J. OPERATOR THEORY
50(2003), 221–247

QUASI-LATTICE ORDERED GROUPS AND TOEPLITZ ALGEBRAS

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Communicated by Norberto Salinas

Abstract. Let (G, G_+) be a quasi-lattice ordered group and $\mathcal{T}^{G_+}$ the corresponding Toeplitz algebra. First, we show that for $G_+ \subseteq E \subseteq G$, the natural C^*-morphism γ^{E,G_+} from $\mathcal{T}^{G_+}$ to $\mathcal{T}^E$ exists if and only if $E = G_+ \cdot H^{-1}$, where H is a hereditary and directed subset of G_+. Next, if E is a semigroup, then necessary and sufficient conditions for a representation of $\mathcal{T}^E$ to be faithful are obtained. By applying these results, diagonal invariant ideals of $\mathcal{T}^{G_+}$ are characterized, conditions under which $\mathcal{T}^{G_+}$ contains a minimal ideal are established, and finally, in the case when E is a semigroup and G is amenable, it is shown that $\mathcal{T}^E$ has the universal property for covariant isometric representations of E.

Keywords: *Toeplitz algebra, quasi-lattice ordered group, hereditary and directed set, covariant isometric representation, diagonal invariant ideal, induced ideal.*

MSC (2000): 47B35.

1. INTRODUCTION

1.1 History. In the past twenty years, Toeplitz algebras or Wiener-Hopf C^*-algebras on various generalized Hardy spaces have been a subject of intense study. One way to study these algebras is using the groupoid approach (see [3], [7], [13] and [20]). When the underlying group is abelian, the theory of Fourier analysis on abelian groups may be applied, as was shown by G. Murphy in his series of works. For example, as an extension of the classical Toeplitz algebra associated with the ordered group $(\mathbb{Z}, \mathbb{Z}_+)$, Murphy constructed a universal Toeplitz algebra related to an abelian partially ordered group, and proved that this universal Toeplitz algebra is equal to the usual Toeplitz algebra when the underlying abelian group is totally ordered (see [8]). However, there are some abelian groups which are not partially ordered but are instead quasi-partially ordered. One way to describe the associated algebras in the latter case is to consider suitable quasi-ordered groups containing the given quasi-partial ordered groups, and to use their associated Toeplitz algebras. This approach, successfully initiated by E. Park in [14], has been generalized in [18] to the case of non-abelian quasi-partially ordered groups. In the non-abelian case, one may also consider Toeplitz algebras on quasi-lattice ordered groups. These Toeplitz algebras were first studied by A. Nica in [12]. Later M. Laca and I. Raeburn rephrased them as crossed products of abelian C^*-algebras by semigroups of endomorphisms (see [6]).

In conclusion, we mention two previously established facts about Toeplitz algebras on quasi-lattice ordered groups which have bearing on our present work, and in fact play an important role in our main results. Let (G, G_+) be a quasi-lattice ordered group and $\mathcal{T}^{G_+}$ be the corresponding Toeplitz algebra (for precise definitions, see Section 2 below). First, in [12] it is shown that $\mathcal{T}^{G_+}$ contains a dense $*$-algebra $\mathcal{T}^\infty(G_+)$, which has a universal property for the covariant isometric representations of G_+, and when G is amenable, $\mathcal{T}^{G_+}$ also has such a universal property. Second, in [6] a necessary and sufficient condition for a representation of $\mathcal{T}^{G_+}$ to be faithful is given (also see Corollary 3.6 below).

1.2 STATEMENT OF RESULTS. In this paper, we study Toeplitz algebras associated with both quasi-lattice ordered and quasi-lattice quasi-ordered groups. Let (G, G_+) be a quasi-lattice ordered group. First, in Section 2 we show that for $G_+ \subseteq E \subseteq G$, the natural C^*-morphism γ^{E,G_+} from $\mathcal{T}^{G_+}$ to $\mathcal{T}^E$ exists if and only if $E = G_+ \cdot H^{-1}$, where H is a hereditary and directed subset of G_+ (see Theorem 2.12). The bulk of the work involved in proving this theorem lies in establishing the reverse implication. Briefly, this is accomplished by finding some natural covariant isometric representations of G_+ which induce $*$-morphisms on $\mathcal{T}^\infty(G_+)$. Then we show that these $*$-morphisms are bounded, and so they may be extended to C^*-morphisms on $\mathcal{T}^{G_+}$ which happen to coincide with the natural morphisms.

Next, in Section 3, we examine faithful representations of Toeplitz algebras. In the case that (G, E) is a quasi-lattice quasi-ordered group, a necessary and sufficient condition for a representation of the Toeplitz algebra $\mathcal{T}^E$ to be faithful is obtained (see Theorem 3.5). The main idea here is to combine the previously mentioned result concerning natural morphisms with the techniques developed by Laca and Raeburn in [6]. In fact, one of the main results of [6] (Theorem 3.7) appears as a special case of our result.

In the remainder of the paper, we apply the results mentioned in the previous paragraphs to certain topics. As a first application, in Section 4 we consider diagonal invariant ideals of the Toeplitz algebra $\mathcal{T}^{G_+}$ associated with a quasi-lattice ordered group (G, G_+). Formerly, diagonal invariant ideals of Toeplitz algebras on ordered groups were studied in [17], and it was proved there that when the underlying group is abelian, an ideal is diagonal invariant if and only if it is invariant in the sense described in [11]. Replacing ordered groups by more general quasi-lattice ordered groups, we characterize diagonal invariant ideals in terms of induced ideals (Theorem 4.4), and we realize certain induced ideals as the kernel of C^*-morphisms γ^{E,G_+} (Proposition 4.10). While we manage to clarify diagonal invariant ideals successfully in terms of induced ideals, one big problem concerning induced ideals still remains to be solved in the future (see Remark 4.5 below).

In Section 5, we study minimal ideals in the Toeplitz algebra $\mathcal{T}^{G_+}$ associated with a quasi-lattice ordered group (G, G_+). We prove that any minimal ideal of $\mathcal{T}^{G_+}$ can induce a minimal invariant ideal of D^{G_+} and vice versa, where D^{G_+} is the diagonal C^*-subalgebra of $\mathcal{T}^{G_+}$. This correspondence will be shown in Proposition 5.2. In the particular case when (G, G_+) is an ordered group, we will show that $\mathcal{T}^{G_+}$ contains a minimal ideal if and only if there exists a minimal non-trivial quasi-ordered group containing (G, G_+). Also, in light of recent work of M. Laca, we can get another view of the minimal ideals in terms of the topological structure of Ω (see Proposition 5.8).

Finally, in the setting of quasi-lattice quasi-ordered groups, in Section 6 we study the universal property of the Toeplitz algebras. In particular, in the case when E is a semigroup and G is amenable, it is shown that $\mathcal{T}^E$ has the universal property for covariant isometric representations of E, thus providing nontrivial generalizations of results stated in [12], [16] and [19].

2. THE NATURAL MORPHISMS BETWEEN TOEPLITZ ALGEBRAS ON DISCRETE GROUPS

In this section we establish necessary and sufficient conditions for the existence of certain natural morphisms between Toeplitz algebras on discrete groups. To begin, we list some relevant definitions regarding these Toeplitz algebras.

Let G be a discrete group and $\{\delta_g : g \in G\}$ be the usual orthonormal basis for $\ell^2(G)$, where

$$\delta_g(h) = \begin{cases} 1 & \text{if } g = h, \\ 0, & \text{otherwise;} \end{cases}$$

for $g, h \in G$. For any $g \in G$, we define a unitary operator u_g on $\ell^2(G)$ by $u_g(\delta_h) = \delta_{gh}$ for $h \in G$. For any subset E of G, let $\ell^2(E)$ be the closed subspace of $\ell^2(G)$ generated by $\{\delta_g : g \in E\}$; the projection from $\ell^2(G)$ onto $\ell^2(E)$ is denoted by p^E.

DEFINITION 2.1. The C^*-algebra generated by $\{T_g^E = p^E u_g p^E : g \in G\}$ is denoted by $\mathcal{T}^E$ and is called the *Toeplitz algebra* with respect to E.

Next we recall some facts about quasi-lattice ordered groups stated in [5], [6] and [12].

Let G be a discrete group, and G_+ a semigroup of G such that $G_+ \cap G_+^{-1} = \{e\}$. There is a partial order on G defined by $x \leqslant y \Leftrightarrow x^{-1}y \in G_+$, which is left invariant in the sense that, if $x \leqslant y$, then $zx \leqslant zy$ for any $x, y, z \in G$.

DEFINITION 2.2. (G, G_+) is said to be a *quasi-lattice ordered group* if every finite subset of G with an upper bound in G_+ has a least upper bound in G_+.

Equivalently, (G, G_+) is a quasi-lattice ordered group if and only if every element of G having an upper bound in G_+ has a least such, and every two elements in G_+ having a common upper bound have a least common upper bound ([12], Section 2.1).

DEFINITION 2.3. Let (G, G_+) be a quasi-lattice ordered group and $H \subseteq G_+$. H is said to be *hereditary* if for any $x, y \in G_+$, $x \leqslant y \in H$ implies that $x \in H$; and H is said to be *directed* if any two elements of H have a common upper bound in H ([12], Section 6.2).

Many examples of quasi-lattice ordered groups are given in ([12], Section 2.3). For instance, *ordered groups* (see Example 4.8 below) are quasi-lattice ordered, and so are direct products of quasi-lattice ordered groups. Further, for certain positive parts G_+ and certain hereditary and directed subsets H of G_+, the product $G_+ \cdot H^{-1}$ is a semigroup of G. In this case $(G, G_+ \cdot H^{-1})$ becomes a *quasi-lattice quasi-ordered group* (for a formal definition, see Definition 3.1). We present two examples.

EXAMPLE 2.4. Let (G_1, G_{1+}) be a quasi-lattice ordered group and (G_2, G_{2+}) an ordered group. Set $G = G_1 \times G_2$, $G_+ = G_{1+} \times G_{2+}$ and $H = \{e\} \times G_{2+}$, where e is the identity of G_1. Then H is both hereditary and directed, and $G_+ \cdot H^{-1} = G_{1+} \times G_2$ is a semigroup of G.

EXAMPLE 2.5. Let (G, G_+) be a quasi-lattice ordered group such that G is abelian. For any $g_0 \in G_+ \setminus \{e\}$, let $H = \{t \in G_+ : \exists\, n \in N \text{ such that} t \leqslant g_0^n\}$. Then H is hereditary and directed, and since G is abelian, $G_+ \cdot H^{-1}$ is a semigroup.

DEFINITION 2.6. Let (G, G_+) be a quasi-lattice ordered group. For any $s, t \in G_+$, if they have a common upper bound in G_+, let $\sigma(s, t)$ denote their least common upper bound. Let B be a unital C^*-algebra and V a map from G_+ to B. V is said to be an *isometric representation* of G_+ if it satisfies the following three conditions:

(i) $V(e) = 1$;
(ii) $V(g)^* V(g) = 1$ for any $g \in G_+$;
(iii) $V(g)V(h) = V(gh)$ for any $g, h \in G_+$.

Moreover, V is said to be *covariant* if for any $s, t \in G_+$,

$$V(s)V(s)^* \cdot V(t)V(t)^* = \begin{cases} V(\sigma(s,t)), & \text{if } s \text{ and } t \text{ have a common upper bound,} \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{T}^\infty(G_+) = \operatorname{span}\{T_g^{G_+}(T_h^{G_+})^* : g, h \in G_+\}.$$

Then $\mathcal{T}^\infty(G_+)$ is a dense unital $*$-algebra of $\mathcal{T}^{G_+}$ ([12], Section 3.2). By [12] we know that $\mathcal{T}^\infty(G_+)$ has a universal property for covariant isometric representations of G_+. More precisely, for any unital C^*-algebra B, and any covariant isometric representation $V : G_+ \to B$, there is a $*$-representation π_V from $\mathcal{T}^\infty(G_+)$ to B such that $\pi_V(T_g^{G_+}) = V(g)$ for any $g \in G_+$. Generally speaking, such a map π_V may fail to be bounded. However, if π_V is bounded, then the Toeplitz algebra $\mathcal{T}^{G_+}$ also has a universal property for covariant isometric representations of G_+.

