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LB ALGEBRAS

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Abstract. We introduce some classes of C^*-algebras with "good" local approximation properties — the class of LB algebras and several subclasses of it — which generalize, among others, the AH algebras, the AD algebras and the separable, simple C^*-algebras with an approximate unit of projections. We initiate the study of these new and rich classes of C^*-algebras, proving results about the ideal property, real rank zero, the projection property, ideal structure, inductive limits, stable isomorphism, hereditary C^*-subalgebras and extensions. Some of our previous results about AH algebras and GAH algebras are generalized.

Keywords: *C^*-algebra,* LB *algebra, special* LB *algebra, ultraspecial* LB *algebra, the ideal property, the projection property, real rank zero, inductive limits, Riesz decomposition property, ideal generated by projections, stable isomorphism, hereditary C^*-subalgebra, extension of two C^*-algebras.*

MSC (2000): 46L05, 46L99.

1. INTRODUCTION

In the recent years, many important results have been obtained for several classes of C^*-algebras including, among others, the AH algebras, the AD algebras and the ASH algebras ([5]). All the above C^*-algebras are GAH algebras, i.e. countable inductive limits of finite direct sums of unital C^*-algebras whose proper ideals have no nonzero projections ([16], [17]). More generally, each countable inductive limit of C^*-algebras defined by continuous fields of unital, simple C^*-algebras over compact spaces with finitely many connected components ([6]) (in particular, each countable inductive limit of finite direct sums of C^*-algebras of the form $C(X, A)$ with X a compact, connected space and A a simple, unital C^*-algebra) is a GAH algebra. In this paper we introduce a class of C^*-algebras, namely the LB algebras (and several important subclasses of them), which generalize the GAH algebras. The LB algebras are C^*-algebras with "good" local approximation properties (see Definition 2.2 below). Finding suitable invariants for LB algebras (and their subclasses introduced here) and studying their properties seems to be a natural extension of Effros' problem on AH algebras ([7]). On the other hand, in the separable, nuclear case, the LB algebras seem to be very pertinent within Elliott's classification program ([8]). The purpose of this paper is to initiate the study of LB algebras (and of their subclasses) which are rich and which seem to be well-behaved. In particular, we shall be interested in studying their behavior with respect to the ideal property. Recall that a C^*-algebra has the ideal property if each ideal (closed, two-sided) is generated (as an ideal) by its projections. The ideal property is very important since all simple, unital C^*-algebras have the ideal property and so do all C^*-algebras of real rank zero ([3]). In the recent years, a lot of results have been proved for C^*-algebras with the ideal property (see e.g. [12]–[20]). In Section 2 we introduce the LB algebras and initiate their study. We prove that if A is a separable LB algebra and if I is an ideal of A, then the following are equivalent:

(a) I is generated by projections;
(b) I has a countable approximate unit of projections;
(c) the canonical extension

$$0 \to I \to A \to A/I \to 0$$

is quasidiagonal;
(d) I is an LB algebra.

(see Theorem 2.3 below). In fact we prove that (a) $\Leftrightarrow$ (d) above is true for an arbitrary LB algebra A (see Theorem 2.9 below). Note that the ideals generated by projections played an important role in the proof given by Dadarlat and Eilers in [4] to the surprising fact that the AH algebras are not closed under inductive limits. The above results immediately imply that a separable LB algebra A has the ideal property if and only if A has the projection property (see Theorem 2.10 below). (A C^*-algebra has the projection property if each of its ideals has an approximate unit of projections; [19].) Note that as it was proved in our paper [19], there are separable C^*-algebras with the ideal property which don't have the projection property. As we have shown jointly with Dadarlat in [15], Theorem 5.1, the ideal property is not closed under extensions. However, we prove here that if

$$0 \to I \to A \to B \to 0$$

is an exact sequence of C^*-algebras such that A is a separable LB algebra, then A has the ideal property if and only if I and B have the ideal property (see Theorem 2.11 below).

In Section 3 we introduce an important subclass of LB algebras: the special LB algebras (see Definition 3.4 below) (note that it is immediate that if A is a special LB algebra, then $M_n(A)$ is a special LB algebra too for any $n \in \mathbb{N}$.) This class in closed under inductive limits (see Proposition 3.7 below) and hence, if A is a special LB algebra and if B is an AF algebra, then $A \otimes B$ is a special LB algebra (see Proposition 3.8 below). We prove that if A is a special LB algebra and I is an ideal of A, then I is generated by projections if and only if I is a special LB algebra (see Theorem 3.6 below). We describe the lattice of the ideals generated by projections of a special LB algebra (see Theorem 3.9 below) and also we describe the partially ordered set of the stably cofinite ideals generated by projections of a separable, special LB algebra A in the case when the projections of $M_\infty(A)$

satisfy the Riesz decomposition property (i.e. if p, q_1, q_2 are projections in $M_\infty(A)$ such that p is Murray-von Neumann equivalent to a subprojection of $q_1 \oplus q_2$, then $p = p_1 \oplus p_2$ for some projections $p_i \in M_\infty(A)$ with p_i Murray-von Neumann equivalent to a subprojection of q_i for $i = 1, 2$) (see Theorem 3.14 below). We also prove that if a special LB algebra is stably isomorphic to a C^*-algebra with the ideal property, then it has the ideal property too (see Corollary 3.12 below).

In Section 4 we introduce and study two important classes of special LB algebras: the ultraspecial LB algebras (see Definition 4.1 below) and the class of separable, ultraspecial LB algebras, denoted by $\mathcal{U}$. For a given C^*-algebra $A \in \mathcal{U}$, we prove that the following are equivalent:

(a) A has real rank zero ($\mathrm{RR}(A) = 0$);
(b) if B is an arbitrary hereditary C^*-subalgebra of A, then $B \in \mathcal{U}$;

(see Theorem 4.8 below) and also that the following are equivalent:

(a′) A has the ideal property;
(b′) if I is an arbitrary ideal of A, the $I \in \mathcal{U}$.

(see Theorem 4.14 below) (in fact, a more general result is proved in Theorem 4.12 below). It is shown that the above class of C^*-algebras $A \in \mathcal{U}$ with $\mathrm{RR}(A) = 0$ is closed under stable isomorphism (see Corollary 4.9 below). Also, we prove that if a C^*-algebra with an approximate unit of projections is stably isomorphic to an ultraspecial LB algebra, then it is an ultraspecial LB algebra (see Proposition 4.10 below). We prove a theorem which implies, in particular, that none of the following six classes of C^*-algebras: $\mathcal{U}$, ultraspecial LB algebras, separable special LB algebras, special LB algebras, separable LB algebras and LB algebras is closed under extensions (see Theorem 4.15 below). This contrasts with a classical result of Brown saying that the AF algebras are closed under extensions ([1]). It seems to us that $\mathcal{U}$ is the most interesting of the subclasses of LB algebras introduced in this paper, since it enjoys most of the properties proved here and, moreover, it has interesting additional features (e.g. the real rank zero situation can be characterized (in terms of hereditary C^*-subalgebras)).

It is worth to mention that some of the results proved in this paper generalize or partially generalize some of our previous theorems on AH and GAH algebras in [12]–[17] and that most of them required completely new ideas of proof, since the methods used in our papers mentioned above didn't work here.

