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NORMS OF SOME SINGULAR INTEGRAL OPERATORS ON WEIGHTED L^2 SPACES

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Abstract. Let α and β be measurable functions on the unit circle T, and let W be a positive function on T such that the Riesz projection P_+ is bounded on the weighted space $L^2(W)$ on T. The singular integral operator $S_{\alpha,\beta}$ is defined by $S_{\alpha,\beta}f = \alpha P_+ f + \beta P_- f$, $f \in L^2(W)$, where $P_- = I - P_+$. Let h be an outer function such that $W = |h|^2$, and let φ be a unimodular function such that $\varphi = \overline{h}/h$. In this paper, the norm of $S_{\alpha,\beta}$ on $L^2(W)$ is calculated in general, using α, β and φ. Moreover, if α and β are constant functions, then we give another proof of the Feldman-Krupnik-Markus theorem. If $\alpha\overline{\beta}$ belongs to the Hardy space H^∞, we give the theorem which is similar to the Feldman-Krupnik-Markus theorem.

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1. INTRODUCTION

Let m denote the normalized Lebesgue measure on the unit circle $T = \{\zeta : |\zeta| = 1\}$. That is, $\mathrm{d}m(\zeta) = \mathrm{d}\theta/2\pi$ for $\zeta = \mathrm{e}^{\mathrm{i}\theta}$. For functions f and g satisfying $fg \in L^1$, we define the quantity (f, g) according to

$$(f,g) = \int_T f(\zeta)\overline{g(\zeta)}\,\mathrm{d}m(\zeta).$$

If $f, g \in L^2$, then this becomes the inner product. Let H^2 (respectively H^∞) be the Hardy space of functions $f \in L^2$ (respectively $f \in L^\infty$) whose negative Fourier coefficients are zero. Let S be the singular integral operator defined by

$$(Sf)(\zeta) = \frac{1}{\pi \mathrm{i}} \int_T \frac{f(\eta)}{\eta - \zeta}\,\mathrm{d}\eta \quad \text{a.e. } \zeta \in T,$$

where the integral is understood in the sense of Cauchy's principal value (cf. [6], p. 11]). If f is in L^1, then $Sf(\zeta)$ exists for almost everywhere ζ on T, and Sf becomes a measurable function on T. For a positive function $W \in L^1$, the norm in $L^2(W)$ is defined by the formula

$$\|f\|_{L^2(W)} = (Wf, f)^{1/2} = \left(\int_T |f(\zeta)|^2 W(\zeta) \, \mathrm{d}m(\zeta) \right)^{1/2}.$$

Let A (respectively $\overline{A}_0$) be the subspace of continuous functions f on T whose negative (respectively positive) Fourier coefficients are zero. Let $A + \overline{A}_0 = \{f_1 + f_2 : f_1 \in A, \ f_2 \in \overline{A}_0\}$. Then $A + \overline{A}_0$ is dense in $L^2(W)$ in norm. Two projections P_+ and P_- are defined by

$$P_+ = (I + S)/2 \quad \text{and} \quad P_- = (I - S)/2,$$

where I denotes the identity operator. P_+ is the Riesz projection. For $\alpha, \beta \in L^\infty$, let $S_{\alpha,\beta}$ be the singular integral operator on $L^2(W)$ defined by

$$S_{\alpha,\beta} f = \alpha P_+ f + \beta P_- f, \quad f \in L^2(W).$$

Then, $S_{1,1} = I$, $S_{1,-1} = S$, $S_{1,0} = P_+$ and $S_{0,1} = P_-$. Let $\|S_{\alpha,\beta}\|_{L^2(W)}$ denote the operator norm of $S_{\alpha,\beta}$ on $L^2(W)$. That is,

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \sup\{\|S_{\alpha,\beta} f\|_{L^2(W)} : f \in L^2(W), \|f\|_{L^2(W)} = 1\}.$$

Let W be a positive function in L^1 on T such that S becomes a bounded operator on $L^2(W)$. The relation between the norms of the operators S, P_+, P_- on the space $L^2(W)$,

$$\|P_+\|_{L^2(W)} = \|P_-\|_{L^2(W)} = \frac{\|S\|_{L^2(W)} + \|S\|^{-1}_{L^2(W)}}{2}$$

was remarked by Spitkovskii ([16]). Let h be an outer function such that $W = |h|^2$, and let φ be an unimodular function such that $\varphi = \overline{h}/h$. Let

$$c = \inf_{k \in H^\infty} \|\varphi - k\|_\infty.$$

Then

$$\|S\|_{L^2(W)} = \|S_{1,-1}\|_{L^2(W)} = \sqrt{\frac{1+c}{1-c}}, \quad \|P_+\|_{L^2(W)} = \|S_{1,0}\|_{L^2(W)} = \frac{1}{\sqrt{1-c^2}},$$

(cf. [4]). For $\zeta_0 \in T$, and $-1 < \delta < 1$, let $W(\zeta) = |\zeta - \zeta_0|^\delta$. Then the equality $\|S\|_{L^2(W)} = \cot \frac{\pi(1-|\delta|)}{4}$ was obtained by Krupnik and Verbitskii ([12]). Hence $\|P_+\|_{L^2(W)} = \frac{1}{\cos(\pi\delta/2)}$. For continuous functions α and β, the essential norm of $S_{\alpha,\beta}$ on $L^2(W)$ was calculated by Krupnik and Avendanio (cf. [10], p. 57, Corollary 6.1, and [1]). For constant functions α, β and a positive function $W \in L^1$ such that $\|P_+\|_{L^2(W)} < \infty$, the equality

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \sqrt{\gamma + \Big(\frac{|\alpha| + |\beta|}{2}\Big)^2} + \sqrt{\gamma + \Big(\frac{|\alpha| - |\beta|}{2}\Big)^2},$$

where

$$\gamma := \Big|\frac{\alpha-\beta}{2}\Big|^2(\|P_+\|^2_{L^2(W)}-1) = \Big|\frac{\alpha-\beta}{2}\Big|^2\Big(\frac{c^2}{1-c^2}\Big)$$

was obtained by Feldman, Krupnik and Markus (cf. [3], [11], [6], Section 13.5, [5], [18]). In this paper, for functions $\alpha, \beta \in L^\infty$, and a positive function $W \in L^1$ on the unit circle T, we will give three formulae of the norm $\|S_{\alpha,\beta}\|_{L^2(W)}$. It follows from the Koosis theorem ([9]) that there exist different functions α and β such that $\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$ if and only if $W^{-1} \in L^1$ (cf. [13]). If $\log W \notin L^1$, then $W^{-1} \notin L^1$. In this case $\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$ implies that $\alpha \equiv \beta$. Hence, $\|S_{\alpha,\beta}\|_{L^2(W)} = \|\alpha I\|_{L^2(W)} = \|\alpha\|_\infty$. Therefore we assume that $\log W \in L^1$. In Section 2, for functions $\alpha, \beta \in L^\infty$, and a positive function $W \in L^1$ on the unit circle T, we will give the first formula (Theorem 2.8) of the norm $\|S_{\alpha,\beta}\|_{L^2(W)}$ using α, β and φ. We will also give the another proof of the Feldman-Krupnik-Markus theorem in Corollary 2.12. In Section 3, we will give the second formula (Theorem 3.1) of the norm $\|S_{\alpha,\beta}\|_{L^2(W)}$. If $\alpha\overline{\beta}$ belongs to the Hardy space H^∞, we give the theorem which is similar to the Feldman-Krupnik-Markus theorem. In Section 4, we will give the third formula (Theorem 4.2) of the norm $\|S_{\alpha,\beta}\|_{L^2(W)}$.