Now we turn to our study of natural morphisms. Throughout this portion of the section, (G, G_+) denotes a quasi-lattice ordered group. We follow the notations as in [6] and [12]. For any $x \in G$, it is easy to show that x has a upper bound in G_+ if and only if $x \in G_+ \cdot G_+^{-1}$, and when $x \in G_+ \cdot G_+^{-1}$, it's least upper bound in G_+ will be denoted by $\sigma(x)$. More generally, for any subset $A \subseteq G_+$, if A has a upper bound in G_+, then it's least upper bound in G_+ will be denoted by $\sigma(A)$. When $x \in G_+ \cdot G_+^{-1}$, x^{-1} also belongs to $G_+ \cdot G_+^{-1}$, and if we let $\tau(x) = x^{-1}\sigma(x)$, then it is easy to show that $\sigma(x) = \tau(x^{-1})$, $\sigma(x^{-1}) = \tau(x)$ and $x = \sigma(x) \cdot \tau(x)^{-1}$. To simplify the notation as in [6], for any pair $x, y \in G_+$, if they have no common upper bound in G_+, then let $\sigma(x, y) = \infty$ and $T_\infty^E = 0$ for any $E \subseteq G$. Thus by ([12], Section 2 and Section 3), we know that

$$T_x^{G_+} = \begin{cases} T_{\sigma(x)}^{G_+} \cdot T_{\tau(x)^{-1}}^{G_+}, & \text{if } x \in G_+ \cdot G_+^{-1}, \\ 0, & \text{if } x \notin G_+ \cdot G_+^{-1}; \end{cases} \tag{2.1}$$

and if $x, y \in G_+$, then

$$T_{x^{-1}}^{G_+} \cdot T_x^{G_+} = 1 \quad \text{and} \quad (T_x^{G_+} \cdot T_{x^{-1}}^{G_+}) \cdot (T_y^{G_+} \cdot T_{y^{-1}}^{G_+}) = T_{\sigma(x,y)}^{G_+} \cdot T_{\sigma(x,y)^{-1}}^{G_+}. \tag{2.2}$$

PROPOSITION 2.7. *Let (G, G_+) be a quasi-lattice ordered group and E a subset of G such that $G_+ \subsetneqq E$. If there is a C^*-morphism $\gamma^{E,G_+} : \mathcal{T}^{G_+} \to \mathcal{T}^E$ satisfying $\gamma^{E,G_+}(T_g^{G_+}) = T_g^E$ for any $g \in G$, then there exists a hereditary and directed subset H of G_+ such that $E = G_+ \cdot H^{-1}$.*

Proof. Since γ^{E,G_+} is a unital C^*-morphism, by (2.1) and (2.2) we know that

$$T_x^E = \begin{cases} T_{\sigma(x)}^E \cdot T_{\tau(x)^{-1}}^E, & \text{if } x \in G_+ \cdot G_+^{-1}, \\ 0, & \text{if } x \notin G_+ \cdot G_+^{-1}; \end{cases} \tag{2.3}$$

and if $x, y \in G_+$, then

$$T_{x^{-1}}^E \cdot T_x^E = 1 \quad \text{and} \quad (T_x^E \cdot T_{x^{-1}}^E) \cdot (T_y^E \cdot T_{y^{-1}}^E) = T_{\sigma(x,y)}^E \cdot T_{\sigma(x,y)^{-1}}^E. \tag{2.4}$$

For any $x \in E$, since $T_x^E \delta_e = \delta_x \neq 0$, by (2.3) we know that $x \in G_+ \cdot G_+^{-1}$. It follows that $E \subseteq G_+ \cdot G_+^{-1}$. Let

$$H = \{x : x \in G_+,\ x^{-1} \in E\}.$$

Observe that $H \neq \{e\}$. In fact, since $G_+ \subsetneqq E$, there exists $x \in E$, $x = \sigma(x) \cdot \tau(x)^{-1}$ such that $\tau(x) \neq e$. By (2.3) we have $T_{\sigma(x)}^E T_{\tau(x)^{-1}}^E \delta_e = \delta_x \neq 0$, so $\tau(x)^{-1} \in E$.

We now prove that H is both directed and hereditary. For any $x, y \in H$, equation (2.4) implies $T_{\sigma(x,y)}^E T_{\sigma(x,y)^{-1}}^E \delta_e = \delta_e \neq 0$, so $\sigma(x, y) \in H$, therefore H is directed. Furthermore, given any $x, y \in G_+$ with $x \leqslant y \in H$, since

$$T_{y^{-1}}^{G_+} = (T_y^{G_+})^* = (T_x^{G_+} T_{x^{-1}y}^{G_+})^* = T_{y^{-1}x}^{G_+} T_{x^{-1}}^{G_+},$$

we know $T_{y^{-1}}^E = T_{y^{-1}x}^E T_{x^{-1}}^E$. It follows that $T_{y^{-1}x}^E T_{x^{-1}}^E \delta_e = \delta_{y^{-1}} \neq 0$, which implies that $x \in H$, and thus H is also hereditary.

Finally we prove that $E = G_+ \cdot H^{-1}$. For any $x \in E$ with $x = \sigma(x) \cdot \tau(x)^{-1}$, equation (2.3) implies $\tau(x) \in H$, therefore $E \subseteq G_+ \cdot H^{-1}$. On the other hand, since $T_{x^{-1}}^E T_x^E = 1$ for any $x \in G_+$, we know that $xy \in E$ for any $x \in G_+$ and $y \in E$, so $G_+ \cdot E \subseteq E$. In particular, $G_+ \cdot H^{-1} \subseteq E$. ∎

Now we work toward proving the converse of Proposition 2.7, culminating in Proposition 2.11 and Theorem 2.12.

LEMMA 2.8. *Suppose (G, G_+) is a quasi-lattice ordered group. Then for any $x \in G_+$ and $y \in G$, $x^{-1}y \in G_+ \cdot G_+^{-1}$ if and only if x, y have a common upper bound in G_+. Furthermore, if they have a common upper bound in G_+, then*

$$\sigma(x^{-1}y) = x^{-1}\sigma(x, y) \quad \text{and} \quad \tau(x^{-1}y) = y^{-1}\sigma(x, y).$$

Proof. If x, y have a common upper bound in G_+, then $x^{-1}y = (x^{-1}\sigma(x, y)) \cdot (y^{-1}\sigma(x, y))^{-1} \in G_+ \cdot G_+^{-1}$.

On the other hand, suppose that $x^{-1}y \in G_+ \cdot G_+^{-1}$. If $x^{-1}y = gh^{-1}$, $g, h \in G_+$, then $x \leqslant yh$. Since $x \in G_+$, we know that $yh = x \cdot (x^{-1}yh) \in G_+$. Obviously $y \leqslant yh$, so yh is a common upper bound of x and y.

Now suppose that x and y have a common upper bound in G_+. We prove that the displayed equations in Lemma 2.8 hold. For any $z \in G_+$, if $z \geqslant x^{-1}y$, then $xz \geqslant y$ and $xz \geqslant x$, so $xz \geqslant \sigma(x, y)$, it follows that $z \geqslant x^{-1}\sigma(x, y)$. Therefore $\sigma(x^{-1}y) \geqslant x^{-1}\sigma(x, y)$. On the other hand, it is clear that $x^{-1}\sigma(x, y) \geqslant \sigma(x^{-1}y)$, hence $x^{-1}\sigma(x, y) = \sigma(x^{-1}y)$. Since $x^{-1}y = \sigma(x^{-1}y) \cdot \tau(x^{-1}y)^{-1}$, we know that $\tau(x^{-1}y) = y^{-1}\sigma(x, y)$. ∎

LEMMA 2.9. *Suppose (G, G_+) is a quasi-lattice ordered group, H a hereditary subset of G_+, and $E = G_+ \cdot H^{-1}$. Then for any $x \in G_+ \cdot G_+^{-1}$, $x \in E$ if and only if $\tau(x) \in H$.*

Proof. Suppose $x \in E$ and write $x = gh^{-1}$, where $g \in G_+$ and $h \in H$. Since $\tau(x) = x^{-1}\sigma(x)$, we obtain $\tau(x)^{-1}h = \sigma(x)^{-1}g$, and furthermore $\sigma(x)^{-1}g \in G_+$ as $x \leqslant g$. We conclude that $\tau(x) \leqslant h$, and thus $\tau(x) \in H$ since H is hereditary. On the other hand, if $\tau(x) \in H$, then clearly $x \in E$ as $x = \sigma(x)\tau(x)^{-1}$. ∎

LEMMA 2.10. *Let (G, G_+) be a quasi-lattice ordered group, H a directed and hereditary subset of G_+, and $E = G_+ \cdot H^{-1}$. Then $V : G_+ \to \mathcal{T}^E$ defined by $V(g) = T_g^E$ for any $g \in G_+$, is a covariant isometric representation of G_+.*

Proof. Since $G_+ \cdot E \subseteq E$, the first three conditions of Definition 2.6 are trivially satisfied. It remains to prove the covariance property

$$T_x^E T_{x^{-1}}^E \cdot T_y^E T_{y^{-1}}^E = T_{\sigma(x,y)}^E T_{\sigma(x,y)^{-1}}^E$$

for any $x, y \in G_+$. It suffices to prove that for any $x, y \in G_+$ and $z \in E$, if $T_x^E T_{x^{-1}}^E \cdot T_y^E T_{y^{-1}}^E \delta_z = \delta_z$, then x, y have a common upper bound in G_+ and that $T_{\sigma(x,y)}^E T_{\sigma(x,y)^{-1}}^E \delta_z = \delta_z$.

First, suppose that $x, y \in G_+, z \in E$ such that $T_x^E T_{x^{-1}}^E \cdot T_y^E T_{y^{-1}}^E \delta_z = \delta_z$. Then by Lemma 2.8, $\sigma(y, z), \sigma(x, z) \in G_+$ with

$$z^{-1}\sigma(y, z) \in H \quad \text{and} \quad z^{-1}\sigma(x, z) \in H.$$

Since H is directed, $\sigma(z^{-1}\sigma(x, z), z^{-1}\sigma(y, z)) \in H$.

We now show that $\sigma(x, y) \in G_+$ and that

$$\sigma(z^{-1}\sigma(x, z), z^{-1}\sigma(y, z)) = z^{-1}\sigma(\sigma(x, y), z).$$

Let $u = \sigma(z^{-1}\sigma(x, z), z^{-1}\sigma(y, z))$. Then $u \geqslant z^{-1}\sigma(x, z)$, so $zu \geqslant \sigma(x, z)$, hence $zu \in G_+$ and $zu \geqslant x$. Similarly, $zu \geqslant y$, so zu is a common upper bound of

x, y and z, and therefore $\sigma(x,y)$ exists in G_+. Furthermore, $zu \geqslant \sigma(\sigma(x,y),z)$, so $u \geqslant z^{-1}\sigma(\sigma(x,y),z)$. On the other hand, obviously we have $u \leqslant z^{-1}\sigma(\sigma(x,y),z)$, so $u = z^{-1}\sigma(\sigma(x,y),z)$.

Finally, since $\sigma(x,y)$ and z have a common upper bound zu and $\tau(\sigma(x,y)^{-1}z) = z^{-1}\sigma(\sigma(x,y),z) \in H$, by Lemma 2.9 we have $\sigma(x,y)^{-1}z \in E$, and thus it follows that $T^E_{\sigma(x,y)}T^E_{\sigma(x,y)^{-1}}\delta_z = \delta_z$. ∎

PROPOSITION 2.11. *Let (G, G_+) be a quasi-lattice ordered group, H a directed and hereditary subset of G_+, and $E = G_+ \cdot H^{-1}$. Then γ^{E,G_+} exists as a C^*-morphism.*

Proof. By Lemma 2.10 and the universal property of $\mathcal{T}^\infty(G_+)$, we know there is a $*$-morphism $\pi_V : \mathcal{T}^\infty(G_+) \to \mathcal{T}^E$ such that $\pi_V(T^{G_+}_g T^{G_+}_{h^{-1}}) = T^E_g T^E_{h^{-1}}$ for any $g, h \in G_+$. The proof of the proposition is accomplished in two steps: First, we prove that π_V is bounded, therefore its domain of definition can be extended to $\mathcal{T}^{G_+}$. Then we show that the extension of π_V agrees with γ^{E,G_+}.

To show that π_V is bounded, we verify that for any $T \in \mathcal{T}^\infty(G_+)$, $\|\pi_V(T)\| \leqslant \|T\|$. In fact, it is enough to show that for any $n \in N$, $\lambda_i \in C$, $g_i, h_i \in G_+$, $i = 1, 2, \ldots, n$, the following inequality holds:

$$\Big\|\sum_{i=1}^n \lambda_i T^E_{g_i} T^E_{h_i^{-1}}\Big\| \leqslant \Big\|\sum_{i=1}^n \lambda_i T^{G_+}_{g_i} T^{G_+}_{h_i^{-1}}\Big\|.$$

Let $T = \sum\limits_{i=1}^n \lambda_i T^{G_+}_{g_i} T^{G_+}_{h_i^{-1}}$. For any $\varepsilon > 0$, there exists some $\xi \in \ell^2(E)$ with finite support such that $\|\xi\| = 1$, and $\|\pi_V(T)\| \leqslant \|\pi_V(T)\xi\| + \varepsilon$. Let $\xi = \sum\limits_{p=1}^m \mu_p \delta_{l_p}$ with $\mu_p \in C$ and $l_p \in E$ for $p = 1, 2, \ldots, m$. Set

$$F = \{l_p : p = 1, \ldots, m\} \cup \{h_i^{-1}l_p : h_i^{-1}l_p \in E, i = 1, \ldots, n, p = 1, \ldots, m\}.$$

Then F is a finite subset of E, so it can be rewritten as $F = \{x_1, x_2, \ldots, x_k\}$ with $\tau(x_j) \in H$ for any $j = 1, 2, \ldots, k$. Since H is directed, we know $g_0 = \sigma\big(\tau(x_1), \tau(x_2), \ldots, \tau(x_k)\big) \in H$ with the property that

$$l_p g_0 \in G_+ \quad \text{and} \quad h_i^{-1}l_p g_0 \in G_+ \text{ for any } h_i^{-1}l_p \in F. \tag{2.5}$$

Therefore, we know that for any pair (i, p),

$$h_i^{-1}l_p \in E \quad \text{if and only if} \quad h_i^{-1}l_p\, g_0 \in G_+. \tag{2.6}$$

In fact, if $h_i^{-1}l_p \in E$, then $h_i^{-1}l_p \in F$, so by (2.5) we have $h_i^{-1}l_p\, g_0 \in G_+$. On the other hand, if $h_i^{-1}l_p\, g_0 \in G_+$, then $h_i^{-1}l_p \in G_+ \cdot g_0^{-1} \subseteq G_+ \cdot H^{-1} = E$.

Now let $\eta = \sum\limits_{p=1}^m \mu_p \delta_{l_p g_0}$. Then $\eta \in \ell^2(G_+)$, $\|\eta\| = \|\xi\| = 1$, and by (2.6) we obtain

$$\|\pi_V(T)\xi\| = \Big\|\Big(\sum_{i=1}^n \lambda_i T^E_{g_i} T^E_{h_i^{-1}}\Big)\xi\Big\| = \Big\|\Big(\sum_{i=1}^n \lambda_i T^{G_+}_{g_i} T^{G_+}_{h_i^{-1}}\Big)\eta\Big\| = \|T\eta\|.$$

Therefore

$$\|\pi_V(T)\| \leqslant \|\pi_V(T)\xi\| + \varepsilon = \|T\,\eta\| + \varepsilon \leqslant \|T\| + \varepsilon.$$

Since ε is arbitrary, $\|\pi_V(T)\| \leqslant \|T\|$, and hence π_V is bounded.