Let A be a C^*-algebra. By an ideal of A will shall mean a closed, two sided ideal of A. If I is an ideal of A we shall say that I is generated by projections if I is generated by projections as an ideal (closed and two-sided) of A. The projections of A will be denoted by $\mathcal{P}(A)$. If $p, q \in \mathcal{P}(A)$ we shall write $p \overset{A}{\sim} q$ or simply $p \sim q$ if p and q are Murray-von Neumann equivalent in A (i.e. there is $v \in A$ such that $v^*v = p$ and $vv^* = q$). $\mathcal{Z}(A)$ will denote the center of A. If I is an ideal of A, by the canonical extension

$$0 \to I \to A \to A/I \to 0$$

we shall mean the extension in which $I \to A$ is the canonical inclusion and $A \to A/I$ is the canonical quotient map. We shall denote by $M(A)$ the multiplier algebra of A and by $M_\infty(A)$ the algebraic inductive limit of matrix algebras $M_n(A)$, $n \in \mathbb{N}$ under the embeddings:

$$M_n(A) \ni a \mapsto a \oplus 0 \in M_{n+1}(A).$$

If $p \in M_\infty(A)$ is a projection, its class in $K_0(A)$ will be denoted by [p]. The real rank of A ([3]) will be denoted by $\mathrm{RR}(A)$. The stable rank of A was defined and studied in [21]. For each $x \in A$ and each $M \subseteq A$, we shall denote $\mathrm{dist}\,(x, M) := \inf\{\|x - m\| : m \in M\}$.

Recall that an extension of C^*-algebras:

$$0 \to I \to A \to B \to 0$$

is called *quasidiagonal* if there is an approximate unit $(p_n)_{n=1}^\infty$ of I consisting of projections, which is quasicentral in A, i.e.

$$\lim_{n\to\infty} \|ap_n - p_n a\| = 0$$

for all $a \in A$.

Recall that an AH algebra is a countable inductive limit of finite direct sums of C^*-algebras of the form $PC(X, M_n)P$, where X is a compact connected metrizable space and $P \in \mathcal{P}(C(X, M_n))$.

We shall denote by $\mathcal{K}$ the C^*-algebra of compact operators on $l^2(\mathbb{N})$.

2. (SEPARABLE) LB ALGEBRAS

In this section we introduce and study two important classes of C^*-algebras: the LB algebras (see Definition 2.2 below) and the separable LB algebras. For a given LB algebra A, we characterize the situation when a fixed ideal of A is generated by projections (in the case when A is separable and also in the general case) (see Theorem 2.3 and Theorem 2.9 below). These theorems give, in a natural way, characterizations of the situation when A has the ideal property (see e.g. Theorem 2.10 below). Also, it is shown that if

$$0 \to I \to A \to B \to 0$$

is an exact sequence of C^*-algebras such that A is a separable LB algebra, then A has the ideal property if and only if I and B have the ideal property (see Theorem 2.11 below).

DEFINITION 2.1. A is called a *basic C^*-algebra* if A is a unital C^*-algebra such that each ideal of A generated (as an ideal) by projections is a direct summand of A.

Note that each finite direct sum of unital C^*-algebras whose proper ideals have no non-zero projections is a basic C^*-algebra. In particular, each finite direct sum of C^*-algebras of the form $PC(X, M_n)P$, where X is a compact connected space and $P \in \mathcal{P}(C(X, M_n))$, is a basic C^*-algebra, and more generally, the C^*-algebra defined by a continuous field of unital, simple C^*-algebras over a compact space with finitely many connected components is a basic C^*-algebra (see [6]). Also, note that each finite direct sum of unital, projectionless C^*-algebras is a basic C^*-algebra.

DEFINITION 2.2. Let A be a C^*-algebra. A is called an LB *algebra* if for each $\varepsilon > 0$ and each finite subset $F \subseteq A$ there exist a basic C^*-algebra B and a $*$-homomorphism $\Phi : B \to A$ such that for each $a \in F$:

$$\operatorname{dist}(a, \Phi(B)) < \varepsilon$$

and such that

$$\operatorname{dist}(p, \Phi(\mathcal{P}(B))) < \varepsilon \quad \text{whenever } p \in F \cap \mathcal{P}(A).$$

Note that each C^*-algebra which is an inductive limit of basic C^*-algebras is an LB algebra. In particular, each AH algebra or, more generally, each GAH algebra ([16]) is an LB algebra. Recall from [16] that a GAH algebra is a C^*-algebra of the form $\varinjlim A_n$, where each A_n, $n \in \mathbb{N}$, is a finite direct sum of unital C^*-algebras whose proper ideals have no non-zero projections. Note also that it is easy to prove that each inductive limit of LB algebras is an LB algebra.

The following result gives several necessary and sufficient conditions for a given ideal of a given separable LB algebra to be generated (as an ideal) by projections.

THEOREM 2.3. *Let A be a separable* LB *algebra and let I be an ideal of A. Then, the following are equivalent:*

(i) *I is generated by projections;*

(ii) *I has a countable approximate unit of projections;*

(iii) *the canonical extension*

$$0 \to I \to A \to A/I \to 0$$

is quasidiagonal;

(iv) *I is an* LB *algebra.*

The proof of the above theorem will need the following four lemmas. The first one is the elaboration of an argument of Hjelmborg and Rørdam in [10].

LEMMA 2.4. (see the proof of Lemma 3.1, [10]) *Let A be a C^*-algebra and let $(p_n)_{n=1}^{\infty}$ be a sequence of projections in A such that $\lim_{n\to\infty} \|p_n x - x\| = 0$ for each $x \in A$. Then, there is an increasing sequence of projections $(q_k)_{k=1}^{\infty}$ in A and there are natural members $n_1 < n_2 < \cdots < n_k < \cdots$ such that $\lim_{k\to\infty} \|q_k - p_{n_k}\| = 0$. (In particular $(q_k)_{k=1}^{\infty}$ is an approximate unit of projections for A.)*

Proof. By the proof of Lemma 3.1, [10], it follows that for each projection $q \in A$, there exists a sequence of projections $(\widetilde{p}_n)_{n=1}^{\infty}$ in A such that $\widetilde{p}_n \geqslant q$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|\widetilde{p}_n - p_n\| = 0$. Using this fact, one can easily construct by mathematical induction an increasing sequence of projections $(q_k)_{k=1}^{\infty}$ in A and a strictly increasing sequence of natural numbers $(n_k)_{k=1}^{\infty}$ such that

$$\|q_k - p_{n_k}\| \leqslant \frac{1}{k}, \quad k \in \mathbb{N}.$$

This implies that $\lim_{k\to\infty} \|q_k - p_{n_k}\| = 0$ and hence that $(q_k)_{k=1}^{\infty}$ is an approximate unit of projections of A. The proof is over. ∎

Lemma 2.5. *Let A be an* LB *algebra. Then, A is generated (as an ideal of A) by projections.*

Proof. Let $a \in A$ and let $\varepsilon > 0$. Then, since A is an LB algebra there are a basic C^*-algebra B, a $*$-homomorphism $\Phi : B \to A$ and $\widetilde{a} \in B$ such that:

$$\|\Phi(\widetilde{a}) - a\| < \varepsilon.$$

Denote $e = \Phi(1_B) \in \mathcal{P}(A)$. Since $\Phi(\widetilde{a}) = \Phi(\widetilde{a}) \cdot e \cdot e$, we have:

$$\|\Phi(\widetilde{a}) \cdot e \cdot e - a\| < \varepsilon.$$

Hence, A is generated (as an ideal) by projections. ∎

Lemma 2.6. *Let E be a separable C^*-subalgebra of an* LB *algebra A. Suppose that I is an ideal of A generated by projections. If $I \subseteq E$, then the canonical extension*

$$0 \to I \to E \to E/I \to 0$$

is quasidiagonal.

