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AN INTRINSIC DIFFICULTY WITH INTERPOLATION ON THE BIDISK

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ABSTRACT. The set of possible values $(w_1, \dots, w_k) = (f(x_1), \dots, f(x_k))$ arising from restricting contractive elements f from some uniform algebra A to a finite set $\{x_1, \dots, x_k\}$ in the domain is called an interpolation body. When the uniform algebra is the bidisk algebra, Cole and Wermer show that the associated interpolation body is a semi-algebraic set and it is in this sense that the interpolation body is "computable". Motivated by the work of Cole and Wermer, Paulsen introduced the notion of the Schur ideal which acts a natural "dual" object for these interpolation bodies. From this "duality" a stronger notion of "computability" follows which will allow us to discuss the intrinsic differences between interpolation on the bidisk and interpolation on the disk.

KEYWORDS: *Interpolation, k-idempotent operator algebras, Schur ideals.*

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1. INTRODUCTION

Let X be compact Hausdorff space and let $C(X)$ denote the continuous complex-valued functions on X. We call $A \subseteq C(X)$ a *uniform algebra* provided that A is uniformly closed, contains the identity, and separates points in X. For fixed points $x_1, \dots, x_k$ in X, B. Cole, K. Lewis, and J. Wermer define in [5] the *interpolation body* associated with A and $x_1, \dots, x_k$, denoted $\mathcal{D}(A;\, x_1, \dots, x_k)$, in the following way: a point $\vec{w} = (w_1, \dots, w_k)$ in $\mathbb{C}^k$ belongs to $\mathcal{D}(A;\, x_1, \dots, x_k)$ if for each $\varepsilon > 0$, there exists f in A with $\|f\|_\infty \leqslant 1 + \varepsilon$ such that $f(x_i) = w_i$ for $i = 1, \dots, k$. Equivalently, a point $(w_1, \dots, w_k)$ in $\mathbb{C}^k$ belongs to $\mathcal{D}(A;\, x_1, \dots, x_k)$ if $\|w_1[f_1] + \cdots + w_k[f_k]\| \leqslant 1$, where $[f_i]$ is in A/I_x, $f_i(x_j) = \delta_{ij}$, and I_x is the ideal of functions in A vanishing at the points $x_1, \dots, x_k$. With this point of view, one can see that an interpolation body $\mathcal{D}(A;\, x_1, \dots, x_k)$ is a natural coordinization of the closed unit ball of the quotient algebra (k-idempotent operator algebra) A/I_x.

G. Pick has solved the classic interpolation problem from $\mathbb{D}$ to $\mathbb{D}$ (open unit disk) in 1916. In [11], G. Pick has showed that $(w_1, \ldots, w_k)$ in $\mathbb{C}^k$ is in $\mathcal{D}(A(\mathbb{D});\, \alpha_1, \ldots, \alpha_k)$ if and only if the matrix $\left(\frac{1-\overline{w}_i w_j}{1-\overline{\alpha}_i \alpha_j}\right)$ is positive semi-definite, where $A(\mathbb{D})$ denotes the disk algebra. Note, that G. Pick's theorem yields the same set of finite conditions for determining if an arbitrary k-tuple $(w_1, \ldots, w_k)$ in $\mathbb{C}^k$ belongs to $\mathcal{D}(A(\mathbb{D});\, \alpha_1, \ldots, \alpha_k)$. The author would like to accentuate that the goal of this paper resides in finding a finite (or possibly infinite) number of conditions which will allow us to determine the ball $\mathcal{D}(A;\, x_1, \ldots, x_k)$ and more specifically the ball $\mathcal{D}(A(\mathbb{D}^2);\, z_1, \ldots, z_k)$, where $A(\mathbb{D}^2)$ denotes the bidisk algebra. For a fixed pair of k-tuples, say $z_1, \ldots, z_k$ in $\mathbb{D}^2$ and $w_1, \ldots, w_k$ in $\mathbb{C}$ it is known that there exist finite methods for determining whether one can interpolate the given pair of k-tuples (i.e, $\forall\, \varepsilon > 0\, \exists\, f \in A(\mathbb{D}^2)$ with $\|f\|_\infty \leqslant 1 + \varepsilon$ such that $f(z_i) = w_i$). For such constructions, see the work of J. Ball and T. Trent ([3]) and/or the work of J. Agler and J. McCarthy ([2]). However, these finite methods yield unique sets of finite conditions for distinct k-tuples, say $(w_1, \ldots, w_k)$ and $(\tilde{w_1}, \ldots, \tilde{w_k})$ in $\mathbb{C}^k$, the target space. This is precisely the intrinsic difficulty we are referring to with interpolation on the bidisk.

In [7], B. Cole and J. Wermer show that $\mathcal{D}(A(\mathbb{D}^2);\, z_1, \ldots, z_k)$ is a *semi-algebraic* set (i.e, a set determined by a finite collection of polynomial inequalities). Thus, one can determine if an arbitrary k-tuple $(w_1, \ldots, w_k)$ in $\mathbb{C}^k$ is in $\mathcal{D}(A(\mathbb{D}^2);\, z_1, \ldots, z_k)$ by checking a finite number of polynomial inequalities (the same set of polynomial inequalities work for each k-tuple). However, in proving that $\mathcal{D}(A(\mathbb{D}^2);\, z_1, \ldots, z_k)$ is a semi-algebraic set B. Cole and J. Wermer appeal to an existential theorem of A. Tarskie and A. Seidenberg and by doing so avoid the construction of the polynomial inequalities which determine $\mathcal{D}(A(\mathbb{D}^2);\, z_1, \ldots, z_k)$.