Next, we prove that the extended C^*-morphism $\pi_V : \mathcal{T}^{G_+} \to \mathcal{T}^E$ satisfies $\pi_V(T_x^{G_+}) = T_x^E$ for any $x \in G$, therefore $\gamma^{E,G_+} = \pi_V$.

First we note that for any $x, y \in H$,

$$x^{-1}y = (x^{-1}\sigma(x,y)) \cdot (y^{-1}\sigma(x,y))^{-1} \in G_+ \cdot G_+^{-1}.$$

It follows that $E \cdot E^{-1} = G_+ \cdot G_+^{-1}$, so if $x \notin G_+ \cdot G_+^{-1}$, then $T_x^E = 0 = T_x^{G_+}$.

On the other hand, if $x \in G_+ \cdot G_+^{-1}$, we prove that (2.3) holds, therefore

$$T_x^E = T_{\sigma(x)}^E T_{\tau(x)^{-1}}^E = \pi_V(T_{\sigma(x)}^{G_+} T_{\tau(x)^{-1}}^{G_+}) = \pi_V(T_x^{G_+}).$$

Clearly, (2.3) holds if and only if the following property holds:

For any $x \in G_+ \cdot G_+^{-1}$ and $y \in E$, $xy \in E$ implies that $\tau(x)^{-1}y \in E$.

So it remains to prove that the above property holds. In fact, if $x \in G_+ \cdot G_+^{-1}$ and $y \in E$ such that $xy \in E$, then $\tau(y) \in H$ and $\tau(xy) \in H$. Let $g_0 = \sigma(\tau(y), \tau(xy))$. Then

$$g_0 \in H \quad \text{and} \quad yg_0 \in G_+,\ xyg_0 \in G_+. \tag{2.7}$$

By (2.7) we know that $x^{-1} \leqslant yg_0 \in G_+$, so $\tau(x) = \sigma(x^{-1}) \leqslant yg_0$, it follows that $\tau(x)^{-1}y \in G_+ \cdot g_0^{-1} \subseteq G_+ \cdot H^{-1} = E$. ∎

We may combine Propositions 2.7 and 2.11 to obtain the following theorem.

THEOREM 2.12. *Let (G, G_+) be a quasi-lattice ordered group, and suppose E is a subset of G such that $G_+ \subseteq E$. Then $\gamma^{E,G_+} : \mathcal{T}^{G_+} \to \mathcal{T}^E$ exists as a C^*-morphism if and only if there is a directed and hereditary subset H of G_+ such that $E = G_+ \cdot H^{-1}$.*

3. THE FAITHFUL REPRESENTATIONS OF TOEPLITZ ALGEBRAS ON QUASI-LATTICE QUASI-ORDERED GROUPS

The faithful representations of Toeplitz algebras on quasi-lattice ordered groups and on quasi-ordered groups are studied in [6] and [16], respectively. The purpose of this section is to extend certain results stated in [6] and [16] to the setting of quasi-lattice quasi-ordered groups, with the purpose of using these new results in the sequel. The key point here is to replace the trivial subgroup $\{e\} \subseteq G_+$ (as in [6]) by some semigroup H contained in G_+.

DEFINITION 3.1. Let (G, G_+) be a quasi-lattice ordered group, H a hereditary and directed subset of G_+, and $E = G_+ \cdot H^{-1}$. If E is a semigroup of G, then (G, E) is referred to as a *quasi-lattice quasi-ordered group.*

Before proceeding with the results of this section, several preliminary remarks are in order.

First, examples of quasi-lattice quasi-ordered groups are given in Examples 2.4 and 2.5.

Second, when (G, E) is a quasi-lattice quasi-ordered group with $E = G_+ \cdot H^{-1}$, E is also denoted by G_H, and we put $G_H^0 := G_H \cap G_H^{-1}$. Observe that $(G \setminus G_H) \cdot G_H^0 \subseteq G \setminus G_H$. Also, using Lemma 2.9 and the fact that H is hereditary, it follows that $G_H^0 = H \cdot H^{-1}$, $G_H \cap G_+^{-1} = H^{-1}$, and that H is a semigroup.

Third, Theorem 2.12 guarantees that the surjective C^*-morphism γ^{G_H, G_+} exists, so if we let

$$\mathcal{T}^\infty(G_H) = \text{span } \{T_g^{G_H} T_{h^{-1}}^{G_H} : g, h \in G_+\} = \gamma^{G_H, G_+}(\mathcal{T}^\infty(G_+)),$$

then $\mathcal{T}^\infty(G_H)$ is a dense $*$-subalgebra of $\mathcal{T}^{G_H}$.

Finally, let $M = G_+$ or G_H, and

$$D^M = \overline{\text{span}\{T_g^M T_{g^{-1}}^M : g \in G_+\}}^{\|\cdot\|}.$$

Note that D^M is a unital commutative C^*-subalgebra of $\mathcal{T}^M$. Denote by D^G the C^*-subalgebra of $B(\ell^2(G))$ consisting of all the operators having diagonal matrix relatively to the canonical basis. It is well-known that there exists a linear and contractive map $\theta^G : B(\ell^2(G)) \to D^G$ determined by the following rule: The idealized matrix of $\theta^G(T)$ (relative to the canonical basis) is obtained from the one for T by replacing with zero all the entries which are not situated on the principal diagonal. By Sections 3.3 and 3.6 of [12] we know that $D^M = \{T \in \mathcal{T}^M : T \text{ has diagonal matrix}\}$ (relative to the canonical basis for $\ell^2(G)$), and $\theta^M = \theta^G | \mathcal{T}^M$ is a faithful bounded linear map from $\mathcal{T}^M$ onto D^M satisfying

$$\theta^M(T_g^M T_{h^{-1}}^M) = \begin{cases} T_g^M T_{h^{-1}}^M, & \text{if } g = h, \\ 0, & \text{if } g \neq h, \end{cases}$$

for any g, h in G_+.

LEMMA 3.2. (cf. Lemma 3.9, [12]) *Let (G, G_+) be a quasi-lattice ordered group. Let $\{L(t) : t \in G_+\}$ be a family of projections of a unital C^*-algebra B satisfying $L(e) = 1$ and*

$$L(s)L(t) = \begin{cases} L(\sigma(s,t)), & \textit{if } s \textit{ and } t \textit{ have a common upper bound,} \\ 0, & \textit{otherwise.} \end{cases}$$

Then for any finite subset $F = \{t_1, t_2, \ldots, t_n\}$ of G_+, any $\lambda_1, \lambda_2, \ldots, \lambda_n \in C$, we have

$$\left\| \sum_{j=1}^n \lambda_j L(t_j) \right\|$$
$$= \max \left\{ \left| \sum_{j \in A} \lambda_j \right| : \emptyset \neq A \subseteq \{1, 2, \ldots, n\}, \prod_{j \in A} L(t_j) \cdot \prod_{k \notin A} (1 - L(t_k)) \neq 0 \right\}.$$

(Note if $A = F$, then the product $\prod_{j\in A} L(t_j)\cdot\prod_{k\notin A}(1-L(t_k))$ should be understood as $\prod_{j\in F} L(t_j)$, and if for every $\emptyset \neq A \subseteq \{1,2,\ldots,n\}$, $\prod_{j\in A} L(t_j)\cdot\prod_{k\notin A}(1-L(t_k)) = 0$, then $\sum_{j=1}^{n}\lambda_j L(t_j) = 0$.)

LEMMA 3.3. *Let (G, G_H) be a quasi-lattice quasi-ordered group such that $H \neq G_+$. Suppose that B is a unital C^*-algebra and π is a unital C^*-morphism from $D^{G_H} \to B$. Let $L(t) = \pi(T_t^{G_H}T_{t^{-1}}^{G_H})$ for any $t \in G_+$. Then π is faithful if and only if*

$$\prod_{i=1}^{n}\big(L(a)-L(t_i)\big)\neq 0 \qquad \textit{whenever } a\in G_+ \textit{ and } a^{-1}t_i\in G_+\setminus H.$$

Proof. First, assume that π is faithful. For any $a \in G_+$ and any $t_1, t_2, \ldots, t_n \in G_+$ satisfying $a^{-1}t_i \in G_+ \setminus H$, note that $t_i^{-1}a \notin G_H$ and so $T_{t_i^{-1}}^{G_H}\delta_a = 0$. Thus it follows that

$$\left(\prod_{i=1}^{n}(T_a^{G_H}T_{a^{-1}}^{G_H} - T_{t_i}^{G_H}T_{t_i^{-1}}^{G_H})\right)\delta_a = \delta_a \neq 0,$$

and then the faithfulness of π leads to the desired conclusion.

Now we turn to the reverse implication. To show that π is faithful, it is sufficient to prove that π is isometric, or equivalently, to prove that $\|\pi(x)\| \geqslant \|x\|$ for any $x \in D^{G_H}$ (because π is a C^*-morphism, it is contractive). Since $\operatorname{span}\{T_g^{G_H}T_{g^{-1}}^{G_H} : g \in G_+\}$ is dense in D^{G_H}, it reduces to prove that, for any finite subset $F = \{t_1, t_2, \ldots, t_n\}$ of G_+ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in C$, the following inequality holds:

$$\left\|\sum_{j=1}^{n}\lambda_j T_{t_j}^{G_H}T_{t_j^{-1}}^{G_H}\right\| \leqslant \left\|\sum_{j=1}^{n}\lambda_j L(t_j)\right\|.$$

By Lemma 3.2, it suffices to prove that for any non-empty subset A of F,

$$\prod_{j\in A}L(t_j)\cdot\prod_{k\notin A}(1-L(t_k))\neq 0 \quad \text{whenever } \prod_{j\in A}T_{t_j}^{G_H}T_{t_j^{-1}}^{G_H}\cdot\prod_{k\notin A}(1-T_{t_k}^{G_H}T_{t_k^{-1}}^{G_H})\neq 0.$$

The argument is now broken into two cases. In the first case, suppose $A = F$. If $\prod_{j\in F}T_{t_j}^{G_H}T_{t_j^{-1}}^{G_H} \neq 0$, then F has a upper bound in G_+ and $\prod_{j\in F}T_{t_j}^{G_H}T_{t_j^{-1}}^{G_H} = T_{\sigma(F)}^{G_H}T_{\sigma(F)^{-1}}^{G_H}$. Choose any $x \in G_+ \setminus H$. By hypothesis $L(\sigma(F)) - L(\sigma(F)x) \neq 0$, so $L(\sigma(F)) \neq 0$.

In the second case, suppose $A \neq F$. Suppose that

$$T = \prod_{j\in A}T_{t_j}^{G_H}T_{t_j^{-1}}^{G_H}\cdot\prod_{k\notin A}(1-T_{t_k}^{G_H}T_{t_k^{-1}}^{G_H})\neq 0.$$

Then A has a upper bound in G_+ and $T_{\sigma(A)}^{G_H}T_{\sigma(A)^{-1}}^{G_H}\cdot\prod_{k\notin A}(1-T_{t_k}^{G_H}T_{t_k^{-1}}^{G_H})\neq 0$.

For any $t \in F \setminus A$, if A and t have no common upper bound in G_+, then

$$T^{G_H}_{\sigma(A)}T^{G_H}_{\sigma(A)^{-1}} \cdot (1 - T^{G_H}_t T^{G_H}_{t^{-1}}) = T^{G_H}_{\sigma(A)}T^{G_H}_{\sigma(A)^{-1}};$$

otherwise,

$$\begin{aligned} T^{G_H}_{\sigma(A)}T^{G_H}_{\sigma(A)^{-1}} \cdot (1 - T^{G_H}_t T^{G_H}_{t^{-1}}) &= T^{G_H}_{\sigma(A)}T^{G_H}_{\sigma(A)^{-1}} - T^{G_H}_{\sigma(A,t)}T^{G_H}_{\sigma(A,t)^{-1}} \\ &= T^{G_H}_{\sigma(A)} \cdot \big(1 - T^{G_H}_{\sigma(A)^{-1}\sigma(A,t)}T^{G_H}_{(\sigma(A)^{-1}\sigma(A,t))^{-1}}\big) \cdot T^{G_H}_{\sigma(A)^{-1}}. \end{aligned}$$

So $T = T^{G_H}_{\sigma(A)}T^{G_H}_{\sigma(A)^{-1}}$ in the case when $\sigma(A,t) = \infty$ for any $t \in F \setminus A$, or

$$\begin{aligned} T &= \prod_{i=1}^{m} \big(T^{G_H}_{\sigma(A)}T^{G_H}_{\sigma(A)^{-1}} - T^{G_H}_{\sigma(A,t_{k_i})}T^{G_H}_{\sigma(A,t_{k_i})^{-1}}\big) \\ &= T^{G_H}_{\sigma(A)} \cdot \prod_{i=1}^{m}(1 - T^{G_H}_{\sigma(A)^{-1}\sigma(A,t_{k_i})}T^{G_H}_{(\sigma(A)^{-1}\sigma(A,t_{k_i}))^{-1}}) \cdot T^{G_H}_{\sigma(A)^{-1}}. \end{aligned}$$

Since $T^{G_H}_{\sigma(A)^{-1}}T^{G_H}_{\sigma(A)} = 1$, in the latter case $T \neq 0$ if and only if

$$\prod_{i=1}^{m}(1 - T^{G_H}_{\sigma(A)^{-1}\sigma(A,t_{k_i})}T^{G_H}_{(\sigma(A)^{-1}\sigma(A,t_{k_i}))^{-1}}) \neq 0,$$

which in turn happens if and only if

$$\sigma(A)^{-1}\sigma(A,t_{k_i}) \in G_+ \setminus H \quad \text{for any } i = 1, 2, \ldots, m.$$

But then by hypothesis we know that

$$\prod_{j\in A} L(t_j) \cdot \prod_{k\notin A}(1 - L(t_k)) = \prod_{i=1}^{m}\big(L(\sigma(A)) - L(\sigma(A,t_{k_i}))\big) \neq 0. \quad ∎$$

COROLLARY 3.4. *Let (G, G_H) be a quasi-lattice quasi-ordered group such that $H \neq G_+$. Suppose that B is a unital C^*-algebra and π is a unital C^*-morphism from $\mathcal{T}^{G_H} \to B$. Let $V(t) = \pi(T^{G_H}_t)$ and $L(t) = \pi(T^{G_H}_t T^{G_H}_{t^{-1}})$ for any $t \in G_+$. Then $\pi|D^{G_H} : D^{G_H} \to B$ is faithful if and only if*

$$\prod_{i=1}^{n}(1 - L(t_i)) \neq 0 \quad \textit{whenever } t_1, t_2, \ldots, t_n \in G_+ \setminus H.$$

We now state the main result of this section. The proof follows the same lines as Lemma 3.3 in [6] or Theorem 2.4 in [16], and an application of this theorem will be given in the proof of Proposition 6.7.

