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K-THEORY OF C^*-ALGEBRAS FROM ONE-DIMENSIONAL GENERALIZED SOLENOIDS

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Abstract. We compute the K-groups of C^*-algebras from one-dimensional generalized solenoids. The results show that Ruelle algebras from one-dimensional generalized solenoids are one-dimensional generalizations of Cuntz-Krieger algebras.

Keywords: *One-dimensional generalized solenoid, Smale space, Ruelle algebra.*

MSC (2000): 46L55, 46L80, 19Kxx, 37D20, 54H20.

1. INTRODUCTION

Ian Putnam and David Ruelle have developed a theory of C^*-algebras for certain hyperbolic dynamical systems ([10], [11], [12], and [15]). These systems include Anosov diffeomorphisms, topological Markov chains and some examples of substitution tiling systems. The corresponding C^*-algebras are modelled as reduced groupoid C^*-algebras for various equivalence relations.

This paper is concerned with C^*-algebras of an orientable one-dimensional generalized solenoid $(\overline{X},\overline{f})$. Naïvely speaking, orientable generalized solenoids are higher dimensional analogues of topological Markov chains ([17]). We consider the principal groupoids of stable and unstable equivalence on $(\overline{X},\overline{f})$, denoted $G_{\rm s}(\overline{X},\overline{f})$ and $G_{\rm u}(\overline{X},\overline{f})$, respectively, with topologies and Haar systems as in [10] and [11]. Then we build their reduced groupoid C^*-algebras $S(\overline{X},\overline{f})$ and $U(\overline{X},\overline{f})$, respectively, as in [13]. The homeomorphism $\overline{f}:\overline{X}\to\overline{X}$ induces automorphisms of $G_{\rm s}(\overline{X},\overline{f})$ and $G_{\rm u}(\overline{X},\overline{f})$, and we form semi-direct products $G_{\rm s}\rtimes\mathbb{Z}$ and $G_{\rm u}\rtimes\mathbb{Z}$. Their groupoid C^*-algebras are denoted $R_{\rm s}(\overline{X},\overline{f})$ and $R_{\rm u}(\overline{X},\overline{f})$, respectively, and are called the *Ruelle algebras* ([11], [12]). In the case of topological Markov chains, the Ruelle algebras are the Cuntz-Krieger algebras, and the stable and unstable equivalence algebras are the corresponding AF-subalgebras of the Cuntz-Krieger algebras.

In this paper, we compute the K-groups of the unstable equivalence algebras and the Ruelle algebras of 1-solenoids to answer the questions posed in Section 4 of [11]. We show that the unstable equivalence algebra of a 1-solenoid $(\overline{X}, \overline{f})$ with an adjacency matrix M is strongly Morita equivalent to the crossed product of a natural Cantor system of $(\overline{X}, \overline{f})$ by $\mathbb{Z}$ so that its K_0-group is order isomorphic to the dimension group of M and its K_1-group is $\mathbb{Z}$. Then we show that the K_0-groups of Ruelle algebras are isomorphic to $\mathbb{Z} \oplus \{\Delta_M / \mathrm{Im}(\mathrm{Id} - \delta_M)\}$ and the K_1-groups are $\mathbb{Z} \oplus \mathrm{Ker}(\mathrm{Id} - \delta_M)$. Thus C^*-algebras from one-dimensional generalized solenoids are one-dimensional analogues of the Cuntz-Krieger algebras.

The outline of the paper is as follow: In Section 2, we recall the axioms of one-dimensional generalized solenoids and their ordered group invariants. In Section 3, we review the definitions of Smale spaces, and show that orientable one-dimensional solenoids are Smale spaces. Then we observe that the K-theory of the unstable equivalence algebras are determined by the adjacency matrices of one-dimensional generalized solenoids. In Section 4, we compute K-groups of unstable and stable Ruelle algebras, and show that they are $*$-isomorphic to each other by the classification theorem of Kirchberg-Phillips.

2. ONE-DIMENSIONAL SOLENOIDS

We review the properties of one-dimensional generalized solenoids of Williams which will be used in later sections. As general references for the notions of one-dimensional generalized solenoids and their ordered group invariants we refer to [17], [18], and [19].

ONE-DIMENSIONAL GENERALIZED SOLENOIDS. Let X be a finite directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, and $f : X \to X$ a continuous map. We define some axioms which might be satisfied by (X, f) ([18]).

AXIOM 0. (*Indecomposability*) (X, f) is indecomposable.

AXIOM 1. (*Nonwandering*) All points of X are nonwandering under f.

AXIOM 2. (*Flattening*) There is $k \geqslant 1$ such that for all $x \in X$ there is an open neighborhood U of x such that $f^k(U)$ is homeomorphic to $(-\varepsilon, \varepsilon)$.

AXIOM 3. (*Expansion*) There are a metric d compatible with the topology and positive constants C and λ with $\lambda > 1$ such that for all $n > 0$ and all points x, y on a common edge of X, if f^n maps the interval $[x, y]$ into an edge, then $d(f^n x, f^n y) \geqslant C\lambda^n d(x, y)$.

AXIOM 4. (*Nonfolding*) $f^n|X - \mathcal{V}$ is locally one-to-one for every positive integer n.

AXIOM 5. (*Markov*) $f(\mathcal{V}) \subseteq \mathcal{V}$.