2. THE FIRST FORMULA OF NORM OF $S_{\alpha,\beta}$ ON $L^2(W)$

The following Theorem 2.8 is the first formula of $\|S_{\alpha,\beta}\|_{L^2(W)}$. We will use Theorem 2.8 to prove Theorems 3.1 and 4.2 in the following sections. We give some lemmas to prove Theorem 2.8.

DEFINITION 2.1. Let $\alpha, \beta \in L^\infty$. For each $\gamma \in L^\infty$, we define the function $G(\gamma) \in L^\infty$ according to

$$(G(\gamma))(\zeta) = \frac{|\alpha(\zeta)|^2+|\beta(\zeta)|^2}{2} + \sqrt{|\gamma(\zeta)|^2 + \Big(\frac{|\alpha(\zeta)|^2-|\beta(\zeta)|^2}{2}\Big)^2}, \quad \zeta \in T.$$

DEFINITION 2.2. Let $\alpha, \beta \in L^\infty$, let h be an outer function in H^2, and let $\varphi = \overline{h}/h$. We define the function F according to

$$F(x) = \inf_{k\in H^\infty} \|G(x-\alpha\overline{\beta}-\overline{\varphi}k)\|_\infty, \quad x \geqslant 0.$$

That is,

$$F(x) = \inf_{k\in H^\infty} \Bigg\| \frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|x-\alpha\overline{\beta}-\overline{\varphi}k|^2 + \Big(\frac{|\alpha|^2-|\beta|^2}{2}\Big)^2} \Bigg\|_\infty.$$

LEMMA 2.3. *For each nonnegative number x, the infimum in the definition of $F(x)$ is attained.*

Proof. Let $\{k_n\}$ be a sequence in H^∞ such that

$$F(x) = \lim_{n\to\infty} \Big\|G(x-\alpha\overline{\beta}-\overline{\varphi}k_n)\Big\|_\infty.$$

For any $\varepsilon > 0$, take a positive integer n_0 with

$$\|G(x - \alpha\overline{\beta} - \overline{\varphi}k_n)\|_\infty \leqslant F(x) + \varepsilon, \quad n \geqslant n_0.$$

If $n \geqslant n_0$, then $\|k_n\|_\infty \leqslant F(x) + \varepsilon + \|x - \alpha\overline{\beta}\|_\infty < \infty$. Since the closed ball of H^∞ is weak-star compact over L^∞ (cf. [8], p. 197]), there exists a subsequence $\{k_{n_j}\}$ and a $k_0 \in H^\infty$ such that

$$\lim_{j\to\infty}(k_{n_j}, g) = (k_0, g), \quad g \in L^1.$$

Then there exists a sequence $\{h_n\}$ in H^∞ such that each h_n is a finite convex linear combination of the k_{n_j} and

$$\lim_{n\to\infty}\|h_n - k_0\|_{L^2} = 0$$

(cf. [17], p. 160, Problem 6). It follows that there exists a subsequence $\{h_{n_j}\}$ such that

$$\lim_{j\to\infty}|k_0(\zeta) - h_{n_j}(\zeta)| = 0, \quad \text{a.e. } \zeta \in T$$

(cf. [15], p. 68, Theorem 3.12). Hence there exist nonnegative numbers $\lambda_{j,1}, \dots$ $\dots, \lambda_{j,m_j}$ such that $\lambda_{j,1} + \cdots + \lambda_{j,m_j} = 1$ and

$$h_{n_j} = \lambda_{j,1}k_{n_1} + \cdots + \lambda_{j,m_j}k_{n_{m_j}}.$$

Since $y = a^2 + \sqrt{t^2 + b^2}$ is a convex function of t, it follows that

$$\begin{aligned} G(x - \alpha\overline{\beta} - \overline{\varphi}h_{n_j}) &= G\Big(x - \alpha\overline{\beta} - \overline{\varphi}\sum_{i=1}^{m_j}\lambda_{j,i}k_{n_i}\Big) = G\Big(\sum_{i=1}^{m_j}\lambda_{j,i}(x - \alpha\overline{\beta} - \overline{\varphi}k_{n_i})\Big) \\ &\leqslant G\Big(\sum_{i=1}^{m_j}\lambda_{j,i}|x - \alpha\overline{\beta} - \overline{\varphi}k_{n_i}|\Big) \\ &\leqslant \sum_{i=1}^{m_j}\lambda_{j,i}G(x - \alpha\overline{\beta} - \overline{\varphi}k_{n_i}) \leqslant \sum_{i=1}^{m_j}\lambda_{j,i}(F(x) + \varepsilon) = F(x) + \varepsilon. \end{aligned}$$

There exists a $g \in L^1$ such that $\|g\|_{L^1} = 1$ and

$$\|G(x - \alpha\overline{\beta} - \overline{\varphi}k_0)\|_\infty \leqslant |(G(x - \alpha\overline{\beta} - \overline{\varphi}k_0), g)| + \varepsilon.$$

By the Lebesgue theorem,

$$\begin{aligned} |(G(x - \alpha\overline{\beta} - \overline{\varphi}k_0), g)| &= \lim_{j\to\infty}|(G(x - \alpha\overline{\beta} - \overline{\varphi}h_{n_j}), g)| \\ &\leqslant \liminf_{j\to\infty}\|G(x - \alpha\overline{\beta} - \overline{\varphi}h_{n_j})\|_\infty \leqslant F(x) + \varepsilon. \end{aligned}$$

Hence,

$$\|G(x - \alpha\overline{\beta} - \overline{\varphi}k_0)\|_\infty \leqslant F(x) + 2\varepsilon.$$

Let $\varepsilon \to 0$. Then the equality holds, and hence the infimum in the definition of $F(x)$ is attained by $k = k_0$. This completes the proof. ∎