Proof. Since E is separable, let $(x_m)_{m=1}^{\infty}$ be a dense sequence in E and let $(y_m)_{m=1}^{\infty}$ be a dense sequence in I. We are going to construct for each $n \in \mathbb{N}$ a projection p_n in I such that:

$$\|p_n x_k - x_k p_n\| < \frac{1}{n}, \quad 1 \leqslant k \leqslant n, \tag{2.1}$$

$$\|y_k - y_k p_n\| < \frac{1}{n}, \quad 1 \leqslant k \leqslant n. \tag{2.2}$$

Obviously, these facts together with Lemma 2.4 will prove our lemma.

Fix now $n \in \mathbb{N}$. Since (by hypothesis) I is generated as an ideal of A by projections, we have that:

$$\Big\|y_k - \sum_{i=1}^{l_n} a_{k,i}^{(n)} \mathrm{e}_{k,i}^{(n)} b_{k,i}^{(n)}\Big\| < \frac{1}{4n}, \quad 1 \leqslant k \leqslant n \tag{2.3}$$

for some $l_n \in \mathbb{N}$, $a_{k,i}^{(n)}$, $b_{k,i}^{(n)} \in A$ and $\mathrm{e}_{k,i}^{(n)} \in \mathcal{P}(I)$, $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant l_n$. Let $0 < \delta < \frac{1}{2n}$ which will be precised later. Since A is an LB algebra, it follows that there are a basic C^*-algebra B, a $*$-homomorphism $\Phi : B \to A$ and $\widetilde{x}_k^{(n)}, \widetilde{a}_{k,i}^{(n)}, \widetilde{b}_{k,i}^{(n)} \in B$, $\widetilde{e}_{k,i}^{(n)} \in \mathcal{P}(B)$, $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant l_n$ such that

$$\|\Phi(\widetilde{x}_k^{(n)}) - x_k\| < \delta, \quad 1 \leqslant k \leqslant n, \tag{2.4}$$

$$\|\Phi(\widetilde{a}_{k,i}^{(n)}) - a_{k,i}^{(n)}\| < \delta, \quad 1 \leqslant k \leqslant n,\ 1 \leqslant i \leqslant l_n, \tag{2.5}$$

$$\|\Phi(\widetilde{e}_{k,i}^{(n)}) - \mathrm{e}_{k,i}^{(n)}\| < \delta, \quad 1 \leqslant k \leqslant n,\ 1 \leqslant i \leqslant l_n, \tag{2.6}$$

$$\|\Phi(\widetilde{b}_{k,i}^{(n)}) - b_{k,i}^{(n)}\| < \delta, \quad 1 \leqslant k \leqslant n,\ 1 \leqslant i \leqslant l_n. \tag{2.7}$$

Let L be the ideal of B generated by the projections $\widetilde{e}_{k,i}^{(n)}$, $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant l_n$. Since B is a basic C^*-algebra, it follows that $L = B\widetilde{p}_n$, where $\widetilde{p}_n$

is a central projection of B. Since $\widetilde{p}_n$ belongs to the center of B, it follows that $\widetilde{x}_k^{(n)}\widetilde{p}_n = \widetilde{p}_n\widetilde{x}_k^{(n)}$, $1 \leqslant k \leqslant n$, which, after denoting $p_n := \Phi(\widetilde{p}_n)$, implies that

$$\Phi(\widetilde{x}_k^{(n)})p_n = p_n\Phi(\widetilde{x}_k^{(n)}), \quad 1 \leqslant k \leqslant n. \tag{2.8}$$

Now, denote for any $1 \leqslant k \leqslant n$, $\widetilde{y}_k^{(n)} = \sum_{i=1}^{l_n} \widetilde{a}_{k,i}^{(n)}\widetilde{e}_{k,i}^{(n)}\widetilde{b}_{k,i}^{(n)}$. Since $\widetilde{e}_{k,i}^{(n)} \in L$ for all the indices, it follows that $\widetilde{y}_k^{(n)} \in L$, $1 \leqslant k \leqslant n$. Hence, since $\widetilde{p}_n$ is the unit of L, it follows that $\widetilde{y}_k^{(n)} = \widetilde{y}_k^{(n)}\widetilde{p}_n$, $1 \leqslant k \leqslant n$, which obviously implies that:

$$\Phi(\widetilde{y}_k^{(n)}) = \Phi(\widetilde{y}_k^{(n)})p_n, \quad 1 \leqslant k \leqslant n. \tag{2.9}$$

Observe that $p_n \in I$ since $p_n = \Phi(\widetilde{p}_n)$, $\widetilde{p}_n \in L$ and $\Phi(L) \subseteq I$. (Indeed, L is generated by the projections $\widetilde{e}_{k,i}^{(n)}$, $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant l_n$ and $\|\Phi(\widetilde{e}_{k,i}^{(n)}) - \mathrm{e}_{k,i}^{(n)}\| < \delta < \frac{1}{2n} < 1$ implies $\Phi(\widetilde{e}_{k,i}^{(n)}) \overset{A}{\sim} \mathrm{e}_{k,i}^{(n)} \in \mathcal{P}(I)$ from which one concludes that $\Phi(\widetilde{e}_{k,i}^{(n)}) \in I$ for all the indices since I is an ideal of A; in conclusion $\Phi(L) \subseteq I$.)

Now suppose that $0 < \delta < \frac{1}{2n}$ is small enough such that the inequalities (2.5), (2.6) and (2.7) will imply that

$$\Big\| \sum_{i=1}^{l_n} a_{k,i}^{(n)}\mathrm{e}_{k,i}^{(n)}b_{k,i}^{(n)} - \Phi(\widetilde{y}_k^{(n)}) \Big\| < \frac{1}{4n}, \quad 1 \leqslant k \leqslant n. \tag{2.10}$$

Observe that (2.3) and (2.10) imply that

$$\|y_k - \Phi(\widetilde{y}_k^{(n)})\| < \frac{1}{4n} + \frac{1}{4n} = \frac{1}{2n}, \quad 1 \leqslant k \leqslant n. \tag{2.11}$$

Using (2.4) and (2.8), one has for any $1 \leqslant k \leqslant n$

$$\begin{aligned} &\|p_nx_k - x_kp_n\| \\ &\quad \leqslant \|p_n\Phi(\widetilde{x}_k^{(n)}) - \Phi(\widetilde{x}_k^{(n)})p_n\| + \|p_n(x_k - \Phi(\widetilde{x}_k^{(n)}))\| + \|(\Phi(\widetilde{x}_k^{(n)}) - x_k)p_n\| \\ &\quad < 0 + \delta + \delta = 2\delta < 2 \cdot \frac{1}{2n} = \frac{1}{n} \end{aligned}$$

which proves (2.1). Finally, to prove (2.2), observe that using (2.9) and (2.11), one has for each $1 \leqslant k \leqslant n$

$$\begin{aligned} \|y_k - y_kp_n\| &\leqslant \|y_k - \Phi(\widetilde{y}_k^{(n)})\| + \|\Phi(\widetilde{y}_k^{(n)}) - \Phi(\widetilde{y}_k^{(n)})p_n\| + \|(\Phi(\widetilde{y}_k^{(n)}) - y_k)p_n\| \\ &< \frac{1}{2n} + 0 + \frac{1}{2n} = \frac{1}{n}. \end{aligned}$$

This ends the proof. ∎

The above lemma generalizes a joint result of Brown and Dadarlat ([2], Proposition 11) and it also generalizes Lemma 2.8, [16] in the separable case.