Let $\overline{X}$ be the inverse limit space

$$\overline{X} = X \stackrel{f}{\leftarrow} X \stackrel{f}{\leftarrow} \cdots = \Big\{(x_0, x_1, x_2, \ldots) \in \prod_0^\infty X : f(x_{n+1}) = x_n\Big\},$$

and $\overline{f} : \overline{X} \to \overline{X}$ the induced homeomorphism defined by

$$(x_0, x_1, x_2, \ldots) \mapsto (f(x_0), f(x_1), f(x_2), \ldots) = (f(x_0), x_0, x_1, \ldots).$$

REMARK 2.1. Williams' construction (6.2, [17]) gives a (unique) measure μ_0 for which there is a constant $\lambda > 1$ such that $\mu_0(X) = 1$ and $\mu_0(f(I)) = \lambda\mu_0(I)$ for every small interval $I \subset X$. Define $d_0(x_0, y_0)$ to be the measure of the smallest interval from x_0 to y_0 in X, and

$$d(x, y) = \sum_{i=0}^\infty \lambda^{-i} d_0(x_i, y_i)$$

for $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2, \ldots)$ in $\overline{X}$. Then $(\overline{X}, d)$ is a compact metric space.

Let Y be a topological space and $g : Y \to Y$ a homeomorphism. We call Y a *one-dimensional generalized solenoid* or 1-*solenoid* and g a *solenoid map* if there exist a directed graph X and a continuous map $f : X \to X$ such that (X, f) satisfies all six axioms and $(\overline{X}, \overline{f})$ is topologically conjugate to (Y, g). We call a point $x \in X$ a *non-branch point* if x has an open neighborhood which is homeomorphic to an open interval, and *branch point* otherwise. An *elementary presentation* (X, f) of a 1-solenoid is such that X is a wedge of circles and f leaves the unique branch point of X fixed.

Recall that a continuous map $\gamma : [0, 1] \to G$, a directed graph, is *orientation preserving* if $e^{-1} \circ \gamma : I \to [0, 1]$ is increasing for every interval $I \subset [0, 1]$ such that $\gamma(I)$ is a subset of a directed edge e. A continuous map $\phi : G_1 \to G_2$ between two directed graphs is *orientation preserving* if, for every orientation preserving map $p : [0, 1] \to G_1$, the map $\phi \circ p : [0, 1] \to G_2$ is orientation preserving ([1]).

When we can give a direction to each edge of X so that the connection map $f : X \to X$ is orientation preserving, we call (X, f) an *orientable presentation.* For a 1-solenoid Y with a solenoid map g, if there exists an orientable presentation (X, f) then Y is called an *orientable* 1-solenoid.

PROPOSITION 2.2. ([1], [17]) *Suppose that* (X, f) *is a presentation of a* 1-*solenoid.*

(i) *The inverse limit spaces of* (X, f) *and* (X, f^n) *are homeomorphic for every positive integer* n.

(ii) *There exists an integer* m *such that* $(\overline{X}, \overline{f}^m)$ *has an elementary presentation.*

Thus, for the purpose of computing invariants of the space $\overline{X}$, there is no loss of generality in replacing (X, f) with (X, f^n) where $n = m \cdot k$ is a positive integer such that $(\overline{X}, \overline{f}^m)$ has an elementary presentation (Z, h) and for every $z \in Z$ there is an open set U_z such that $h^k(U_z)$ is an open interval by the Flattening Axiom. Hence we can assume that every point $x \in X$ has a neighborhood U_x such that $f(U_x)$ is an interval.

STANDING ASSUMPTION. In this paper, we always assume that (X, f) is an orientable elementary presentation such that every point $x \in X$ has a neighborhood U_x such that $f(U_x)$ is an interval.

NOTATION 2.3. Suppose that (X, f) is a presentation of a 1-solenoid, and that $\mathcal{E} = \{e_1, \dots, e_n\}$ is the edge set of the directed graph X. For each edge $e_i \in \mathcal{E}$, we can give e_i the *partition* $\{I_{i,j}\}$, $1 \leqslant j \leqslant l(i)$, such that:

(i) the initial point of $I_{i,1}$ is the initial point of e_i;
(ii) the terminal point of $I_{i,j}$ is the initial point of $I_{i,j+1}$ for $1 \leqslant j < l(i)$;
(iii) the terminal point of $I_{i,l(i)}$ is the terminal point of e_i;
(iv) $f|\mathrm{Int} I_{i,j}$ is injective;
(v) $f(I_{i,j}) = e_{i,j}^{s(i,j)}$ where $e_{i,j} \in \mathcal{E}$, $s(i,j) = 1$ if the direction of $f(I_{i,j})$ agree with that of $e_{i,j}$, and $s(i,j) = -1$ if the direction of $f(I_{i,j})$ is reverse to that of $e_{i,j}$.

The *wrapping rule* $\check{f} : \mathcal{E} \to \mathcal{E}^*$ associated with f is given by

$$\check{f} : e_i \mapsto e_{i,1}^{s(i,1)} \cdots e_{i,l(i)}^{s(i,l(i))},$$

and the *adjacency matrix* M of $(\mathcal{E}, \check{f})$ is given by

$$M(i,k) = \#\{I_{i,j} : f(I_{i,j}) = e_k^{\pm 1}\}.$$

REMARK 2.4. (6.2, [17]) The measure μ_0 in Remark 2.1 is given as follows: Suppose that λ is the Perron-Frobenius eigenvalue of the adjacency matrix M and that $\mathbf{v} = (v_1, \dots, v_n)$ is the corresponding Perron eigenvector such that $\sum_{i=1}^{n} v_i = 1$. For edges e_i, e_j of X and an interval I of e_i such that $f^n(I) = e_j$ and $f^n|\mathrm{Int} I$ is injective, let

$$\mu_0(e_i) = v_i \quad \text{and} \quad \mu_0(I) = \lambda^{-n} v_j.$$

Then μ_0 is extended to a regular Borel measure on X by the standard procedure.