In this paper we will show that for three particular points z_1, z_2, z_3 in $\mathbb{D}^2$ that $\mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)$ fails to be "computable" in a stricter sense, namely, that the largest (affiliated) *Schur ideal* is not finitely generated. In Section 2 we will describe tersely the motivation for studying Schur ideals and make the terminology precise. In Section 3 we will bring forward the notion of an interpolation problem being "strongly computable", as well as, compare and contrast this new notion of computability with the work of B. Cole and J. Wermer in [7]. In Section 4 we will present an example which is not *strongly computable* by using some results from V. Paulsen's paper [9]. Last, in Section 5 we will give an alternate proof of J. Agler's bidisk interpolation formula, [1], due to V. Paulsen.

2. PRELIMINARIES

Throughout the paper M_k will denote the $k \times k$ matrices with complex entries and M_k^+ will denote the closed cone of positive semi-definite matrices in M_k. If P is in M_k, then it will be understood that P is the matrix $(p_{ij})_{i,j=1}^k$ simply written as (p_{ij}). If P, Q are in M_k, then the Schur product of the matrices P and Q, denoted as $P * Q$, is defined to be $(p_{ij}) * (q_{ij}) = (p_{ij}q_{ij})$. The matrix P being positive semi-definite will be denoted as $P \geqslant 0$.

The following theorem is due to B. Cole and J. Wermer ([6]), in which they give conditions on a point $(w_1, \dots, w_k)$ in $\mathbb{C}^k$ in order that $(w_1, \dots, w_k)$ belong to the interpolation body $\mathcal{D} = \mathcal{D}(A;\, x_1, \dots, x_k)$ where $A \subseteq C(X)$ is an arbitrary uniform algebra and $x_1, \dots, x_k$ in X.

THEOREM 2.1. (Cole-Wermer) *If $A \subseteq C(X)$ is a uniform algebra and $x_1, \dots, x_k \in X$, then there exists a set $\mathcal{S} \subseteq M_k^+$ such that*

$$(w_1, \dots, w_k) \in \mathcal{D} \textit{ if and only if } ((1 - \overline{w}_i w_j)p_{ij}) \geqslant 0 \textit{ for all } (p_{ij}) \in \mathcal{S}.$$

In [9], V. Paulsen observes that Theorem 2.1 suggests the following "dualities" between subsets of the closed k-polydisk and subsets of M_k^+. Given a non-empty set $\mathcal{S} \subseteq M_k^+$ define

$$\mathcal{S}^\perp = \{(w_1, \dots, w_k) \in \mathbb{C}^k : ((1 - \overline{w}_i w_j)p_{ij}) \geqslant 0 \ \forall \ (p_{ij}) \in \mathcal{S}\}.$$

Similarly, given a subset $\mathcal{D}$ of the closed k-polydisk with $0 \in \mathcal{D}$ define

$$\mathcal{D}^\perp = \{(p_{ij}) : ((1 - \overline{w}_i w_j)p_{ij}) \geqslant 0 \ \forall \ (w_1, \dots, w_k) \in \mathcal{D}\}.$$

Since we insist that $0 \in \mathcal{D}$ then $\mathcal{D}^\perp$ is always a set of positive semi-definite matrices. Observing further that the set $\mathcal{D}^\perp$ has certain properties, V. Paulsen ([9]) introduces the concept of a Schur ideal, defined below. By studying this duality between hyperconvex sets and Schur ideals and using some results from the theory of abstract operator algebras, V. Paulsen is able to generalize J. Agler's scalar-valued interpolation results ([1]) for the bidisk to more general product domains ([9]). For another approach to such problems see the paper of A. Tomerlin ([14]).

DEFINITION 2.2. Let $\mathcal{I} \subseteq {M_k}^+$ be a non-empty set. Then $\mathcal{I}$ will be called a *Schur ideal* provided that:

(i) $A, B \in \mathcal{I} \Rightarrow \ A + B \in \mathcal{I}$.

(ii) $A \in \mathcal{I}$, $P \in {M_k}^+ \Rightarrow A * P \in \mathcal{I}$.

If $\mathcal{D}$ is a subset of the closed k-polydisk with $0 \in \mathcal{D}$, then $\mathcal{D}^\perp$ is a Schur ideal. A consequence of Theorem 2.1 is that if $\mathcal{D} = \mathcal{D}(A;\, x_1, \dots, x_k)$, then $\mathcal{D} = \mathcal{D}^{\perp\perp}$.

DEFINITION 2.3. A Schur ideal $\mathcal{I} \subseteq M_k^+$ is said to be *finitely generated* provided that there exist $P_1, \dots, P_m \in M_k^+$ such that

$$\mathcal{I} = \langle P_1, \dots, P_m \rangle = \Big\{ \sum_{i=1}^m P_i * R_i : \ R_i \in M_k^+ \Big\}.$$

Definition 2.4. Let $A \subseteq C(X)$, $x_1, \ldots, x_k \in X$, and $\mathcal{I} \subseteq M_k^+$ be a Schur ideal. Then the Schur ideal $\mathcal{I}$ is said to be *affiliated* with the interpolation body $\mathcal{D}(A;\, x_1, \ldots, x_k)$ associated with A and $x_1, \ldots, x_k$ provided that $\mathcal{I}^\perp = \mathcal{D}(A;\, x_1, \ldots, x_k)$.

Another consequence of Theorem 2.1 is that each interpolation body $\mathcal{D}(A;\, x_1, \ldots, x_k)$ has an affiliated Schur ideal, namely $\mathcal{D}(A;\, x_1, \ldots, x_k)^\perp$. Note, we have that $\left\langle \left(\frac{1}{1-\overline{\alpha}_i \alpha_j} \right) \right\rangle$ is the unique affiliated Schur ideal for the interpolation body $\mathcal{D}(A(\mathbb{D});\, \alpha_1, \ldots, \alpha_k)$ ([13]). In Definition 2.4 we could have easily defined the notion of an *affiliated set* instead of *an affiliated Schur ideal*. However, the interpolation body $\mathcal{D}(A(\mathbb{D});\, \alpha_1, \ldots, \alpha_k)$ has infinitely many affiliated sets.

