce from the projection property for $W_{\mathrm{e}}(\,\cdot\,)$ that $0 \in W_{\mathrm{e}}(X)$; a contradiction. ∎

The perturbation R in Proposition 1.3 can be taken real if A is real. We mention that the set of operators with nonclosed numerical ranges is not dense in $L(\mathcal{H})$. Proposition 1.3 improves the following result of I.D. Berg and B. Sims ([3]): operators which attain their numerical radius are norm dense in $L(\mathcal{H})$. A motivation for Berg and Sims was the following fact: given an arbitrary operator A, a small rank one perturbation of A yields an operator which attains its norm. Indeed, the polar decompositon allows us to assume that A is positive, an easy case when reasoning as in the proof of Proposition 1.3.

Let us say that a convex set in $\mathbb{C}$ is relatively open if either it is a single point, an open segment or an usual open set. Using similar methods as in the previous proof, or applying Propositions 1.2 and 1.3, we obtain:

Proposition 1.4. *For an operator A the following assertions are equivalent:*
(i) $W(A)$ *is relatively open;*
(ii) $\Delta(A) = W(A)$.

From the previous results we may derive some information about $W(\,\cdot\,)$, $W_{\mathrm{e}}(\,\cdot\,)$ and $\Delta(\,\cdot\,)$ for various classes of operators:

(a) Let S be either the unilateral or bilateral shift, then $\Delta(S) = W(S)$ is the open unit disc. More generally Stout showed ([10]) that weighted periodic shifts S have open numerical ranges; therefore $\Delta(S) = W(S)$.

(b) There exist a number of Toeplitz operators with open numerical range. See the papers by E.M. Klein ([7]) and by J.K. Thukral ([11]).

(c) Let X be an operator lying in a C^*-subalgebra of $L(\mathcal{H})$ with no finite dimensional projections. Then for any real θ, $\overline{W}(\operatorname{Re} \mathrm{e}^{\mathrm{i}\theta} X) = W_{\mathrm{e}}(\operatorname{Re} \mathrm{e}^{\mathrm{i}\theta} X)$. From the projection property for $W(\,\cdot\,)$ and $W_{\mathrm{e}}(\,\cdot\,)$ we infer that $W_{\mathrm{e}}(X) = \overline{W}(X)$.

(d) Let X be an essentially normal operator i.e. $X^*X - XX^*$ is compact. It is known that $W_{\mathrm{e}}(X) = \operatorname{coSp}_{\mathrm{e}}(X)$. Indeed, for such an operator the essential norm equals the essential spectral radius i.e. $\|X\|_{\mathrm{e}} = \rho_e(X)$. Denoting by $W_{\mathrm{e}}(X)$ the essential numerical radius of X we deduce that $\|X\|_{\mathrm{e}} = W_{\mathrm{e}}(X) = \rho_e(X)$. Note that $\mathrm{e}^{\mathrm{i}\theta} X + \mu I = Y$ is also an essentially normal operator for any $\theta \in \mathbf{R}$ and $\mu \in \mathbb{C}$. Let z be an extremal point of $W_{\mathrm{e}}(X)$. With suitable θ and μ we have $\mathrm{e}^{\mathrm{i}\theta} z + \mu = W_{\mathrm{e}}(Y) = \max\{|y| : y \in W_{\mathrm{e}}(Y)\}$, the maximum being attained at the single point $\mathrm{e}^{\mathrm{i}\theta} z + \mu$. Since $\operatorname{co}\operatorname{Sp}_{\mathrm{e}}(Y) \subset W_{\mathrm{e}}(Y)$ and $\rho_{\mathrm{e}}(Y) = W_{\mathrm{e}}(Y)$, this implies that $\mathrm{e}^{\mathrm{i}\theta} z + \mu \in \operatorname{Sp}_{\mathrm{e}}(Y)$. Hence $z \in \operatorname{Sp}_{\mathrm{e}}(Y)$, so that $W_{\mathrm{e}}(X) = \operatorname{co}\operatorname{Sp}_{\mathrm{e}}(X)$.

2. THE PINCHING THEOREM

Recall that one way to define the essential numerical range of an operator A is:

$$W_{\mathrm{e}}(A) = \{\lambda : \text{there is an orthonormal system } \{e_n\}_{n=1}^{\infty} \text{ with } \lim\langle e_n, Ae_n\rangle = \lambda\}.$$

It is then easy to check that $W_{\mathrm{e}}(A)$ is a compact convex set. Moreover $W_{\mathrm{e}}(A)$ contains the open unit disc $\mathcal{D}$ if and only if there is a basis $\{e_n\}_{n=1}^{\infty}$ such that $\mathrm{co}\{\langle e_k, Ae_k\rangle : k > n\} \supset \mathcal{D}$ for all n.