EXAMPLES 2.5. (i) Suppose that X is the unit circle and that $f : X \to X$ is given by $z \mapsto z^n$. Then the adjacency matrix is (n).

(ii) Suppose that Y is a wedge of two circles a and b and that $g : Y \to Y$ is a continuous map such that its corresponding wrapping rule $\check{g}$ is given by

$$a \mapsto aab \quad \text{and} \quad b \mapsto ab.$$

Then (Y, g) is an elementary presentation of a solenoid, and the adjacency matrix is

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The Perron-Frobenious eigenvalue of M is $\frac{3+\sqrt{5}}{2}$, and the corresponding Perron eigenvector is

$$\mathbf{v} = \Big(\frac{1+\sqrt{5}}{3+\sqrt{5}}, \frac{2}{3+\sqrt{5}}\Big).$$

Hence the measure μ_0 on Y is given by

$$\mu_0(a) = \frac{1+\sqrt{5}}{3+\sqrt{5}} \quad \text{and} \quad \mu_0(b) = \frac{2}{3+\sqrt{5}}.$$

NOTATION 2.6. Given an $n \times n$ nonnegative integer matrix A we denote the *dimension group* of A,

$$\varinjlim(\mathbb{Z}^n, A) = \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \cdots,$$

by $\big(\Delta_A, \Delta_A^+\big)$.

THEOREM 2.7. ([6], [20]) *Suppose that* $(\overline{X}, \overline{f})$ *is a* 1*-solenoid. Then there exists a uniquely ergodic flow* ϕ *whose phase space is* $\overline{X}$*.*

Suppose that (X, f) is a presentation of a 1-solenoid and that μ_0 is the measure given on X as in Remark 2.4. For a measurable set I in X, we let $U_n(I) = \{(x_0, \ldots, x_n, \ldots) \in \overline{X} : x_n \in I\}$, and define a measure μ on $\overline{X}$ by

$$\mu\left(U_n(I)\right) = \mu_0(I).$$

Then μ is extended to a regular Borel measure on $\overline{X}$ in the standard way. It is not difficult to verify that μ is the unique ϕ-invariant measure on $\overline{X}$ where ϕ is the flow on $\overline{X}$ given in Theorem 2.7.

A closed subset K of a phase space Y of a flow ψ is called a *cross section* if the mapping $\psi : K \times \mathbb{R} \to Y$ defined by $(p, t) \mapsto p \cdot t$ is a local homeomorphism onto Y. The *return time map* $r_K : K \to K$ of a cross section K is defined by $x \mapsto y = x \cdot t_x$ where $x \in K$ and t_x is the smallest positive number such that $x \cdot t_x = y \in K$. It is a crucial fact that the return time map r_K of a cross section K is a homeomorphism, and Y is the standard suspension space of (K, r_K).

PROPOSITION 2.8. ([19], [20]) *Suppose that* $(\overline{X}, \overline{f})$ *is a* 1*-solenoid with the corresponding adjacency matrix* M*. Then there is a cross section with the return time map* (K, r_K) *of* $\overline{X}$ *such that:*

(i) $\mathrm{K}_1(C(K) \times_{r_K} \mathbb{Z}) = \mathbb{Z}$*;*

(ii) $\mathrm{K}_0(C(K) \times_{r_K} \mathbb{Z})$ *is order isomorphic to* Δ_M*.*

3. Smale spaces and C^*-algebras from solenoids

Smale spaces ([10], [15]). Suppose that (Y,d) is a compact metric space and φ is a homeomorphism of Y. Assume that we have constants $0<\lambda_0<1$, $\varepsilon_0>0$ and a continuous map $(x,y)\in\{(x,y)\in Y\times Y : d(x,y)\leqslant 2\varepsilon_0\}\mapsto [x,y]\in Y$ satisfying the following:

$$[x,x]=x,\quad [\,[x,y],z]=[x,z],\quad [x,[y,z]\,]=[x,z],\quad [\varphi(x),\varphi(y)]=\varphi([x,y])$$

for $x,y,z\in Y$ whenever both sides of the equation are defined. For every $0<\varepsilon\leqslant\varepsilon_0$ let

$$V^{\mathrm{s}}(x,\varepsilon)=\{y\in Y : [x,y]=y \text{ and } d(x,y)<\varepsilon\},$$
$$V^{\mathrm{u}}(x,\varepsilon)=\{y\in Y : [y,x]=y \text{ and } d(x,y)<\varepsilon\}.$$

We assume that

$$d\left(\varphi(y),\varphi(z)\right)\leqslant\lambda_0 d(y,z)\quad y,z\in V^{\mathrm{s}}(x,\varepsilon),$$
$$d\left(\varphi^{-1}(y),\varphi^{-1}(z)\right)\leqslant\lambda_0 d(y,z)\quad y,z\in V^{\mathrm{u}}(x,\varepsilon).$$

Then (Y,d,φ) is called a *Smale space.*

Suppose that $\left(\overline{X},\overline{f}\right)$ is a 1-solenoid with the metric d given in Remark 2.1. Let $\lambda_0=\varepsilon_0=\frac{1}{\lambda}$ and define $[\,\cdot\,,\cdot\,]:\overline{X}\times\overline{X}\to\overline{X}$ by $[x,y]\mapsto z=(z_0,\dots,z_n,\dots)$ where $z_0=x_0$ and z_n is the unique element contained in the λ_0^{n+1}-neighborhood of y_n such that $f^n(z_n)=x_0$. Then it is not difficult to show that $\left(\overline{X},\overline{f},d\right)$ satisfies the above conditions. Therefore we have the following:

Proposition 3.1. *One-dimensional generalized solenoids are Smale spaces.*

Groupoids. ([11], [13]) For a Smale space (Y,d,φ), define

$$G_{\mathrm{s},0}=\{(x,y)\in Y\times Y : y\in V^{\mathrm{s}}(x,\varepsilon_0)\}\quad G_{\mathrm{u},0}=\{(x,y)\in Y\times Y : y\in V^{\mathrm{u}}(x,\varepsilon_0)\}$$

and let

$$G_{\mathrm{s}}=\bigcup_{n=0}^{\infty}(\varphi\times\varphi)^{-n}\left(G_{\mathrm{s},0}\right),\quad G_{\mathrm{u}}=\bigcup_{n=0}^{\infty}(\varphi\times\varphi)^{n}\left(G_{\mathrm{u},0}\right).$$

Then G_{s} and G_{u} are equivalence relations on Y, called *stable* and *unstable* equivalence. Each $(\varphi\times\varphi)^{-n}\left(G_{\mathrm{s},0}\right)$, $(\varphi\times\varphi)^{-n}\left(G_{\mathrm{u},0}\right)$ is given the relative topology of $Y\times Y$, and G_{s} and G_{u} are given the inductive limit topology. It is not difficult to verify that G_{s} and G_{u} are locally compact Hausdorff principal groupoids. The Haar systems $\{\mu^x_{\mathrm{s}} : x\in Y\}$ and $\{\mu^x_{\mathrm{u}} : x\in Y\}$ for G_{s} and G_{u}, respectively, are described in 3.c of [11]. The groupoid C^*-algebras of G_{s} and G_{u} are denoted $S(Y,\varphi)$ and $U(Y,\varphi)$, respectively.

The map $\varphi\times\varphi$ acts as an automorphism of G_{s} and G_{u}. We form the semidirect products

$$G_{\mathrm{s}}\rtimes\mathbb{Z}=\{(x,n,y) : n\in\mathbb{Z} \text{ and } (\overline{f}^{\,n}(x),y)\in G_{\mathrm{s}}\}$$
$$G_{\mathrm{u}}\rtimes\mathbb{Z}=\{(x,n,y) : n\in\mathbb{Z} \text{ and } (\overline{f}^{\,n}(x),y)\in G_{\mathrm{u}}\}$$

with groupoid operations

$$(x,n,y)\cdot(u,m,v)=(x,n+m,v) \text{ if } y=u,\quad\text{and}\quad (x,n,y)^{-1}=(y,-n,x).$$

The product topology of $G_* \times \mathbb{Z}$ is transfered to $G_* \rtimes \mathbb{Z}$ by the bijective map $\eta : (x, y, n) \mapsto (x, n, \varphi(y))$. And a Haar system on $G_* \rtimes \mathbb{Z}$ is given by $\mu_*^x \circ \eta^{-1}$ where μ_*^x is the Haar system on G_*. The groupoid C^*-algebras $C^*(G_{\mathrm{s}} \rtimes \mathbb{Z})$ and $C^*(G_{\mathrm{u}} \rtimes \mathbb{Z})$ are denoted $R_{\mathrm{s}}(Y, \varphi)$ and $R_{\mathrm{u}}(Y, \varphi)$ and are called the *Ruelle algebras.*

For general properties of these C^*-algebras, we refer to [3], [10], [11], and [12].

Unstable equivalence algebras. Suppose that $\left(\overline{X}, \overline{f}\right)$ is an orientable solenoid and that ϕ is the flow on $\overline{X}$ given in Theorem 2.7. Then there exists a cross section with return time map (K, r) such that $\overline{X}$ is the suspension space of (K, r) by Proposition 2.8.

Sublemma 3.2. ([13]) (i) $(\overline{X}, \mathbb{R}, \phi)$ *and* $(K, \mathbb{Z}, r)$ *are groupoids;*

(ii) *the groupoid algebras of* $(\overline{X}, \mathbb{R}, \phi)$ *and* $(K, \mathbb{Z}, r)$ *are isomorphic to* $C(\overline{X}) \times_\phi \mathbb{R}$ *and* $C(K) \times_r \mathbb{Z}$*, respectively.*

Lemma 3.3. ([2], [11]) *Suppose that* $\left(\overline{X}, \overline{f}\right)$ *is an orientable solenoid, and that* (K, r) *is a cross section with the return time map of the flow* ϕ*. Then:*

(i) $U(\overline{X}, \overline{f}) \simeq C(\overline{X}) \times_\phi \mathbb{R}$*;*

(ii) $C(\overline{X}) \times_\phi \mathbb{R}$ *is strongly Morita equivalent to* $C(K) \times_r \mathbb{Z}$*.*

Proof. (i) Suppose $x = (x_0, x_1, \ldots), y = (y_0, y_1, \ldots) \in \overline{X}$ and $(x, y) \in G_{\mathrm{u}}$. Then $d\left(\overline{f}^n(x), \overline{f}^n(y)\right) \to 0$ as $n \to -\infty$ implies $d_0\left(x_n, y_n\right) \to 0$ as $n \to \infty$ and that there exists a $t \in \mathbb{R}$ such that $y = \phi_t(x)$. Let $\alpha : (\overline{X}, \mathbb{R}, \phi) \to G_{\mathrm{u}}$ be given by $(x, t) \mapsto (x, \phi_t(x))$. Then it is not difficult to see that α is an isomorphism. Therefore $U(\overline{X}, \overline{f})$ is isomorphic to $C(\overline{X}) \times_\phi \mathbb{R}$ by Sublemma 3.2.

