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ON MEASURABLE OPERATOR VALUED INDEFINITE FUNCTIONS WITH A FINITE NUMBER OF NEGATIVE SQUARES

RAMÓN BRUZUAL and MARISELA DOMÍNGUEZ

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Abstract. Let f be a κ-indefinite function defined on a locally compact group with values in the space of the continuous linear operators of a Kreĭn space. We prove that if f is weakly measurable then $f = f^{\mathrm{c}} + f^0$, where f^{c} is a κ-indefinite and weakly continuous function and f^0 is a positive definite function which is zero locally almost everywhere. We also prove that if f is weakly continuous then f is strongly continuous. As an application we obtain that a weakly measurable group of unitary operators, on a separable Pontryagin space and with parameter on a locally compact group, is strongly continuous.

Keywords: *Weakly measurable, operators on Kreĭn spaces, groups of linear operators, strong continuity.*

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1. INTRODUCTION

Let f be a complex valued positive definite function, measurable in the Lebesgue sense on a Euclidean space. Riesz ([14]) proved that $f = f^{\mathrm{c}} + f^0$ where f^{c} is a continuous positive definite function and f^0 is zero almost everywhere. Crum ([6]) established the interesting result that f^0 is also positive definite.

In [16] the following comment appears: According to a remark in Kreĭn ([11]), already Artjomenko, who lost his life during the second world war, knew that the remainder function f^0 is also positive definite, but he never published his proof. These results have been extended for complex valued measurable positive definite functions defined on locally compact groups (see [8] and [13]). Crum's result was generalized by Devinatz ([7]) for functions, with values in the set of bounded operators on a separable Hilbert space, which are weakly measurable and positive definite on a locally compact group with a left invariant measure. Langer ([12])

proved a similar result for scalar valued measurable functions with a finite number of negative squares defined on an interval of $\mathbb{R}$. In the book of Sasvári ([16]), an extension of Crum's result was obtained for scalar valued measurable functions, with a finite number of negative squares, defined on a locally compact group with a left invariant measure.

The aim of this paper is to establish the analogous version of this result for Kreĭn space operator valued indefinite functions with a finite number of negative squares, on a locally compact group.

More precisely: let κ be a nonnegative integer and let f be a κ-indefinite function defined on a locally compact group, with values in the space of the continuous linear operators of a Kreĭn space. We will prove that if f is weakly measurable then $f = f^{c} + f^{0}$ wheref^{c} is a κ-indefinite and weakly continuous function and f^{0} is a positive definite function that is zero locally almost everywhere. We also prove that if f is weakly continuous then f is strongly continuous.

As an application we obtain that a weakly measurable group of unitary operators on a separable Pontryagin space and with parameter on a locally compact group, is strongly continuous.

Some of our proofs are inspired in techniques developed in [7] and [16].

2. Preliminary notions

Some familiarity with operator theory on Kreĭn spaces is assumed. For the theory of indefinite inner product spaces (see [2], [4], [5], [9], [10]).

We recall some basic notions and results from the theory of Kreĭn spaces and operators on them. The pair $(\mathcal{R}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{R}})$ is an *inner product space* if $\mathcal{R}$ is a linear space over $\mathbb{C}$ and $\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{R}} : \mathcal{R}\times\mathcal{R} \mapsto \mathbb{C}$ is a sesquilinear mapping, called the *inner product.* If $\mathcal{R}$ is an inner product space we call $(\mathcal{R}, -\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{R}})$ its *antispace.* A *Kreĭn space* $(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$ is an inner product space which can be written as an orthogonal direct sum

$$\mathcal{K} = \mathcal{K}^{+} \oplus \mathcal{K}^{-}$$

where $\mathcal{K}^{+}$ is a Hilbert space and $\mathcal{K}^{-}$ is the antispace of a Hilbert space, which will be denoted by $|\mathcal{K}^{-}|$. Such a representation is called a *fundamental decomposition.*

Let $\mathcal{K}$ be a Krein space. A fundamental decomposition is not necessarily unique. However, if $\mathcal{K} = \mathcal{K}^{+} \oplus \mathcal{K}^{-}$ and $\mathcal{K} = \mathcal{K}_1^{+} \oplus \mathcal{K}_1^{-}$ are two fundamental decompositions, then $\dim \mathcal{K}^{+} = \dim \mathcal{K}_1^{+}$ and $\dim \mathcal{K}^{-} = \dim \mathcal{K}_1^{-}$. A *Pontryagin space* is a Kreĭn space $\mathcal{K}$ such that $\dim \mathcal{K}^{-}$ is finite. A topology is induced on $\mathcal{K}$ by any fundamental decomposition $\mathcal{K} = \mathcal{K}^{+} \oplus \mathcal{K}^{-}$. First form the *associated Hilbert space*

$$|\mathcal{K}| = \mathcal{K}^{+} \oplus |\mathcal{K}^{-}|.$$

This Hilbert space has an associated quadratic norm $\|\cdot\|_{|\mathcal{K}|}$. Two quadratic norms arising from different fundamental decompositions can be shown to be equivalent. Therefore the norm topologies resulting from two fundamental decompositions are the same. All notions of continuity and convergence are understood with respect to this common topology, which is called the *strong topology* of $\mathcal{K}$.

For $h_1, h_2 \in \mathcal{K}$, it holds that

$$|\langle h_1, h_2\rangle_{\mathcal{K}}| \leqslant \|h_1\|_{|\mathcal{K}|}\, \|h_2\|_{|\mathcal{K}|}.$$

Let $L(\mathcal{K})$ stand for the space of the continuous linear operators on $\mathcal{K}$. If $T \in L(\mathcal{K})$ its *adjoint* is the unique operator $T^* \in L(\mathcal{K})$ such that, for $h_1, h_2 \in \mathcal{K}$

$$\langle Th_1, h_2\rangle_{\mathcal{K}} = \langle h_1, T^* h_2\rangle_{\mathcal{K}}.$$

Let $T \in L(\mathcal{K})$ be an operator: T is *selfadjoint* if $T = T^*$, T is a *projection* if it is selfadjoint and $T^2 = T$ and T is *unitary* if $T^*T = TT^* = I$.

3. THE PONTRYAGIN SPACE AND THE UNITARY GROUP ASSOCIATED TO A κ-INDEFINITE FUNCTION

Let $(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$ be a Kreĭn space and fix a fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$, also let $(\Omega, \cdot\,)$ be a locally compact group with left invariant Haar measure $\mathrm{d}x$.