THEOREM 3.5. *Let $(G, G_H), B, \pi, V(t)$ and $L(t)$ be as in Corollary 3.4. Then $\pi : \mathcal{T}^{G_H} \to B$ is faithful if and only if, for any finite collection $g_{11}, g_{12}, \ldots, g_{1m} \in G_+ \setminus H$, $g_{01}, g_{02}, \ldots, g_{0n} \in G^0_H \setminus \{e\}$, and $\lambda_0, \lambda_1, \ldots, \lambda_n \in C$, the following inequality holds:*

$$|\lambda_0| \leqslant \Big\| \prod_{j=1}^{m}\big(1 - L(g_{1j})\big)\Big(\lambda_0 + \sum_{i=1}^{n}\lambda_i V(g_{0i})\Big)\prod_{j=1}^{m}\big(1 - L(g_{1j})\big)\Big\|.$$

Note when H reduces to $\{e\}$, the preceding theorem simplifies to the main result of Section 3 of [6].

COROLLARY 3.6. (cf. Theorem 3.7, [6]) *Let (G,G_+) be a quasi-lattice ordered group, B a unital C^*-algebra and π a unital C^*-morphism from $\mathcal{T}^{G_+} \to B$. Denote by $V(t) = \pi(T_t^{G_H})$ and $L(t) = \pi(T_t^{G_H} T_{t^{-1}}^{G_H})$ for any $t \in G_+$. Then π is faithful if and only if for any finite collection $g_1, g_2, \ldots, g_n \in G_+ \setminus \{e\}$,*

$$\prod_{j=1}^{n} \big(1 - L(g_j)\big) \neq 0.$$

4. DIAGONAL INVARIANT IDEALS OF TOEPLITZ ALGEBRAS ON QUASI-LATTICE ORDERED GROUPS

In this section we study diagonal invariant ideals of Toeplitz algebras. Specifically, we characterize diagonal invariant ideals in terms of induced ideals (Theorem 4.4), and we realize certain induced ideals as kernels of the C^*-morphisms γ^{E,G_+} (Proposition 4.10).

Throughout this section, (G,G_+) denotes a quasi-lattice ordered group. By an ideal of $\mathcal{T}^{G_+}$ we mean that it is non-trivial, proper, closed, and two-sided.

Let I be an ideal of $\mathcal{T}^{G_+}$. Since the quotient morphism $\pi : \mathcal{T}^{G_+} \to \mathcal{T}^{G_+}/I$ is not faithful, by Corollary 3.6 there exist $g_1, g_2, \ldots, g_n \in G_+ \setminus \{e\}$ satisfying $\prod\limits_{i=1}^{n} \big(1 - \pi(T_{g_i}^{G_+} T_{g_i^{-1}}^{G_+})\big) = 0$, or equivalently $\prod\limits_{i=1}^{n} (1 - T_{g_i}^{G_+} T_{g_i^{-1}}^{G_+}) \in I$. Thus every ideal I meets D^{G_+} nontrivially. Let $K(I)$ be the ideal of $\mathcal{T}^{G_+}$ generated by

$$\left\{ \prod_{i=1}^{n} (1 - T_{g_i}^{G_+} T_{g_i^{-1}}^{G_+}) : g_i \in G_+ \setminus \{e\},\ \prod_{i=1}^{n} (1 - T_{g_i}^{G_+} T_{g_i^{-1}}^{G_+}) \in I \right\}.$$

Then for any $g_1, g_2, \ldots, g_n \in G_+$,

$$\prod_{i=1}^{n} (1 - T_{g_i}^{G_+} T_{g_i^{-1}}^{G_+}) \in K(I) \Leftrightarrow \prod_{i=1}^{n} (1 - T_{g_i}^{G_+} T_{g_i^{-1}}^{G_+}) \in I. \tag{4.1}$$

Since $K(I) \subseteq I$, the forward part of (4.1) holds. The other direction follows from the definition of $K(I)$.

LEMMA 4.1. *Let (G,G_+) be a quasi-lattice ordered group and I an ideal of $\mathcal{T}^{G_+}$. Then*

$$I \cap D^{G_+} = \overline{I \cap \mathcal{T}^\infty(G_+) \cap D^{G_+}}^{\|\cdot\|} = K(I) \cap D^{G_+}.$$

Proof. First we note that since $\mathcal{T}^\infty(G_+) \cdot \mathcal{T}^\infty(G_+) = \mathcal{T}^\infty(G_+)$ and since $D^{G_+} \cap \mathcal{T}^\infty(G_+)$ is dense in D^{G_+}, $\overline{I \cap \mathcal{T}^\infty(G_+) \cap D^{G_+}}^{\|\cdot\|}$ is also an ideal of D^{G_+}. Let

$$J_1 = I \cap D^{G_+}, \quad J_2 = \overline{I \cap \mathcal{T}^\infty(G_+) \cap D^{G_+}}^{\|\cdot\|}, \quad \text{and} \quad J_3 = K(I) \cap D^{G_+}.$$

Obviously $J_2, J_3 \subseteq J_1$, and by (4.1) we know that for any $s_1, s_2, \ldots, s_n \in G_+$,

$$\prod_{i=1}^{n} (1 - T_{s_i}^{G_+} T_{s_i^{-1}}^{G_+}) \in J_1 \Leftrightarrow \prod_{i=1}^{n} (1 - T_{s_i}^{G_+} T_{s_i^{-1}}^{G_+}) \in J_2 \Leftrightarrow \prod_{i=1}^{n} (1 - T_{s_i}^{G_+} T_{s_i^{-1}}^{G_+}) \in J_3. \tag{4.2}$$

Let $\pi_1 : \mathcal{T}^{G_+} \to \mathcal{T}^{G_+}/I$ be the quotient morphism, and $\pi_1\big|D^{G_+} : D^{G_+} \to D^{G_+}/J_1$ the restriction of π_1. Similarly define π_2, π_3 and $\pi_2\big|D^{G_+}, \pi_3\big|D^{G_+}$.

For any $t \in G_+$, let $L_i(t) = \pi_i(T_t^{G_+} T_{t^{-1}}^{G_+})$. Since $J_2, J_3 \subseteq J_1$, there are two C^*-morphisms $\Lambda_{21} : D^{G_+}/J_2 \to D^{G_+}/J_1$ and $\Lambda_{31} : D^{G_+}/J_3 \to D^{G_+}/J_1$ such that $\Lambda_{i1} \cdot \pi_i\big|D^{G_+} = \pi_1\big|D^{G_+}$ for $i = 2, 3$. Observe that for $g \in G_+$, $\Lambda_{i1} L_i(g) = L_1(g)$.

We achieve the desired result showing that the mappings Λ_{i1} are injective. Specifically, given $g_1, \ldots, g_n \in G_+$ and $\lambda_1, \ldots, \lambda_n$, it suffices to show that

$$\Big\| \sum_{k=1}^{n} \lambda_k L_1(g_k) \Big\| = \Big\| \sum_{k=1}^{n} \lambda_k L_2(g_k) \Big\| = \Big\| \sum_{k=1}^{n} \lambda_k L_3(g_k) \Big\|. \tag{4.3}$$

We proceed to use Lemma 3.2 to verify (4.3). For any $g \in G_+$, $t_1, t_2, \ldots, t_n \in G_+$, and $i \in \{1, 2, 3\}$, let

$$T_i = L_i(g) \cdot \prod_{k=1}^{n} (1 - L_i(t_k)).$$

Then

$$T_1 = \pi_1(T_g^{G_+}) \cdot \prod_{k=1}^{n} \big(1 - L_1\big(g^{-1}\sigma(g, t_k)\big)\big) \cdot \pi_1(T_{g^{-1}}^{G_+}).$$

Since $T_{g^{-1}}^{G_+} T_g^{G_+} = 1$, we know that

$$\begin{aligned} T_1 \neq 0 &\Leftrightarrow \prod_{k=1}^{n} \big(1 - L_1(g^{-1}\sigma(g, t_k))\big) \neq 0 \\ &\Leftrightarrow \prod_{k=1}^{n} \big(1 - T_{g^{-1}\sigma(g,t_k)}^{G_+} T_{(g^{-1}\sigma(g,t_k))^{-1}}^{G_+}\big) \notin J_1. \end{aligned}$$

Now, by (4.2) we know that $T_1 \neq 0$ if and only if $T_i \neq 0$ for $i = 2$ or 3, and thus (4.3) holds by Lemma 3.2. Therefore, both Λ_{21} and Λ_{31} are injective. ∎

DEFINITION 4.2. Let (G, G_+) be a quasi-lattice ordered group.

(i) An ideal I of $\mathcal{T}^{G_+}$ is said to be *diagonal invariant* if $\theta^{G_+}(I) \subseteq I$.

(ii) For any $s, t \in G_+$, define $\alpha_{s,t} : D^{G_+} \to D^{G_+}$ by

$$\alpha_{s,t}(x) = (T_s^{G_+} T_{t^{-1}}^{G_+}) \cdot x \cdot (T_s^{G_+} T_{t^{-1}}^{G_+})^*$$

for any $x \in D^{G_+}$.

(iii) For any ideal J of D^{G_+}, put

$$\operatorname{Ind} J = \big\{ T \in \mathcal{T}^{G_+} : \alpha_{s,t}\big(\theta^{G_+}(T^*T)\big) \in J, \ \forall\, s, t \in G_+ \big\}.$$

$\operatorname{Ind} J$ is called the *induced ideal* associated to J.

We make several observations concerning this definition. First, since $\operatorname{span}(I_+) = I$, I is diagonal invariant if and only if $\theta^{G_+}(I_+) \subseteq I$, where I_+ is the positive part of I. Second, $\operatorname{Ind} J$ is indeed an ideal of $\mathcal{T}^{G_+}$, and in fact the induced ideals are diagonal invariant ([12], Section 6). Furthermore, if $J = I \cap D^{G_+}$, where I is an ideal of $\mathcal{T}^{G_+}$, then $\operatorname{Ind} J$ can be simplified as

$$\operatorname{Ind} J = \big\{ T \in \mathcal{T}^{G_+} : \theta^{G_+}(T^*T) \in J \big\}. \tag{4.4}$$

Finally, observe that if I is diagonal invariant, then $I \subseteq \operatorname{Ind}(I \cap D^{G_+})$.

LEMMA 4.3. *Let I be an ideal of $\mathcal{T}^{G_+}$. Then*

$$K(I) = \overline{\operatorname{Ind}(I \cap D^{G_+}) \cap \mathcal{T}^{\infty}(G_+)}^{\|\cdot\|} \subseteq I.$$

Proof. First we prove that $\overline{\operatorname{Ind}(I \cap D^{G_+}) \cap \mathcal{T}^{\infty}(G_+)}^{\|\cdot\|} \subseteq I$. Since I is closed, it is sufficient to prove that $\operatorname{Ind}(I \cap D^{G_+}) \cap \mathcal{T}^{\infty}(G_+) \subseteq I$. Let $S \in \mathcal{T}^{\infty}(G_+), S = \sum_{i=1}^{n} S_{x_i}$ with $x_i \neq x_j$ when $i \neq j$ and

$$S_{x_i} \in \operatorname{span}\{T_g^{G_+} T_{h^{-1}}^{G_+} : g, h \in G_+ , gh^{-1} = x_i\} \quad \text{for } i = 1, 2, \ldots, n.$$

Now, since

$$(S_{x_i} + S_{x_j})^*(S_{x_i} + S_{x_j}) + (S_{x_i} - S_{x_j})^*(S_{x_i} - S_{x_j}) = 2(S_{x_i}^* S_{x_i} + S_{x_j}^* S_{x_j}),$$

it follows that $(S_{x_i} + S_{x_j})^*(S_{x_i} + S_{x_j}) \leqslant 2(S_{x_i}^* S_{x_i} + S_{x_j}^* S_{x_j})$, and thus

$$S^* S \leqslant 2^{n-1} \left(\sum_{i=1}^{n} S_{x_i}^* S_{x_i} \right) = 2^{n-1} \theta^{G_+}(S^* S). \tag{4.5}$$

Furthermore, if $S \in \operatorname{Ind}(I \cap D^{G_+})$, then $\theta^{G_+}(S^* S) \in I$, and hence by (4.5) we know that $S \in I$.