(ii) Since $\overline{X}$ is the suspension of (K, r), for every $x \in \overline{X}$ there exist unique $z_x \in K$ and $\tau_x \in [0, 1)$ such that $x = \phi_{\tau_x}(z_x)$. Define

$$I = \left\{(x, n - \tau_x) : x \in \overline{X}, n \in \mathbb{Z}\right\},$$

and let $\mathcal{C}(I)$ be the completion of $C_{\mathrm{c}}(I)$. Then, by the Theorem in Section 4 of [11], $\mathcal{C}(I)$ is a $C(\overline{X}) \times_\phi \mathbb{R}$-$C(K) \times_r \mathbb{Z}$ imprimitivity bimodule. For completeness, we write down the module structures and the inner products. ∎

Module structures. Suppose that $\alpha \in C_{\mathrm{c}}(I)$, $g \in C_{\mathrm{c}}(\overline{X}, \mathbb{R}, \phi)$ and $h \in C_{\mathrm{c}}(K, \mathbb{Z}, r)$. Then

$$\left(g \cdot \alpha\right)(x, n - \tau_x) = \int g(x, t) \cdot \alpha(\phi_t(x), n - \tau_x - t) \, \mathrm{d}\mu^{[x]}(t)$$

and

$$\left(\alpha \cdot h\right)(x, n - \tau_x) = \sum_m \alpha(x, m - \tau_x) \cdot h(r^m(z_x), n - m)$$

give that $\mathcal{C}(I)$ is a left $C(\overline{X}) \times_\phi \mathbb{R}$ and right $C(K) \times_r \mathbb{Z}$ bimodule with $(\widetilde{g} \cdot \widetilde{\alpha}) \cdot \widetilde{h} = \widetilde{g} \cdot (\widetilde{\alpha} \cdot \widetilde{h})$ for every $\widetilde{\alpha} \in \mathcal{C}(I)$, $\widetilde{g} \in C(\overline{X}) \times_\phi \mathbb{R}$ and $\widetilde{h} \in C(K) \times_r \mathbb{Z}$.

INNER PRODUCTS. Define $\langle\cdot,\cdot\rangle_L : C_c(I)\times C_c(I)\to C_c(\overline{X},\mathbb{R},\phi)$ and $\langle\cdot,\cdot\rangle_R$: $C(I)\times C(I)\to C_c(K,\mathbb{Z},r)$ by

$$\langle\alpha,\beta\rangle_L(x,t)=\sum\alpha(x,m-\tau_x)\cdot\overline{\beta(x,m-\tau_x)}$$

and

$$\langle\alpha,\beta\rangle_R(z,k)=\int\overline{\alpha\left(\phi_t(z),k-t\right)}\cdot\beta\left(\phi_t(z),k-t\right)\,\mathrm{d}\mu^{[\phi_t(z)]}(t). \quad ∎$$

Therefore we have the following proposition from Proposition 2.8 and the above lemma.

PROPOSITION 3.4. (i) $\mathrm{K}_1\left(U(\overline{X},\overline{f})\right)=\mathbb{Z}$;
(ii) $\mathrm{K}_0\left(U(\overline{X},\overline{f})\right)$ *is order isomorphic to* Δ_M *where* M *is the adjacency matrix of* $(\overline{X},\overline{f})$.

Recall that the flow ϕ on $\overline{X}$ is uniquely ergodic without rest point (Theorem 2.7). So $C(\overline{X})\times_\phi\mathbb{R}$ has the unique trace τ_μ induced by the unique ϕ-invariant measure μ (3.3.10, [16]). Thus τ_μ^*, the induced state on $\mathrm{K}_0(C(\overline{X})\times_\phi\mathbb{R})$, is the unique state.

PROPOSITION 3.5. *If* $(\overline{X},\overline{f})$ *is a* 1*-solenoid and* M *is the corresponding adjacency matrix with the normalized Perron eigenvector* $\mathbf{v}=(v_1,\dots,v_n)$, *then*

$$\tau_\mu^*\left(\mathrm{K}_0(U(\overline{X},\overline{f}),\mathrm{K}_0(U(\overline{X},\overline{f}))_+\right)=\left\langle(\Delta_M,\Delta_M^+),\mathbf{v}\right\rangle.$$

Proof. Suppose that $\mathcal{E}_k=\mathcal{E}$ is the edge set of the kth coordinate space of $\overline{X}$. Then by Proposition 2.8

$$\left(\mathrm{K}_0(U(\overline{X},\overline{f})),\mathrm{K}_0(U(\overline{X},\overline{f}))_+\right)\cong(\varinjlim C(\mathcal{E}_k,\mathbb{Z}),\varinjlim C_+(\mathcal{E}_k,\mathbb{Z}))\cong(\Delta_M,\Delta_M^+).$$

For $g\in C(\mathcal{E}_k,\mathbb{Z})$, $x=(x_0,\dots,x_k,\dots)\in\overline{X}$ with $x_k=\mathrm{e}^{2\pi\mathrm{i}s}\in e_i\in\mathcal{E}_k$ and the canonical projection to the kth coordinate space $\pi_k:\overline{X}\to X$, define $g_k\in C(X_k,S^1)$ and $\widetilde{g}\in C(\overline{X},S^1)$ by

$$g_k : x_k\mapsto\exp(2\pi\mathrm{i}g(e_i)s)\quad\text{and}\quad\widetilde{g}:x\to g_k\circ\pi_k(x).$$

Then every $\widetilde{g}$ is a unitary element in $C(\overline{X})$, and $\mathrm{K}_0(U(\overline{X},\overline{f}))\cong\mathrm{K}_1(C(\overline{X}))$ is generated by $\widetilde{g}$. If we denote g as $(g(e_1),\dots,g(e_n))$, then by Theorem 2.2 of [8]

$$\begin{aligned}\tau_\mu^*(\widetilde{g})&=\frac{1}{2\pi\mathrm{i}}\int_{\overline{X}}\frac{\widetilde{g}'}{\widetilde{g}}\,\mathrm{d}\mu=\int_{X_k}g'\,\mathrm{d}\mu_0=\sum_{i=1}^n g(e_i)\mu_0(e_i)=\sum_{i=1}^n g(e_i)v_i\\&=\langle(g(e_1),\dots,g(e_n)),\mathbf{v}\rangle.\quad ∎\end{aligned}$$

4. RUELLE ALGEBRAS FOR SOLENOIDS

We compute K-groups of Ruelle algebras for 1-solenoids to show that they are $*$-isomorphic.