LEMMA 2.7. *Let A be an* LB *algebra and let I be an ideal of A. Suppose that I is generated by projections. Then I is an* LB *algebra.*

Proof. Let $\varepsilon > 0$, $x_1, x_2, \dots, x_n \in I$ and $f_1, f_2, \dots, f_m \in \mathcal{P}(I)$. By the hypothesis on I, it follows that there are $a_{k,i}, b_{k,i} \in A$, $e_{k,i} \in P(I)$, $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant l$ such that

$$\Big\| x_k - \sum_{i=1}^{l} a_{k,i} e_{k,i} b_{k,i} \Big\| < \frac{\varepsilon}{2}, \quad 1 \leqslant k \leqslant n. \tag{2.12}$$

Let $0 < \delta < 1$ be a number which will be precised later. Since A is an LB algebra, it follows that there are a basic C^*-algebra B, a $*$-homomorphism $\Phi : B \to A$ and $\widetilde{a}_{k,i}, \widetilde{b}_{k,i} \in B$, $\widetilde{e}_{k,i} \in \mathcal{P}(B)$ with $1 \leqslant k \leqslant n$ and $1 \leqslant i \leqslant l$, $\widetilde{f}_j \in \mathcal{P}(B)$, $1 \leqslant j \leqslant m$ such that

$$\|\Phi(\widetilde{a}_{k,i}) - a_{k,i}\| < \delta, \quad 1 \leqslant k \leqslant n, 1 \leqslant i \leqslant l, \tag{2.13}$$

$$\|\Phi(\widetilde{e}_{k,i}) - e_{k,i}\| < \delta, \quad 1 \leqslant k \leqslant n, 1 \leqslant i \leqslant l, \tag{2.14}$$

$$\|\Phi(\widetilde{b}_{k,i}) - b_{k,i}\| < \delta, \quad 1 \leqslant k \leqslant n, 1 \leqslant i \leqslant l, \tag{2.15}$$

$$\|\Phi(\widetilde{f}_j) - f_j\| < \delta, \quad 1 \leqslant j \leqslant m. \tag{2.16}$$

Let M be the ideal of B generated by the projections $\widetilde{e}_{k,i}, \widetilde{f}_j$, $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant m$. Observe that

$$\Phi(M) \subseteq I. \tag{2.17}$$

Indeed, it is enough to prove that $\Phi(\widetilde{e}_{k,i}), \Phi(\widetilde{f}_j) \in I$ for each $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant m$. Since for all the indices we have by (2.14) that $\|\Phi(\widetilde{e}_{k,i}) - e_{k,i}\| < \delta < 1$, it follows that $\Phi(\widetilde{e}_{k,i}) \overset{A}{\sim} e_{k,i} \in I$, which implies that $\Phi(\widetilde{e}_{k,i}) \in I$, for all k and i since I is an ideal of A. Similarly, since by (2.16), we have $\|\Phi(\widetilde{f}_j) - f_j\| < \delta < 1$ we have that $\Phi(\widetilde{f}_j) \in I$, $1 \leqslant j \leqslant m$ (since $f_j \in \mathcal{P}(I)$ for $1 \leqslant j \leqslant m$).

Now, suppose that $0 < \delta < 1$ is small enough such that $\delta < \varepsilon$ and (2.13), (2.14) and (2.15) imply that:

$$\Big\| \sum_{i=1}^{l} a_{k,i} e_{k,i} b_{k,i} - \Phi\Big(\sum_{i=1}^{l} \widetilde{a}_{k,i} \widetilde{e}_{k,i} \widetilde{b}_{k,i} \Big) \Big\| < \frac{\varepsilon}{2}, \quad 1 \leqslant k \leqslant n. \tag{2.18}$$

Define a $*$-homomorphism $\Psi : M \to I$ by $\Psi(x) := \Phi(x)$, $x \in M$. By (2.17), this definition is correct. Note also that since B is a basic C^*-algebra and M is an ideal of B generated by projections, it follows that M is also a basic C^*-algebra. Observe that since $\widetilde{e}_{k,i} \in M$, $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant l$ and M is an ideal of B, it follows that $\widetilde{x}_k := \sum\limits_{i=1}^{l} \widetilde{a}_{k,i} \widetilde{e}_{k,i} \widetilde{b}_{k,i} \in M$, $1 \leqslant k \leqslant n$. Now, by (2.12) and (2.18) we have

$$\begin{aligned} \|x_k - \Psi(\widetilde{x}_k)\| = \|x_k - \Phi(\widetilde{x}_k)\| &\leqslant \Big\| x_k - \sum_{i=1}^{l} a_{k,i} e_{k,i} b_{k,i} \Big\| + \Big\| \sum_{i=1}^{l} a_{k,i} e_{k,i} b_{k,i} - \Phi(\widetilde{x}_k) \Big\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad 1 \leqslant k \leqslant n. \end{aligned}$$

Also, $\widetilde{f}_j \in \mathcal{P}(M), 1 \leqslant j \leqslant m$ and by (2.16) we have:

$$\|\Psi(\widetilde{f}_j) - f_j\| = \|\Phi(\widetilde{f}_j) - f_j\| < \delta < \varepsilon$$

for each $1 \leqslant j \leqslant m$.

The proof is over. ∎

Proof of Theorem 2.3. The proof of the implication (i) ⇒ (iii) follows from Lemma 2.6 and the proof of the implication (i) ⇒ (iv) follows from Lemma 2.7. Since the proofs of the implications (iii) ⇒ (ii), (ii) ⇒ (i) are obvious and since the proof of the implication (iv) ⇒ (i) follows from Lemma 2.5 , the proof of the theorem is over. ∎

The equivalence (i) ⇔ (ii) in the above Theorem 2.3 generalizes, in the separable case, the equivalence (1) ⇔ (3) in Theorem 2.2, [16]. Note that the proof of Theorem 2.3 needed new type of arguments.

PROPOSITION 2.8. *Any separable* LB *algebra has a countable approximate unit of projections.*

Proof. Let A be a separable a LB algebra. By Lemma 2.5, it follows that A is generated as an ideal of A by projections. Hence, by the implication (i) ⇒ (ii) in Theorem 2.3 above, A has a countable approximate unit of projections. ∎

Combining Lemma 2.5 with Lemma 2.7 above, one can prove the following:

THEOREM 2.9. *Let A be an* LB *algebra. Let I be an ideal of A. Then, the following are equivalent:*

(i) *I is generated by projections;*

(ii) *I is an* LB *algebra.*

Recall that a C^*-algebra is said to have the *ideal property* if each ideal is generated by projections and is said to have the *projection property* ([19]) if each ideal has an approximate unit of projections. The above Theorem 2.3 implies immediately the following:

THEOREM 2.10. *Let A be a separable* LB *algebra. Then, the following are equivalent:*

(i) *A has the ideal property;*

(ii) *each ideal of A has a countable approximate unit of projections;*

(ii′) *A has the projection property;*

(iii) *for each ideal I of A , the canonical extension*

$$0 \to I \to A \to A/I \to 0$$

is quasidiagonal;

(iv) *each ideal of A is an* LB *algebra.*

The above equivalence (i) ⇔ (ii) generalizes the equivalence (i) ⇔ (iv) in Theorem 3.1, [14], and in the separable case, the equivalence (1) ⇔ (3) in Corollary 2.4, [16].