Suppose that κ is a nonnegative integer. A function $f : \Omega \to L(\mathcal{K})$ is said to be κ-*indefinite* if:

(a) $f(x) = f(x^{-1})^*$ for all $x \in \Omega$,

(b) for any finite set of points $x_1, \dots, x_n \in \Omega$ and vectors $h_1, \dots, h_n \in \mathcal{K}$, the hermitian matrix

$$(\langle f(x_i^{-1}x_j)h_i, h_j\rangle_{\mathcal{K}})_{i,j=1}^n$$

has at most κ negative eigenvalues, counted according to their multiplicities, and at least one such matrix has exactly κ negative eigenvalues.

The function f is said to be *positive definite* if this property holds with $\kappa = 0$, that is the above matrix is nonnegative in every case.

Let $f : \Omega \to L(\mathcal{K})$ be a κ-indefinite function. According to the definition given in p. 6 of [1], the hermitian $L(\mathcal{K})$-valued kernel on $\Omega \times \Omega$, given by

$$F(x, y) = f(x^{-1}y)$$

has κ negative squares. For $x \in \Omega$, let $f_x : \Omega \to L(\mathcal{K})$ be the function defined by

$$f_x(y) = f(x^{-1}y).$$

Observe that $f_x(\cdot) = F(x, \cdot)$.

By Theorem 1.1.3 of [1] there exists a unique Pontryagin space $(\mathcal{E}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{E}})$ of $\mathcal{K}$-valued functions on Ω, with reproducing kernel F, such that $\mathrm{ind}_-(\mathcal{E}) = \kappa$. The theory of reproducing kernels was introduced in [3] for the positive definite case.

Recall that $\mathcal{E}$ has the following properties:

(i) The elements of $\mathcal{E}$ are $\mathcal{K}$-valued functions on Ω.

(ii) For each $x \in \Omega$ and $h \in \mathcal{K}$ the function $f_x(\,\cdot\,)h$ belongs to $\mathcal{E}$.

(iii) For every $\varphi \in \mathcal{E}$ and $h \in \mathcal{K}$

$$\langle \varphi(\cdot), f_x(\cdot)h\rangle_{\mathcal{E}} = \langle \varphi(x), h\rangle_{\mathcal{K}}.$$

(iv) If $\mathcal{M}$ is the space of the functions $u \in \mathcal{E}$ such that

$$u(\cdot) = \sum_{i=1}^n f_{x_i}(\cdot)h_i$$

where $n \in \mathbb{N}$, $x_1, \dots, x_n \in \Omega$ and $h_1, \dots, h_n \in \mathcal{K}$, then $\mathcal{M}$ is a dense subspace of $\mathcal{E}$.

(v) If $u, v \in \mathcal{M}$ are such that

$$u(\cdot) = \sum_{i=1}^{n} f_{x_i}(\cdot)h_i, \quad v(\cdot) = \sum_{j=1}^{m} f_{y_j}(\cdot)h'_j$$

where $x_1, \dots, x_n, y_1, \dots, y_m \in \Omega, h_1, \dots, h_n, h'_1, \dots, h'_m \in \mathcal{K}$, then

$$\langle u, v\rangle_{\mathcal{E}} = \sum_{i=1}^{n}\sum_{j=1}^{m} \langle f(x_i^{-1}y_j)h_i \, , h'_j\rangle_{\mathcal{K}}.$$

In particular, if $x, y \in \Omega$ and $h, h' \in \mathcal{K}$

$$\langle f_x(\cdot)h, f_y(\cdot)h'\rangle_{\mathcal{E}} = \langle f(x^{-1}y)h \, , h'\rangle_{\mathcal{K}}.$$

The space $\mathcal{E}$ will be called the *Pontryagin space associated to* f and $\mathcal{M}$ will be called the *pre-Pontryagin space associated to* f. Since $\mathcal{M}$ is a dense linear manifold on $\mathcal{E}$ there exists a fundamental decomposition of $\mathcal{E}$

$$\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-,$$

where $\mathcal{E}^-$ is generated by a set of functions $\{u_1, \dots, u_\kappa\} \subset \mathcal{M}$, such that

$$\langle u_i(\cdot), u_j(\cdot)\rangle_{\mathcal{E}} = -\delta_{ij} \tag{3.1}$$

for $i, j = 1, \dots, \kappa$.

Let $|\mathcal{E}^-|$ be the antispace of $\mathcal{E}^-$, that is $|\mathcal{E}^-|$ is the Hilbert space $(\mathcal{E}^-, -\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{E}})$. The norm induced on $\mathcal{E}$ by this fundamental decomposition is the norm in the Hilbert space $|\mathcal{E}| = \mathcal{E}^+ \oplus |\mathcal{E}^-|$ given by

$$\|\varphi\|^2_{|\mathcal{E}|} = \langle \varphi, \varphi\rangle_{\mathcal{E}} + 2\sum_{j=1}^{\kappa} |\langle \varphi, u_j\rangle_{\mathcal{E}}|^2 \tag{3.2}$$

for $\varphi \in \mathcal{E}$.

In the sequel:

(a) $f : \Omega \to L(\mathcal{K})$ is a κ-indefinite function, $\mathcal{E}$ and $\mathcal{M}$ are the Pontryagin and the pre-Pontryagin spaces associated to f.

(b) $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ is a fundamental decomposition of $\mathcal{E}$, $\{u_1, \dots, u_\kappa\} \subset \mathcal{M}$ is a total set on $\mathcal{E}^-$ such that equality (3.1) holds.

(c) $\|\cdot\|_{|\mathcal{E}|}$ is the norm given by formula (3.2).

THEOREM 3.1. *Let $\mathcal{K}$ be a Kreĭn space, let $f : \Omega \to L(\mathcal{K})$ be a κ-indefinite function and let $\mathcal{E}$ be the Pontryagin space associated to f then:*

(a) *The elements of $\mathcal{E}$ are $\mathcal{K}$-valued functions on Ω and $f(\cdot)h \in \mathcal{E}$ for all $h \in \mathcal{K}$.*

(b) *$\mathcal{E}$ is invariant under translations in the following sense: if $\varphi \in \mathcal{E}$, $\omega \in \Omega$ and ψ is the function defined by $\psi(x) = \varphi(\omega x)$ then ψ is in $\mathcal{E}$.*

(c) *If $(U_\omega\varphi)(x) = \varphi(\omega x)$ for $\varphi \in \mathcal{E}$, $x, \omega \in \Omega$, then $(U_\omega)_{\omega\in\Omega}$ is a unitary representation of Ω in $\mathcal{E}$ with cyclic subspace $\bigvee\{f(\cdot)h : h \in \mathcal{K}\}$.*

(d) *The linear operator $\tau : \mathcal{K} \to \mathcal{E}$ defined by $\tau h = f(\cdot)h$ is bounded, and*

$$f(\omega) = \tau^* U_\omega \tau$$

for all $\omega \in \Omega$.