THEOREM 2.1. *Let A be an operator with $W_{\mathrm{e}}(A) \supset \mathcal{D}$ and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of operators such that $\sup\limits_n \|A_n\| < 1$. Then, we have a pinching*

$$\mathcal{P}(A) = \bigoplus_{n=1}^{\infty} A_n.$$

(*If A and $\{A_n\}_{n=1}^{\infty}$ are real, then we may take a real pinching*).

Proof. It suffices to solve the following problem:

PROBLEM (P). Let A be an operator, with $\|A\| \leqslant \gamma$ and $W_{\mathrm{e}}(A) \supset \mathcal{D}$, let h be a norm one vector and X a strict contraction, $\|X\| < \rho < 1$. Find a projection E, and a constant $\varepsilon > 0$ only depending on γ and ρ such that:

(i) $\dim E = \infty$ and $A_E = X$;

(ii) $\dim \mathrm{E}^{\perp} = \infty$, $W_{\mathrm{e}}(A_{E^{\perp}}) \supset \mathcal{D}$ and $\|Eh\| \geqslant \varepsilon$.

Let us explain why it is sufficient to solve Problem (P). Take $\gamma = \|A\|$ and fix a dense sequence $\{h_n\}$ in the unit sphere of $\mathcal{H}$. We claim that (i) and (ii) ensure that there exists a sequence of mutually orthogonal projections $\{E_j\}$ such that, setting $F_n = \sum\limits_{j\leqslant n} E_j$, we have for all integers n:

($*$) $A_n = A_{E_n}$ and $W_{\mathrm{e}}(A_{F_n^{\perp}}) \supset \mathcal{D}$ (so $\dim F_n^{\perp} = \infty$);

($**$) $\|F_n h_n\| \geqslant \varepsilon$.

This is true for $n = 1$ by (i). Suppose this holds for an $N \geqslant 1$. Let $\nu(N) \geqslant N+1$ be the first integer for which $F_N^{\perp} h_{\nu(N)} \neq 0$. Note that $\|A_{F_N^{\perp}}\| \leqslant \gamma$. We apply (i) and (ii) to $A_{F_N^{\perp}}$, A_{N+1} and $F_N^{\perp} h_{\nu(N)}/\|F_N^{\perp} h_{\nu(N)}\|$ in place of A, X and h. We then deduce that ($*$) and ($**$) are still valid for $N+1$. Therefore ($*$) and ($**$) hold for all n. Denseness of $\{h_n\}$ and ($**$) show that F_n strongly increases to the identity I so that $\sum\limits_{j=1}^{\infty} E_j = I$ as required.

We first solve Problem (P) restricted to condition (i), consisting in representing A as a dilation of X. Next, we solve Problem (P) completely.

2.1. PRELIMINARIES.

We shall use a sequence $\{V_k\}_{k\geqslant 1}$ of orthogonal matrices acting on spaces of dimensions 2^k. This sequence is built up by induction:

$$V_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{then} \quad V_k = \frac{1}{\sqrt{2}}\begin{pmatrix} V_{k-1} & V_{k-1} \\ -V_{k-1} & V_{k-1} \end{pmatrix} \quad \text{for } k \geqslant 2.$$

Given a Hilbert space $\mathcal{G}$ and a decomposition

$$\mathcal{G} = \bigoplus_{j=1}^{2^k} \mathcal{H}_j \quad \text{with } \mathcal{H}_1 = \cdots = \mathcal{H}_{2^k} = \mathcal{H},$$

we may consider the unitary (orthogonal) operator on $\mathcal{G}$: $W_k = V_k \bigotimes I$, where I denotes the identity on $\mathcal{H}$.

Now, let $B : \mathcal{G} \to \mathcal{G}$ be an operator which, relatively to the above decomposition of $\mathcal{G}$, is written with a block diagonal matrix

$$B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_{2^k} \end{pmatrix}.$$

We observe that the block matrix representation of $W_k B W_k^*$ has its diagonal entries all equal to

$$\frac{1}{2^k}\left(B_1 + \cdots + B_{2^k}\right).$$

So, the orthogonal operators W_k allow us to pass from a block diagonal matrix representation to a block matrix representation in which the diagonal entries are all equal.

2.2. SOLUTION OF PROBLEM (P) (i). The contraction $Y = (1/\|X\|)X$ can be dilated in a unitary

$$U = \begin{pmatrix} Y & -(I - YY^*)^{1/2} \\ (I - Y^*Y)^{1/2} & Y^* \end{pmatrix}$$

thus X can be dilated in a normal operator $N = \|X\|U$ with $\|N\| < \rho$. This permits to restrict to the case when X is a normal contraction, $\|X\| < \rho < 1$. Thus we set the following problem:

PROBLEM (Q). Let X be a normal contraction, $\|X\| < \rho < 1$. Find a projection E, $\dim E = \infty$, such that $A_E = X$.