While the class of the C^*-algebras with the ideal property is not closed under extensions (as it follows from an example constructed jointly with Dadarlat in [15] and which also implies that the class of LB algebras with the ideal property is not closed extensions as well as the class of separable LB algebras with the ideal property) one has the following:

Theorem 2.11. *Let $0 \to I \to A \to B \to 0$ be an exact sequence of C^*-algebras. Suppose that A is a separable* LB *algebra. Then, the following are equivalent:*

(i) *A has the ideal property;*

(ii) *I and B have the ideal property.*

The above theorem generalizes Theorem 3.1, [13] and it also generalizes Theorem 2.6, [16] in the separable case.

Proof of Theorem 2.11. It is similar with the proof of Theorem 2.6, [16] and uses the above Lemma 2.6, Lemma 3.9 (1), [13] and Lemma 2.10, [16]. ∎

3. (SEPARABLE) SPECIAL LB ALGEBRAS

In this section we introduce and study two important classes of LB algebras: the special LB algebras (see Definition 3.4 below) and the separable special LB algebras. We prove that if A is a special LB algebra and if I is an ideal of A, then I is generated by projections if and only if I is a special LB algebra (see Theorem 3.6 below). We describe the lattice of the ideals generated by projections of a special LB algebra (see Theorem 3.9 below) and also we describe the partially ordered set of the stably cofinite ideals generated by projections of a separable, special LB algebra A when the projections in $M_\infty(A)$ satisfy the Riesz decomposition property (see Theorem 3.14 below). We also prove that if a special LB algebra is stably isomorphic to a C^*-algebra with the ideal property, then it has the ideal property (see Corollary 3.12 below).

Definition 3.1. *A C^*-*algebra A is called a *stably basic C^*-algebra* if for each $n \in \mathbb{N}$, $M_n(A)$ is a basic C^*-algebra.

Remark 3.2. Note that the C^*-algebra defined by a continuous field of unital, simple C^*-algebras over a compact space with finitely many connected components is a stably basic C^*-algebra. In particular, each C^*-algebra which is a finite direct sum of C^*-algebras of the form $PC(X, M_n)P$ where X is a compact, connected space and $P \in \mathcal{P}(C(X, M_n))$ or each C^*-algebra which is a finite direct sum of C^*-algebras of the form $C(X, A)$, where X is a compact, connected space and A is a simple, unital C^*-algebra, is a stably basic C^*-algebra. ∎

Lemma 3.3. *Let A be a C^*-algebra. Then the following are equivalent:*

(i) *A is a stably basic C^*-algebra;*

(ii) *A is unital and for each $n \in \mathbb{N}$ and each ideal I of $M_n(A)$ generated by projections, $I = M_n(J)$ where J is a direct summand of A.*

Proof. (i) ⇒ (ii) Suppose that A is a stably basic C^*-algebra. The fact that A is unital follows from the fact that A is a basic C^*-algebra. Let $n \in \mathbb{N}$ and let I be an ideal of $M_n(A)$ generated by projections. By hypothesis, $I = M_n(A)p$, where $p \in \mathcal{P}(\mathcal{Z}(M_n(A))$. But

$$\mathcal{Z}(M_n(A)) = \mathcal{Z}(M_n \otimes A) = \mathcal{Z}(M_n) \otimes \mathcal{Z}(A) = \mathbb{C} \otimes \mathcal{Z}(A) = \mathbb{1}_n \otimes \mathcal{Z}(A),$$

where $\mathbb{1}_n$ is the unit of M_n. Hence, since $p \in \mathcal{P}(\mathcal{Z}(M_n(A))$, it follows that $p = \mathbb{1}_n \otimes r$, where $r \in \mathcal{P}(\mathcal{Z}(A))$. Hence

$$I = M_n(A)p = (M_n \otimes A)(\mathbb{1}_n \otimes r) = M_n \otimes Ar = M_n(Ar) = M_n(J),$$

where $J := Ar$ is a direct summand of A (since $r \in \mathcal{P}(\mathcal{Z}(A))$.

(ii) $\Rightarrow$ (i) The proof of this implication is obvious. ∎

DEFINITION 3.4. Let A be a C^*-algebra. A is called a *special LB algebra* if for each $\varepsilon > 0$, each $k \in \mathbb{N}$ and each finite subset $F \subseteq M_k(A)$ there exist a stably basic C^*-algebra B and a $*$-homomorphism $\Phi : B \to A$ such that for any $a \in F$, $\operatorname{dist}(a, (\Phi \otimes \operatorname{id}_{M_k})(M_k(B))) < \varepsilon$ and such that

$$\operatorname{dist}(p, (\Phi \otimes \operatorname{id}_{M_k})(\mathcal{P}(M_k(B)))) < \varepsilon \quad \text{whenever } p \in F \cap (\mathcal{P}(M_k(A)).$$

REMARK 3.5. Obviously, any special LB algebra is an LB algebra. Moreover, if A is a special LB algebra and $n \in \mathbb{N}$, then $M_n(A)$ is a special LB algebra.

THEOREM 3.6. *Let A be a special* LB *algebra. Let I be an ideal of A. Then, the following are equivalent:*

(i) *I is generated by projections;*

(ii) *I is a special* LB *algebra.*

Proof. It is similar with the proof of Theorem 2.9 above and uses the above Lemma 3.3 and the fact that if B is a stably basic C^*-algebra then each ideal of B generated by projections is a stably basic C^*-algebra too. ∎

PROPOSITION 3.7. *Let $A = \varinjlim A_\lambda$ where each A_λ, for $\lambda \in \Lambda$, is a special* LB *algebra. Then, A is a special* LB *algebra.*

Proof. It is standard and it is left to the reader. ∎

PROPOSITION 3.8. *Let A be a special* LB *algebra and let B be an* AF *algebra. Then, $A \otimes B$ is a special* LB *algebra.*

Proof. Observe that by the above Remark 3.5, for each $n \in \mathbb{N}$, $M_n(A)$ is a special LB algebra. Since the class of special LB algebras is obviously closed under finite direct sums, it follows that $A \otimes F$ is a special LB algebra for any finite dimensional C^*-algebra F. Now, the result follows using Proposition 3.7. ∎

One of the main results of this section is the following:

THEOREM 3.9. *Let A be a special* LB *algebra. Then, there is a canonical lattice isomorphism:*

$$\{I : I \textit{ is an ideal of } A \textit{ generated by projections}\}$$
$$\cong \{J : J \textit{ is an ideal of } D(A \otimes \mathcal{K})\}.$$

In the above theorem we used the standard notation $D(B)$, where B is a C^*-algebra, to denote the abelian local semigroup of Murray-von Neumann equivalence classes of projections in B (the addition of two classes is defined when they have orthogonal representatives). Also, recall that an ideal in $D(B)$ is a nonempty hereditary subset which is closed under addition, where defined.