3. Semi-algebraic versus strongly computable

In this section we will introduce the notion of an interpolation problem being "strongly computable" as well as compare and contrast this new notion with the work of B. Cole and J. Wermer in [7].

Throughout this section we will assume that $\mathcal{D} = \mathcal{D}(A;\, x_1, \ldots, x_k)$ for some uniform algebra A and for fixed points $x_1, \ldots, x_k$ in X. This assumption will allow us to make use of the fact that $\mathcal{D}^{\perp\perp} = \mathcal{D}$.

Definition 3.1. A Schur ideal $\mathcal{I}$ is said to be *affiliated* with the set $\mathcal{D}$ provided that
$$\mathcal{I}^\perp = \mathcal{D}.$$

One can verify that if $\mathcal{I}$ is any other affiliated Schur ideal with respect to the set $\mathcal{D}$, then $\mathcal{I} \subseteq \mathcal{D}^\perp$. In this sense, $\mathcal{D}^\perp$ is the largest affiliated Schur ideal with respect to the set $\mathcal{D}$. This motivated the following definition.

Definition 3.2. The set $\mathcal{D}$ is *strongly computable* provided that $\mathcal{D}^\perp$ is a finitely generated Schur ideal.

Remark 3.3. If $\mathcal{D}$ is strongly computable, then there exist $P_1, \ldots, P_m \in M_k^+$ such that $\mathcal{D}^\perp = \langle P_1, \ldots, P_m \rangle$. Let $(w_1, \ldots, w_k) \in \mathbb{C}^k$ and consider the following implications:
$$\begin{aligned}
&(1 - \overline{w}_i w_j) * P_l \geqslant 0 \quad \text{for } l = 1, \ldots, m \\
&\quad \Rightarrow \forall\, Q \in M_k^+, \quad (1 - \overline{w}_i w_j) * Q * P_l \geqslant 0 \quad \text{for } l = 1, \ldots, m \\
&\quad \Rightarrow \forall\, Q_1, \ldots, Q_m \in M_k^+, \quad (1 - \overline{w}_i w_j) * \sum_{l=1}^{m} Q_l * P_l \geqslant 0 \\
&\quad \Rightarrow \forall\, P \in \mathcal{D}^\perp, \quad (1 - \overline{w}_i w_j) * P \geqslant 0.
\end{aligned}$$
Since $\mathcal{D}^{\perp\perp} = \mathcal{D}$ we have that $(w_1, \ldots, w_k) \in \mathcal{D}$. Thus, if $\mathcal{D}^\perp$ is finitely generated, then it is sufficient to check finitely many positivity conditions to determine if $(w_1, \ldots, w_k) \in \mathcal{D}$.

It turns out that, if $\mathcal{D} = \mathcal{D}(A(\mathbb{D});\, \alpha_1, \ldots, \alpha_k)$, then $\mathcal{D}^\perp = \left\langle \left(\frac{1}{1-\overline{\alpha}_i \alpha_j} \right) \right\rangle$, see Proposition 4.1 in Section 4. Hence, $\mathcal{D} = \mathcal{D}(A(\mathbb{D});\, \alpha_1, \ldots, \alpha_k)$ is strongly computable. Moreover, if $\mathcal{I}$ is any other Schur ideal affiliated with $\mathcal{D} = \mathcal{D}(A(\mathbb{D}); \alpha_1,$

$\ldots, \alpha_k)$, then $\mathcal{I}$ is the Schur ideal $\left\langle \left(\frac{1}{1-\overline{\alpha}_i \alpha_j} \right) \right\rangle$; see [13]. However, for a general uniform algebra A, it is still not known whether $\mathcal{I}^\perp = \mathcal{D}(A;\, x_1, \ldots, x_k)$ implies that $\mathcal{I} = \mathcal{D}(A;\, x_1, \ldots, x_k)^\perp$; we leave this issue for future work. In Section 4 we will show that for three particular points z_1, z_2, z_3 in $\mathbb{D}^2$ the ball $\mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)$ is not strongly computable.

B. Cole and J. Wermer prove the following theorem in [7].

Theorem 3.4. (Cole-Wermer) *If* $z_1 = (\alpha_1, \beta_1), \ldots, z_k = (\alpha_k, \beta_k) \in \mathbb{D}^2$, *then* $\mathcal{D}(A(\mathbb{D}^2);\, z_1, \ldots, z_k)$ *is a semi-algebraic set in* $\mathbb{C}^k$.

Loosely speaking, a set $X \subseteq \mathbb{R}^k$ is *semi-algebraic* provided that X is determined by a finite collection of polynomial inequalities (for a brief exposition of the theory of semi-algebraic sets see [7]). Thus, despite our example where $\mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)$ is not strongly computable, it seems plausible to assume that there may exists a finitely generated Schur ideal, say $\mathcal{I}$, such that $\mathcal{I} \subseteq \mathcal{D}^\perp$ and $\mathcal{I}^\perp = \mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)$. This was the motivation behind the next definition.

Definition 3.5. The set $\mathcal{D}$ is said to be *computable* provided there exists a finitely generated Schur ideal $\mathcal{I}_0$ with $\mathcal{I}_0^\perp = \mathcal{D}$.

Clearly, if a set $\mathcal{D}$ is strongly computable, then the set $\mathcal{D}$ is computable. In the next proposition we will show that if a set $\mathcal{D}$ is computable, then the set $\mathcal{D}$ is semi-algebraic.