LEMMA 2.4. *$F(x)$ is a convex function of x. (Hence it is continuous.)*

Proof. Let λ and μ be nonnegative numbers such that $\lambda+\mu=1$. Since $y=a^2+\sqrt{t^2+b^2}$ is a nonnegative, convex, increasing function of $t, t\geqslant 0$, it follows that

$$\begin{aligned}\lambda F(x)+\mu F(y)&=\lambda\inf_{k_1\in H^\infty}\|G(x-\alpha\overline{\beta}-\overline{\varphi}k_1)\|_\infty+\mu\inf_{k_2\in H^\infty}\|G(y-\alpha\overline{\beta}-\overline{\varphi}k_2)\|_\infty\\&\geqslant\inf_{k_1\in H^\infty}\inf_{k_2\in H^\infty}\|\lambda G(x-\alpha\overline{\beta}-\overline{\varphi}k_1)+\mu G(y-\alpha\overline{\beta}-\overline{\varphi}k_2)\|_\infty\\&\geqslant\inf_{k_1\in H^\infty}\inf_{k_2\in H^\infty}\|G(\lambda|x-\alpha\overline{\beta}-\overline{\varphi}k_1|+\mu|y-\alpha\overline{\beta}-\overline{\varphi}k_2|)\|_\infty\\&\geqslant\inf_{k_1\in H^\infty}\inf_{k_2\in H^\infty}\|G(\lambda x+\mu y-\alpha\overline{\beta}-\overline{\varphi}(\lambda k_1+\mu k_2))\|_\infty\\&\geqslant\inf_{k\in H^\infty}\|G(\lambda x+\mu y-\alpha\overline{\beta}-\overline{\varphi}k)\|_\infty\\&=F(\lambda x+\mu y).\end{aligned}$$

This completes the proof. ∎

LEMMA 2.5. *If $x\geqslant\max\{|\alpha|^2,|\beta|^2\}$, then*

$$G\Big(\sqrt{x-|\alpha|^2}\sqrt{x-|\beta|^2}\Big)=x.$$

Proof. Since $x\geqslant\max\{|\alpha|^2,|\beta|^2\}\geqslant(|\alpha|^2+|\beta|^2)/2$ and

$$\Big(x-\frac{|\alpha|^2+|\beta|^2}{2}\Big)^2-\Big(\frac{|\alpha|^2-|\beta|^2}{2}\Big)^2=(x-|\alpha|^2)(x-|\beta|^2),$$

it follows that

$$\begin{aligned}x&=\frac{|\alpha|^2+|\beta|^2}{2}+\sqrt{(x-|\alpha|^2)(x-|\beta|^2)+\Big(\frac{|\alpha|^2-|\beta|^2}{2}\Big)^2}\\&=G\Big(\sqrt{x-|\alpha|^2}\sqrt{x-|\beta|^2}\Big).\end{aligned}$$

This completes the proof. ∎

LEMMA 2.6. *$F(x)\leqslant x$ if and only if $x\geqslant\|S_{\alpha,\beta}\|^2_{L^2(W)}$.*

Proof. We prove the "if" part. Suppose $x\geqslant\|S_{\alpha,\beta}\|^2_{L^2(W)}$. Then,

$$\|S_{\alpha,\beta}f\|^2_{L^2(W)}\leqslant x\|f\|^2_{L^2(W)},\quad f\in A+\overline{A}_0.$$

Hence,

$$\|\alpha f_1+\beta f_2\|^2_{L^2(W)}\leqslant x\|f_1+f_2\|^2_{L^2(W)},\quad f_1\in A,\ f_2\in\overline{A}_0.$$

Let $W_1=(x-|\alpha|^2)W$, $W_2=(x-|\beta|^2)W$, $W_3=(x-\alpha\overline{\beta})W$, then for any $f_1\in A$ and $f_2\in\overline{A}_0$,

$$(W_1f_1,f_1)+(W_2f_2,f_2)+2\mathrm{Re}(W_3f_1,f_2)\geqslant 0.$$

By the Cotlar-Sadosky lifting theorem ([2]), $W_1\geqslant 0, W_2\geqslant 0$ and there exists a $g\in H^1$ such that $|W_3-g|^2\leqslant W_1W_2$. This implies that $x\geqslant\max\{|\alpha|^2,|\beta|^2\}$, and $|(x-\alpha\overline{\beta})W-g|^2\leqslant(x-|\alpha|^2)(x-|\beta|^2)W^2$. Hence,

$$\Big|x-\alpha\overline{\beta}-\frac{g}{W}\Big|^2\leqslant(x-|\alpha|^2)(x-|\beta|^2).$$

Then,

$$\frac{g}{W} = \frac{g}{|h|^2} = \frac{h}{\overline{h}}\frac{g}{h^2} = \overline{\varphi}\frac{g}{h^2}.$$

Let $k = g/h^2$. Then $k \in H^\infty$, and $|x - \alpha\overline{\beta} - \overline{\varphi}k|^2 \leqslant (x - |\alpha|^2)(x - |\beta|^2)$. It follows from Lemma 2.5 that

$$G(x - \alpha\overline{\beta} - \overline{\varphi}k) \leqslant G\Big(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}\Big) = x.$$

This implies that $F(x) \leqslant x$. We prove the "only if" part. Suppose $F(x) \leqslant x$. By Lemma 2.3, the infimum in the definition of $F(x)$ is attained. Hence, there exists a $k \in H^\infty$ such that $G(x - \alpha\overline{\beta} - \overline{\varphi}k) \leqslant x$. It follows from Lemma 2.5 that

$$G(x - \alpha\overline{\beta} - \overline{\varphi}k) \leqslant G\Big(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}\Big).$$

Hence, $|x - \alpha\overline{\beta} - \overline{\varphi}k| \leqslant \sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}$. Since $\varphi = \overline{h}/h$ and $W = |h|^2$, it follows that $|(x - \alpha\overline{\beta})W - h^2k| \leqslant \sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W$. Since $h^2k \in H^1$, it follows that

$$\begin{aligned}
x\|f_1 + f_2\|^2_{L^2(W)} &- \|\alpha f_1 + \beta f_2\|^2_{L^2(W)} \\
&= x(W(f_1 + f_2), f_1 + f_2) - (W(\alpha f_1 + \beta f_2), \alpha f_1 + \beta f_2) \\
&= ((x - |\alpha|^2)Wf_1, f_1) + ((x - |\beta|^2)Wf_2, f_2) + 2\mathrm{Re}((x - \alpha\overline{\beta})Wf_1, f_2) \\
&\geqslant 2\Big(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W|f_1|, |f_2|\Big) - 2|((x - \alpha\overline{\beta})Wf_1, f_2)| \\
&= 2\Big(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W|f_1|, |f_2|\Big) - 2|(((x - \alpha\overline{\beta})W - h^2k)f_1, f_2)| \\
&\geqslant 2\Big(\sqrt{x - |\alpha|^2}\sqrt{x - |\beta|^2}W|f_1|, |f_2|\Big) - 2(|(x - \alpha\overline{\beta})W - h^2k||f_1|, |f_2|) \geqslant 0,
\end{aligned}$$

where $f_1 \in A$, $f_2 \in \overline{A}_0$. Hence, $x \geqslant \|S_{\alpha,\beta}\|^2_{L^2(W)}$. This completes the proof. ∎