We remark with the Berg-Weyl-von Neumann theorem ([2]) that a normal contraction X, $\|X\| < \rho < 1$, can be written

$$X = D + K \tag{2.1}$$

where D is normal diagonalizable, $\|D\| = \|X\| < \rho$, and K is compact with an arbitrarily small norm. Let $K = \operatorname{Re} K + \mathrm{i} \operatorname{Im} K$ be the cartesian decomposition of K. We can manage to have an integer l, a real α and a real β so that decomposition (2.1) satisfies:

(a) the operators αD, $\beta \operatorname{Re} K$, $\beta \operatorname{Im} K$ are majorized in norm by ρ;

(b) there are positive integers m, n with $2^l = m + 2n$ and

$$X = \frac{1}{2^l}(m\alpha D + n\beta \operatorname{Re} K + n\beta \mathrm{i} \operatorname{Im} K). \tag{2.2}$$

More precisely we can take any l such that $[2^l/(2^l - 2)] \cdot \|X\| < \rho$. Next, assuming $\|K\| < \rho/2^l$, we can take $m = 2^l - 2$, $n = 1$, $\alpha = 2^l/(2^l - 2)$ and $\beta = 2^l$.

Let then T be the diagonal normal operator acting on the space

$$\mathcal{G} = \bigoplus_{j=1}^{2^l} \mathcal{H}_j \quad \text{with } \mathcal{H}_1 = \cdots = \mathcal{H}_{2^l} = \mathcal{H},$$

and defined by

$$T = \Big(\bigoplus_{j=1}^{m} D_j\Big) \oplus \Big(\bigoplus_{j=m+1}^{m+n} R_j\Big) \oplus \Big(\bigoplus_{j=m+n+1}^{2^l} S_j\Big)$$

where $D_j = \alpha D$, $S_j = \beta \mathrm{Re}\, K$ and $S_j = \beta \mathrm{i}\, \mathrm{Im} K$.

We note that $\|T\| < \rho < 1$ and that the operator $W_l T W_l^*$, represented in the preceding decomposition of $\mathcal{G}$, has its diagonal entries all equal to X by (i). Thus to solve Problem (Q) it suffices to solve the following problem.

PROBLEM (R). Given a diagonal normal operator T, $\|T\| < \rho < 1$, find a projection E, $\dim E = \infty$, such that $A_E = T$.

Solution of Problem (R). Let $\{\lambda_n(T)\}_{n\geqslant 1}$ be the eigenvalues of T repeated according to their multiplicities. Since $|\lambda_n(T)| < 1$ for all n and that $W_{\mathrm{e}}(A) \supset \mathcal{D}$, we have a norm one vector e_1 such that $\langle e_1, Ae_1\rangle = \lambda_1(T)$. Let

$$F_1 = [\mathrm{span}\{e_1, Ae_1, A^*e_1\}]^{\perp}.$$

As F_1 is of finite codimension, $W_{\mathrm{e}}(A_{F_1}) \supset \mathcal{D}$. So, there exists a norm one vector $e_2 \in F_1$ such that $\langle e_2, Ae_2\rangle = \lambda_2(T)$. Next, we set

$$F_2 = [\mathrm{span}\{e_1, Ae_1, A^*e_1, e_2, Ae_2, A^*e_2\}]^{\perp}, \ldots.$$

If we go on like this, we exhibit an orthonormal system $\{e_n\}_{n\geqslant 1}$ such that, setting $E = \mathrm{span}\{e_n\}_{n\geqslant 1}$, we have $A_E = T$. ∎

2.3. SOLUTION OF PROBLEM (P) (i) AND (ii). We take an arbitrary norm one vector h. We can show, using the same reasoning as that applied to solve Problem (R), that we have an orthonormal system $\{f_n\}_{n\geqslant 0}$, with $f_0 = h$, such that:

(a) $\langle f_{2j}, Af_{2j}\rangle = 0$ for all $j \geqslant 1$;
(b) $\{\langle f_{2j+1}, Af_{2j+1}\rangle\}_{j\geqslant 0}$ is a dense sequence in $\mathcal{D}$;
(c) if $F = \mathrm{span}\{f_j\}_{j\geqslant 0}$, then A_F is the normal operator

$$\sum_{j\geqslant 0} \langle f_j, Af_j\rangle f_j \otimes f_j.$$

Setting $F_0 = \mathrm{span}\{f_{2j}\}_{j\geqslant 0}$ and $F_0' = \mathrm{span}\{f_{2j+1}\}_{j\geqslant 0}$, we have then:

(a) Relatively to the decomposition $F = F_0 \bigoplus F_0'$, A_F can be written

$$A_F = \begin{pmatrix} A_{F_0} & 0 \\ 0 & A_{F_0'} \end{pmatrix}.$$

(b) $W_{\mathrm{e}}(A_{F_0'}) \supset \mathcal{D}$ and $h \in F_0$.