Proposition 3.6. *If* $\mathcal{D}$ *is computable, then* $\mathcal{D}$ *is a semi-algebraic set.*

Proof. We note that a matrix $P \in M_k$ is positive semi-definite ([8]) if and only if $P = P^*$ and the principal minors of P are non-negative, where the principal minors of P are the scalars $\Delta_l(P)$ defined by

$$\Delta_l(P) = \det \begin{bmatrix} p_{11} & \cdots & p_{1l} \\ \vdots & & \vdots \\ p_{l1} & \cdots & p_{ll} \end{bmatrix}, \quad 1 \leqslant l \leqslant k.$$

The set $\mathcal{D}$ being computable implies that there exists a finitely generated Schur ideal $\mathcal{I}_0$ with $\mathcal{I}_0^\perp = \mathcal{D}$. Thus, we can choose $P_1, \ldots, P_m \in M_k^+$ such $\mathcal{I}_0 = \langle P_1, \ldots, P_m \rangle$. But,

$$\begin{aligned} (w_1, \ldots, w_k) \in \mathcal{D} \quad &\Leftrightarrow \quad ((1 - \overline{w}_i w_j) p_{ij}^l) \geqslant 0 && \text{for } l = 1, \ldots, m, \\ &\Leftrightarrow \quad \Delta_1((1 - \overline{w}_i w_j) p_{ij}^l) \geqslant 0 && \text{for } l = 1, \ldots, m, \\ &\Leftrightarrow \quad \Delta_2((1 - \overline{w}_i w_j) p_{ij}^l) \geqslant 0 && \text{for } l = 1, \ldots, m, \\ &\ \ \vdots \\ &\Leftrightarrow \quad \Delta_k((1 - \overline{w}_i w_j) p_{ij}^l) \geqslant 0 && \text{for } l = 1, \ldots, m. \end{aligned}$$

Thus, the real polynomials $\Delta_n^l(w_1, \ldots, w_k, \overline{w}_1, \ldots, \overline{w}_k)$ for $n = 1, \ldots, k$ and $l = 1, \ldots, m$ determine the set $\mathcal{D}$. Hence the set $\mathcal{D}$ is semi-algebraic in $\mathbb{C}^k$. ∎

We mentioned previously that $\mathcal{D}^\perp$ is the largest affiliated Schur ideal with respect to the set $\mathcal{D}$. Thus, if $\mathcal{D}^\perp$ is finitely generated, then the set $\mathcal{D}$ is semi-algebraic. Hence, a set being strongly computable (and/or computable) is a stricter notion of computability than a set being semi-algebraic.

4. EXAMPLE

In this section we will show that for three particular points z_1, z_2, z_3 in $\mathbb{D}^2$ that the interpolation body $\mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)$ is not strongly computable. Proving that $\mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)^\perp$ is infinitely generated does not rely on any particularly deep theorems in analysis but is rather a straight forward argument. The subtlety lies in determining what form the Schur ideal $\mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)^\perp$ takes. Theorem 4.2 below does exactly that for any fixed set of points $z_1, \dots, z_k$ in $\mathbb{D}^2$, where k is arbitrary. Theorem 4.2 is a special case of a more general theorem, due to V. Paulsen in [9]. Theorem 4.2 does even more than tell us what form $\mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)^\perp$ takes! By using some elementary facts about dual cones and a factorization theorem for polynomials in two complex-variables we will see in Section 5 that J. Agler's bidisk interpolation formula ([1]) follows as a corollary to Theorem 4.2 quite naturally. We will begin this section with a proposition about *Pick bodies* (i.e, $\mathcal{D}(A(\mathbb{D});\, \alpha_1, \dots, \alpha_k)$).

Proposition 4.1. *Let $\alpha_1, \dots, \alpha_k \in \mathbb{D}$. Then*

$$\Big\langle \Big(\frac{1}{1-\overline{\alpha}_i \alpha_j} \Big) \Big\rangle = \mathcal{D}(A(\mathbb{D});\, \alpha_1, \dots, \alpha_k)^\perp = \Big\{ \Big(\frac{1}{1-\overline{\alpha}_i \alpha_j} \Big) \Big\}^{\perp\perp}.$$

The proof of Proposition 4.1 will be omitted. However, the key ingredient in proving Proposition 4.1 is Pick's theorem. Note, throughout the remainder of the paper we will often interchange $\Big\langle \Big(\frac{1}{1-\overline{\alpha}_i \alpha_j} \Big) \Big\rangle$ with $\mathcal{D}(A(\mathbb{D});\, \alpha_1, \dots, \alpha_k)^\perp$ and vice versa when convenient.

Theorem 4.2. (Paulsen) *Let $z_1 = (\alpha_1, \beta_1), \dots, z_k = (\alpha_k, \beta_k) \in \mathbb{D}^2$. Then*

$$\mathcal{D}(A(\mathbb{D}^2);\, z_1, \dots, z_k) = \Big(\Big\langle \Big(\frac{1}{1-\overline{\alpha_i} \alpha_j} \Big) \Big\rangle \cap \Big\langle \Big(\frac{1}{1-\overline{\beta_i} \beta_j} \Big) \Big\rangle \Big)^\perp.$$

A proof of Theorem 4.2 will be provided in Section 5.

Theorem 4.3. *Let $z_1 = (0,0)$, $z_2 = \Big(\frac{1}{\sqrt{2}}, 0\Big)$, $z_3 = \Big(0, \frac{1}{\sqrt{2}}\Big)$. Then the Schur ideal*

$$\mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)^\perp = \left\langle \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right\rangle \cap \left\langle \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \right\rangle$$

is infinitely generated.