Lemma 2.7. *If $x \geqslant 0$, then*

$$F(x) \leqslant x \inf_{k \in H^\infty} \|\varphi - k\|_\infty + 2\max\{\|\alpha\|^2_\infty, \|\beta\|^2_\infty\}.$$

Proof. Since $\sqrt{|a|^2 + |b|^2} \leqslant |a| + |b|$, it follows that

$$\begin{aligned}
F(x) &= \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|x - \alpha\overline{\beta} - \overline{\varphi}k|^2 + \Big(\frac{|\alpha|^2 - |\beta|^2}{2}\Big)^2} \right\|_\infty \\
&\leqslant \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + |x - \alpha\overline{\beta} - \overline{\varphi}k| + \left|\frac{|\alpha|^2 - |\beta|^2}{2}\right| \right\|_\infty \\
&\leqslant \inf_{k \in H^\infty} \|x - \overline{\varphi}k\|_\infty + \|\max\{|\alpha|^2, |\beta|^2\} + |\alpha\beta|\,\|_\infty \\
&\leqslant x \inf_{k \in H^\infty} \|\varphi - k\|_\infty + 2\|\max\{|\alpha|, |\beta|\}\|^2_\infty.
\end{aligned}$$

This completes the proof. ∎

THEOREM 2.8. *Let $\alpha, \beta \in L^\infty$. Let φ and W be functions such that there exists an outer function $h \in H^2$ satisfying $\varphi = \overline{h}/h$ and $W = |h|^2$. Then*

(i) $F(\|S_{\alpha,\beta}\|^2_{L^2(W)}) = \|S_{\alpha,\beta}\|^2_{L^2(W)}$;

(ii) *if* $\inf_{k \in H^\infty} \|\varphi - k\|_\infty < 1$ *then* $x = \|S_{\alpha,\beta}\|^2_{L^2(W)}$ *is the unique solution of the equation* $F(x) = x$.

Proof. We prove (i). Let $s = \|S_{\alpha,\beta}\|^2_{L^2(W)}$. We prove that $x = s$ is the solution of the equation $F(x) = x$. It follows from Lemma 2.6 that $s \geqslant F(s)$. It follows from Lemma 2.4 that $F(x)$ is a continuous function of x. It follows from Lemma 2.6 that $F(x) > x$, $x < s$. Hence,

$$s \geqslant F(s) = \lim_{x \to s} F(x) = \lim_{x \to s-0} F(x) \geqslant \lim_{x \to s-0} x = s.$$

Therefore, $F(s) = s$. We prove (ii). Suppose there exists a t such that $t \neq s$ and $F(t) = t$. It follows from Lemma 2.6 that $t > s$. Let x be any number satisfying $x > t$. Since $s < t < x$, it follows that there exist positive numbers λ, μ such that $\lambda + \mu = 1$ and $t = \lambda s + \mu x$. It follows from Lemma 2.4 that

$$F(t) = F(\lambda s + \mu x) \leqslant \lambda F(s) + \mu F(x).$$

Since $F(s) = s$ and $F(t) = t$, it follows that $t \leqslant \lambda s + \mu F(x)$. Hence, $\mu x = t - \lambda s \leqslant \mu F(x)$. Since $\mu > 0$, this implies that $x \leqslant F(x)$. Since $x > s$, it follows from Lemma 2.6 that $F(x) \leqslant x$. Therefore, $F(x) = x$, $x > t$. By Lemma 2.7, this implies that

$$x \leqslant x \inf_{k \in H^\infty} \|\varphi - k\|_\infty + 2 \max\{\|\alpha\|^2_\infty, \|\beta\|^2_\infty\}, \quad x > t.$$

Hence,

$$\inf_{k \in H^\infty} \|\varphi - k\|_\infty \geqslant 1.$$

This is a contradiction. Therefore, there does not exist a t such that $t \neq s$ and $F(t) = t$. This completes the proof. ∎

If W is a constant function, then $F(x)$ becomes a constant function. In this case, the formula of $\|S_{\alpha,\beta}\|^2_{L^2}$ follows easily from Theorem 2.8 (i) as Corollary 2.9. In the preceding paper ([14]), we gave Corollary 2.9.

COROLLARY 2.9. *Let $\alpha, \beta \in L^\infty$. Then*

$$\|S_{\alpha,\beta}\|^2_{L^2} = \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\overline{\beta} - k|^2 + \Big(\frac{|\alpha|^2 - |\beta|^2}{2}\Big)^2} \right\|_\infty.$$

The infimum is attained.

Proof. In the statement of Theorem 2.8, if $W = 1$, then h and φ are constants. If $k \in H^\infty$, then $k - \varphi x \in H^\infty$. Hence,

$$\begin{aligned} F(x) &= \inf_{k \in H^\infty} \|G(x - \alpha\overline{\beta} - \overline{\varphi}k)\|_\infty = \inf_{k \in H^\infty} \|G(0 - \alpha\overline{\beta} - \overline{\varphi}(k - \varphi x))\|_\infty \\ &= \inf_{k \in H^\infty} \|G(0 - \alpha\overline{\beta} - \overline{\varphi}k)\|_\infty = F(0). \end{aligned}$$

Hence $F(x)$ is a constant function of x. It follows from Theorem 2.8 that

$$\|S_{\alpha,\beta}\|^2_{L^2} = F(\|S_{\alpha,\beta}\|^2_{L^2}) = F(0) = \inf_{k \in H^\infty} \Big\|G(\alpha\overline{\beta} - k)\Big\|_\infty.$$

By Lemma 2.3, the infimum is attained. This completes the proof. ∎

COROLLARY 2.10. *Let $\alpha, \beta \in L^\infty$. Then*
$$\max\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\} \leqslant \|S_{\alpha,\beta}\|_{L^2}^2 \leqslant \max\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\} + \inf_{k\in H^\infty} \|\alpha\overline{\beta} - k\|_\infty.$$
If $\alpha\overline{\beta} \in H^\infty$, then the equality holds.