(e) *If* $\varphi \in \mathcal{E}$, $\omega \in \Omega$ *and* $h \in \mathcal{K}$ *then*

$$\langle \varphi(\omega), h\rangle_{\mathcal{K}} = \langle U_\omega \varphi, \tau h\rangle_{\mathcal{E}} = \langle \varphi, U_\omega^{-1}\tau h\rangle_{\mathcal{E}}.$$

(f) *Weak convergence of a net* $\{\varphi_\alpha\}$ *in* $\mathcal{E}$ *implies weak convergence in* $\mathcal{K}$ *of the net* $\{\varphi_\alpha(\omega)\}$ *for all* $\omega \in \Omega$.

Proof. Note that (a) follows immediately from the properties of the space $\mathcal{E}$. For $\omega, x \in \Omega$ let $V_\omega : \mathcal{M} \to \mathcal{M}$ be the linear operator defined by

$$(V_\omega u)(x) = u(\omega x).$$

It turns out that V_ω is an unitary operator and

$$V_\omega f_x = f_{\omega^{-1}x}, \quad V_\omega^{-1} f_x = f_{\omega x}.$$

Since $\mathcal{M}$ is a pre-Pontryagin space with completion $\mathcal{E}$, V_ω has a unitary extension $U_\omega : \mathcal{E} \to \mathcal{E}$.

Let $\varphi \in \mathcal{E}$, $x \in \Omega$, $h \in \mathcal{K}$, then

$$\begin{aligned} \langle (U_\omega\varphi)(x), h\rangle_{\mathcal{K}} &= \langle U_\omega \varphi, f_x(\cdot)h\rangle_{\mathcal{E}} = \langle \varphi, U_\omega^{-1} f_x(\cdot)h\rangle_{\mathcal{E}} = \langle \varphi, V_\omega^{-1} f_x(\cdot)h\rangle_{\mathcal{E}} \\ &= \langle \varphi(\cdot), f_{\omega x}(\cdot)h\rangle_{\mathcal{E}} = \langle \varphi(\omega x), h\rangle_{\mathcal{K}}. \end{aligned} \tag{3.3}$$

From (iv) it follows that $\bigvee_{\omega\in\Omega} U_\omega\Big(\bigvee\{f(\cdot)h : h \in \mathcal{K}\}\Big)$ spans $\mathcal{E}$. Thus we have proved (b) and (c).

Now we will prove (d). Let $\tau : \mathcal{K} \to \mathcal{E}$ defined by $\tau h = f(\cdot)h$. For all $h \in \mathcal{K}$,

$$\begin{aligned} \|\tau h\|^2_{|\mathcal{E}|} &= \langle \tau h, \tau h\rangle_{\mathcal{E}} + 2\sum_{j=1}^{\kappa} |\langle \tau h, u_j\rangle_{\mathcal{E}}|^2 = \langle f(\cdot)h, f(\cdot)h\rangle_{\mathcal{E}} + 2\sum_{j=1}^{\kappa} |\langle u_j, f(\cdot)h\rangle_{\mathcal{E}}|^2 \\ &= \langle f(1)h, h\rangle_{\mathcal{K}} + 2\sum_{j=1}^{\kappa} |\langle u_j(1), h\rangle_{\mathcal{K}}|^2 \leqslant \Big(\|f(1)\| + 2\sum_{j=1}^{\kappa} \|u_j(1)\|^2_{|\mathcal{K}|}\Big)\|h\|^2_{|\mathcal{K}|}. \end{aligned}$$

Given $h, h' \in \mathcal{K}$, $\omega \in \Omega$

$$\begin{aligned} \langle f(\omega)h, h'\rangle_{\mathcal{K}} &= \langle f_{\omega^{-1}}(\cdot)h, f_1(\cdot)h'\rangle_{\mathcal{E}} = \langle U_\omega f(\cdot)h, f(\cdot)h'\rangle_{\mathcal{E}} \\ &= \langle U_\omega \tau h, \tau h'\rangle_{\mathcal{E}} = \langle \tau^* U_\omega \tau h, h'\rangle_{\mathcal{K}}. \end{aligned}$$

So (d) is proved.

From formula (3.3) it follows that

$$\langle U_\omega \varphi, f_x(\cdot)h\rangle_{\mathcal{E}} = \langle \varphi, U_\omega^{-1} f_x(\cdot)h\rangle_{\mathcal{E}} = \langle \varphi(\omega x), h\rangle_{\mathcal{K}}.$$

Taking $x = 1$ and observing that, for all $h \in \mathcal{K}$, $f_1(\cdot)h = \tau h$ we obtain (e).

Finally (f) follows from (e). ∎

In the following $(U_\omega)_{\omega\in\Omega}$ will be the unitary representation of Ω on $\mathcal{E}$ given by Theorem 3.1 and $\tau : \mathcal{K} \to \mathcal{E}$ will be the linear operator given by $\tau h = f(\cdot)h$.

PROPOSITION 3.2. *If f is locally bounded then:*
(a) *the function $\omega \mapsto \|U_\omega\|$ is locally bounded;*
(b) *all the elements of $\mathcal{E}$ are locally bounded.*

Proof. Since Ω is locally compact, a function is locally bounded if and only if it is bounded on compact subsets of Ω. So we will show that $\omega \mapsto \|U_\omega\|$ is bounded on compact sets.

STEP 1. For each pair $i, j \in \{1, \dots, \kappa\}$ the function $\omega \mapsto |\langle U_\omega u_i, u_j\rangle_{\mathcal{E}}|$ from Ω to $\mathbb{C}$ is bounded on compact sets. In fact, for each $i \in \{1, \dots, \kappa\}$ there exists $x_l^i \in \Omega$, $h_l^i \in \mathcal{K}$, $l = 1, \dots, m_i$, such that

$$u_i = \sum_{l=1}^{m_i} f_{x_l^i}(\cdot) h_l^i.$$

Hence

$$|\langle U_\omega u_i, u_j\rangle_{\mathcal{E}}| = \Big| \sum_{l=1}^{m_i} \sum_{p=1}^{m_j} \langle f_{\omega^{-1}x_l^i}(\cdot) h_l^i, f_{x_p^j}(\cdot) h_p^j\rangle_{\mathcal{E}} \Big| = \Big| \sum_{l=1}^{m_i} \sum_{p=1}^{m_j} \langle f((x_l^i)^{-1}\omega x_p^j) h_l^i, h_p^j\rangle_{\mathcal{K}} \Big|$$

which is clearly bounded on compact sets.