Next we show that $K(I) = \overline{\operatorname{Ind}(I \cap D^{G_+}) \cap \mathcal{T}^{\infty}(G_+)}^{\|\cdot\|}$. By the definiton of $K(I)$, one readily concludes that $K(I) \subseteq \overline{\operatorname{Ind}(I \cap D^{G_+}) \cap \mathcal{T}^{\infty}(G_+)}^{\|\cdot\|}$. On the other hand, by Lemma 4.1 and the first part above (replace I by $K(I)$), we have

$$\overline{\operatorname{Ind}(I \cap D^{G_+}) \cap \mathcal{T}^{\infty}(G_+)}^{\|\cdot\|} = \overline{\operatorname{Ind}(K(I) \cap D^{G_+}) \cap \mathcal{T}^{\infty}(G_+)}^{\|\cdot\|} \subseteq K(I).$$

Therefore, $K(I) = \overline{\operatorname{Ind}(I \cap D^{G_+}) \cap \mathcal{T}^{\infty}(G_+)}^{\|\cdot\|}$. ∎

THEOREM 4.4. *Let I be an ideal of $\mathcal{T}^{G_+}$. Then I is diagonal invariant if and only if there exists some ideal R of $\mathcal{T}^{G_+}$ such that*

$$\overline{\operatorname{Ind}(R \cap D^{G_+}) \cap \mathcal{T}^{\infty}(G_+)}^{\|\cdot\|} \subseteq I \subseteq \operatorname{Ind}(R \cap D^{G_+}). \tag{4.6}$$

Proof. Suppose that I is a diagonal invariant ideal of $\mathcal{T}^{G_+}$. Then, we obtain by Lemma 4.3 and the observation immediately preceding it

$$\overline{\operatorname{Ind}(I \cap D^{G_+}) \cap \mathcal{T}^{\infty}(G_+)}^{\|\cdot\|} \subseteq I \subseteq \operatorname{Ind}(I \cap D^{G_+}).$$

Suppose there exists some ideal R of $\mathcal{T}^{G_+}$ such that (4.6) holds. Then by Lemma 4.1 we know that for any $x \in I$,

$$\begin{aligned} \theta^{G_+}(x^* x) \in R \cap D^{G_+} &= \overline{R \cap \mathcal{T}^{\infty}(G_+) \cap D^{G_+}}^{\|\cdot\|} \\ &\subseteq \overline{\operatorname{Ind}(R \cap D^{G_+}) \cap \mathcal{T}^{\infty}(G_+)}^{\|\cdot\|} \subseteq I. \end{aligned}$$

So I is diagonal invariant. ∎

Remark 4.5. (i) One may wonder whether $\overline{\mathrm{Ind}\,(I\cap D^{G_+})\cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|} = \mathrm{Ind}\,(I\cap D^{G_+})$ for every ideal I of $\mathcal{T}^{G_+}$. If equality always holds, then by Theorem 4.4 I is diagonal invariant if and only if $I = \mathrm{Ind}(I\cap D^{G_+})$. In other words, the induced ideals are in fact the totality of all diagonal invariant ideals of $\mathcal{T}^{G_+}$.

Note that equality does hold in the special case when G is amenable. For, if G is amenable, then by Proposition 6.1, [12], we know that $\mathrm{Ind}(I\cap D^{G_+})$ is the ideal of $\mathcal{T}^{G_+}$ generated by $I\cap D^{G_+}$. Then Lemma 4.1 implies $\mathrm{Ind}(I\cap D^{G_+})\subseteq K(I) = \overline{\mathrm{Ind}\,(I\cap D^{G_+})\cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|}$, so $\mathrm{Ind}(I\cap D^{G_+}) = \overline{\mathrm{Ind}\,(I\cap D^{G_+})\cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|}$.

(ii) The maximal induced ideals were recently studied by M. Laca, see Proposition 4.3, [5].

We now turn to study reduced ideals of $\mathcal{T}^{G_+}$. First we give the definition.

Definition 4.6. An ideal I of $\mathcal{T}^{G_+}$ is said to be *reduced*, if $\prod_{i=1}^{n}(1-T^{G_+}_{g_i}T^{G_+}_{g_i^{-1}}) \in I$ for any $g_1, g_2, \ldots, g_n \in G_+$ implies there exists g_{i_0} such that $1-T^{G_+}_{g_{i_0}}T^{G_+}_{g_{i_0}^{-1}} \in I$.

Example 4.7. Let (G,G_+) be a quasi-lattice ordered group and (G,E) a quasi-lattice quasi-ordered group such that $G_+ \subsetneqq E = G_+\cdot H^{-1}$. Let $I = \mathrm{Ker}\,\gamma^{E,G_+}$, where the existence of γ^{E,G_+} is guaranteed by Theorem 2.12. Then I is a reduced diagonal invariant ideal of $\mathcal{T}^{G_+}$. In fact, for any $g_1, g_2, \ldots, g_n \in G_+$, if $\Big(\prod_{i=1}^{n}(1-T^{E}_{g_i}T^{E}_{g_i^{-1}})\Big)\delta_e = 0$, then there exists g_{i_0} such that $g_{i_0}\in H$. Since E is a semigroup, we know that $1-T^{E}_{g_{i_0}}T^{E}_{g_{i_0}^{-1}} = 0$, so I is reduced. Moreover, since

$$\gamma^{E,G_+}\big(\theta^{G_+}(x^*x)\big) = \theta^{E}\big(\gamma^{E,G_+}(x^*x)\big) \quad \text{for any } x\in \mathcal{T}^{G_+}, \tag{4.7}$$

it follows that I is diagonal invariant.

Example 4.8. Recall a pair (G,G_+) is called an ordered group, if G is a discrete group and G_+ a semigroup of G, such that $G_+\cap G_+^{-1} = \{e\}$ and $G = G_+\cup G_+^{-1}$. Since G is totally ordered, any ideal of $\mathcal{T}^{G_+}$ is reduced.

Lemma 4.9. *Let (G,G_+) be a quasi-lattice ordered group and I a reduced ideal of $\mathcal{T}^{G_+}$. Let*

$$E = \{g : g\in G,\ \textit{such that } 1-T^{G_+}_{g^{-1}}T^{G_+}_{g}\in I\}.$$

Then (G,E) is a quasi-lattice quasi-ordered group.

Proof. Let $H = G_+\cap E^{-1}$, or equivalently,

$$H = \{g : g\in G_+,\ \text{such that } 1-T^{G_+}_{g}T^{G_+}_{g^{-1}}\in I\}.$$

First we show that H is directed and hereditary. Corollary 3.6 and the fact that I is reduced imply $H\neq\{e\}$. Let π be the quotient morphism from $\mathcal{T}^{G_+}\to \mathcal{T}^{G_+}/I$. Then for any $g\in G_+$, $g\in H$ if and only if $\pi(T^{G_+}_{g})\pi(T^{G_+}_{g^{-1}}) = 1$. Let $g_1, g_2\in H$. The fact that π is multiplicative together with (2.2) gives

$$1 = \pi(T^{G_+}_{g_1}T^{G_+}_{g_1^{-1}}T^{G_+}_{g_2}T^{G_+}_{g_2^{-1}}) = \pi(T^{G_+}_{\sigma(g_1,g_2)}T^{G_+}_{\sigma(g_1,g_2)^{-1}}),$$

which implies $\sigma(g_1,g_2)\in H$, and so H is directed. Now if $x,y\in G_+$ such that $x\leqslant y\in H$, then

$$\pi(T_x^{G_+})\,\pi(T_{x^{-1}y}^{G_+})\pi(T_{x^{-1}y}^{G_+})^*\,\pi(T_x^{G_+})^*=\pi(T_y^{G_+})\,\pi(T_{y^{-1}}^{G_+})=1. \tag{4.8}$$

Multiplying by $\pi(T_{x^{-1}}^{G_+})$ on the left and by $\pi(T_x^{G_+})$ on the right gives

$$\pi(T_{x^{-1}y}^{G_+})\cdot\pi(T_{x^{-1}y}^{G_+})^*=1. \tag{4.9}$$

By (4.8) and (4.9) we know that $\pi(T_x^{G_+})\cdot\pi(T_{x^{-1}}^{G_+})=1$, and so $x\in H$. Therefore H is hereditary.

Next we show that $E=G_+\cdot H^{-1}$. For any $g\in G_+$ and $h\in H$, using the facts that $T_{gh^{-1}}^{G_+}T_h^{G_+}=T_g^{G_+}$ and $T_{g^{-1}}^{G_+}T_g^{G_+}=1$, we obtain

$$\pi(T_{h^{-1}}^{G_+})\,\pi(T_{hg^{-1}}^{G_+})\,\pi(T_{gh^{-1}}^{G_+})\,\pi(T_h^{G_+})=1.$$

Since $\pi(T_h^{G_+})\pi(T_{h^{-1}}^{G_+})=1$, we know that $\pi(T_{hg^{-1}}^{G_+})\,\pi(T_{gh^{-1}}^{G_+})=1$, i.e., $gh^{-1}\in E$, so $G_+\cdot H^{-1}\subseteq E$. Also, for any $x\in E$, x must be in $G_+\cdot G_+^{-1}$, otherwise $T_x^{G_+}=0$, which implies that $1\in I$, which is a contradiction. Then by (2.1) we know that

$$\pi\big(T_{\tau(x)}^{G_+}\,T_{\sigma(x)^{-1}}^{G_+}\big)\cdot\pi\big(T_{\sigma(x)}^{G_+}\,T_{\tau(x)^{-1}}^{G_+}\big)=1,$$

or equivalently,

$$\pi\big(T_{\tau(x)}^{G_+}\big)\cdot\pi\big(T_{\tau(x)^{-1}}^{G_+}\big)=1.$$

Therefore, $\tau(x)\in H$, so $x=\sigma(x)\tau(x)^{-1}\in G_+\cdot H^{-1}$.

Finally we show that $H^{-1}\cdot E\subseteq E$, which implies E is a semigroup of G. For any $h\in H\subseteq G_+$ and $g\in E$,

$$\pi(T_{g^{-1}h}^{G_+})\cdot\pi(T_{h^{-1}g}^{G_+})=\pi(T_{g^{-1}}^{G_+})\cdot\pi(T_h^{G_+})\cdot\pi(T_{h^{-1}}^{G_+})\cdot\pi(T_g^{G_+})\pi(T_{g^{-1}}^{G_+})\cdot\pi(T_g^{G_+})=1.$$

Therefore $h^{-1}g\in E$, so $H^{-1}\cdot E\subseteq E$. ∎

PROPOSITION 4.10. *Let (G,G_+) be a quasi-lattice ordered group, I a reduced ideal of $\mathcal{T}^{G_+}$, and $E=G_+\cdot H^{-1}$ as in Lemma* 4.9. *Then* $\operatorname{Ind}(I\cap D^{G_+})=\operatorname{Ker}\gamma^{E,G_+}$.

Proof. First we prove that $K(I)=K\big(\operatorname{Ker}\gamma^{E,G_+}\big)$. In fact, since I is reduced, $K(I)$ is generated by $\{1-T_g^{G_+}T_{g^{-1}}^{G_+}:g\in H\}$. Similarly, since $\operatorname{Ker}\gamma^{E,G_+}$ is reduced and $E\cap G_+^{-1}=H^{-1}$, $K\big(\operatorname{Ker}\gamma^{E,G_+}\big)$ is also generated by $\{1-T_g^{G_+}T_{g^{-1}}^{G_+}:g\in H\}$. Therefore $K(I)=K\big(\operatorname{Ker}\gamma^{E,G_+}\big)$.

Next since θ^E is faithful, by (4.7) we know for any $x\in\mathcal{T}^{G_+}$,

$$x^*x\in\operatorname{Ker}\gamma^{E,G_+}\Leftrightarrow\theta^{G_+}(x^*x)\in\operatorname{Ker}\gamma^{E,G_+}. \tag{4.10}$$

By Lemma 4.1 and (4.10), we know that for any $x\in\mathcal{T}^{G_+}$,

$$\begin{aligned}x\in\operatorname{Ind}(I\cap D^{G_+})&\Leftrightarrow\theta^{G_+}(x^*x)\in I\cap D^{G_+}\Leftrightarrow\theta^{G_+}(x^*x)\in K(I)\cap D^{G_+}\\&\Leftrightarrow\theta^{G_+}(x^*x)\in K\big(\operatorname{Ker}\gamma^{E,G_+}\big)\cap D^{G_+}\Leftrightarrow\theta^{G_+}(x^*x)\in\operatorname{Ker}\gamma^{E,G_+}\cap D^{G_+}\\&\Leftrightarrow\theta^{G_+}(x^*x)\in\operatorname{Ker}\gamma^{E,G_+}\Leftrightarrow x^*x\in\operatorname{Ker}\gamma^{E,G_+}\Leftrightarrow x\in\operatorname{Ker}\gamma^{E,G_+}.\end{aligned}$$

Therefore, $\operatorname{Ind}\,(I\cap D^{G_+})=\operatorname{Ker}\gamma^{E,G_+}$. ∎

REMARK 4.11. Let (G, E) be a quasi-lattice quasi-ordered group and I an ideal of $\mathcal{T}^{G_+}$ such that $\overline{\operatorname{Ker}\gamma^{E,G_+} \cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|} \subseteq I \subseteq \operatorname{Ker}\gamma^{E,G_+}$. Then by (4.10) we have $\operatorname{Ind}(\operatorname{Ker}\gamma^{E,G_+} \cap D^{G_+}) = \operatorname{Ker}\gamma^{E,G_+}$, thus I is diagonal invariant by Theorem 4.4. Moreover, for any $g_1, g_2, \ldots, g_n \in G_+$,

$$\prod_{i=1}^{n}\big(1 - T_{g_i}^{G_+}T_{g_i^{-1}}^{G_+}\big) \in I \Leftrightarrow \prod_{i=1}^{n}\big(1 - T_{g_i}^{G_+}T_{g_i^{-1}}^{G_+}\big) \in \operatorname{Ker}\gamma^{E,G_+}.$$

It follows that I is also reduced. Applying Theorem 4.4 and Proposition 4.10, we obtain the following corollary:

COROLLARY 4.12. *Let (G, G_+) be a quasi-lattice ordered group, I an ideal of $\mathcal{T}^{G_+}$. Then I is reduced and diagonal invariant if and only if there exists some quasi-lattice quasi-ordered group (G, E), such that $\overline{\operatorname{Ker}\gamma^{E,G_+} \cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|} \subseteq I \subseteq \operatorname{Ker}\gamma^{E,G_+}$.*

5. THE MINIMAL IDEALS OF TOEPLITZ ALGEBRAS ON QUASI-LATTICE ORDERED GROUPS

In this section we study minimal ideals of Toeplitz algebras. Specifically, in the case that (G, G_+) is an ordered group, we characterize the existence of a minimal ideal in $\mathcal{T}^{G_+}$ in terms of the existence of a minimal quasi-ordered group containing (G, G_+). Further, we show that these minimal ideals are "almost" kernels of some natural morphisms between Toeplitz algebras. This extends the results of [8] and [19] to the non-abelian setting. Finally, we discuss the situation when (G, G_+) is a quasi-lattice ordered group, studying minimal ideals in terms of the topological space Ω consisting of all directed and hereditary subsets of G_+.