We can then write a decomposition of F_0', $F_0' = \bigoplus_{j=1}^{\infty} F_j$ where for each index j, F_j commutes with A_F and $W_{\mathrm{e}}(A_{F_j}) \supset \mathcal{D}$; so that the decomposition $F = \bigoplus_{j=0}^{\infty} F_j$ yields a representation of A_F as a block diagonal matrix,

$$A_F = \bigoplus_{j=0}^{\infty} A_{F_j}.$$

Since $W_{\mathrm{e}}(A_{F_j}) \supset \mathcal{D}$ when $j \geqslant 1$, the same reasoning as that used in the solution of Problem (R) entails that for any sequence $\{X_j\}_{j\geqslant 0}$ of strict contractions we have decompositions

$$F_j = G_j \bigoplus G_j' \tag{$\dagger$}$$

allowing us to write for $j \geqslant 1$

$$A_{F_j} = \begin{pmatrix} X_j & * \\ * & * \end{pmatrix}.$$

Since $\|X\| < \rho < 1$, we can find an integer l only depending on ρ and γ, as well as strict contractions $X_1, \dots, X_{2^l}$, such that

$$X = \frac{1}{2^l}\Big(A_{F_0} + \sum_{j=1}^{2^l-1} X_j\Big). \tag{2.3}$$

We come back to decompositions ($\dagger$) and we set

$$G = F_0 \oplus \Big(\bigoplus_{j=1}^{2^l-1} G_j\Big).$$

Relatively to this decomposition,

$$A_G = \begin{pmatrix} A_{F_0} & & & \\ & X_1 & & \\ & & \ddots & \\ & & & X_{2^l-1} \end{pmatrix}.$$

Then we deduce from (2.3) that the block matrix $W_l A_G W_l^*$ has its diagonal entries all equal to X.

SUMMARY: $h \in G$ and there exists a decomposition $G = \bigoplus_{j=1}^{2^l} E_j$ such that $A_{E_j} = X$ for each j. Thus we have an integer j_0 such that, setting $E_{j_0} = E$, we have

$$A_E = X \quad \text{and} \quad \|Eh\| \geqslant \frac{1}{\sqrt{2^l}}.$$

The proof is finished. ∎

COROLLARY 2.2. *Let A be an operator with $W_e(A) \supset \mathcal{D}$. For any strict contraction X, there is an isometry V such that $X = V^*AV$.*

COROLLARY 2.3. *Let A be an operator with $W_e(A) \supset \mathcal{D}$. For any contraction X, there is a sequence $\{U_n\}$ of unitary operators such that $U_n^*AU_n \to X$ in the weak operator topology.*

We use the strict inclusion notation $X \subset\subset Y$ for subsets X, Y of $\mathbb{C}$ to mean that there is an $\varepsilon > 0$ such that $\{x + z : x \in X, |z| < \varepsilon\} \subset Y$.

THEOREM 2.4. *Let A be an operator and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of normal operators. If $\bigcup_{n=1}^{\infty} W(A_n) \subset W_e(A)$ then we have a pinching*

$$\mathcal{P}(A) = \bigoplus_{n=1}^{\infty} A_n.$$

(For self-adjoint operators, this result holds with the strict inclusion of $\mathbb{R}$.)

Sketch of proof. Let N be a normal operator with $W(N) \subset\subset W_e(A)$. If N is diagonalizable, reasonning as in the proof of Theorem 2.1, we deduce that N can be realized as a compression of A. If N is not diagonalizable we may assume that $0 \in W_e(A)$. Thanks to the Berg-Weyl-von Neumann theorem and still reasonning as in the proof of Theorem 2.1 we again deduce that N is a compression of A. Finally, the strict containment assumption allows us to get the wanted pinching.

To finish this section, we mention that we can not drop the assumption that the strict contractions A_n of Theorem 2.1 are uniformly bounded in norm by a real < 1. This observation is equivalent to the fact that we can not delete the strict containment assumption in Theorem 2.4:

Let P be a halving projection ($\dim P = \dim P^{\perp} = \infty$), so $W_e(P) = [0, 1]$. Then the sequence $\{1 - 1/n^2\}_{n \geqslant 1}$ can not be realized as the entries of the main diagonal of a matrix representation of P. To check that, we note that the positive operator $I - P$ would be in the trace-class: a contradiction. (Recall that a positive operator with a summable diagonal is trace class.)

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