The proof of the above Theorem 3.9 will use the following result, which generalizes Lemma 4.5, [15] and Lemma 2.14, [16] (see also (the proof of) Proposition 4.3, [17]) and requires a different idea of proof:

Lemma 3.10. *Let A be a special* LB *algebra. Then, the map*

$$\Phi : \{I : I \text{ is an ideal of } A\} \to \{J : J \text{ is an ideal of } A \otimes \mathcal{K}\}$$

given by $\Phi(I) = I \otimes \mathcal{K}$ for any ideal I of A is a lattice isomorphism such that:

$$\Phi(\{I : I \text{ is an ideal of } A \text{ generated by projections}\}) \\ = \{J : J \text{ is an ideal of } A \otimes \mathcal{K} \text{ generated by projections }\}.$$

In particular, A has the ideal property if and only if $A \otimes \mathcal{K}$ has the ideal property and Φ induces a canonical lattice isomorphism

$$\{I : I \text{ is an ideal of } A \text{ generated by projections}\} \\ \cong \{J : J \text{ is an ideal of } A \otimes \mathcal{K} \text{ generated by projections}\}.$$

Proof. Let J be an ideal of $A \otimes \mathcal{K}$. It is known that then there is a unique ideal I of A such that $J = I \otimes \mathcal{K}$. To prove our lemma it will be enough to show that if J is generated by projections, then I is generated by projections (since the remaining part of the proof is trivial by Remark 4.3, [15]) (see also the final part of the proof of Lemma 4.5, [15]). Hence, let us assume from now on that J is generated by projections. Let $A \otimes \mathcal{K} = \varinjlim(M_n(A), \Phi_n)$ where each $\Phi_n : M_n(A) \to M_{n+1}(A)$ is given by $a \mapsto a \oplus 0$, $a \in M_n(A)$. For each $m, n \in \mathbb{N}$, $n \leqslant m$, we shall identify $M_n(A)$ with its canonical image in $A \otimes \mathcal{K}$ which is $\Phi_{n,\infty}(M_n(A))$ and we shall simply write $M_n(A) \subseteq A \otimes \mathcal{K}$ and we shall identify $M_n(A)$ with its canonical image in $M_m(A)$ and we shall simply write $M_n(A) \subseteq M_m(A)$.

Let $\varepsilon > 0$ and let $x \in I \subseteq I \otimes \mathcal{K} = \varinjlim(M_n(I), \Phi_{n|M_n(I)})$. Then, since $I \otimes \mathcal{K}$ is generated by projections, there is $m \in \mathbb{N}$, $l \in \mathbb{N}$, $a_i, b_i \in M_m(A)$, $p_i \in \mathcal{P}(M_m(I))$, $1 \leqslant i \leqslant l$, such that

$$\Big\| x - \sum_{i=1}^{l} a_i p_i b_i \Big\| < \frac{\varepsilon}{2}. \tag{3.1}$$

Since A is a special LB algebra it follows that for a given $0 < \delta < 1$ (which will be precised later) there are a stably basic C^*-algebra B, a $*$-homomorphism $\Phi : B \to A$ and $\widetilde{a}_i, \widetilde{b}_i \in M_m(B)$, $\widetilde{p}_i \in \mathcal{P}(M_m(B))$ for $1 \leqslant i \leqslant l$ such that if we denote $\Psi := \Phi \otimes \mathrm{id}_{M_m} : M_m(B) \to M_m(A)$, we have

$$\|\Psi(\widetilde{p}_i) - p_i\| < \delta, \quad \|\Psi(\widetilde{a}_i) - a_i\| < \delta, \quad \|\Psi(\widetilde{b}_i) - b_i\| < \delta, \quad 1 \leqslant i \leqslant l.$$

Let J be the ideal of $M_m(B)$ generated by the projections $\widetilde{p}_1, \widetilde{p}_2, \dots, \widetilde{p}_l$. By Lemma 3.3 it follows that $J = M_m(K)$, where K is a direct summand of B. Observe that $\Psi(J) \subseteq M_m(I)$ (indeed, $\|\Psi(\widetilde{p}_i) - p_i\| < \delta < 1$, $1 \leqslant i \leqslant l$ implies that for each $1 \leqslant i \leqslant l$, $\Psi(\widetilde{p}_i) \overset{M_m(A)}{\sim} p_i \in \mathcal{P}(M_m(I))$ and hence $\Psi(\widetilde{p}_i) \in \mathcal{P}(M_m(I))$ since $M_m(I)$ is an ideal of $M_m(A)$).

Hence, we can define $\Psi_1 : M_m(K) \to M_m(I)$ by $\Psi_1(x) := \Psi(x)$ $(= (\Phi \otimes \mathrm{id}_{M_m})(x))$, for $x \in M_m(K)$. It follows that

$$\Phi(K) \subseteq I. \tag{3.2}$$

Observe that $\sum_{i=1}^{l} \widetilde{a}_i \widetilde{p}_i \widetilde{b}_i \in J = M_m(K)$ (since $\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_l \in J$) and that if $0 < \delta < 1$ is small enough, then

$$\Big\| \sum_{i=1}^{l} a_i p_i b_i - \Psi_1\Big(\sum_{i=1}^{l} \widetilde{a}_i \widetilde{p}_i \widetilde{b}_i\Big)\Big\| < \frac{\varepsilon}{2}. \tag{3.3}$$

But (3.1) and (3.3) obviously imply

$$\Big\| x - \Psi_1\Big(\sum_{i=1}^{l} \widetilde{a}_i \widetilde{p}_i \widetilde{b}_i\Big)\Big\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{3.4}$$

Let $\{e_{i,j}\}_{i,j=1}^{m}$ be the canonical system of matrix units of M_m. Then, there are $y_{i,j} \in K$, $1 \leqslant i, j \leqslant m$, such that

$$\Psi_1\Big(\sum_{i=1}^{l} \widetilde{a}_i \widetilde{p}_i \widetilde{b}_i\Big) = \sum_{i,j=1}^{m} \Phi(y_{ij}) \otimes e_{ij}.$$

Then (3.4) can be written

$$\Big\| x \otimes e_{11} - \sum_{i,j=1}^{m} \Phi(y_{ij}) \otimes e_{ij} \Big\| < \varepsilon$$

which implies

$$\begin{aligned} \|x - \Phi(y_{11})\| &= \Big\| (1 \otimes e_{11}) \cdot \Big(x \otimes e_{11} - \sum_{i,j=1}^{m} \Phi(y_{ij}) \otimes e_{ij} \Big) \cdot (1 \otimes e_{11}) \Big\| \\ &\leqslant \Big\| x \otimes e_{11} - \sum_{i,j=1}^{m} \Phi(y_{ij}) \otimes e_{ij} \Big\| < \varepsilon. \end{aligned} \tag{3.5}$$

Let $e \in \mathcal{P}(\mathcal{Z}(B))$ such that $K = Be$. By (3.2) it follows that $\Phi(e) \in \mathcal{P}(I)$ and since $y_{11} = e \cdot e \cdot y_{11}$ (because $y_{11} \in K$) then, $\Phi(y_{11}) = \Phi(e)\Phi(e)\Phi(y_{11}) \in I$ and (3.5) becomes

$$\|x - \Phi(e) \cdot \Phi(e) \cdot \Phi(y_{11})\| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that x belongs to the ideal of A generated by the projections of I. Hence I is generated by its projections. This ends the proof. ∎

Proof of Theorem 3.9. The proof follows using the proof of Lemma 4.2, [15], Lemma 4.3, [15] and the above Lemma 3.10. ∎

REMARK 3.11. The above Theorem 3.9 generalizes Theorem 2.13, [16].

COROLLARY 3.12. *Let A be a special* LB *algebra which is stably isomorphic to a C^*-algebra B. If B has the ideal property, then A has the ideal property.*

Proof. By Proposition 2.4, [15], it follows that $B \otimes \mathcal{K}$ has the ideal property. Hence $A \otimes \mathcal{K}$ has the ideal property and, by Lemma 3.10, it follows that A has the ideal property. ∎

Let us recall the following:

DEFINITION 3.13. An ideal I in a C^*-algebra A is said to be *stably cofinite* if the C^*-algebra A/I is stably finite, i.e. there do not exist projections $p, q \in M_\infty(A/I)$ such that $p \oplus q \sim q$ and $p \neq 0$.