Corollary 4.4. *The set $\mathcal{D}\Big(A(\mathbb{D}^2);\, (0,0), \Big(\frac{1}{\sqrt{2}}, 0\Big), \Big(0, \frac{1}{\sqrt{2}}\Big)\Big)$ is not strongly computable.*

This example exhibits the intrinsic difference between interpolation on $\mathbb{D}^2$ and interpolation on $\mathbb{D}$ (i.e, $\mathcal{D}(A(\mathbb{D}); \alpha_1, \dots, \alpha_k)^\perp$ is necessarily finitely generated and $\mathcal{D}(A(\mathbb{D}^2);\, z_1, \dots, z_k)^\perp$ is not necessarily finitely generated). However, $\mathcal{D}\Big(A(\mathbb{D}^2); (0,0), \Big(\frac{1}{\sqrt{2}}, 0\Big), \Big(0, \frac{1}{\sqrt{2}}\Big)\Big)$ may be computable since $\mathcal{D}(A(\mathbb{D}^2);\, z_1, \dots, z_k)$ is necessarily a semi-algebraic set ([7]); we leave this issue for future work.

We will need three lemmas in order to prove Theorem 4.3. The proofs of the first two lemmas will be omitted since they are straight ahead calculations. Throughout the remainder of this section $z_1 = (0,0)$, $z_2 = \left(\frac{1}{\sqrt{2}}, 0\right)$, and $z_3 = \left(0, \frac{1}{\sqrt{2}}\right)$.

Lemma 4.5. *If $\theta \in [0, 2\pi]$, then $S_\theta \in \mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)^\perp$ where*

$$S_\theta = \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 1 & \frac{1+\mathrm{e}^{\mathrm{i}\theta}}{3} \\ \frac{1}{\sqrt{3}} & \frac{1+\mathrm{e}^{-\mathrm{i}\theta}}{3} & 1 \end{pmatrix}.$$

Lemma 4.6. *Let $P = (p_{ij})_{i,j=1}^3 \geqslant 0$. Let*

$$v_1 = \begin{pmatrix} \frac{1-2\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{3}} \\ 2\mathrm{e}^{\mathrm{i}\theta} \\ -1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} \frac{2-\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{3}} \\ \mathrm{e}^{\mathrm{i}\theta} \\ -2 \end{pmatrix} \quad \text{for a fixed } \theta.$$

If $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} * Pv_1 = 0$ *and* $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} \end{pmatrix} * Pv_2 = 0$, *then $P = tS_\theta$ where $t \geqslant 0$ and* $S_\theta = \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 1 & \frac{1+\mathrm{e}^{\mathrm{i}\theta}}{3} \\ \frac{1}{\sqrt{3}} & \frac{1+\mathrm{e}^{-\mathrm{i}\theta}}{3} & 1 \end{pmatrix}.$

Lemma 4.7. *Let $Q_1, \ldots, Q_m \in \mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)^\perp$ and*

$$S_\theta = \sum_{i=1}^m P_i * Q_i$$

where $P_1, \ldots, P_m \geq 0$. Then $P_i * Q_i = t_i S_\theta$ where $t_i \geqslant 0$ for $i = 1, \ldots, m$.

Proof.

$$S_\theta = \sum_{i=1}^m P_i * Q_i \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} * S_\theta = \sum_{i=1}^m \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} * P_i * Q_i$$

$$\text{and} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} \end{pmatrix} * S_\theta = \sum_{i=1}^m \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} \end{pmatrix} * P_i * Q_i\,.$$

Now for $i = 1, \ldots, m$ we will let

$$R_i = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} * P_i * Q_i \quad \text{and} \quad T_i = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} \end{pmatrix} * P_i * Q_i\,.$$

For $i = 1, \ldots, m$ each $Q_i \in \mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)^{\perp} \Rightarrow \exists\, U_i \geq 0$ such that

$$Q_i = U_i * \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Hence, $R_i = P_i * U_i \geq 0$, for $i = 1, \ldots, m$. Similarly, $T_i \geq 0$, for $i = 1, \ldots, m$. Let v_1, v_2 be as in Lemma 4.6. Then

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} * S_\theta v_1 = 0 \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} \end{pmatrix} * S_\theta v_2 = 0.$$

But this implies

$$\Big\langle \sum_{i=1}^{m} R_i v_1, v_1 \Big\rangle = 0 \Rightarrow R_i v_1 = 0 \quad \text{for } i = 1, \ldots, m \tag{4.1}$$

and

$$\Big\langle \sum_{i=1}^{m} T_i v_2, v_2 \Big\rangle = 0 \Rightarrow T_i v_2 = 0 \quad \text{for } i = 1, \ldots, m. \tag{4.2}$$

Thus, by Lemma 4.6 and (4.1), (4.2) above we have that $P_i * Q_i = t_i S_\theta$ and $t_i \geqslant 0$ for $i = 1, \ldots, k$. Note, that $\sum_{i=1}^{m} t_i = 1$. ∎

Prof of Theorem 4.3. Suppose that $\mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)^{\perp}$ is finitely generated. Then there exists $Q_1, \ldots, Q_m \in M_3^+$, such that $\mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)^{\perp} = \langle Q_1, \ldots, Q_m \rangle$. For each $\theta \in [0, 2\pi]$ we can choose $P_1, \ldots, P_m \geqslant 0$ such that

$$\begin{aligned} S_\theta &= \sum_{i=1}^{m} P_i * Q_i \\ &\Rightarrow P_i * Q_i = t_i S_\theta,\ t_i \geqslant 0 \text{ and } \sum_{i=1}^{m} t_i = 1 \text{ for } i = 1, \ldots, m \text{ (Lemma 4.7)}, \\ &\Rightarrow \exists\, k \in \{1, \ldots, m\} \text{ such that } t_k \neq 0, \\ &\Rightarrow S_\theta = t_k^{-1} P_k * Q_k. \end{aligned}$$