As in Section 4, throughout this section all ideals are intended to be non-trivial, proper, closed, and two-sided.

DEFINITION 5.1. ([12], Section 6]) Let (G, G_+) be a quasi-lattice ordered group. An ideal J of D^{G_+} is said to be *invariant* if $\alpha_{s,t}(J) \subseteq J$ for any $s, t \in G_+$, where $\alpha_{s,t} : D^{G_+} \to D^{G_+}$ is defined by $\alpha_{s,t}(x) = (T_s^{G_+}T_{t^{-1}}^{G_+}) \cdot x \cdot (T_s^{G_+}T_{t^{-1}}^{G_+})^*$ for any $x \in D^{G_+}$.

As stated in Section 4, any invariant ideal J of D^{G_+} induces an ideal $\operatorname{Ind} J$ of $\mathcal{T}^{G_+}$, where $\operatorname{Ind} J = \big\{T \in \mathcal{T}^{G_+} : \theta^{G_+}(T^*T) \in J\big\}$.

PROPOSITION 5.2. *Let (G, G_+) be a quasi-lattice ordered group. Then $\mathcal{T}^{G_+}$ contains a minimal ideal if and only if D^{G_+} contains a minimal invariant ideal. Furthermore there is a correspondence between these two minimal ideals, which can be stated as follows:*

(i) *If I is the minimal ideal of $\mathcal{T}^{G_+}$, then $I \cap D^{G_+}$ is the minimal invariant ideal of D^{G_+}.*

(ii) *If J is the minimal invariant of D^{G_+}, then $\overline{\operatorname{Ind} J \cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|}$ is the minimal ideal of $\mathcal{T}^{G_+}$.*

Proof. Let I be the minimal ideal of $\mathcal{T}^{G_+}$. Corollary 3.6 implies that $I \cap D^{G_+}$ is not trivial, therefore it is an invariant ideal of D^{G_+}. Since I is minimal, we know $I \subseteq \operatorname{Ind} J$ for any invariant ideal J of D^{G_+}. So for any $x \in I \cap D^{G_+}$, $x^*x = \theta^{G_+}(x^*x) \in J$, therefore $x \in J$. It follows that $I \cap D^{G_+}$ is the minimal invariant ideal of D^{G_+}.

On the other hand, let J be the minimal invariant ideal of D^{G_+}. Then for any ideal I of $\mathcal{T}^{G_+}$, $J \subseteq I \cap D^{G_+}$, so

$$\overline{\operatorname{Ind} J \cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|} \subseteq \overline{\operatorname{Ind}(I \cap D^{G_+}) \cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|} \subseteq I.$$

Therefore, $\overline{\operatorname{Ind} J \cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|}$ is the minimal ideal of $\mathcal{T}^{G_+}$. ∎

DEFINITION 5.3. ([18], Section 1) Let G be a discrete group, E a semigroup of G such that $e \in E$. If $G = E \cup E^{-1}$, then (G, E) is called a *quasi-ordered group*. Further, (G, E) is said to be *non-trivial* if $E \cap E^{-1} \neq \{e\}$.

REMARK 5.4. (i) When $E \cap E^{-1} = \{e\}$, then the quasi-ordered group (G, E) is an ordered group. On the other hand, if (G, G_+) is an ordered group and E is a semigroup of G such that $G_+ \subseteq E$, then (G, E) is a quasi-ordered group.

(ii) For any two quasi-ordered groups (G, E_1) and (G, E_2) with $E_1 \subseteq E_2$, by Proposition 1.4 of [18] we know that the natural morphism γ^{E_2, E_1} exists.

(iii) Let (G, G_+) be an ordered group. For any $g \in G_+ \setminus \{e\}$, let

$$G_F = \bigcap_{g \in G_+ \setminus \{e\}} G_g,$$

where G_g is the semigroup of G generated by G_+ and g^{-1}. If $G_F \neq G_+$, then (G, G_F) will be the minimal non-trivial quasi-ordered group containing (G, G_+).

PROPOSITION 5.5. *Let (G, G_+) be an ordered group, and G_F as in part* (iii) *of Remark* 5.4. *The following are equivalent:*

(i) *$\mathcal{T}^{G_+}$ has a minimal ideal;*

(ii) *There is a minimal non-trivial quasi-ordered group containing (G, G_+);*

(iii) *$G_F \neq G_+$.*

Proof. Assume that $\mathcal{T}^{G_+}$ has minimal ideal I. Observe that I must also be reduced. Thus, by Lemma 4.3 and Proposition 4.10, we know that

$$I = \overline{\operatorname{Ind}(I \cap D^{G_+}) \cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|} = \overline{\operatorname{Ker} \gamma^{E, G_+} \cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|}$$

for some non-trivial quasi-ordered group (G, E).

Let (G, M) be any non-trivial quasi-ordered group such that $G_+ \subset M$. For any $x \in E$,

$$1 - T^{G_+}_{x^{-1}} T^{G_+}_x \in \operatorname{Ker} \gamma^{E, G_+} \cap \mathcal{T}^\infty(G_+) \subseteq I \subseteq \operatorname{Ker} \gamma^{M, G_+}$$

since I is minimal. So $1 - T^M_{x^{-1}} T^M_x = 0$, that is, $x \in M$ and therefore $E \subseteq M$. So (G, E) is the minimal non-trivial quasi-ordered group, and hence (i) implies (ii), which in turn implies (iii) by part (iii) of Remark 5.4.

Now, for (iii) implies (i), suppose that $G_F \neq G_+$. For any ideal I of $\mathcal{T}^{G_+}$, there exists a non-trivial quasi-ordered group (G, E) such that

$$\overline{\operatorname{Ker} \gamma^{E, G_+} \cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|} = \overline{\operatorname{Ind}(I \cap D^{G_+}) \cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|} \subseteq I.$$

Since $\gamma^{E, G_F} \circ \gamma^{G_F, G_+} = \gamma^{E, G_+}$, we know that $\operatorname{Ker} \gamma^{G_F, G_+} \subseteq \operatorname{Ker} \gamma^{E, G_+}$, and thus $\overline{\operatorname{Ker} \gamma^{G_F, G_+} \cap \mathcal{T}^\infty(G_+)}^{\|\cdot\|}$ is the minimal ideal of $\mathcal{T}^{G_+}$. ∎

Throughout the rest of this section, (G, G_+) is a quasi-lattice ordered group.

We now clarify minimal ideals in terms of invariant subsets of Ω, where

$$\Omega = \{A : A \text{ is a hereditary and directed subset of } G_+\}.$$

When endowed with the topology inherited from $\{0,1\}^{G_+}$ by identifying subsets of G_+ with their characteristic functions, Ω becomes a compact Hausdorff space, and a net $\{A_\lambda\}$ converges to A if and only if $\chi_{A_\lambda}(t)$ converges to $\chi_A(t)$ for any $t \in G_+$ ([12], Section 6).

Let $\widehat{D}^{G_+}$ be the maximal ideal space of D^{G_+}. For any $\gamma \in \widehat{D}^{G_+}$, let

$$A_\gamma = \{t : t \in G_+, \text{ such that } \gamma(T_t^{G_+} T_{t^{-1}}^{G_+}) = 1\}.$$

Then $A_\gamma \in \Omega$, and by Proposition 6.2 of [12] we know that $\Delta : \widehat{D}^{G_+} \to \Omega, \Delta(\gamma) = A_\gamma$ is a homeomorphism. Moreover for any $t \in G_+$, $\Delta(\langle \cdot \delta_t, \delta_t \rangle) = [e,t] = \{s \in G_+ : s \leqslant t\}$, and the subset $\{[e,t] : t \in G_+\}$ is dense in Ω.

Since $\widehat{D}^{G_+} \cong \Omega$, naturally we can identify D^{G_+} and $C(\Omega)$ in the sense that for any $x \in D^{G_+}$ and $A \in \Omega$, $x(A) = \big(\Delta^{-1}(A)\big)(x)$. In particular, $(T_t^{G_+} T_{t^{-1}}^{G_+})(A) = \chi_A(t)$ for any $t \in G_+$ and $A \in \Omega$.

Now because $D^{G_+} \cong C(\Omega)$, every ideal of D^{G_+} corresponds with a closed subset of Ω, and when the ideal is invariant, the closed subset should also be "invariant", which can be defined as follows:

DEFINITION 5.6. (Section 6 in [12] or Sections 2 and 3 of [1]) For any $t \in G_+$ and $A \in \Omega$, let

$$\theta_t(A) = [e, tA] \stackrel{\triangle}{=} \{y \in G_+ : \exists\, a \in A, \text{ such that } y \leqslant ta\}.$$

A subset K of Ω is said to be *invariant* if $\theta_t(K) \subseteq K$ and $\theta_t(\Omega \setminus K) \subseteq \Omega \setminus K$ for any $t \in G_+$.

REMARK 5.7. (i) It is easy to check that $\theta_t(A) \in \Omega$ for any $t \in G_+$ and $A \in \Omega$. For any $t \in G_+$, let $\Omega_t = \{B \in \Omega : t \in B\}$. Then Ω_t is a closed and open subset of Ω, and if Ω_t is endowed with the induced topology, then θ_t is a homeomorphism from Ω onto Ω_t with inverse given by $\theta_t^{-1}(B) = \{t^{-1}\sigma(t,b) : b \in B\}$ ([5], Proposition 2.2).

(ii) The proof of Theorem 3.7 from [5] implies that if K is an invariant subset of Ω, then its closure $\overline{K}$ will also be an invariant subset of Ω.

(iii) By Proposition 3.2 of [5] we know that a closed subset K of Ω is invariant if and only if the associated ideal $I_K = \{x \in D^{G_+} : x(A) = 0, \forall\, A \in K\}$ is invariant. So D^{G_+} contains a minimal invariant ideal if and only if Ω contains a maximal closed invariant proper subset.

For any $A \in \Omega$, let

$$S(A) = \{\theta_{s_n}^{-1} \circ \theta_{t_n} \circ \cdots \circ \theta_{s_1}^{-1} \circ \theta_{t_1}(A) : \text{whenever } \theta_{s_i}^{-1} \text{ is meaningful}\}.$$

Note when s_n or t_n equals to e, $\theta_{s_n}^{-1}$ or θ_{t_n} will be the identity morphism. $S(A)$ is the smallest invariant subset of Ω containing A.

PROPOSITION 5.8. *Let (G, G_+) be a quasi-lattice ordered group. Denote by $M(\Omega) = \{A : \overline{S(A)} \neq \Omega\}$. Then Ω contains a maximal closed invariant proper subset if and only if $\emptyset \neq M(\Omega) \subseteq \overline{M(\Omega)} \neq \Omega$.*

Proof. Suppose Γ is maximal closed invariant proper subset of Ω. For any $A \in \Gamma$, we have $\overline{S(A)} \subseteq \overline{\Gamma} = \Gamma \neq \Omega$, and so $M(\Omega) \neq \emptyset$.

Next, for any $A \in M(\Omega)$, $\overline{S(A)}$ is a closed invariant proper subset of Ω, so $A \in S(A) \subseteq \overline{S(A)} \subseteq \Gamma$, therefore $\overline{M(\Omega)} \subseteq \Gamma \neq \Omega$.

Now suppose that $\emptyset \neq M(\Omega) \subseteq \overline{M(\Omega)} \neq \Omega$. For any $A \in \Omega$, $t \in G_+$ and $s \in A$, we know that $S(\theta_t(A)) = S(A) = S(\theta_s^{-1}(A))$. It follows that $M(\Omega)$ is an invariant subset of Ω, which implies that $\overline{M(\Omega)}$ is a closed invariant proper subset of Ω.

Now let K be any closed invariant proper subset of Ω. For any $B \in \Omega$, if $B \notin M(\Omega)$, i.e., $\overline{S(B)} = \Omega$, then $B \notin K$. Therefore $K \subseteq M(\Omega)$, so in this case $M(\Omega) = \overline{M(\Omega)}$ is in fact the maximal closed invariant proper subset of Ω. ∎

REMARK 5.9. (i) For any $s, t \in G_+$, $\theta_s([e, t]) = [e, st]$, so $S([e, e]) = \{[e, t] : t \in G_+\}$. Since $S([e, e])$ is dense in Ω, $[e, t] \notin M(\Omega)$ for any $t \in G_+$.

(ii) If $[e, e]$ is open, equivalently, $[e, t]$ is open for any $t \in G_+$ (since θ_t is a homeomorphism), then $\{[e, t] : t \in G_+\} = \bigcup_{t \in G_+} \{[e, t]\}$ is open, so $\Omega \setminus \{[e, t] : t \in G_+\}$ is the maximal closed invariant proper subset of Ω. By Proposition 6.3 from [12] we know that in this case the minimal ideal of $\mathcal{T}^{G_+}$ equals to the ideal of compact operators on $\ell^2(G_+)$.