Now we shall prove the following:

THEOREM 3.14. *Let A be a separable, special* LB *algebra such that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property. Then, there is an order isomorphism*

$$\{I : I \text{ is a stably cofinite ideal of } A \text{ generated by projections}\} \\ \cong \{J : J \text{ is an ideal of } K_0(A)\}.$$

More precisely, there are order-preseving inverse isomorphisms sending each stably cofinite ideal I of A generated by projections to the kernel of K_0 of the quotient map $A \to A/I$ and sending each ideal J of $K_0(A)$ to the ideal of A generated by those projections $p \in A$ for which $[p] \in J$. (Here, by an ideal of $K_0(A)$ we mean a subgroup H of $K_0(A)$ such that $H^+ := H \cap K_0(A)^+$ is hereditary (i.e. if $0 \leqslant g \leqslant h$ for some $g \in K_0(A)$ and $h \in H^+$, then $g \in H$) and $H = H^+ - H^+$).

The proof of the above theorem will use the following:

LEMMA 3.15. *Let A be a separable, special* LB *algebra and let I be an ideal of A generated by projections. Then, for each $n \in \mathbb{N}$, all the projections in $M_n(A/I)$ lift to projections in $M_n(A)$.*

Proof. Let $n \in \mathbb{N}$ be an arbitrary, fixed number. Since $M_n(I)$ is an ideal generated by projections of the separable (special) LB algebra $M_n(A)$, by the implication (i) $\Rightarrow$ (iii) in Theorem 2.3 above it follows that the extension

$$0 \to M_n(I) \to M_n(A) \to M_n(A/I) \to 0$$

is quasidiagonal. Now, by Lemma 3.9 (1), [13], it follows that all the projections in $M_n(A/I)$ lift to projections in $M_n(A)$. ∎

Proof of Theorem 3.14. The argument of the proof is similar with the one given in the proof of Lemma 4.10, [15] or in the proof of Theorem 2.16, [16]. The proof uses the implication (i) $\Rightarrow$ (ii) in Theorem 2.3 above, the six term exact sequence in K-theory, the above Lemma 3.15 and the argument in Lemma 10.8 (a), [9]. ∎

4. (SEPARABLE) ULTRASPECIAL LB ALGEBRAS

In this section we introduce and study two important classes of special LB algebras: the ultraspecial LB algebras (see Definition 4.1 below) and the class of separable, ultraspecial LB algebras, denoted by $\mathcal{U}$. We prove that for a given C^*-algebra A in $\mathcal{U}$, A has real rank zero if and only if each hereditary C^*-subalgebra of A belongs to $\mathcal{U}$ (see Theorem 4.8 below) and that A has the ideal property if and only if each ideal of A belongs to $\mathcal{U}$ (see Theorem 4.14 below). In fact, given an ultraspecial LB algebra B, it is proved that B has the ideal property if and only if each ideal of B is an ultraspecial LB algebra (see Theorem 4.12 below). Also, it is shown that the above class of C^*-algebras $A \in \mathcal{U}$ with $\mathrm{RR}(A) = 0$ is closed under stable isomorphism (see Corollary 4.9 below). It is proved that if a C^*-algebra with an approximate unit of projections is stably isomorphic to an ultraspecial LB algebra, then it is an ultraspecial LB algebra (see Proposition 4.10 below). It is worth to mention that as an easy consequence of Theorem 4.15 below, it follows that none of the following six classes of C^*-algebras: $\mathcal{U}$, ultraspecial LB algebras, separable special LB algebras, special LB algebras, separable LB algebras and LB algebras is closed under extensions.

DEFINITION 4.1. Let A be a special LB algebra. A is an *ultraspecial* LB *algebra* if $pM_n(A)p$ is a special LB algebra for each $n \in \mathbb{N}$ and each $p \in \mathcal{P}(M_n(A))$.

REMARK 4.2. Each C^*-algebra of the form $\varinjlim A_\lambda$, where each A_λ with $\lambda \in \Lambda$ is a stably basic C^*-algebra, is an ultraspecial LB algebra. This follows using the fact that $\varinjlim A_\lambda$ is a special LB algebra by Proposition 3.7 above (since each $A_\lambda, \lambda \in \Lambda$, is a special LB algebra), using the fact that if B is a stably basic C^*-algebra, $n \in \mathbb{N}$ and $p \in \mathcal{P}(M_n(B))$, then $pM_n(B)p$ is also a stably basic C^*-algebra and hence a special LB algebra (if A is a basic C^*-algebra, $m \in \mathbb{N}$ and $q \in \mathcal{P}(M_m(A))$, then $qM_m(A)q$ is a basic C^*-algebra) and using also the fact that an inductive limit of special LB algebras is a special LB algebra (Proposition 3.7 above).

PROPOSITION 4.3. *Let A be an ultraspecial* LB *algebra and let $n \in \mathbb{N}$ and $p \in \mathcal{P}(M_n(A))$. Then $pM_n(A)p$ and $M_n(A)$ are ultraspecial* LB *algebras.*

Proof. Let $m \in \mathbb{N}$ and $q \in \mathcal{P}(M_m(pM_n(A)p))$. Let $p^{(m)} \in \mathcal{P}(M_m(M_n(A))) = \mathcal{P}(M_{mn}(A))$ be the direct sum of m copies of p. Then, we have:

$$qM_m(pM_n(A)p)q = q(p^{(m)}M_{mn}(A)p^{(m)})q = qM_{mn}(A)q.$$

Since, by hypothesis, $pM_n(A)p$ and $qM_{mn}(A)q$ are special LB algebras, it follows that $pM_n(A)p$ is an ultraspecial LB algebra.

Let $r \in \mathcal{P}(M_m(M_n(A))) = \mathcal{P}(M_{mn}(A))$. Then

$$rM_m(M_n(A))r = rM_{mn}(A)r.$$

Since, by hypothesis, $rM_{mn}(A)r$ is a special LB algebra, and since, obviously, $M_n(A)$ is a special LB algebra since A is, it follows that $M_n(A)$ is an ultraspecial LB algebra. ∎

PROPOSITION 4.4. *Let A be a C^*-algebra of the form $A = \varinjlim A_\lambda$, where each A_λ, $\lambda \in \Lambda$ is an ultraspecial* LB *algebra. Then, A is an ultraspecial* LB *algebra.*

Proof. Since each $A_\lambda, \lambda \in \Lambda$, is a special LB algebra (by hypothesis), it follows by Proposition 3.7 above that A is a special LB algebra. Let $n \in \mathbb{N}$ and $p \in \mathcal{P}(M_n(A))$. Then, by a standard argument, we may suppose that $pM_n(A)p \cong \varinjlim p_\lambda M_n(A_\lambda)p_\lambda$, where for each $\lambda \in \Lambda, p_\lambda$ is some projection in $M_n(A_\lambda)$. By hypothesis, each $p_\lambda M_n(A_\lambda)p_\lambda$, $\lambda \in \Lambda$ is a special LB algebra. Now, the proof ends using the fact that the set of special LB algebras is closed under inductive limits (Proposition 3.7 above).

NOTATION 4.5. Let $\mathcal{U}$ be the class of separable, ultraspecial LB algebras.