STEP 2. For each $i \in \{1, \dots, \kappa\}$ the function $\omega \mapsto \|U_\omega u_i\|_{|\mathcal{E}|}$ from Ω to $\mathbb{C}$ is bounded on compact sets. Simply observe that

$$\|U_\omega u_i\|_{|\mathcal{E}|}^2 = \langle u_i, u_i\rangle_{\mathcal{E}} + 2\sum_{j=1}^{\kappa} |\langle U_\omega u_i, u_j\rangle_{\mathcal{E}}|^2 = -1 + 2\sum_{j=1}^{\kappa} |\langle U_\omega u_i, u_j\rangle_{\mathcal{E}}|^2.$$

STEP 3. The function $\omega \mapsto \|U_\omega\|$ is locally bounded. Indeed, if $\varphi \in \mathcal{E}$ then

$$\begin{aligned}\|U_\omega \varphi\|_{|\mathcal{E}|}^2 &= \langle U_\omega\varphi, U_\omega\varphi\rangle_{\mathcal{E}} + 2\sum_{j=1}^{\kappa} |\langle U_\omega\varphi, u_j\rangle_{\mathcal{E}}|^2 = \langle \varphi, \varphi\rangle_{\mathcal{E}} + 2\sum_{j=1}^{\kappa} |\langle \varphi, U_\omega^{-1} u_j\rangle_{\mathcal{E}}|^2 \\ &\leqslant \Big(1 + 2\sum_{j=1}^{\kappa} \|U_\omega^{-1} u_j\|_{|\mathcal{E}|}^2\Big) \|\varphi\|_{|\mathcal{E}|}^2.\end{aligned}$$

We conclude that

$$\|U_\omega\| \leqslant 1 + 2\sum_{j=1}^{\kappa} \|U_\omega^{-1} u_j\|_{|\mathcal{E}|}^2.$$

To prove (b) let $\varphi \in \mathcal{E}$. If $\omega \in \Omega$ and $h \in \mathcal{K}$ then

$$|\langle \varphi(\omega), h\rangle_{\mathcal{K}}| = |\langle U_\omega \varphi, \tau h\rangle_{\mathcal{E}}| \leqslant \|U_\omega\|\,\|\tau\|\,\|\varphi\|_{|\mathcal{E}|} \|h\|_{|\mathcal{K}|}.$$

Taking $h = \varphi(\omega)$ the result follows. ∎

REMARK 3.3. From the proof of Proposition 3.2 it also follows that: If f is bounded then:
(a) the function $\omega \mapsto \|U_\omega\|$ is bounded;
(b) every $\varphi \in \mathcal{E}$ is bounded.

3.1. THE WEAKLY CONTINUOUS CASE. Recall that a function $g : \Omega \to L(\mathcal{K})$ is *weakly continuous* if for all $h_1, h_2 \in \mathcal{K}$ the complex-valued function $\omega \mapsto \langle g(\omega) h_1, h_2\rangle_{\mathcal{K}}$ is continuous and g is *strongly continuous* if for all $h \in \mathcal{K}$ the $\mathcal{K}$-valued function $\omega \mapsto g(\omega) h$ is continuous.

Also recall that a function $\varphi : \Omega \to \mathcal{K}$ is *weakly continuous* if for every $h \in \mathcal{K}$ the complex valued function $\omega \mapsto \langle \varphi(\omega), h\rangle_{\mathcal{K}}$ is continuous.

Proposition 3.4. *If f is weakly continuous then:*

(a) *for every $u \in \mathcal{M}$ the function from Ω to $\mathcal{E}$ given by $\omega \mapsto U_\omega u$ is continuous;*

(b) *all the elements of $\mathcal{E}$ are weakly continuous.*

Proof. Since every $u \in \mathcal{M}$ is a linear combination of functions of the form $U_\omega \tau h$, with $\omega \in \Omega$ and $h \in \mathcal{K}$, to prove (a) it is enough to show that for each $\omega \in \Omega$ and $h \in \mathcal{K}$ the function $x \mapsto U_x U_\omega \tau h$ is continuous. It is clear that all the elements of $\mathcal{M}$ are weakly continuous, in particular $u_1, \ldots, u_k$ are weakly continuous.

Let $x, x_0, \omega \in \Omega$ then

$$\begin{aligned}
&\|U_{x\omega}\tau h - U_{x_0\omega}\tau h\|^2_{|\mathcal{E}|}\\
&\quad = \langle U_{x\omega}\tau h - U_{x_0\omega}\tau h, U_{x\omega}\tau h - U_{x_0\omega}\tau h\rangle_{\mathcal{E}} + 2\sum_{j=1}^{\kappa} |\langle U_{x\omega}\tau h - U_{x_0\omega}\tau h, u_j\rangle_{\mathcal{E}}|^2\\
&\quad = 2\langle f(\cdot)h, f(\cdot)h\rangle_{\mathcal{E}} - 2\mathrm{Re}\langle U_{(x\omega)^{-1}x_0\omega} f(\cdot)h, f(\cdot)h\rangle_{\mathcal{E}}\\
&\qquad\qquad + 2\sum_{j=1}^{\kappa} |\langle u_j, U_{x\omega}\tau h\rangle_{\mathcal{E}} - \langle u_j, U_{x_0\omega}\tau h\rangle_{\mathcal{E}}|^2\\
&\quad = 2\langle f(1)h, h\rangle_{\mathcal{K}} - 2\mathrm{Re}\langle f(\omega^{-1}x^{-1}x_0\omega)h, h\rangle_{\mathcal{K}}\\
&\qquad\qquad + 2\sum_{j=1}^{\kappa} |\langle u_j(\omega^{-1}x^{-1}), h\rangle_{\mathcal{K}} - \langle u_j(\omega^{-1}{x_0}^{-1}), h\rangle_{\mathcal{K}}|^2.
\end{aligned}$$

Since $f(1)^* = f(1)$ we have that $\langle f(1)h, h\rangle_{\mathcal{K}}$ is a real number. Thus (a) follows from the weak continuity of f and u_j.