6. THE UNIVERSAL PROPERTY OF TOEPLITZ ALGEBRAS ON QUASI-LATTICE QUASI-ORDERED GROUPS

In this section, we study the universal property of Toeplitz algebras. Previous work on this topic may be found in [8], [12] and [19] in cases when the underlying groups are abelian ordered groups, quasi-lattice ordered groups and quasi-ordered groups, respectively.

Throughout this section, (G, G_+) is a quasi-lattice ordered group, H is a hereditary and directed subset of G_+, and $E = G_+ \cdot H^{-1}$ such that (G, E) is a quasi-lattice quasi-ordered group. We also assume that $\{e\} \subsetneqq H \subsetneqq G_+$. Since the natural morphism γ^{E,G_+} is a surjective C^*-morphism, $\mathcal{T}^\infty(E) = \operatorname{span}\{T_g^E T_{h^{-1}}^E : g, h \in G_+\}$ is a dense $*$-algebra of $\mathcal{T}^E$. In this section, we will prove that $\mathcal{T}^\infty(E)$ has a universal property for the covariant isometric representations of E. In the particular case when G is amenable, we prove that $\mathcal{T}^E$ also has such a universal property, and we clarify a particular covariant isometric representation of E, see Proposition 6.7.

PROPOSITION 6.1 (cf. Proposition 3.2, [12]) *The operators* $\{T^E_s T^E_{t^{-1}} : s, t \in G_+\}$ *are linear independent in the sense that, if*

$$\sum_j \lambda_j T^E_{s_j} T^E_{t_j^{-1}} = 0 \quad \text{with } T^E_{s_{j_1}} T^E_{t_{j_1}^{-1}} \neq T^E_{s_{j_2}} T^E_{t_{j_2}^{-1}} \text{ whenever } j_1 \neq j_2,$$

then $\lambda_j = 0$ *for all* j.

Proof. Let $E^0 = E \cap E^{-1} = H \cdot H^{-1}$. For any $s_1, s_2, t_1, t_2 \in G_+$, it is easy to show that $T^E_{s_1} T^E_{t_1^{-1}} = T^E_{s_2} T^E_{t_2^{-1}}$ if and only if $t_1^{-1} t_2 \in E^0$ and $s_1 t_1^{-1} = s_2 t_2^{-1}$.

Now suppose there exist $\lambda_1, \lambda_2, \ldots, \lambda_n \in C \setminus \{0\}$ such that $\sum\limits_j \lambda_j T^E_{s_j} T^E_{t_j^{-1}} = 0$ with the property that $T^E_{s_{j_1}} T^E_{t_{j_1}^{-1}} \neq T^E_{s_{j_2}} T^E_{t_{j_2}^{-1}}$ whenever $j_1 \neq j_2$. Since $F = \{j : j = 1, 2, \ldots, n\}$ is finite, there exists j_0, such that $t_j^{-1} t_{j_0} \in E$ implies $t_j^{-1} t_{j_0} \in E^0$ for all $j = 1, 2, \ldots, n$. Since $\Big(\sum\limits_j \lambda_j T^E_{s_j} T^E_{t_j^{-1}}\Big)\delta_{t_{j_0}} = 0$ and $\lambda_{t_{j_0}} \neq 0$, we know that there must be some k, such that $t_k^{-1} t_{j_0} \in E$ and $s_k t_k^{-1} = s_{j_0} t_{j_0}^{-1}$. Therefore, $T^E_{s_k} T^E_{t_k^{-1}} = T^E_{s_{j_0}} T^E_{t_{j_0}^{-1}}$, which is a contradiction. ∎

DEFINITION 6.2. Let B be a unital C^*-algebra and $V : G_+ \to B$ a covariant isometric representation of G_+. If $V(g)V(g)^* = 1$ for any $g \in H$, then V is said to be a *covariant isometric representation of* E.

PROPOSITION 6.3. *Let* $V : G_+ \to B$ *be a covariant isometric representation of* E. *Then there is a natural* $*$*-morphism* π_V *from* $\mathcal{T}^\infty(E)$ *to* B *such that* $\pi_V(T^E_s T^E_{t^{-1}}) = V(s)V(t)^*$ *for any* $s, t \in G_+$.

Proof. First we prove that π_V is well-defined. More precisely, for any $s_1, s_2, t_1, t_2 \in G_+$ such that $t_1^{-1} t_2 \in E^0$ and $s_1 t_1^{-1} = s_2 t_2^{-1}$, we prove that $V(s_1)V(t_1)^* = V(s_2)V(t_2)^*$. Put $x = s_1 t_t^{-1} = s_2 t_2^{-1}$. Since $x = s_1 t_1^{-1} \leqslant s_1$, it follows that $s_1 = \sigma(x)\, a$ and $t_1 = \tau(x)\, a$ for some $a \in G_+$. Similarly, there exists some $b \in G_+$ such that $s_2 = \sigma(x)\, b$ and $t_2 = \tau(x)\, b$. Let $t_1^{-1} t_2 = h_1 h_2^{-1}$ for some $h_1, h_2 \in H$. Then $t_1 h_1 = t_2 h_2$, so

$$\begin{aligned}
V(a)V(a)^* &= V(\tau(x))^*\big[V(\tau(x))V(a)V(a)^*V(\tau(x))^*\big]V(\tau(x)) \\
&= V(\tau(x))^*\big[V(t_1)V(t_1)^*\big]V(\tau(x)) \\
&= V(\tau(x))^*\big[V(t_1)V(h_1)V(h_1)^*V(t_1)^*\big]V(\tau(x)) \\
&= V(\tau(x))^*\big[V(t_1h_1)V(t_1h_1)^*\big]V(\tau(x)) \\
&= V(\tau(x))^*\big[V(t_2h_2)V(t_2h_2)^*\big]V(\tau(x)) \\
&= V(\tau(x))^*\big(V(t_2)V(t_2)^*\big)V(\tau(x)) \\
&= V(\tau(x))^*\big[V(\tau(x))V(b)V(b)^*V(\tau(x))^*\big]V(\tau(x)) \\
&= V(b)V(b)^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V(s_1)V(t_1)^* &= V(\sigma(x))V(a)V(a)^*V(\tau(x))^* \\
&= V(\sigma(x))V(b)V(b)^*V(\tau(x))^* \\
&= V(s_2)V(t_2)^*.
\end{aligned}$$

By Proposition 6.1 we know there is a linear operator π_V from $\mathcal{T}^\infty(E)$ to B, such that

$$\pi_V\left(\sum_j \lambda_j T^E_{s_j} T^E_{t_j^{-1}}\right) = \sum_j \lambda_j V(s_j)V(t_j)^*$$

for any $\lambda_j \in C$ and $s_j, t_j \in G_+$. Obviously, π_V preserves the $*$-operation, and the proof will be finished if we show that $\pi_V\big(T^E_{s_1}T^E_{t_1^{-1}} \cdot T^E_{s_2}T^E_{t_2^{-1}}\big) = \pi_V\big(T^E_{s_1}T^E_{t_1^{-1}}\big) \cdot \pi_V\big(T^E_{s_2}T^E_{t_2^{-1}}\big)$ for any $s_i, t_i \in G_+$, $i = 1, 2$. Suppose $a = s_1 t_1^{-1}\sigma(t_1, s_2)$, $b = t_2 s_2^{-1}\sigma(t_1, s_2) \in G_+ \cup \{\infty\}$. By Section 3.2 from [12] we know that

$$T^E_{s_1}T^E_{t_1^{-1}} \cdot T^E_{s_2}T^E_{t_2^{-1}} = T^E_a T^E_{b^{-1}},$$

so it is sufficient to show that

$$V(s_1)V(t_1)^* \cdot V(s_2)V(t_2)^* = V(a)V(b)^*.$$

Suppose first that t_1 and s_2 have a common upper bound in G_+, then by (2.4),

$$\begin{aligned}
&V(s_1)V(t_1)^*V(s_2)V(t_2)^*\\
&\quad= V(s_1)V(t_1)^*\big(V(t_1)V(t_1)^*V(s_2)V(s_2)^*\big)V(s_2)V(t_2)^*\\
&\quad= V(s_1)V(t_1)^*V(\sigma(t_1, s_2))V(\sigma(t_1, s_2))^*V(s_2)V(t_2)^*\\
&\quad= V(s_1)V(t_1)^*V(t_1)V(t_1^{-1}\sigma(t_1, s_2))V(s_2^{-1}\sigma(t_1, s_2))^*V(s_2)^*V(s_2)V(t_2)^*\\
&\quad= V(s_1)V(t_1^{-1}\sigma(t_1, s_2))V(s_2^{-1}\sigma(t_1, s_2))^*V(t_2)^*\\
&\quad= V(a)V(b)^*.
\end{aligned}$$

On the other hand, if t_1 and s_2 have no common upper bound in G_+, then again by (2.4), $V(t_1)^*V(s_2) = V(t_1)^*\big(V(t_1)V(t_1)^*V(s_2)V(s_2)^*\big)V(s_2) = 0$. ∎

Note that by Proposition 6.3 we can define, as in Section 4, [12], a universal C^*-algebra $C^*(G, E)$ for such a pair (G, E). Also, the morphism π_V discussed above may not be bounded. However, the following theorem indicates that if G is amenable, then π_V may be extended as a C^*-morphism from $\mathcal{T}^E$ to B.

THEOREM 6.4. *Suppose that G is amenable. Then for any covariant isometric representation $V : G_+ \to B$ of E, there is a C^*-morphism π_V from $\mathcal{T}^E$ to B such that $\pi_V(T^E_s T^E_{t^{-1}}) = V(s)V(t)^*$ for any $s, t \in G_+$.*

Proof. Since G is amenable and (V, B) is also a covariant isometric representation of G_+, by Section 4 from [12] we know there is a C^*-morphism $\theta_V : \mathcal{T}^{G_+} \to B$ such that $\theta_V(T^{G_+}_s T^{G_+}_{t^{-1}}) = V(s)V(t)^*$ for any $s, t \in G_+$. Since $\operatorname{Ker}\gamma^{E,G_+}$ is reduced, by Proposition 4.10, Remark 4.5 and Lemma 4.3, we know that $\operatorname{Ker}\gamma^{E,G_+}$ is generated by $\{1 - T^{G_+}_g T^{G_+}_{g^{-1}} : g \in H\}$. By assumption $V(g)V(g)^* = 1$ for any $g \in H$, so $\operatorname{Ker}\gamma^{E,G_+} \subseteq \operatorname{Ker}\theta_V$. Therefore, $\pi_V : \mathcal{T}^E \longrightarrow \mathcal{T}^{G_+}/\operatorname{Ker}\gamma^{E,G_+} \xrightarrow{\overline{\theta}_V} B$ is bounded. ∎

We now construct a particular covariant isometric representation of E. Let $E^0 = E \cap E^{-1} = H \cdot H^{-1}$ and $G_1 = (E \setminus E^0) \cup \{e\}$. Then since $E \cdot (E \setminus E^0) \subseteq E \setminus E^0$ and $(E \setminus E^0) \cdot E \subseteq E \setminus E^0$, we know that G_1 is a semigroup. Let $K(\ell^2(G_1))$ be the ideal of compact operators on $\ell^2(G_1)$.

First we show that $K(\ell^2(G_1)) \subseteq \mathcal{T}^{G_1}$. Let $T \in B(\ell^2(G_1))$ such that $TS = ST$ for any $S \in \mathcal{T}^{G_1}$, we prove that $T = \lambda$ for some $\lambda \in C$, therefore $\mathcal{T}^{G_1}$ is irreducible. In fact, for any $t \in G_1 \setminus \{e\}$, $T^{G_1}_{t^{-1}} T\delta_e = TT^{G_1}_{t^{-1}}\delta_e = 0$, so $T\delta_e = \lambda\, \delta_e$ for some $\lambda \in C$. It follows that $T\delta_g = TT^{G_1}_g \delta_e = T^{G_1}_g T\delta_e = \lambda\, \delta_g$ for any $g \in G_1$, so $T = \lambda$. Choose any $x \in H \setminus \{e\}$, then obviously $1 - T^{G_1}_x T^{G_1}_{x^{-1}}$ is a projection of rank one. Since $\mathcal{T}^{G_1}$ is irreducible, we know that $K(\ell^2(G_1)) \subseteq \mathcal{T}^{G_1}$.

Next define $V : G_+ \to \mathcal{T}^{G_1} / K(\ell^2(G_1))$ by $V(g) = \left[T^{G_1}_g\right]$ for any $g \in G_+$. Note $G_+ \cdot (G_1 \setminus \{e\}) \subseteq E \cdot (E \setminus E^0) \subseteq G_1$, V is an isometric representation of G_+. Obviously, $V(h)V(h)^* = 1$ for any $h \in H$, so V will be a covariant isometric representation of E if for any $g, h \in G_+$,

$$(T^{G_1}_g T^{G_1}_{g^{-1}}) \cdot (T^{G_1}_h T^{G_1}_{h^{-1}}) - T^{G_1}_{\sigma(g,h)} T^{G_1}_{\sigma(g,h)^{-1}} \in K(\ell^2(G_1)). \tag{6.1}$$

LEMMA 6.5. *Suppose that H is infinite. Then equation* (6.1) *holds if and only if for any $x, y \in G_+$ with $x^{-1}y \in G_+ \cdot G_+^{-1}$, either $\tau(x^{-1}y) \in H$ or $\sigma(x^{-1}y) \in H$.*

Proof. First, we note that $(E \setminus E^0) \cdot E \subseteq E \setminus E^0$, and $E \cdot (E \setminus E^0) \subseteq E \setminus E^0$. This will be used frequently in the proof.