Therefore, for all $\theta \in [0, 2\pi]$, there exists $k_\theta \in \{1, \ldots, m\}$ and $R_\theta \geqslant 0$ such that

$$S_\theta = R_\theta * Q_{k_\theta}.$$

Note, if $\theta \neq \pi$, then Q_{k_θ} has all non-zero entries. Now, define $\Gamma_k = \{\theta : k_\theta = k\}$ for $k = 1, \ldots, m$. Since the $\Gamma_{k's}$ cover the unit circle, we can choose $k_0 \in \{1, \ldots, m\}$ and $\{\theta_n\} \subseteq \Gamma_{k_0}$ such that $\theta_n \to \pi$ as $n \to \infty$ and for each $n \in \mathbb{N}$, $\theta_n \neq \pi$. Thus, for each $n \in \mathbb{N}$ we can choose $R_n \geqslant 0$ such that

$$S_{\theta_n} = R_n * Q_{k_0}.$$

The fact that there exists an $n \in \mathbb{N}$ such that $\theta_n \neq \pi$ implies that $Q_{k_0} = (q_{ij}^{k_0})$ has all non-zero entries. Thus, we can define

$$\widehat{Q}_{k_0} = \left(\frac{1}{q_{ij}^{k_0}} \right).$$

But for each $n \in \mathbb{N}$ we have that $S_{\theta_n} = R_n * Q_{k_0}$, this implies for each $n \in \mathbb{N}$ that $R_n = \widehat{Q}_{k_0} * S_{\theta_n}$ and

$$S_{\theta_n} \to S_\pi \text{ as } n \to \infty \Rightarrow R_n \to R_\pi \text{ as } n \to \infty \Rightarrow S_\pi = R_\pi * Q_{k_0}.$$

Since Q_{k_0} has all non-zero entries we may assume that

$$Q_{k_0} = \begin{pmatrix} 1 & a & b \\ \overline{a} & 1 & z \\ \overline{b} & \overline{z} & 1 \end{pmatrix}, \quad \text{where } a, b, z \in \mathbb{C}.$$

But $Q_{k_0} \in \mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)^\perp$ implies that the following two matrices

$$\begin{pmatrix} 1 & a & b \\ \overline{a} & \frac{1}{2} & z \\ \overline{b} & \overline{z} & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & a & b \\ \overline{a} & 1 & z \\ \overline{b} & \overline{z} & \frac{1}{2} \end{pmatrix}$$

are positive semi-definite. Thus, $|a|^2 \leqslant \frac{1}{2}$ and $|b|^2 \leqslant \frac{1}{2}$. We also have that

$$S_\pi = \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad R_\pi = \begin{pmatrix} 1 & \frac{1}{a\sqrt{3}} & \frac{1}{b\sqrt{3}} \\ \frac{1}{\overline{a}\sqrt{3}} & 1 & 0 \\ \frac{1}{\overline{b}\sqrt{3}} & 0 & 1 \end{pmatrix}.$$

Thus, we have the following two implications,

$$R_\pi \geqslant 0 \Rightarrow 1 - \frac{1}{3|a|^2} - \frac{1}{3|b|^2} \geqslant 0 \tag{4.3}$$

and

$$|a|^2 \leqslant \frac{1}{2} \Rightarrow \frac{1}{3|a|^2} \geqslant \frac{2}{3}. \tag{4.4}$$

But (4.3) and (4.4) imply that $|b|^2 \geqslant 1$, which contradicts the fact that $|b|^2 \leqslant \frac{1}{2}$. Hence, $\mathcal{D}(A(\mathbb{D}^2);\, z_1, z_2, z_3)^\perp$ is not finitely generated. ∎

5. AGLER'S BIDISK INTERPOLATION FORMULA

In this section we will first state a factorization theorem for polynomials in two complex variables with norm strictly less than 1, due to D. Blecher and V. Paulsen ([4]). This theorem together with the matrix-valued version of Pick's theorem ([12]), and the fact that $\mathcal{D}^{\perp\perp} = \mathcal{D}$ for an arbitrary interpolation body will enable us to prove Theorem 4.2. Again, Theorem 4.2 is a special case of a more general theorem in [9]. The proof of Theorem 4.2 is much simpler than the proof of the general theorem and is due to V. Paulsen.

Following the proof of Theorem 4.2 we state two corollaries (5.2 and 5.3) which follow from Theorem 4.2 (not from Theorem 5.1). We will omit the proof of Corollary 5.2 (V. Paulsen's bidisk interpolation formula) and before proving Corollary 5.3 (J. Agler's bidisk interpolation formula) we will discuss how the two corollaries tell us when we can or can't interpolate two sets of k-tuples. We will then recall some elementary facts about dual cones and end the paper with an alternate proof of J. Agler's bidisk interpolation formula, due to V. Paulsen.

Theorem 5.1. (Blecher-Paulsen) *Let $p(\alpha,\beta)$ be a polynomial in two complex variables with $\|p\| = \sup\{|p(\alpha,\beta)| : |\alpha|, |\beta| \leqslant 1\} < 1$. Then p factors as follows:*

$$p(\alpha,\beta) = \underset{1\times l_1}{F_1(\alpha)}\ \underset{l_1\times l_2}{G_1(\beta)}\ \underset{l_2\times l_3}{F_2(\alpha)}\underset{l_3\times l_4}{G_2(\beta)}\underset{\cdots}{\cdots} F_m(\alpha)\ \underset{l_{2m-1}\times 1}{G_m(\beta)}$$

where $\|F_i\|, \|G_i\| < 1$ for all i and entries of F_i and G_i are polynomials.