UNSTABLE EQUIVALENCE RUELLE ALGEBRAS. Suppose that $(\overline{X}, \overline{f})$ is an oriented 1-solenoid and that $G_{\mathrm{u}} \simeq (\overline{X}, \mathbb{R}, \phi)$ is the unstable equivalence groupoid on $\overline{X}$. Recall that for $x, y \in \overline{X}$ such that $y = \phi_t(x)$, $t \in \mathbb{R}$, we have $\overline{f}^{-1}(y) = \phi_{t\lambda^{-1}} \circ \overline{f}^{-1}(x)$.

DEFINITION 4.1. (Section 4, [11]) Let α_{u} be an automorphism on $U(\overline{X}, \overline{f})$ defined by

$$\alpha_{\mathrm{u}}(g)(x,t) = \lambda^{-1} g(\overline{f}^{-1}(x), t\lambda^{-1}) \quad \text{for } g \in C_{\mathrm{c}}(\overline{X}, \mathbb{R}, \phi) \text{ and } (x,t) \in (\overline{X}, \mathbb{R}).$$

The *unstable equivalence Ruelle algebra* $R_{\mathrm{u}}(\overline{X}, \overline{f})$ is the crossed product

$$R_{\mathrm{u}}(\overline{X}, \overline{f}) = U(\overline{X}, \overline{f}) \times_{\alpha_{\mathrm{u}}} \mathbb{Z} = \big(C(\overline{X}) \times_\phi \mathbb{R}\big) \times_{\alpha_{\mathrm{u}}} \mathbb{Z}.$$

REMARKS 4.2. (i) Let A be an $n \times n$ integer matrix and Δ_A the dimension group of A. The *dimension group automorphism* δ_A of A is the restriction of A to Δ_A so that $\delta_A(\mathbf{v}) = A\mathbf{v}$ (7.5.1, [5]). Then $\Delta_A/\mathrm{Im}(\mathrm{Id} - \delta_A)$ is isomorphic to $\mathbb{Z}^n/(\mathrm{Id} - A)\mathbb{Z}^n$.

(ii) For $g \in C(\mathcal{E}_k, \mathbb{Z})$, let $g_k \in C(X_k, S^1)$ be as in the proof of Proposition 3.5. The wrapping rule $\check{f} : \mathcal{E}_{k+1} \to \mathcal{E}_k$ induces a map $f^* : C(\mathcal{E}_k, \mathbb{Z}) \to C(\mathcal{E}_{k+1}, \mathbb{Z})$ by $g \mapsto g \circ \check{f}$ where $(g \circ \check{f})(e) = \sum_{i=1}^{j} g(e_i)$ such that $\check{f}(e) = e_1 \cdots e_j$. Then $g_k \circ f \circ \pi_k$ is homotopic to $(g \circ f^*)_{k+1} \circ \pi_{k+1}$ (3.6, [19]).

PROPOSITION 4.3. *Suppose that $(\overline{X}, \overline{f})$ is a 1-solenoid with the adjacency matrix M and corresponding dimension group automorphism δ_M. Then*

$$\mathrm{K}_0(R_{\mathrm{u}}(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \{\Delta_M/\mathrm{Im}(\mathrm{Id} - \delta_M)\} \quad \textit{and} \quad \mathrm{K}_1(R_{\mathrm{u}}(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \mathrm{Ker}(\mathrm{Id} - \delta_M).$$

Proof. We have the following Pimsner-Voiculescu exact sequence:

$$\begin{array}{ccccc}
\mathrm{K}_0(U(\overline{X},\overline{f})) & \xrightarrow{1-\alpha_{\mathrm{u}*}} & \mathrm{K}_0(U(\overline{X},\overline{f})) & \xrightarrow{\iota_*} & \mathrm{K}_0(R_{\mathrm{u}}(\overline{X},\overline{f})) \\
\uparrow & & & & \downarrow \\
\mathrm{K}_1(R_{\mathrm{u}}(\overline{X},\overline{f})) & \xleftarrow[\iota_*]{} & \mathrm{K}_1(U(\overline{X},\overline{f})) & \xleftarrow[1-\alpha_{\mathrm{u}*}]{} & \mathrm{K}_1(U(\overline{X},\overline{f}))
\end{array}$$

We consider $\alpha_{\mathrm{u}*} : \mathrm{K}_0(U(\overline{X}, \overline{f})) = \mathrm{K}_0\big(C(\overline{X}) \times_\phi \mathbb{R}\big) \to \mathrm{K}_0\big(C(\overline{X}) \times_\phi \mathbb{R}\big)$ as the automorphism $\widehat{\alpha}_{\mathrm{u}*} : \mathrm{K}_1(C(\overline{X})) \to \mathrm{K}_1(C(\overline{X}))$ given by the Thom isomorphism of Connes. Define $\beta : C(\overline{X}) \to C(\overline{X})$ by $h \mapsto h \circ \overline{f}^{-1}$ for $h \in C(\overline{X})$. Then the induced automorphism $\beta_* : \mathrm{K}_1(C(\overline{X})) \to \mathrm{K}_1(C(\overline{X}))$ is the required isomorphism.