Assertion (b) is obtained from (a) and from (e) of Theorem 3.1. ∎

Theorem 3.5. *If f is weakly continuous then the representation $(U_\omega)_{\omega\in\Omega}$ is strongly continuous.*

Proof. Let $\varphi \in \mathcal{E}$, we must prove that the function $\omega \mapsto U_\omega \varphi$ is continuous. Observe that given $\omega, \omega_0 \in \Omega$ and $u \in \mathcal{M}$

$$\begin{aligned}
\|U_\omega\varphi - U_{\omega_0}\varphi\|_{\mathcal{E}} &\leqslant \|U_\omega\varphi - U_\omega u\|_{\mathcal{E}} + \|U_\omega u - U_{\omega_0}u\|_{\mathcal{E}} + \|U_{\omega_0}u - U_{\omega_0}\varphi\|_{\mathcal{E}}\\
&\leqslant \|U_\omega\|\,\|\varphi - u\|_{\mathcal{E}} + \|U_\omega u - U_{\omega_0}u\|_{\mathcal{E}} + \|U_{\omega_0}\|\,\|\varphi - u\|_{\mathcal{E}}.
\end{aligned}$$

Since $\mathcal{M}$ is dense in $\mathcal{E}$, using the continuity of $\omega \mapsto U_\omega u$ (Proposition 3.4) and the local boundedness of $\omega \mapsto \|U_\omega\|$ (Proposition 3.2) the result follows. ∎

Corollary 3.6. *Let $\mathcal{K}$ be a Kreĭn space and let $f : \Omega \to L(\mathcal{K})$ be a κ-indefinite function. If f is weakly continuous then f is strongly continuous.*

Proof. Since $f(\omega) = \tau^* U_\omega \tau$ the result follows. ∎

3.2. THE WEAKLY MEASURABLE CASE. Let us consider the left invariant Haar measure on the locally compact group Ω.

Recall that a function $g : \Omega \to L(\mathcal{K})$ is *weakly measurable* if for every $h_1, h_2 \in \mathcal{K}$ the complex valued function $\omega \mapsto \langle g(\omega)h_1, h_2\rangle_{\mathcal{K}}$ is measurable. Also recall that a function $\varphi : \Omega \to \mathcal{K}$ is *weakly measurable* if for every $h \in \mathcal{K}$ the complex valued function $\omega \mapsto \langle \varphi(\omega), h\rangle_{\mathcal{K}}$ is measurable.

PROPOSITION 3.7. *If f is weakly measurable then f is locally bounded.*

Proof. It is enough to show that f is bounded on each compact subset of Ω. By a straightforward application of the Banach-Steinhaus theorem it is enough to prove that for each $h_1, h_2 \in \mathcal{K}$ the function $\omega \mapsto \langle f(\omega)h_1, h_2\rangle_{\mathcal{K}}$ is bounded on each compact subset of Ω.

If $h \in \mathcal{K}$ then the function g given by $g(\omega) = \langle f(\omega)h, h\rangle_{\mathcal{K}}$ is κ_0-indefinite, where $\kappa_0 \leqslant \kappa$. By Theorem 1 of [15] (see also Theorem 5.3.1 of [16]), g is locally bounded. From the identity

$$\begin{aligned}\langle f(\omega)h_1, h_2\rangle_{\mathcal{K}} = \frac{1}{4}\{&\langle f(\omega)(h_1 + h_2), h_1 + h_2\rangle_{\mathcal{K}} - \langle f(\omega)(h_1 - h_2), h_1 - h_2\rangle_{\mathcal{K}}\} \\ &+ \frac{\mathrm{i}}{4}\{\langle f(\omega)(h_1 + \mathrm{i}h_2), h_1 + \mathrm{i}h_2\rangle_{\mathcal{K}} - \langle f(\omega)(h_1 - \mathrm{i}h_2), h_1 - \mathrm{i}h_2\rangle_{\mathcal{K}}\}\end{aligned}$$

the result follows. ∎

THEOREM 3.8. *If f is weakly measurable then the representation $(U_\omega)_{\omega\in\Omega}$ is weakly measurable.*

Proof. It is clear that all the elements of $\mathcal{M}$ are weakly measurable. From the definition of the inner product on $\mathcal{E}$ it follows that: If $u, v \in \mathcal{M}$ then the function $w \mapsto \langle u, U_\omega v\rangle_{\mathcal{E}}$ is measurable.

If $\varphi, \psi \in \mathcal{E}$ then there exist sequences $\{u_n\}$, $\{v_n\}$ in $\mathcal{M}$ such that

$$\varphi = \lim_{n\to+\infty} u_n \quad \text{and} \quad \psi = \lim_{n\to+\infty} v_n.$$

Thus $\langle \varphi, U_\omega \psi\rangle_{\mathcal{E}} = \lim_{n\to+\infty} \langle u_n, U_\omega v_n\rangle_{\mathcal{E}}$. Therefore the function

$$w \to \langle \varphi, U_\omega \psi\rangle_{\mathcal{E}} \tag{3.4}$$

is measurable. ∎

COROLLARY 3.9. *If f is weakly measurable then all the elements of $\mathcal{E}$ are weakly measurable.*

Proof. Let $\varphi \in \mathcal{E}$ and take $\psi = \tau h$ in equation (3.4). From (e) of Theorem 3.1 it follows that φ is weakly measurable. ∎

4. THE MAIN RESULT

THEOREM 4.1. *Let κ be a nonnegative integer, let $\mathcal{K}$ be a separable Kreĭn space and let $(\Omega, \cdot)$ be a locally compact group. If $f : \Omega \to L(\mathcal{K})$ is a weakly measurable κ-indefinite function, then there exist two functions f^0 and f^{c} from Ω to $L(\mathcal{K})$ such that:*

(a) $f = f^{\mathrm{c}} + f^0$;

(b) f^{c} *is κ-indefinite and weakly continuous;*

(c) f^0 *is positive definite and it is zero locally almost everywhere.*

In order to prove this theorem we need some previous results.

Let $C_{00}(\Omega)$ denote the set of the complex valued function defined on Ω with compact support. For $\alpha \in C_{00}(\Omega)$, let $\alpha^*(x) = \overline{\alpha}(x^{-1})$.

PROPOSITION 4.2. *For each $\alpha \in C_{00}(\Omega)$ there exists a linear operator $A_\alpha \in L(\mathcal{E})$ such that:*

(a) *For $\varphi \in \mathcal{E}$, $\omega \in \Omega$ and $h \in \mathcal{K}$,*

$$\langle (A_\alpha \varphi)(\omega), h\rangle_{\mathcal{K}} = (\alpha * \langle \varphi(\cdot), h\rangle_{\mathcal{K}})(\omega) = \int_\Omega \alpha(y) \langle \varphi(\omega y^{-1}), h\rangle_{\mathcal{K}} \, \mathrm{d}y.$$

(b) $A_\alpha^* = A_{\alpha^*}$.