Proof. It follows from Corollary 2.9 that
$$\begin{aligned}\|\max\{|\alpha|^2, |\beta|^2\}\|_\infty \leqslant \|S_{\alpha,\beta}\|_{L^2}^2 &= \inf_{k\in H^\infty} \|G(\alpha\overline{\beta} - k)\|_\infty \\ &\leqslant \inf_{k\in H^\infty} \|\max\{|\alpha|^2, |\beta|^2\} + |\alpha\overline{\beta} - k|\|_\infty \\ &\leqslant \|\max\{|\alpha|^2, |\beta|^2\}\|_\infty + \inf_{k\in H^\infty} \|\alpha\overline{\beta} - k\|_\infty.\end{aligned}$$
This completes the proof. ∎

COROLLARY 2.11. *Let $\alpha, \beta \in L^\infty$. Let φ and W be functions such that there exists an outer function $h \in H^2$ satisfying $\varphi = \overline{h}/h$ and $W = |h|^2$.*

(i) *If $|\alpha|, |\beta|$ are constants, then*
$$\inf_{k\in H^\infty} \big\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\overline{\beta} - \overline{\varphi}k\big\|_\infty = \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2}\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2}.$$
(ii) *If $|\alpha|, |\beta|$ are constants and $\alpha\overline{\beta} \in H^\infty$, then*
$$\inf_{k\in H^\infty} \big\|(\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\overline{\beta})(\varphi - k)\big\|_\infty = \sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2}\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2}.$$

Proof. We prove (i). Since $|\alpha|, |\beta|$ are constants, it follows from Theorem 2.8 that
$$\begin{aligned}&\|S_{\alpha,\beta}\|_{L^2(W)}^2 \\ &\quad= \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\inf_{k\in H^\infty} \big\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\overline{\beta} - \overline{\varphi}k\big\|_\infty^2 + \Big(\frac{|\alpha|^2 - |\beta|^2}{2}\Big)^2} \\ &\quad= G\big(\inf_{k\in H^\infty} \big\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\overline{\beta} - \overline{\varphi}k\big\|_\infty\big).\end{aligned}$$
Since $\|S_{\alpha,\beta}\|_{L^2(W)}^2 \geqslant \max\{|\alpha|^2, |\beta|^2\}$, it follows from Lemma 2.5 that
$$\|S_{\alpha,\beta}\|_{L^2(W)}^2 = G\Big(\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2}\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2}\Big).$$
Therefore,
$$\begin{aligned}&G\big(\inf_{k\in H^\infty} \big\| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\overline{\beta} - \overline{\varphi}k\big\|_\infty\big) \\ &\qquad= G\Big(\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\alpha|^2}\sqrt{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - |\beta|^2}\Big).\end{aligned}$$
This proves (i).

We prove (ii). Since $\alpha\overline{\beta} \in H^\infty$ and $\mathrm{Re}(\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\overline{\beta}) \geqslant 0$, it follows that $\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\overline{\beta}$ is an outer function or a zero function. Hence,
$$\begin{aligned}&\inf_{k\in H^\infty} \| \|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\overline{\beta} - \overline{\varphi}k\|_\infty \\ &\quad= \inf_{k\in H^\infty} \Big\|(\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\overline{\beta})\Big(\varphi - \frac{k}{\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\overline{\beta}}\Big)\Big\|_\infty \\ &\quad= \inf_{k\in H^\infty} \big\|(\|S_{\alpha,\beta}\|_{L^2(W)}^2 - \alpha\overline{\beta})(\varphi - k)\big\|_\infty.\end{aligned}$$

By (i), this implies (ii). This completes the proof. ∎

The statement (iii) in the following corollary was given by Feldman, Krupnik and Markus (cf. [3]). By the Helson-Szegö theorem ([7]), $\|P_+\|_{L^2(W)} < \infty$ if and only if $\inf_{k\in H^\infty} \|\varphi - k\|_\infty < 1$.

COROLLARY 2.12. *Let α and β be complex numbers satisfying $\alpha \neq \beta$. Let φ and W be functions such that there exists an outer function $h \in H^2$ satisfying $\varphi = \overline{h}/h$ and $W = |h|^2$. Let $c = \inf_{k\in H^\infty} \|\varphi - k\|_\infty$. If $c < 1$, then the following equalities hold:*

(i) $c\big|\, \|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\overline{\beta}\big| = \sqrt{\|S_{\alpha,\beta}\|^2_{L^2(W)} - |\alpha|^2}\sqrt{\|S_{\alpha,\beta}\|^2_{L^2(W)} - |\beta|^2}$;

(ii) $\big|\, \|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\overline{\beta}\big| = |\alpha-\beta|\, \|S_{\alpha,\beta}\|_{L^2(W)}\|P_+\|_{L^2(W)} = \frac{|\alpha-\beta|\, \|S_{\alpha,\beta}\|_{L^2(W)}}{\sqrt{1-c^2}}$;

(iii) $\|S_{\alpha,\beta}\|_{L^2(W)} = \sqrt{\gamma + \Big(\frac{|\alpha|+|\beta|}{2}\Big)^2} + \sqrt{\gamma + \Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}$, *where*

$$\gamma := \Big|\frac{\alpha-\beta}{2}\Big|^2\Big(\frac{c^2}{1-c^2}\Big) = \Big|\frac{\alpha-\beta}{2}\Big|^2(\|P_+\|^2_{L^2(W)} - 1).$$

Proof. Since $\|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\overline{\beta}$ is a constant, it follows from Corollary 2.11 (ii) that (i) holds.

We prove (ii). Since $P_+ = S_{1,0}$, it follows from (i) that

$$c\|P_+\|^2_{L^2(W)} = \|P_+\|_{L^2(W)}\sqrt{\|P_+\|^2_{L^2(W)} - 1}.$$

Hence, $(1 - c^2)\|P_+\|^2_{L^2(W)} = 1$. It follows from (i) that

$$\begin{aligned} c^2\big|\, \|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\overline{\beta}\big|^2 &= (\|S_{\alpha,\beta}\|^2_{L^2(W)} - |\alpha|^2)(\|S_{\alpha,\beta}\|^2_{L^2(W)} - |\beta|^2) \\ &= \big|\, \|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\overline{\beta}\big|^2 - |\alpha-\beta|^2\|S_{\alpha,\beta}\|^2_{L^2(W)}. \end{aligned}$$

Hence, $(1-c^2)\big|\, \|S_{\alpha,\beta}\|^2_{L^2(W)} - \alpha\overline{\beta}\big|^2 = |\alpha-\beta|^2\|S_{\alpha,\beta}\|^2_{L^2(W)}$. Since $(1-c^2)\|P_+\|^2_{L^2(W)} = 1$, this implies (ii).