REMARK 4.6. Observe that if A is an AH algebra, then $A \in \mathcal{U}$. ∎

PROPOSITION 4.7. *Let A be an ultraspecial* LB *algebra and let B be an* AF *algebra. Then, $A \otimes B$ is an ultraspecial* LB *algebra. Moreover, if $A \in \mathcal{U}$, then $A \otimes B \in \mathcal{U}$.*

Proof. If A is an ultraspecial LB algebra then, combining Proposition 4.3 above with the fact that the class of ultraspecial LB algebras is obvioulsy closed under finite direct sums and with Proposition 4.4 above, we get that then $A \otimes B$ is an ultraspecial LB algebra. This obviously implies that if $A \in \mathcal{U}$, then $A \otimes B \in \mathcal{U}$. ∎

THEOREM 4.8. *Let $A \in \mathcal{U}$. Then, the following are equivalent:*
(i) $\mathrm{RR}(A) = 0$;
(ii) *if B is an arbitrary hereditary C^*-subalgebra of A, then $B \in \mathcal{U}$.*

Proof. (i) $\Rightarrow$ (ii) Assume that $\mathrm{RR}(A) = 0$. Let B be a hereditary C^*-subalgebra of A. Then, by Theorem 2.6, [3], B has an approximate unit (not necessarily increasing) of projections. Since B is separable, by Theorem 6, [11], it follows that B has an (increasing) approximate unit $(p_n)_{n \in \mathbb{N}}$ of projections. Then, it is not difficult to see that:

$$B = \overline{\bigcup_{n=1}^{\infty} p_n A p_n} = \varinjlim p_n A p_n.$$

Since $p_n A p_n \in \mathcal{U}$ for each $n \in \mathbb{N}$ by Proposition 4.3 above, using Proposition 4.4 above it follows that $B \in \mathcal{U}$.

(ii) $\Rightarrow$ (i) Let B be an arbitrary fixed hereditary C^*-subalgebra of A. By hypothesis, B is a separable LB algebra (since $B \in \mathcal{U}$). By Proposition 2.8 above, it follows that B has a countable approximate unit of projections. By Theorem 2.6, [3], it follows that $\mathrm{RR}(A) = 0$. ∎

COROLLARY 4.9. *The set of all* $A \in \mathcal{U}$ *with* $\mathrm{RR}(A) = 0$ *is closed under stable isomorphism.*

Proof. Let B be a C^*-algebra stably isomorphic to a C^*-algebra A with $A \in \mathcal{U}$ and $\mathrm{RR}(A) = 0$. By [3] it follows that $\mathrm{RR}(B) = 0$ and $\mathrm{RR}(B \otimes \mathcal{K}) = 0$. By Proposition 4.7 above we deduce that $A \otimes \mathcal{K} \in \mathcal{U}$ (since $A \in \mathcal{U}$). Hence, $B \otimes \mathcal{K} \in \mathcal{U}$ and, as we noted above, $\mathrm{RR}(B \otimes \mathcal{K}) = 0$. Since B is a hereditary C^*-subalgebra of $B \otimes \mathcal{K}$ ($B = (1 \otimes e)(B \otimes \mathcal{K})(1 \otimes e)$ where $1 \in M(B)$ and $e \neq 0$ is a minimal projection of $\mathcal{K}$), by the implication (i) $\Rightarrow$ (ii) in the above Theorem 4.8, it follows that $B \in \mathcal{U}$. ∎

PROPOSITION 4.10. *Let A be a C^*-algebra with an approximate unit of projections. If A is stably isomorphic to an ultraspecial* LB *algebra B, then A is an ultraspecial* LB *algebra.*

Proof. By Proposition 4.7 above it follows that $B \otimes \mathcal{K}$ is an ultraspecial LB algebra. Hence $A \otimes \mathcal{K}$ is an ultraspecial LB algebra. Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate unit of projections of A and let $e \neq 0$ be a minimal projection of $\mathcal{K}$. Then, we have

$$\overline{\bigcup_{\lambda \in \Lambda} (e_\lambda \otimes e)(A \otimes \mathcal{K})(e_\lambda \otimes e)} = \overline{\bigcup_{\lambda \in \Lambda} (e_\lambda A e_\lambda \otimes e\mathcal{K}e)} \cong \overline{\bigcup_{\lambda \in \Lambda} e_\lambda A e_\lambda} = A.$$

Hence

$$A \cong \varinjlim (e_\lambda \otimes e)(A \otimes \mathcal{K})(e_\lambda \otimes e).$$

Since $A \otimes \mathcal{K}$ is an ultraspecial LB algebra, by Proposition 4.3 above it follows that $(e_\lambda \otimes e)(A \otimes \mathcal{K})(e_\lambda \otimes e)$ is an ultraspecial LB algebra for each $\lambda \in \Lambda$. Now, by Proposition 4.4 above, it follows that A is an ultraspecial LB algebra. ∎

THEOREM 4.11. *Let A be an ultraspecial* LB *algebra A. Let I be an ideal of A. Then, the following are equivalent:*

(i) *I is generated by projections;*

(ii) *I is an ultraspecial* LB *algebra.*

Proof. (i) $\Rightarrow$ (ii) Let us assume that I is generated by projections. Since A is an ultraspecial LB algebra, A is in particular a special LB algebra. Then, by Theorem 3.6 above, I is a special LB algebra. Let $n \in \mathbb{N}$ and $p \in \mathcal{P}(M_n(I))$. Then, $pM_n(I)p$ is obviously an ideal of $pM_n(A)p$ (because $M_n(I)$ is an ideal of $M_n(A)$) and $pM_n(I)p$ is generated by projections (since for any $a \in pM_n(I)p$, we have: $a = pap \cdot p \cdot p$, and obviously $pap = a \in pM_n(I)p$ and $p \in \mathcal{P}(pM_n(I)p)$).

But, since A is an ultraspecial LB algebra, $pM_n(A)p$ is a special LB algebra. Then, by Theorem 3.6 above, $pM_n(I)p$ is a special LB algebra. In conclusion, I is an ultraspecial LB algebra.

(ii) $\Rightarrow$ (i) Let us assume that I is an ultraspecial LB algebra. Then I is an LB algebra, and the conclusion follows using Lemma 2.5. ∎

The above theorem easily implies the following:

THEOREM 4.12. *Let A be an ultraspecial* LB *algebra. Then, the following are equivalent:*

(i) *A has the ideal property;*
(ii) *every ideal I of A is an ultraspecial* LB *algebra.*

The last two results obviously imply the following two theorems:

THEOREM 4.13. *Let $A \in \mathcal{U}$. Let I be an ideal of A. Then, the following are equivalent:*

(i) *I is generated by projections;*
(ii) *$I \in \mathcal{U}$.*

THEOREM 4.14. *Let $A \in \mathcal{U}$. Then, the following are equivalent:*

(i) *A has the ideal property;*
(ii) *if I is an arbitrary ideal of A, then $I \in \mathcal{U}$.*

The next theorem answers, in particular, several natural questions. It shows that none of the following six classes of C^*-algebras:

(1) $\mathcal{U}$;
(2) ultraspecial LB algebras;
(3) separable special LB algebras;
(4) special LB algebras;
(5) separable LB algebras;
(6) LB algebras

is closed under extensions.

THEOREM 4.15. *There is an exact sequence of C^*-algebras:*

$$0 \to I \to A \to B \to 0$$

such that I and B are AH *algebras (in particular $I, B \in \mathcal{U}$) of real rank zero and stable rank one and A is a nuclear, stably finite C^*-algebra of real rank zero and stable rank one which is not an ideal of an* LB *algebra.*

Proof. The proof is similar with the proof of Proposition 13, [2], using Lemma 2.7 above and using the implication (i) $\Rightarrow$ (iii) in Theorem 2.3 above instead of Proposition 11, [2]. (We used also the fact that an extension of separable C^*-algebras is separable.) ∎

REMARK 4.16. The above Theorem 4.15 generalizes Theorem 3.4, [17].

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