Proof. For $\alpha \in C_{00}(\Omega)$, consider the sesquilinear functional B_α from $\mathcal{E} \times \mathcal{E}$ to $\mathbb{C}$ defined by

$$B_\alpha(\varphi_1, \varphi_2) = \int_\Omega \alpha(y) \langle \varphi_1, U_y \varphi_2\rangle_{\mathcal{E}} \, \mathrm{d}y$$

for $\varphi_1, \varphi_2 \in \mathcal{E}$. Since $\|U_\omega\|$ is locally bounded we have that B is continuous. Therefore there exists a continuous linear operator A_α from $\mathcal{E}$ to $\mathcal{E}$ such that $B_\alpha(\varphi_1, \varphi_2) = \langle A_\alpha \varphi_1, \varphi_2\rangle_{\mathcal{E}}$.

Let $\varphi \in \mathcal{E}$, $\omega \in \Omega$, and $h \in \mathcal{K}$

$$\begin{aligned}\langle (A_\alpha \varphi)(\omega), h\rangle_{\mathcal{K}} &= \langle U_\omega A_\alpha \varphi, \tau h\rangle_{\mathcal{E}} = \langle A_\alpha \varphi, U_\omega^{-1} \tau h\rangle_{\mathcal{E}} = \int_\Omega \alpha(y) \langle \varphi, U_y U_\omega^{-1} \tau h\rangle_{\mathcal{E}} \, \mathrm{d}y \\ &= \int_\Omega \alpha(y) \langle \varphi(\omega y^{-1}), h\rangle_{\mathcal{K}} \, \mathrm{d}y.\end{aligned}$$

Let $\varphi_1, \varphi_2 \in \mathcal{E}$,

$$\begin{aligned}\langle A_\alpha \varphi_1, \varphi_2\rangle_{\mathcal{E}} &= \int_\Omega \alpha(y) \langle \varphi_1, U_y \varphi_2\rangle_{\mathcal{E}} \, \mathrm{d}y = \overline{\int_\Omega \overline{\alpha(y)} \langle \varphi_2, U_{y^{-1}} \varphi_1\rangle_{\mathcal{E}} \, \mathrm{d}y} \\ &= \overline{\int_\Omega \alpha^*(y) \langle \varphi_2, U_y \varphi_1\rangle_{\mathcal{E}} \, \mathrm{d}y} = \overline{\langle A_{\alpha^*} \varphi_2, \varphi_1\rangle_{\mathcal{E}}} = \langle \varphi_1, A_{\alpha^*} \varphi_2\rangle_{\mathcal{E}}.\end{aligned}$$

Thus $A_\alpha^* = A_{\alpha^*}$. ∎

Proposition 4.3. (a) *If $\alpha \in C_{00}(\Omega)$ and $\varphi \in \mathcal{E}$ then $A_\alpha\varphi$ is weakly continuous.*

(b) *If $\varphi \in \mathcal{E}$ and $A_\alpha\varphi = 0$ for all $\alpha \in C_{00}(\Omega)$ then $\varphi(\omega) = 0$ locally almost everywhere.*

Proof. If $h \in \mathcal{K}$ and $\omega \in \Omega$ then

$$\langle (A_\alpha\varphi)(\omega), h\rangle_{\mathcal{K}} = \int\limits_{\Omega} \alpha(y)\ \langle \varphi(\omega y^{-1}), h\rangle_{\mathcal{K}}\,\mathrm{d}y = \int\limits_{\Omega} \alpha(\omega y^{-1})\ \langle \varphi(y), h\rangle_{\mathcal{K}}\,\mathrm{d}y.$$

Since φ is locally bounded and α is continuous with compact support it follows that the function $\omega \mapsto \langle (A_\alpha\varphi)(\omega), h\rangle_{\mathcal{K}}$ is continuous.

Let $\varphi \in \mathcal{E}$ such that $A_\alpha\varphi = 0$ for all $\alpha \in C_{00}(\Omega)$. Then for each $h \in \mathcal{K}$ we have that

$$\int\limits_{\Omega} \alpha(\omega y^{-1})\ \langle \varphi(y), h\rangle_{\mathcal{K}}\,\mathrm{d}y = 0$$

for all $\alpha \in C_{00}(\Omega)$. Thus $\langle \varphi(\omega), h\rangle_{\mathcal{K}} = 0$ locally almost everywhere.

Let $\{e_n\}_{n\geqslant 1}$ be a complete orthogonal subset of $\mathcal{K}$. For each $n \in \mathbb{N}$ there exists a locally null set $A_n \subset \Omega$ such that $\langle \varphi(\omega), e_n\rangle_{\mathcal{K}} = 0$ if $\omega \notin A_n$.

Let $A = \bigcup\limits_{n\geqslant 1} A_n$. Then A is locally null and $\langle \varphi(\omega), h\rangle_{\mathcal{K}} = 0$ if $\omega \notin A$ and $h \in \mathcal{K}$. Therefore $\varphi(\omega) = 0$ locally almost everywhere. ∎

Consider $\mathcal{C} = \{\varphi \in \mathcal{E} : \varphi \text{ is weakly continuous}\}$.

As usual, let $\mathcal{C}^\perp = \{\varphi \in \mathcal{E} : \langle \varphi, \psi\rangle_{\mathcal{E}} = 0 \text{ for all } \psi \in \mathcal{E}\}$.

Proposition 4.4. (a) *The space $\mathcal{C}$ is a closed linear space.*

(b) *Every function of $\mathcal{C}^\perp$ is zero locally almost everywhere.*

(c) $\mathcal{C} \cap \mathcal{C}^\perp = \{0\}$.