We prove (iii). Let $s = \|S_{\alpha,\beta}\|_{L^2(W)}$. It follows from (i) that $c^2|s - \alpha\overline{\beta}|^2 = (s - |\alpha|^2)(s - |\beta|^2)$. Then

$$(1 - c^2)s^2 - \{|\alpha - \beta|^2 + 2(1 - c^2)\mathrm{Re}(\alpha\overline{\beta})\}s + (1 - c^2)|\alpha\beta|^2 = 0.$$

Since $c < 1$, it follows that $s^2 - \big\{\frac{|\alpha-\beta|^2}{1-c^2} + 2\mathrm{Re}(\alpha\overline{\beta})\big\}s + |\alpha\beta|^2 = 0$. Since $s \geqslant \max\{|\alpha|^2, |\beta|^2\} \geqslant |\alpha\beta|$, it follows that $s = t + \sqrt{t^2 - |\alpha\beta|^2}$, where $t := \frac{|\alpha-\beta|^2}{2(1-c^2)} + \mathrm{Re}(\alpha\overline{\beta})$. Hence,

$$\begin{aligned} \|S_{\alpha,\beta}\|_{L^2(W)} = \sqrt{s} = \sqrt{t + \sqrt{t^2 - |\alpha\beta|^2}} &= \sqrt{\frac{t + |\alpha\beta|}{2}} + \sqrt{\frac{t - |\alpha\beta|}{2}} \\ &= \sqrt{\frac{|\alpha-\beta|^2}{4(1-c^2)} + \frac{\mathrm{Re}(\alpha\overline{\beta}) + |\alpha\beta|}{2}} + \sqrt{\frac{|\alpha-\beta|^2}{4(1-c^2)} + \frac{\mathrm{Re}(\alpha\overline{\beta}) - |\alpha\beta|}{2}} \\ &= \sqrt{\gamma + \Big(\frac{|\alpha| + |\beta|}{2}\Big)^2} + \sqrt{\gamma + \Big(\frac{|\alpha| - |\beta|}{2}\Big)^2}, \end{aligned}$$

where $\gamma := \left|\frac{\alpha-\beta}{2}\right|^2\left(\frac{c^2}{1-c^2}\right)$. Then,

$$\|P_+\|_{L^2(W)} = \|S_{1,0}\|_{L^2(W)} = 2\sqrt{\frac{c^2}{4(1-c^2)}+\frac{1}{4}} = \frac{1}{\sqrt{1-c^2}}.$$

Hence,

$$\gamma = \left|\frac{\alpha-\beta}{2}\right|^2(\|P_+\|^2_{L^2(W)}-1).$$

This completes the proof. ∎

3. THE SECOND FORMULA OF $S_{\alpha,\beta}$ ON $L^2(W)$

The following Theorem 3.1 is the second formula of $\|S_{\alpha,\beta}\|_{L^2(W)}$. It is the generalization of the Feldman-Krupnik-Markus theorem. Since the formula is symmetric with respect to α and β, it follows that if $\alpha\overline{\beta}$ is a constant then

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \|S_{\beta,\alpha}\|_{L^2(W)}.$$

By Theorem 3.1, if $W(\zeta)=1$ and $\alpha\overline{\beta}\in H^\infty$, then $\|S_{\alpha,\beta}\|_{L^2} = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$. If $\alpha(\zeta)=\zeta, \beta(\zeta)=1, W(\zeta)=1$, then $\|S_{\zeta,1}\|_{L^2}=1$. It follows from Corollary 2.9 that

$$\|S_{1,\zeta}\|_{L^2} = \inf_{k\in H^\infty}\left\|1+|\overline{\zeta}-k|\right\|_\infty^{1/2} = \sqrt{1+\inf_{k\in H^\infty}\|\overline{\zeta}-k\|_\infty} = \sqrt{2} \neq \|S_{\zeta,1}\|_{L^2}.$$

THEOREM 3.1. *Let $\alpha,\beta\in L^\infty$. Let φ and W be functions such that there exists an outer function $h\in H^2$ satisfying $\varphi=\overline{h}/h$ and $W=|h|^2$. If $\alpha\overline{\beta}$ belongs to H^∞ and $|\alpha-\beta|>0$, then*

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \inf_{\substack{k\in H^\infty\\|\varphi-k|<1}}\left\|\sqrt{\gamma_k+\Big(\frac{|\alpha|+|\beta|}{2}\Big)^2}+\sqrt{\gamma_k+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}\right\|_\infty,$$

where $\gamma_k := \left|\frac{\alpha-\beta}{2}\right|^2\left(\frac{|\varphi-k|^2}{1-|\varphi-k|^2}\right)$.

Proof. Let $s=\|S_{\alpha,\beta}\|^2_{L^2(W)}$. It is well known that $\max\{|\alpha|^2,|\beta|^2\}\leqslant s$. For any $k\in H^\infty$ satisfying $|\varphi-k|<1$, we define the quantity N_k according to

$$N_k = \left\|\sqrt{\gamma_k+\Big(\frac{|\alpha|+|\beta|}{2}\Big)^2}+\sqrt{\gamma_k+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}\right\|_\infty.$$

We prove that $\|S_{\alpha,\beta}\|_{L^2(W)}\geqslant \inf\{N_k : k\in H^\infty, |\varphi-k|<1\}$. It follows from Theorem 2.8 that $F(s)=s$. Hence there exists a function $g\in H^\infty$ such that $G(s-\alpha\overline{\beta}-\overline{\varphi}g)\leqslant s$. It follows from Lemma 2.5 that $G\big(\sqrt{s-|\alpha|^2}\sqrt{s-|\beta|^2}\big)=s$. Hence, $G(s-\alpha\overline{\beta}-\overline{\varphi}g)\leqslant G\big(\sqrt{s-|\alpha|^2}\sqrt{s-|\beta|^2}\big)$. Hence,

$$|s-\alpha\overline{\beta}-\overline{\varphi}g|^2\leqslant (s-|\alpha|^2)(s-|\beta|^2).$$

Suppose $g=0$. Then, $|s-\alpha\overline{\beta}|^2 \leqslant (s-|\alpha|^2)(s-|\beta|^2)$. Hence $|\alpha-\beta|^2 s \leqslant 0$. Since $\max\{|\alpha|^2,|\beta|^2\} \leqslant s$ and $|\alpha-\beta|>0$, it follows that $s>0$. Hence $\alpha\equiv\beta$. This contradiction implies $g\neq 0$. Since

$$|g|-|s-\alpha\overline{\beta}| \leqslant |s-\alpha\overline{\beta}-\overline{\varphi}g| \leqslant \sqrt{s-|\alpha|^2}\sqrt{s-|\beta|^2} \leqslant |s-\alpha\overline{\beta}|,$$

it follows that

$$0<|g|<2|s-\alpha\overline{\beta}|.$$

Since $\mathrm{Re}(s-\alpha\overline{\beta}) \geqslant s-\max\{|\alpha|^2,|\beta|^2\} \geqslant 0$, and $\alpha\overline{\beta}\in H^\infty$, it follows that $s-\alpha\overline{\beta}$ is an outer function. We define the function k according to $k=\frac{g}{s-\alpha\overline{\beta}}$. Since $|\alpha-\beta|>0$, it follows that

$$|\varphi-k|^2=\Big|\varphi-\frac{g}{s-\alpha\overline{\beta}}\Big|^2 \leqslant \frac{(s-|\alpha|^2)(s-|\beta|^2)}{|s-\alpha\overline{\beta}|^2}=1-\frac{|\alpha-\beta|^2 s}{|s-\alpha\overline{\beta}|^2}<1.$$