For $g \in C(\mathcal{E}_k, \mathbb{Z})$, let $\widetilde{g} \in C(\overline{X}, S^1)$ be the induced unitary element as in the proof of Proposition 3.5. Then $\beta^{-1}(\widetilde{g}) = \widetilde{g} \circ \overline{f} = g_k \circ \pi_k \circ \overline{f} = g_k \circ f \circ \pi_k$ is homotopic to $(g \circ f^*)_{k+1} \circ \pi_{k+1}$. Hence if we denote g as $(g(e_1), \ldots, g(e_n)) \in \mathbb{Z}^n$, then $g \circ f^*$ is given by Mg and the induced automorphism $\beta_*^{-1} : \mathrm{K}_1(C(\overline{X})) \to \mathrm{K}_1(C(\overline{X}))$ is the dimension group automorphism δ_M of the adjacency matrix M. Therefore β_* is the inverse of δ_M, and $1 - \alpha_{\mathrm{u}*} : \mathrm{K}_0(U(\overline{X}, \overline{f})) \to \mathrm{K}_0(U(\overline{X}, \overline{f}))$ is the same as $\mathrm{Id} - \delta_M^{-1} : \Delta_M \to \Delta_M$.

Since $\mathrm{K}_1(U(\overline{X}, \overline{f}))$ is isomorphic to $\mathbb{Z}$, $\alpha_{\mathrm{u}*} : \mathbb{Z} \to \mathbb{Z}$ is trivially the identity map. Thus the six-term exact sequence is divided into the following two short exact sequences:

$$0 \to \Delta_M/\mathrm{Im}(\mathrm{Id} - \delta_M^{-1}) \to \mathrm{K}_0(R_{\mathrm{u}}(\overline{X}, \overline{f})) \to \mathbb{Z} \to 0,$$
$$0 \to \mathbb{Z} \to \mathrm{K}_1(R_{\mathrm{u}}(\overline{X}, \overline{f})) \to \mathrm{Ker}(\mathrm{Id} - \delta_M^{-1}) \to 0.$$

Because $\mathbb{Z}$ and $\mathrm{Ker}(\mathrm{Id} - \delta_M^{-1})$ are free groups, these sequences split. Therefore we conclude that

$$\mathrm{K}_0(R_{\mathrm{u}}(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \{\Delta_M/\mathrm{Im}(\mathrm{Id} - \delta_M^{-1})\} \cong \mathbb{Z} \oplus \{\Delta_M/\mathrm{Im}(\mathrm{Id} - \delta_M)\}$$

and

$$\mathrm{K}_1(R_{\mathrm{u}}(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \mathrm{Ker}(\mathrm{Id} - \delta_M^{-1}) \cong \mathbb{Z} \oplus \mathrm{Ker}(\mathrm{Id} - \delta_M). \quad \blacksquare$$

REMARK 4.4. Although the above short exact sequences are *natural*, they split *unnaturally*. Hence the isomorphisms of Proposition 4.3 are *unnatural*.

STABLE EQUIVALENCE RUELLE ALGEBRAS. We use K-theoretic duality of the Ruelle algebras and the Universal Coefficient Theorem to compute K-groups of $R_{\mathrm{s}}(\overline{X}, \overline{f})$.

LEMMA 4.5. ([11], [14]) *Suppose that* $(\overline{X}, \overline{f})$ *is a* 1*-solenoid. Then:*
(i) $\mathrm{K}_*(R_{\mathrm{s}}(\overline{X}, \overline{f}))$ *is isomorphic to* $\mathrm{K}^{*+1}(R_{\mathrm{u}}(\overline{X}, \overline{f}))$*;*
(ii) *there are short exact sequences*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(R_{\mathrm{u}}(\overline{X}, \overline{f})), \mathbb{Z}) \to \mathrm{K}^1(R_{\mathrm{u}}(\overline{X}, \overline{f})) \to \mathrm{Hom}(\mathrm{K}_1(R_{\mathrm{u}}(\overline{X}, \overline{f})), \mathbb{Z}) \to 0,$$
$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(R_{\mathrm{u}}(\overline{X}, \overline{f})), \mathbb{Z}) \to \mathrm{K}^0(R_{\mathrm{u}}(\overline{X}, \overline{f})) \to \mathrm{Hom}(\mathrm{K}_0(R_{\mathrm{u}}(\overline{X}, \overline{f})), \mathbb{Z}) \to 0.$$

Therefore K-groups of the stable equivalence Ruelle algebra are determined by Ext- and Hom-groups of $K_*(R_{\mathrm{u}}(\overline{X}, \overline{f}))$.

PROPOSITION 4.6. *Suppose that* $(\overline{X}, \overline{f})$ *is a* 1*-solenoid. Then*

$$\mathrm{K}_0(R_{\mathrm{s}}(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \{\Delta_M/\mathrm{Im}(\mathrm{Id} - \delta_M)\} \quad \textit{and} \quad \mathrm{K}_1(R_{\mathrm{s}}(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \mathrm{Ker}(\mathrm{Id} - \delta_M).$$

Proof. Transform $\mathrm{Id} - M$ to the Smith form

$$\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}$$

where $d_i \geqslant 0$ and d_i divides d_{i+1} (Section 7.4, [5]). Then $\Delta_M/\mathrm{Im}(\mathrm{Id}-\delta_M)$ is isomorphic to $\bigoplus_{i=1}^{n} \mathbb{Z}/d_i\mathbb{Z}$, and the dimension of $\mathrm{Ker}(\mathrm{Id}-\delta_M)$ is equal to the number of zeros in the diagonal of the Smith form. Suppose $d_1 = \cdots = d_m = 0$ and $d_{m+1} \neq 0$. Then we have