(d) *The space $\mathcal{C}$ is a nondegenerate subspace of $\mathcal{E}$. That is, $\mathcal{C}$ is a Pontryagin space with the indefinite inner product $\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{E}}$.*

(e) *$\mathcal{C}$ and $\mathcal{C}^\perp$ are invariant under $(U_\omega)_{\omega\in\Omega}$.*

Proof. It is clear that $\mathcal{C}$ is a linear space. Let $\{\varphi_n\} \subset \mathcal{E}$ and $\varphi \in \mathcal{E}$ be such that $\lim\limits_{n\to\infty} \|\varphi_n - \varphi\|_{|\mathcal{E}|} = 0$. For $\omega \in \Omega$ and $h \in \mathcal{K}$

$$\begin{aligned}|\langle \varphi_n(\omega) - \varphi(\omega), h\rangle_{\mathcal{K}}| &= |\langle \varphi_n - \varphi, U_\omega^{-1}\tau h\rangle_{\mathcal{E}}| \leqslant \|\varphi_n - \varphi\|_{|\mathcal{E}|}\|U_\omega^{-1}\tau h\|_{|\mathcal{E}|}\\ &\leqslant \|\varphi_n - \varphi\|_{|\mathcal{E}|}\|U_\omega^{-1}\|\ \|\tau h\|_{|\mathcal{E}|}.\end{aligned}$$

From Propositions 3.2 and 3.7 we have that $\|U_\omega^{-1}\|$ is locally bounded. Then

$$\lim_{n\to\infty} \langle \varphi_n(\omega), h\rangle_{\mathcal{K}} = \langle \varphi(\omega), h\rangle_{\mathcal{K}}$$

uniformly on compact subsets of Ω, thus $\varphi \in \mathcal{C}$. Therefore $\mathcal{C}$ is closed.

By Proposition 4.3, if $\psi \in \mathcal{E}$, $A_\alpha\psi \in \mathcal{C}$ for all $\alpha \in C_{00}(\Omega)$. Therefore if $\varphi \in \mathcal{C}^\perp$ then

$$\langle A_\alpha\varphi, \psi\rangle_{\mathcal{E}} = \langle \varphi, A_{\alpha^*}\psi\rangle_{\mathcal{E}} = 0 \quad \text{for all } \psi \in \mathcal{E}.$$

Again by Proposition 4.3, $\varphi = 0$ locally almost everywhere.

If $\varphi \in \mathcal{C} \cap \mathcal{C}^\perp$ then φ must be weakly continuous and zero locally almost everywhere, so (c) and (d) follow.

Clearly, $\mathcal{C}$ is invariant under $(U_\omega)_{\omega\in\Omega}$. Finally (e) follows from

$$U_\omega^* = U_\omega^{-1} = U_{\omega^{-1}}. \quad ∎$$

The following result is an extension of Lemma 5.3.4 given in [16].

LEMMA 4.5. *Let f be κ-indefinite where $\kappa \geqslant 1$. If $x_1,\dots,x_n \in \Omega$ and $h_1,\dots,h_n \in \mathcal{K}$, are such that*

$$M = \big(\langle f(x_i^{-1}x_j)h_i, h_j\rangle_{\mathcal{K}}\big)_{i,j=1}^n$$

has exactly κ negative eigenvalues, counted according to their multiplicity, then

$$\Omega = \bigcup_{i,j=1}^n x_i S(f,h_i,h_j)x_j^{-1}$$

where $S(f,h,h') = \{\omega\in\Omega : \langle f(\omega)h,h'\rangle_{\mathcal{K}} \neq 0\}$ for $h,h'\in\mathcal{K}$.

Proof. If the result were not true there would exist $z\in\Omega$ such that $z\notin x_iS(f,h_i,h_j)x_j^{-1}$, $i,j=1,\dots,n$, so we would have that

$$\langle f(x_i^{-1}zx_j)h_i,h_j\rangle_{\mathcal{K}} = 0.$$

Let $y_1,\dots,y_{2n}\in\Omega$ and $h_1',\dots,h_{2n}'\in\mathcal{K}$ be defined by

$$y_i = x_i,\quad y_{n+i} = zx_i,\quad h_i' = h_i,\quad h_{n+i}' = h_i,$$

for $i=1,\dots,n$. Then

$$(\langle f(y_i^{-1}y_j)h_i',h_j'\rangle_{\mathcal{K}})_{i,j=1}^{2n} = \begin{pmatrix} M & 0\\ 0 & M\end{pmatrix}$$

which has 2κ negative eigenvalues. This is a contradiction. ∎

COROLLARY 4.6. *If f is κ-indefinite and f vanish locally almost everywhere then f is positive definite.*

Proof. If $f=0$ locally almost everywhere, then every set of the form $x_i^{-1}S(f,h_i,h_j)x_j^{-1}$, $x_1,\dots,x_n\in\Omega$ and $h_1,\dots,h_n\in\mathcal{K}$ would have locally measure zero. Therefore if $\kappa\geqslant 1$, the whole group Ω would be locally null. Thus $\kappa = 0$. ∎

Now we prove Theorem 4.1.

Since $\mathcal{C}$ is orthocomplemented the orthogonal projection of $\mathcal{E}$ on $\mathcal{C}$, $P = P_{\mathcal{C}}^{\mathcal{E}}$, is a bounded linear operator.

Given $\omega\in\Omega$ set

$$f^{\mathrm{c}}(\omega) = \tau^* U_\omega P\tau.$$

Then $f^{\mathrm{c}}(\omega)\in L(\mathcal{K})$.

Let $h_1,h_2\in\mathcal{K}$, then

$$\langle f^{\mathrm{c}}(\omega)h_1,h_2\rangle_{\mathcal{K}} = \langle \tau^*U_\omega P\tau h_1,h_2\rangle_{\mathcal{K}} = \langle U_\omega P\tau h_1,\tau h_2\rangle_{\mathcal{E}} = \langle P\tau(\omega)h_1,h_2\rangle_{\mathcal{K}}.$$

Since $P\tau$ is weakly continuous it follows that f^{c} is weakly continuous.

Now we shall prove that f^{c} has a finite number of negative squares. By (e) of Proposition 4.4, $(U_\omega)_{\omega\in\Omega}$ and P commute, therefore

$$\langle f^{\mathrm{c}}(\omega)h_1,h_2\rangle_{\mathcal{K}} = \langle \tau^*U_\omega P\tau h_1,h_2\rangle_{\mathcal{K}} = \langle U_\omega P^2\tau h_1,\tau h_2\rangle_{\mathcal{E}} = \langle U_\omega P\tau h_1,P\tau h_2\rangle_{\mathcal{E}}.$$

Thus

$$\begin{aligned}
&\sum_i \sum_j \lambda_i \bar{\lambda}_j \langle f^{\mathrm{c}}(\omega_i^{-1}\omega_j) h_i, h_j \rangle_{\mathcal{K}} \\
&\quad = \sum_i \sum_j \lambda_i \bar{\lambda}_j \langle U_{\omega_i^{-1}\omega_j} P\tau h_i, P\tau h_j \rangle_{\mathcal{E}} = \sum_i \sum_j \lambda_i \bar{\lambda}_j \langle U^{-1}_{\omega_j^{-1}\omega_i} P\tau h_i, P\tau h_j \rangle_{\mathcal{E}} \\
&\quad = \sum_i \sum_j \lambda_i \bar{\lambda}_j \langle U^{-1}_{\omega_i} P\tau h_i, U^{-1}_{\omega_j} P\tau h_j \rangle_{\mathcal{E}} = \Big\langle \sum_i \lambda_i U^{-1}_{\omega_i} P\tau h_i, \sum_j \lambda_j U^{-1}_{\omega_j} P\tau h_j \Big\rangle_{\mathcal{E}}.
\end{aligned}$$

Then f^{c} is κ_{c}-indefinite, where $\kappa_{\mathrm{c}} \leqslant \kappa$.