Hence, $k\in H^\infty$, $|\varphi-k|<1$, and $|s-\alpha\overline{\beta}|^2-\frac{|\alpha-\beta|^2}{1-|\varphi-k|^2}s \geqslant 0$. Therefore

$$s^2-\Big(\frac{|\alpha-\beta|^2}{1-|\varphi-k|^2}+2\mathrm{Re}(\alpha\overline{\beta})\Big)s+|\alpha\beta|^2 \geqslant 0.$$

Hence,

$$0\leqslant s\leqslant t-\sqrt{t^2-|\alpha\beta|^2}, \quad \text{or} \quad s\geqslant t+\sqrt{t^2-|\alpha\beta|^2},$$

where $t:=\Big(\frac{|\alpha-\beta|^2}{2}\frac{1}{1-|\varphi-k|^2}+\mathrm{Re}(\alpha\overline{\beta})\Big)$. Hence,

$$\sqrt{s}\leqslant\sqrt{t-\sqrt{t^2-|\alpha\beta|^2}}=\sqrt{\frac{t+|\alpha\beta|}{2}}-\sqrt{\frac{t-|\alpha\beta|}{2}},$$

or

$$\sqrt{s}\geqslant\sqrt{t+\sqrt{t^2-|\alpha\beta|^2}}=\sqrt{\frac{t+|\alpha\beta|}{2}}+\sqrt{\frac{t-|\alpha\beta|}{2}}.$$

Suppose $\sqrt{s}\leqslant\sqrt{\frac{t+|\alpha\beta|}{2}}-\sqrt{\frac{t-|\alpha\beta|}{2}}$ on some measurable subset E of T. Then

$$\sqrt{s}\leqslant\sqrt{\gamma_k+\Big(\frac{|\alpha|+|\beta|}{2}\Big)^2}-\sqrt{\gamma_k+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}$$

on E. Since $\max\{|\alpha|,|\beta|\}\leqslant\sqrt{s}$, it follows that

$$\begin{aligned}\frac{|\alpha|+|\beta|}{2}+\Big|\frac{|\alpha|-|\beta|}{2}\Big|+\sqrt{\gamma_k+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2} &\leqslant \sqrt{s}+\sqrt{\gamma_k+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}\\ &\leqslant\sqrt{\gamma_k+\Big(\frac{|\alpha|+|\beta|}{2}\Big)^2}\leqslant\sqrt{\gamma_k+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}+\frac{|\alpha|+|\beta|}{2}\end{aligned}$$

on E. Hence $|\alpha|=|\beta|$ on E. Hence $|\alpha|+\sqrt{\gamma_k}=\sqrt{\gamma_k+|\alpha|^2}$ on E. Hence $\alpha\gamma_k=0$ on E. Hence $\gamma_k=0$ on E. Therefore,

$$\sqrt{s}\geqslant\sqrt{\gamma_k+\Big(\frac{|\alpha|+|\beta|}{2}\Big)^2}+\sqrt{\gamma_k+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}$$

on T. Hence, $\|S_{\alpha,\beta}\|_{L^2(W)}\geqslant\inf\limits_{\substack{k\in H^\infty\\|\varphi-k|<1}}N_k$.

Next we prove the reverse inequality. This is the easy direction of the theorem. Since

$$N_k \geqslant \sqrt{\gamma_k + \Big(\frac{|\alpha| + |\beta|}{2}\Big)^2} + \sqrt{\gamma_k + \Big(\frac{|\alpha| - |\beta|}{2}\Big)^2},$$

squaring both sides, it follows that

$$N_k^2 \geqslant 2\gamma_k + \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\Big(2\gamma_k + \frac{|\alpha|^2 + |\beta|^2}{2}\Big)^2 - |\alpha\beta|^2}.$$

Hence, $N_k^4 - (4\gamma_k + |\alpha|^2 + |\beta|^2)N_k^2 + |\alpha\beta|^2 \geqslant 0$. Hence, $4\gamma_k N_k^2 \leqslant (N_k^2 - |\alpha|^2) \cdot (N_k^2 - |\beta|^2)$. Hence,

$$N_k^2 |\alpha - \beta|^2 \Big(\frac{|\varphi - k|^2}{1 - |\varphi - k|^2}\Big) \leqslant (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2).$$

Since $N_k^2|\alpha - \beta|^2 = |N_k^2 - \alpha\overline{\beta}|^2 - (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2)$, it follows that

$$(|N_k^2 - \alpha\overline{\beta}|^2 - (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2))\Big(\frac{|\varphi - k|^2}{1 - |\varphi - k|^2}\Big) \leqslant (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2).$$

Hence, $|N_k^2 - \alpha\overline{\beta}|^2|\varphi - k|^2 \leqslant (N_k^2 - |\alpha|^2)(N_k^2 - |\beta|^2)$. It follows from Lemma 2.5 that

$$G(|N_k^2 - \alpha\overline{\beta}|\,|\varphi - k|) \leqslant G\Big(\sqrt{N_k^2 - |\alpha|^2}\sqrt{N_k^2 - |\beta|^2}\Big) = N_k^2.$$

Since

$$|N_k^2 - \alpha\overline{\beta}|\,|\varphi - k| = |(N_k^2 - \alpha\overline{\beta})(1 - \overline{\varphi}k)| = |N_k^2 - \alpha\overline{\beta} - \overline{\varphi}(N_k^2 - \alpha\overline{\beta})k|$$

and $(N_k^2 - \alpha\overline{\beta})k \in H^\infty$, it follows that $F(N_k^2) \leqslant N_k^2$. It follows from Lemma 2.6 that $N_k^2 \geqslant \|S_{\alpha,\beta}\|^2_{L^2(W)}$. This completes the proof. ∎

COROLLARY 3.2. *If* $\inf_{k\in H^\infty} \|\varphi - k\|_\infty = c < 1$, $\alpha\overline{\beta} \in H^\infty$ *and* $|\alpha - \beta| > 0$, *then*

(i) $\|S_{\alpha,\beta}\|_{L^2(W)} \geqslant \inf_{k\in H^\infty} \Big\|\sqrt{\max\{|\alpha|^2, |\beta|^2\} + \frac{|\alpha-\beta|^2|\varphi-k|^2}{1-|\varphi-k|^2}}\Big\|_\infty$,

(ii) $\|S_{\alpha,\beta}\|_{L^2(W)} \leqslant \Bigg\|\sqrt{\gamma + \Big(\frac{|\alpha|+|\beta|}{2}\Big)^2} + \sqrt{\gamma + \Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}\Bigg\|_\infty$,

(iii) $\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \leqslant \|S_{\alpha,\beta}\|_{L^2(W)} \leqslant \|\max\{|\alpha|, |\beta|\} + 2\sqrt{\gamma}\|_\infty$,

where $\gamma := \Big|\frac{\alpha-\beta}{2}\Big|^2\Big(\frac{c^2}{1-c^2}\Big) = \Big|\frac{\alpha-\beta}{2}\Big|^2\Big(\|P_+\|^2_{L^2(W)} - 1\Big)$.