Proof of Theorem 4.2. Let $\mathcal{D} = \mathcal{D}(A(\mathbb{D}^2);\, z_1,\dots,z_k)$, where $z_i = (\alpha_i,\beta_i)$. Looking at functions constant in β implies that $\mathcal{D}(A(\mathbb{D});\, \alpha_1,\dots,\alpha_k) \subseteq \mathcal{D}$ which implies that $\mathcal{D}^\perp \subseteq \mathcal{D}(A(\mathbb{D});\, \alpha_1,\dots,\alpha_k)^\perp$. Similarly, $\mathcal{D}^\perp \subseteq \mathcal{D}(A(\mathbb{D});\, \beta_1,\dots,\beta_k)^\perp$. Therefore,

$$\mathcal{D}^\perp \subseteq \mathcal{D}(A(\mathbb{D});\, \alpha_1,\dots,\alpha_k)^\perp \cap \mathcal{D}(A(\mathbb{D});\, \beta_1,\dots,\beta_k)^\perp$$

which implies that

$$\Big(\Big\langle\Big(\frac{1}{1-\overline{\alpha}_i\alpha_j}\Big)\Big\rangle \cap \Big\langle\Big(\frac{1}{1-\overline{\beta}_i\beta_j}\Big)\Big\rangle\Big)^\perp \subseteq \mathcal{D}^{\perp\perp} = \mathcal{D}.$$

To show the other containment let $Q = (q_{ij}) \in \mathcal{D}(A(\mathbb{D});\, \alpha_1,\dots,\alpha_k)^\perp \cap \mathcal{D}(A(\mathbb{D});\, \beta_1,\dots,\beta_k)^\perp$ and let $p(\alpha,\beta)$ be a polynomial such that $\|p\| < 1$. Let $w_i = p(\alpha_i,\beta_i)$ and suppose $((1-\overline{w_i}w_j)q_{ij}) \geqslant 0$. Then we have the following:

$$\begin{aligned}
&((1-\overline{w}_iw_j)q_{ij}) \geqslant 0\\
&\quad\Rightarrow (q_{ij}) \in \mathcal{D}^\perp\\
&\quad\Rightarrow \mathcal{D}(A(\mathbb{D});\, \alpha_1,\dots,\alpha_k)^\perp \cap \mathcal{D}(A(\mathbb{D});\, \beta_1,\dots,\beta_k)^\perp \subseteq \mathcal{D}^\perp\\
&\quad\Rightarrow \mathcal{D}^{\perp\perp} = \mathcal{D} \subseteq (\mathcal{D}(A(\mathbb{D});\, \alpha_1,\dots,\alpha_k)^\perp \cap \mathcal{D}(A(\mathbb{D});\, \beta_1,\dots,\beta_k)^\perp)^\perp.
\end{aligned}$$

Thus, we need only prove that $((1-\overline{w}_iw_j)q_{ij}) \geqslant 0$. Assume $p(\alpha,\beta) = F(\alpha)G(\beta)$.

$$\begin{aligned}
1-\overline{w}_iw_j &= 1-\overline{p(\alpha_i,\beta_i)}p(\alpha_j,\beta_j)\\
&= 1-G^*(\beta_i)F^*(\alpha_i)F(\alpha_j)G(\beta_j)\\
&= 1-G^*(\beta_i)G(\beta_j) + G^*(\beta_i)[I-F^*(\alpha_i)F(\alpha_j)]G(\beta_j).
\end{aligned}$$

Now Schur product $(1-\overline{w}_iw_j)$ with (q_{ij}) where $(q_{ij}) = \Big(\frac{p^1_{ij}}{1-\overline{\alpha}_i\alpha_j}\Big)$ or $(q_{ij}) = \Big(\frac{p^2_{ij}}{1-\overline{\beta}_i\beta_j}\Big)$ where $(p^1_{ij}), (p^2_{ij}) \geqslant 0$, to get

$$\begin{aligned}
((1-\overline{w}_iw_j)q_{ij}) &= ((1-G^*(\beta_i)G(\beta_j))q_{ij}) + (G^*(\beta_i)[(I-F^*(\alpha_i)F(\alpha_j))q_{ij}]G(\beta_j))\\
&= \Big(\frac{1-G^*(\beta_i)G(\beta_j)}{1-\overline{\beta}_i\beta_j}p^2_{ij}\Big) + \Big(G^*(\beta_i)\Big[\frac{I-F^*(\alpha_i)F(\alpha_j)}{1-\overline{\alpha}_i\alpha_j}p^1_{ij}\Big]G(\beta_j)\Big) \geqslant 0.
\end{aligned}$$

To prove the above inequality we are using three separate facts. First, we are using the matrix-valued version of Pick's theorem ([12]). Next, we are using two facts from matrix analysis. First, the Schur product of two positive semi-definite matrices is positive semi-definite and if P is positive semi-definite, then X^*PX is positive semi-definite for $X \in M_{k,m}$, m arbitrary ([8]). Now, by induction on $p(\alpha_i,\beta_i) = F(\alpha_i)H(\alpha_i,\beta_i)$, where $H(\alpha_i,\beta_i)$ has $n-1$ factors, we are done. ∎

Now that we have proven Theorem 4.2 the following corollaries follow easily.

Corollary 5.2. (Paulsen) *Let* $z_1 = (\alpha_1, \beta_1), \dots, z_k = (\alpha_k, \beta_k) \in \mathbb{D}^2$ *and* $w_1, \dots, w_k \in \mathbb{D}$. *Then* $(w_1, \dots, w_k) \in \mathcal{D}(A(\mathbb{D}^2); z_1, \dots, z_k)$ *if and only if* $(1 - \overline{w}_i w_j) * Q \geqslant 0 \,\forall\, Q$ *satisfying:*

(i) $(1 - \overline{\alpha}_i \alpha_j) * Q \geqslant 0$;

(ii) $(1 - \overline{\beta}_i \beta_j) * Q \geqslant 0$.