Let us suppose equation (6.1) holds. Then for any $x, y \in G_+$ with $x^{-1}y \in G_+ \cdot G_+^{-1}$, we show either $\tau(x^{-1}y) \in H$ or $\sigma(x^{-1}y) \in H$. In fact, if it is not true, then by Lemma 2.8 we know that $x^{-1}\sigma(x,y) \in G_+ \setminus H$ and $y^{-1}\sigma(x,y) \in G_+ \setminus H$. So $\sigma(x,y) = x \cdot (x^{-1}\sigma(x,y)) \subseteq E \cdot (E \setminus E^0) \subseteq E \setminus E^0$. Choose $\{h_1, h_2, \ldots\} \subseteq H \setminus \{e\}$ and let $z_n = \sigma(x,y) \cdot h_n^{-1}$ for $n \in N$. Then $z_n \in (E \setminus E^0) \cdot H^{-1} \subseteq E \setminus E^0 \subseteq G_1$. Let

$$T = (T^{G_1}_x T^{G_1}_{x^{-1}}) \cdot (T^{G_1}_y T^{G_1}_{y^{-1}}) - T^{G_1}_{\sigma(x,y)} T^{G_1}_{\sigma(x,y)^{-1}}. \tag{6.2}$$

Then $T\, \delta_{z_n} = \delta_{z_n}$ for any $n \in N$, so $T \notin K(\ell^2(G_1))$, which is a contradiction.

We now prove the sufficient part. For any $x, y \in G_+$, let T be as (6.2). We prove $T \in K(\ell^2(G_1))$. Note if x, y have no common upper bound in G_+, then for any $z \in G_1 \subseteq E$, $(T^{G_1}_x T^{G_1}_{x^{-1}}) \cdot (T^{G_1}_y T^{G_1}_{y^{-1}})\, \delta_z = 0$, otherwise, $(T^E_x T^E_{x^{-1}}) \cdot (T^E_y T^E_{y^{-1}})\, \delta_z = \delta_z$, which by Lemma 2.10 implies that x, y should have a common upper bound in G_+. So in this case $T = 0$.

Now suppose that $x, y \in G_+$ have a common upper bound in G_+. By assumption, either $\tau(x^{-1}y) \in H$ or $\sigma(x^{-1}y) \in H$. Since

$$T^{G_1}_x T^{G_1}_{x^{-1}} T^{G_1}_y T^{G_1}_{y^{-1}} = T^{G_1}_y T^{G_1}_{y^{-1}} T^{G_1}_x T^{G_1}_{x^{-1}}$$

and $\sigma(x,y) = \sigma(y,x)$, without loss of generality we may assume that $\tau(x^{-1}y) \in H$. We prove that T is contained in $K(\ell^2(G_1))$ by showing that $T\delta_q = 0$ for all but finitely many $q \in G_1$.

First, we observe from Lemma 2.8 that

$$T = T^{G_1}_x T^{G_1}_{x^{-1}} T^{G_1}_y T^{G_1}_{y^{-1}} - T^{G_1}_{x\sigma(x^{-1}y)} T^{G_1}_{\tau(x^{-1}y)^{-1}y^{-1}}. \tag{6.3}$$

Also, given $q \in E \setminus E^0$, the facts that $E \cdot (E \setminus E^0) \subseteq (E \setminus E^0)$ and $\sigma(x^{-1}y) \cdot \tau(x^{-1}y)^{-1} = x^{-1}y$ imply

$$\tau(x^{-1}y)^{-1}y^{-1}q \in E \setminus E^0 \Leftrightarrow y^{-1}q \in E \setminus E^0 \text{ and } x^{-1}q \in E \setminus E^0. \tag{6.4}$$

From (6.3), (6.4), and the fact that $G_1 = \{e\} \cup (E \setminus E^0)$, we see that if $T\delta_q \neq 0$, then $q \in \{y\tau(x^{-1}y), y, x\}$. Therefore T is compact. ∎

LEMMA 6.6. *For any finite subset F of $G_+ \setminus H$, there exists some element $x \in F$, such that for any $g \in F$ and $h \in H \setminus \{e\}$, $g^{-1}xh^{-1} \notin E \setminus E^0$.*

Proof. We prove this property by induction. If F just contains one point, then since $(H \setminus \{e\})^{-1} \cap G_1 = \emptyset$, we know in this case, the conclusion holds. Suppose this property holds for any subset of $G_+ \setminus H$ with $n-1$ elements ($n \geqslant 2$). Let $F = \{g_1, g_2, \ldots, g_n\}$ be any subset of $G_+ \setminus H$ with n elements. First we note that if there exist two elements x, y in F such that $x^{-1}y \in E^0$, then by induction we know there exists $z \in F \setminus \{x\}$, such that $g^{-1}zh^{-1} \notin E \setminus E^0$ for any $g \in F \setminus \{x\}$ and $h \in H \setminus \{e\}$. In particular, $y^{-1}zh^{-1} \notin E \setminus E^0$, so $x^{-1}zh^{-1} = (x^{-1}y) \cdot y^{-1}zh^{-1} \notin E \setminus E^0$. For, if $y^{-1}zh^{-1} \notin E$, then $x^{-1}zh^{-1} \in E^0 \cdot (G \setminus E) \subseteq G \setminus E$; otherwise, $y^{-1}zh^{-1} \in E^0$, so $x^{-1}zh^{-1} \in E^0 \cdot E^0 = E^0$. Therefore, we may assume that $x^{-1}y \notin E^0$ for any $x, y \in F$.

Case 1 $(n-1:0)$. $g_2^{-1}g_1 \notin E, g_3^{-1}g_1 \notin E, \ldots, g_n^{-1}g_1 \notin E$. In this case, if we choose $x = g_1$, then since $(G \setminus E) \cdot E^0 \subseteq G \setminus E$, the conclusion holds.

Case 2 $(n-2:1)$. $g_2^{-1}g_1 \notin E, g_3^{-1}g_1 \notin E, \ldots, g_{n-1}^{-1}g_1 \notin E$, $g_n^{-1}g_1 \in E$. By assumption $g_1^{-1}g_n \notin E$ (otherwise, $g_1^{-1}g_n \in E^0$), and moreover since E is a semigroup, $g_2^{-1}g_n \notin E$ (otherwise, $g_2^{-1}g_1 = (g_2^{-1}g_n)(g_n^{-1}g_1) \in E$, a contradiction). Similarly, $g_3^{-1}g_n \notin E, \ldots, g_{n-1}^{-1}g_n \notin E$. By Case 1, the conclusion holds.

Case 3 $(n-3:2)$. $g_2^{-1}g_1 \notin E, g_3^{-1}g_1 \notin E, \ldots, g_{n-2}^{-1}g_1 \notin E$, $g_{n-1}^{-1}g_1 \in E$, $g_n^{-1}g_1 \in E$. In this case, $g_1^{-1}g_{n-1} \notin E$, $g_2^{-1}g_{n-1} \notin E, \ldots, g_{n-2}^{-1}g_{n-1} \notin E$. If $g_n^{-1}g_{n-1} \notin E$, then reduces to Case 1; otherwise reduces to Case 2, so the conclusion holds.

$\vdots$

Case $n-1$ $(1:n-2)$. $g_2^{-1}g_1 \notin E, g_3^{-1}g_1 \in E, \ldots, g_n^{-1}g_1 \in E$. By the former process, it can eventually reduce to Case 1, so the conclusion holds. ∎

PROPOSITION 6.7. *Suppose that G is amenable, H is infinite, and condition* (6.1) *is satisfied. Then the induced C^*-morphism $\pi_V : \mathcal{T}^E \to \mathcal{T}^{G_1}/K(\ell^2(G_1))$ is injective.*

Proof. By Theorem 6.4, there is a C^*-morphism $\pi_V : \mathcal{T}^E \to \mathcal{T}^{G_1}/K(\ell^2(G_1))$, such that $\pi_V(T_s^E T_{t^{-1}}^E) = [T_s^{G_1} T_{t^{-1}}^{G_1}]$ for any $s, t \in G_+$. Let us first show that $\pi_V(T_x^E) = [T_x^{G_1}]$ for any $x \in E$. In fact, since $E \cdot (E \setminus E^0) \subseteq E \setminus E^0$, we know that for any $x \in E$, $T_x^{G_1} - T_{\sigma(x)}^{G_1} T_{\tau(x)^{-1}}^{G_1} \in K(\ell^2(G_1))$. By Proposition 2.11, we know that $T_x^E = T_{\sigma(x)}^E T_{\tau(x)^{-1}}^E$, so $\pi_V(T_x^E) = [T_{\sigma(x)}^{G_1} T_{\tau(x)^{-1}}^{G_1}] = [T_x^{G_1}]$.

Next in order to show that π_V is injective, by Theorem 3.5, we need to verify that for any finite collection $g_{11}, g_{12}, \ldots, g_{1m} \in G_+ \setminus H$, $g_{01}, g_{02}, \ldots, g_{0n} \in E^0 \setminus \{e\}$, and $\lambda_0, \lambda_1, \ldots, \lambda_n \in C$, the following inequality holds:

$$|\lambda_0| \leqslant \Big\| \prod_{j=1}^{m} \big(1 - L(g_{1j})\big)\Big(\lambda_0 + \sum_{i=1}^{n} \lambda_i V(g_{0i})\Big) \prod_{j=1}^{m} \big(1 - L(g_{1j})\big)\Big\|.$$

It is equivalent to show that

$$|\lambda_0| \leqslant \inf_{k \in K(\ell^2(G_1))} \|T + k\|,$$

where

$$T = \prod_{j=1}^{m} \big(1 - T^{G_1}_{g_{1j}} T^{G_1}_{g_{1j}^{-1}}\big)\Big(\lambda_0 + \sum_{i=1}^{n} \lambda_i T^{G_1}_{g_{0i}}\Big) \prod_{j=1}^{m} \big(1 - T^{G_1}_{g_{1j}} T^{G_1}_{g_{1j}^{-1}}\big).$$

Let $F = \{g_{11}, g_{12}, \ldots, g_{1m}\} \subseteq G_+ \setminus H$. By Lemma 6.6 we know there exists $x \in F$, such that $g^{-1}xh^{-1} \notin E \setminus E^0$ for any $g \in F$ and $h \in H \setminus \{e\}$. Since F is finite, but $H \setminus \{e\}$ is infinite, we can choose a countable infinite subset $\{h_p : p \in N\} \subseteq H \setminus \{e\}$ such that for any $g \in F$, $g^{-1}xh_p^{-1} \notin G_1$. Let $x_p = xh_p^{-1}$ for $p \in N$. Then

$$\begin{aligned}
\|T\,\delta_{x_p}\| &= \Big\|\lambda_0 \delta_{x_p} + \prod_{j=1}^{m} \big(1 - T^{G_1}_{g_{1j}} T^{G_1}_{g_{1j}^{-1}}\big)\Big(\sum_{i=1}^{n} \lambda_i \delta_{g_{0i}x_p}\Big)\Big\| \\
&= \Big\|\lambda_0 \delta_{x_p} + \sum_{i=1}^{n} \lambda_i \Big[\prod_{j=1}^{m} \big(1 - T^{G_1}_{g_{1j}} T^{G_1}_{g_{1j}^{-1}}\big)\Big] \delta_{g_{0i}x_p}\Big\| \\
&= \Big\|\lambda_0 \delta_{x_p} + \sum_{i'} \lambda_{i'} \delta_{g_{0i'}x_p}\Big\| \geqslant |\lambda_0|,
\end{aligned}$$

where $i' \in \{1, 2, \ldots, n\}$ such that $g_{1j}^{-1} g_{0i'} x_p \notin G_1$ for all $j = 1, 2, \ldots, m$. So for any $k \in K(\ell^2(G_1))$,

$$\|(T + k)(\delta_{x_p})\| \geqslant \|T(\delta_{x_p})\| - \|k(\delta_{x_p})\| \geqslant |\lambda_0| - \|k(\delta_{x_p})\|.$$

Since $\delta_{x_p} \to 0$ weakly in $\ell^2(G_1)$, by the compactness of k we know that $\|k(\delta_{x_p})\| \to 0$. It follows that for all $k \in K\,(\ell^2(G_1))$

$$\|T + k\| \geqslant \sup\,\{\|(T + k)(\delta_{x_p})\| : p \in N\} \geqslant |\lambda_0|,$$

so $|\lambda_0| \leqslant \inf_{k \in K(\ell^2(G_1))} \|T + k\|$. ∎

REMARK 6.8. A special case of the preceding proposition was obtained in Theorem 3.2, [16], which can be understood as follows:

Suppose that G is abelian, then by Lemma 6.5 we know that (6.1) holds if and only if, for any $x \in G_+ \cdot G_+^{-1}$, either $\tau(x) \in H$ or $\sigma(x) \in H$, if and only if $G_+ \cdot G_+^{-1} = E \cup E^{-1}$. Moreover if (6.1) holds, then for any $x \in G_+ \cdot G_+^{-1}$, $T \in K(\ell^2(G_1))$, where $T = T^{G_1}_x - T^{G_1}_{\sigma(x)} T^{G_1}_{\tau(x)^{-1}}$. For, if $\tau(x) \in H$, then for any $z \in G_1 \setminus \{e\}$, $\tau(x)^{-1}z \in G_1$, so $T\delta_z = 0$, therefore $T \in K(\ell^2(G_1))$. If, however, $\sigma(x) \in H$, then $T^* \in K(\ell^2(G_1))$, so $T \in K(\ell^2(G_+))$. So in this case, if the induced C^*-morphism π_V exists, then it is surjective. Therefore, by Proposition 6.7, we know if G is abelian, H is infinite, and $(G_+ \cdot G_+^{-1}, E)$ is a quasi-ordered group, then the induced C^*-morphism $\pi_V : \mathcal{T}^E \to \mathcal{T}^{G_1}/K, \ell^2(G_1)$, is an isomorphism.

Acknowledgements. This work was completed while the second author was visiting the Ball State University. He thanks Ball State University for its hospitality.

The second author's research is supported by the National Natural Science Foundation of China (no. 19901019), and the Science and Technology Foundation of Shanghai Higer Education (no. 01D01-3).

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Received May 20, 2001.