Set $f^0(\omega) = f(\omega) - f^{\mathrm{c}}(\omega)$ then $f^0(\omega) = \tau^* U_\omega (I-P)\tau$. As before it follows that f^0 is κ_0-indefinite, where $\kappa_0 \leqslant \kappa$.

Let $h_1, h_2 \in \mathcal{K}$. Since $I-P$ is the orthogonal projection on $\mathcal{C}^\perp$, from (b) of Proposition 4.4 it follows that $(I-P)\tau(\omega)h_1 = 0$ locally almost everywhere. Therefore

$$\begin{aligned}
\langle f^0(\omega)h_1, h_2\rangle_{\mathcal{K}} &= \langle \tau^* U_\omega (I-P)\tau h_1, h_2\rangle_{\mathcal{K}} = \langle U_\omega (I-P)\tau h_1, \tau h_2\rangle_{\mathcal{E}} \\
&= \langle (I-P)\tau(\omega) h_1, h_2\rangle_{\mathcal{K}} = 0,
\end{aligned}$$

locally almost everywhere.

Let $\{e_n\}_{n\geqslant 1}$ be a complete orthogonal subset of $\mathcal{K}$. For each $n, m \in \mathbb{N}$ there exists a locally null set $A_{nm} \subset \Omega$ such that

$$\langle f^0(\omega)e_n, e_m\rangle_{\mathcal{K}} = 0 \quad \text{if } \omega \notin A_{nm}.$$

Let $A = \bigcup_{n,m\geqslant 1} A_{nm}$. Then A is locally null and, for all $h_1, h_2 \in \mathcal{K}$,

$$\langle f^0(\omega)h_1, h_2\rangle_{\mathcal{K}} = 0 \quad \text{if } \omega \notin A.$$

Thus $f^0(\omega) = 0$ locally almost everywhere.

Finally from Corollary 4.6 it follows that f^0 is positive definite and therefore $\kappa_{\mathrm{c}} = \kappa$.

5. ON MEASURABLE UNITARY GROUPS

The following theorem is an extension for groups of unitary operators on Pontryagin spaces, of a result given in [7].

THEOREM 5.1. *Let $(\Omega, \cdot)$ be a locally compact group and let $\mathcal{G}$ be a separable Pontryagin space. If $(T_\omega)_{\omega\in\Omega} \subset L(\mathcal{G})$ is a weakly measurable group of unitary operators, then $(T_\omega)_{\omega\in\Omega}$ is strongly continuous.*

Proof. Let κ be the index of $\mathcal{G}$. Since $(T_\omega)_{\omega\in\Omega}$ is a unitary group, it is a κ-indefinite function.

By Theorem 4.1 we have that

$$T_\omega = T(\omega) = V(\omega) + R(\omega),$$

where $V(\omega)$ is a κ-indefinite weakly continuous function and $R(\omega)$ is positive definite and equal to zero locally almost everywhere.

For $x \in \Omega$ let

$$\Omega_x = \{\omega \in \Omega : R(\omega\, x) = 0\}.$$

The set $\Omega_1^{\mathbb{C}}$ is locally null and it is clear that $\Omega_x = \Omega_1 x^{-1}$, therefore $\Omega_x^{\mathbb{C}}$ is also locally null.

Let $\omega_0 \in \Omega$ be such that $R(\omega_0) = 0$. If $\omega \in \Omega_1 \cap \Omega_{\omega_0}$ then

$$R(\omega\omega_0) = R(\omega) = R(\omega_0) = 0$$

and

$$\begin{aligned} V(\omega\omega_0) &= V(\omega\omega_0) + R(\omega\omega_0) = T(\omega\omega_0) = T(\omega)\, T(\omega_0) \\ &= (V(\omega) + R(\omega))(V(\omega_0) + R(\omega_0)) = V(\omega)V(\omega_0). \end{aligned}$$

We have that V is weakly continuous, the set $(\Omega_1 \cap \Omega_{\omega_0})^{\mathbb{C}} = \Omega_1^{\mathbb{C}} \cup \Omega_{\omega_0}^{\mathbb{C}}$ is locally null and $R(\omega) = 0$ locally almost everywhere. Therefore

$$V(\omega\omega') = V(\omega)V(\omega')$$

for all $\omega, \omega' \in \Omega$. Since R is positive definite, we have that $R(x) = R^*(x^{-1})$ for all $x \in \Omega$.

Let $\omega \in \Omega$ be such that $R(\omega) = 0$, then

$$R(\omega^{-1}) = 0, \quad T(\omega) = V(\omega) \quad \text{and} \quad T(\omega^{-1}) = V(\omega^{-1}).$$

So

$$V(1) = V(\omega)V(\omega^{-1}) = T(\omega)T(\omega^{-1}) = T(1) = I.$$

Therefore $R(1) = 0$. Since R is positive definite it follows that $R \equiv 0$.

So we have that $(T_\omega)_{\omega\in\Omega}$ is weakly continuous. From Corollary 3.6 it follows that $(T_\omega)_{\omega\in\Omega}$ is strongly continuous. ∎

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RAMÓN BRUZUAL
Escuela de Matemática
Faculdad Ciencias
Universidad Central de Venezuela
Caracas
VENEZUELA

E-mail: rbruzual@euler.ciens.ucv.ve
rbruzual@reacciun.ve

Current address:

Apartado Postal 47686
Caracas 1041–A
VENEZUELA

MARISELA DOMÍNGUEZ
Escuela de Matemática
Faculdad Ciencias
Universidad Central de Venezuela
Caracas
VENEZUELA

E-mail: mdomin@euler.ciens.ucv.ve

Current address:

Apartado Postal 47159
Caracas 1041–A
VENEZUELA

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