Corollary 5.3. (Agler) *Let* $z_1 = (\alpha_1, \beta_1), \dots, z_k = (\alpha_k, \beta_k) \in \mathbb{D}^2$ *and* $w_1, \dots, w_k \in \mathbb{D}$. *Then* $(w_1, \dots, w_k) \in \mathcal{D}(A(\mathbb{D}^2); z_1, \dots, z_k)$ *if and only if there exists positive semi-definite matrices* P *and* Q *such that*

$$(1 - \overline{w}_i w_j) = ((1 - \overline{\alpha}_i \alpha_j) p_{ij}) + ((1 - \overline{\beta}_i \beta_j) q_{ij}).$$

The above two corollaries tell us the following. If one chooses the correct $P, Q \geqslant 0$ satisfying

$$(1 - \overline{w}_i w_j) = ((1 - \overline{\alpha}_i \alpha_j) p_{ij}) + ((1 - \overline{\beta}_i \beta_j) q_{ij}),$$

then J. Agler's formula tells us we can interpolate $(z_1, \dots, z_k)$ and $(w_1, \dots, w_k)$. However, one would have to exhaust an infinite number of possible $P, Q \geqslant 0$ to show one can't interpolate $(w_1, \dots, w_k)$ using Corollary 5.3.

Conversely, suppose Q satisfies

$$(1 - \overline{\alpha}_i \alpha_j) * Q \geqslant 0 \quad \text{and} \quad (1 - \overline{\beta}_i \beta_j) * Q \geqslant 0.$$

If $(1 - \overline{w}_i w_j) * Q$ is not positive semi-definite, then V. Paulsen's formula tells us we can't interpolate $(w_1, \dots, w_k)$. On the other hand, to show that we can interpolate $(w_1, \dots, w_k)$ we would need to verify $(1 - \overline{w}_i w_j) * Q \geqslant 0$ for all Q satisfying conditions (i) and (ii) in Corollary 5.2. But by Theorem 4.3 we have that for three particular points in the bidisk that there are infinitely many Q's satisfying conditions (i) and (ii) in Corollary 5.2.

Before proving Agler's bidisk interpolation formula (Corollary 5.3) we need to recall some facts about dual cones.

Given a set S let $S^+ = \{y : 0 \leqslant y \cdot x, \,\forall\, x \in S\}$.

(i) S^+ is a cone.

(ii) $S \subseteq S^{++}$.

(iii) $S^{++} = \text{cone}(S)$ — smallest closed cone containing S (Krein-Milman).

Theorem 5.4. *Let* C_1 *and* C_2 *be closed cones. Then*

$$(C_1 \cap C_2)^+ = (C_1^+ + C_2^+)^-.$$

Consider the space of $k \times k$ Hermitian matrices M_k^h as a real Hilbert space with pairing $A \odot B = \sum_{i,j=1}^{k} a_{ij} b_{ij} = \langle A * Be, e \rangle$ where e is the vector of all ones and $\langle \cdot\,, \cdot \rangle$ denotes the usual inner product on $\mathbb{C}^k$.

PROPOSITION 5.5. *Let $\mathcal{I} \subseteq M_k^+$ be a Schur ideal. Then $\mathcal{I}$ is a cone and*

$$\mathcal{I}^+ = \{H \in M_k^h : H \odot P \geqslant 0,\ \forall P \in \mathcal{I}\} = \{H \in M_k^h : H * P \geqslant 0,\ \forall P \in \mathcal{I}\}.$$

Moreover, $(w_1, \dots, w_k) \in \mathcal{I}^\perp$ if and only if $(1 - \overline{w}_i w_j) \in \mathcal{I}^+$.

We are now able to prove Agler's bidisk interpolation formula.

Proof of Corollary 5.3. Let $\mathcal{D} = \mathcal{D}(A(\mathbb{D}^2); z_1, \dots, z_k)$. By Theorem 4.2

$$\mathcal{D} = \Big(\Big\langle\Big(\frac{1}{1-\overline{\alpha}_i\alpha_j}\Big)\Big\rangle \cap \Big\langle\Big(\frac{1}{1-\overline{\beta}_i\beta_j}\Big)\Big\rangle\Big)^\perp.$$

But by Proposition 5.5

$$(w_1, \dots, w_k) \in \mathcal{D} \Leftrightarrow (1 - \overline{w}_i w_j) \in \Big(\Big\langle\Big(\frac{1}{1-\overline{\alpha}_i\alpha_j}\Big)\Big\rangle \cap \Big\langle\Big(\frac{1}{1-\overline{\beta}_i\beta_j}\Big)\Big\rangle\Big)^+.$$

But by Theorem 5.4

$$\Big(\Big\langle\Big(\frac{1}{1-\overline{\alpha}_i\alpha_j}\Big)\Big\rangle \cap \Big\langle\Big(\frac{1}{1-\overline{\beta}_i\beta_j}\Big)\Big\rangle\Big)^+ = \Big\langle\Big(\frac{1}{1-\overline{\alpha}_i\alpha_j}\Big)\Big\rangle^+ + \Big\langle\Big(\frac{1}{1-\overline{\beta}_i\beta_j}\Big)\Big\rangle^+.$$

Next observe that

$$\Big\langle\Big(\frac{1}{1-\overline{\alpha}_i\alpha_j}\Big)\Big\rangle^+ = \Big\{H \in M_k^h : H * \Big(\frac{1}{1-\overline{\alpha}_i\alpha_j}\Big) \geqslant 0\Big\} = \{(1-\overline{\alpha}_i\alpha_j) * P : P \geqslant 0\}.$$

This completes the proof of Corollary 5.3. ∎

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