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(R_{\mathrm{u}}(\overline{X},\overline{f})),\mathbb{Z}) &= \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/d_{m+1}\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/d_n\mathbb{Z},\mathbb{Z})\\ &= \mathbb{Z}/d_{m+1}\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/d_n\mathbb{Z}\end{aligned}$$

and

$$\mathrm{Hom}(\mathrm{K}_1(R_{\mathrm{u}}(\overline{X},\overline{f})),\mathbb{Z}) = \mathbb{Z}^{m+1}.$$

Hence we have

$$\begin{aligned}K^1(R_{\mathrm{u}}(\overline{X},\overline{f})) &\cong \mathrm{Hom}(\mathrm{K}_1(R_{\mathrm{u}}(\overline{X},\overline{f})),\mathbb{Z})\oplus\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(R_{\mathrm{u}}(\overline{X},\overline{f})),\mathbb{Z})\\ &= \mathbb{Z}\oplus\mathbb{Z}^m\oplus\mathbb{Z}/d_{m+1}\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/d_n\mathbb{Z}\\ &\cong \mathbb{Z}\oplus\{\Delta_M/\mathrm{Im}(\mathrm{Id}-\delta_M)\}.\end{aligned}$$

Recall that $\mathrm{Ker}(\mathrm{Id}-\delta_M)$ is isomorphic to $\mathbb{Z}^m$ so that

$$\mathrm{K}_1(R_{\mathrm{u}}(\overline{X},\overline{f})) \cong \mathbb{Z}\oplus\mathrm{Ker}(\mathrm{Id}-\delta_M) \cong \mathbb{Z}^{m+1}.$$

Thus we have $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(R_{\mathrm{u}}(\overline{X},\overline{f})),\mathbb{Z}) = 0$ and

$$K^0(R_{\mathrm{u}}(\overline{X},\overline{f})) \cong \mathrm{Hom}(\mathrm{K}_0(R_{\mathrm{u}}(\overline{X},\overline{f})),\mathbb{Z}).$$

Then $\mathrm{K}_0(R_{\mathrm{u}}(\overline{X},\overline{f})) \cong \mathbb{Z}\bigoplus_{i=1}^{n}\mathbb{Z}/d_i\mathbb{Z}$ implies

$$\begin{aligned}\mathrm{Hom}(\mathrm{K}_0(R_{\mathrm{u}}(\overline{X},\overline{f})),\mathbb{Z}) &\cong \mathrm{Hom}\Big(\mathbb{Z}\bigoplus_{i=1}^{n}\mathbb{Z}/d_i\mathbb{Z},\mathbb{Z}\Big) \cong \mathbb{Z}\bigoplus_{i=1}^{m}\mathbb{Z}\\ &\cong \mathbb{Z}\oplus\mathrm{Ker}(\mathrm{Id}-\delta_M). \quad \blacksquare\end{aligned}$$

REMARK 4.7. The isomorphisms in Proposition 4.6 are *unnatural* as the short exact sequences in the Universal Coefficient Theorem split unnaturally.

Recall that the unstable and stable equivalence Ruelle algebras of a 1-solenoid are nuclear, purely infinite, separable, simple and stable C^*-algebras ([3]). Then the classification theorem of Kirchberg-Phillips ([4], [9]) implies the following proposition.

PROPOSITION 4.8. *$R_{\mathrm{u}}(\overline{X},\overline{f})$ is $*$-isomorphic to $R_{\mathrm{s}}(\overline{X},\overline{f})$.*

EXAMPLES 4.9. (i) Suppose that X is the unit circle and that $f : X \to X$ is given by $z \mapsto z^n$, $n \geqslant 2$. Then the adjacency matrix is (n), $\mathrm{K}_0(U(\overline{X},\overline{f})) = \mathbb{Z}[\frac{1}{n}]$ and $\mathrm{K}_1(U(\overline{X},\overline{f})) = \mathbb{Z}$. Since $\delta^{-1}_{(n)}$ is multiplication by $\frac{1}{n}$, we have

$$\mathrm{K}_0(R_{\mathrm{u}}(\overline{X},\overline{f})) = \mathrm{K}_0(R_{\mathrm{s}}(\overline{X},\overline{f})) = \mathbb{Z}\oplus\{\mathbb{Z}/(n-1)\mathbb{Z}\}$$

and

$$\mathrm{K}_1(R_{\mathrm{u}}(\overline{X},\overline{f})) = \mathrm{K}_1(R_{\mathrm{s}}(\overline{X},\overline{f})) = \mathbb{Z}.$$

(ii) Suppose that Y is a wedge of two circles a and b and that $g : Y \to Y$ is given by $a \mapsto aab$ and $b \mapsto ab$. Then the adjacency matrix is $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Since M is an invertible matrix, we have $\mathrm{K}_0(U(\overline{Y}, \overline{g})) = \mathbb{Z} \oplus \mathbb{Z}$, $\mathrm{K}_1(U(\overline{Y}, \overline{g})) = \mathbb{Z}$ and that $1 - \alpha_{\mathrm{u}*} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is an isomorphism. Hence we obtain

$$\mathrm{K}_0(R_{\mathrm{u}}(\overline{Y}, \overline{g})) = \mathrm{K}_1(R_{\mathrm{u}}(\overline{Y}, \overline{g})) = \mathrm{K}_0(R_{\mathrm{s}}(\overline{Y}, \overline{g})) = \mathrm{K}_1(R_{\mathrm{s}}(\overline{Y}, \overline{g})) = \mathbb{Z}.$$

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