Proof. We prove (1). Let $\gamma_k = \Big|\frac{\alpha-\beta}{2}\Big|^2\Big(\frac{|\varphi-k|^2}{1-|\varphi-k|^2}\Big)$. Then

$$\begin{aligned}
&\Bigg(\sqrt{\gamma_k + \Big(\frac{|\alpha| + |\beta|}{2}\Big)^2} + \sqrt{\gamma_k + \Big(\frac{|\alpha| - |\beta|}{2}\Big)^2}\Bigg)^2 \\
&\quad = 2\gamma_k + \frac{|\alpha|^2 + |\beta|^2}{2} + 2\sqrt{\gamma_k + \Big(\frac{|\alpha| + |\beta|}{2}\Big)^2}\sqrt{\gamma_k + \Big(\frac{|\alpha| - |\beta|}{2}\Big)^2} \\
&\quad \geqslant 2\gamma_k + \frac{|\alpha|^2 + |\beta|^2}{2} + 2\gamma_k + \Big|\frac{|\alpha|^2 - |\beta|^2}{2}\Big| = \max\{|\alpha|^2, |\beta|^2\} + 4\gamma_k.
\end{aligned}$$

By Theorem 3.1, this implies (i).

We prove (ii). Since

$$\gamma_k = \Big|\frac{\alpha-\beta}{2}\Big|^2\Big(\frac{|\varphi-k|^2}{1-|\varphi-k|^2}\Big) \leqslant \Big|\frac{\alpha-\beta}{2}\Big|^2\Big(\frac{\|\varphi-k\|_\infty^2}{1-\|\varphi-k\|_\infty^2}\Big)$$
$$\leqslant \Big|\frac{\alpha-\beta}{2}\Big|^2\Big(\frac{c^2}{1-c^2}\Big) = \gamma,$$

(ii) follows from Theorem 3.1.

Since

$$\sqrt{\gamma+\Big(\frac{|\alpha|+|\beta|}{2}\Big)^2}+\sqrt{\gamma+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2} \leqslant \sqrt{\gamma}+\frac{|\alpha|+|\beta|}{2}+\sqrt{\gamma}+\Big|\frac{|\alpha|-|\beta|}{2}\Big|$$
$$= \max\{|\alpha|,|\beta|\}+2\sqrt{\gamma},$$

(iii) follows from (ii). This completes the proof. ∎

EXAMPLE 3.3. Let $\zeta = e^{i\theta}$, $\alpha(\zeta) = \zeta+1$, $\beta(\zeta) = 1$ and $W(\zeta) = |\zeta+1|^{1/2}$. Then $\alpha\overline{\beta} \in H^\infty, \varphi(e^{i\theta}) = e^{-i\theta/4}, \zeta = e^{i\theta}, -\pi \leqslant \theta < \pi$. Hence, $|\varphi - \frac{1}{\sqrt{2}}| \leqslant \frac{1}{\sqrt{2}} < 1$. By Theorem 3.1,

$$2 \leqslant \|S_{\alpha,\beta}\|_{L^2(W)} \leqslant \left\|\sqrt{\gamma_{1/\sqrt{2}}+\Big(\frac{|\alpha|+|\beta|}{2}\Big)^2}+\sqrt{\gamma_{1/\sqrt{2}}+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}\right\|_\infty,$$

where $\gamma_{1/\sqrt{2}} = \frac{1}{4}\Big(\frac{1}{1-|\varphi-\frac{1}{\sqrt{2}}|^2}-1\Big)$. Hence,

$$2 \leqslant \|S_{\alpha,\beta}\|_{L^2(W)} \leqslant \sup_{-\pi\leqslant\theta<\pi}\left\{\sqrt{\frac{3-2\sqrt{2}\cos\frac{\theta}{4}}{4(2\sqrt{2}\cos\frac{\theta}{4}-1)}+\Big(\frac{\sqrt{2(1+\cos\theta)}+1}{2}\Big)^2}\right.$$
$$\left.+\sqrt{\frac{3-2\sqrt{2}\cos\frac{\theta}{4}}{4(2\sqrt{2}\cos\frac{\theta}{4}-1)}+\Big(\frac{\sqrt{2(1+\cos\theta)}-1}{2}\Big)^2}\right\}$$
$$= \sqrt{\frac{29+2\sqrt{2}}{14}}+\sqrt{\frac{1+2\sqrt{2}}{14}} < 2.04.$$

Since $\|P_+\|_{L^2(W)} = \frac{1}{\cos(\pi/4)} = \sqrt{2}$ (cf. [12]),

$$\gamma = \Big|\frac{\alpha-\beta}{2}\Big|^2(\|P_+\|_{L^2(W)}^2-1) = \frac{1}{4},$$

and hence

$$\left\|\sqrt{\gamma+\Big(\frac{|\alpha|+|\beta|}{2}\Big)^2}+\sqrt{\gamma+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}\right\|_\infty$$
$$= \sqrt{\frac{1}{4}+\Big(\frac{3}{2}\Big)^2}+\sqrt{\frac{1}{4}+\Big(\frac{1}{2}\Big)^2} = \frac{\sqrt{10}+\sqrt{2}}{2} > 2.28.$$

This example shows that it is not able to change the function γ_k by the function γ in Theorem 3.1, and that

$$2 \leqslant \|S_{\alpha,\beta}\|_{L^2(W)} < 2.04.$$

COROLLARY 3.4. *If* $|\alpha(\zeta)| > 0$, *then*

$$\|\alpha P_+\|_{L^2(W)} = \|\alpha P_-\|_{L^2(W)} = \inf_{k\in H^\infty} \left\| \frac{\alpha}{\sqrt{1-|\varphi-k|^2}} \right\|_\infty .$$

COROLLARY 3.5. *Let* $\alpha \in H^\infty$. *Then*

(i) *if* $W^{-1} \in L^1$, *then there exists* $\varepsilon_n \in L^\infty, \varepsilon_n > 0, \|\varepsilon_n\|_\infty \to 0$, $n \to \infty$, *such that*

$$\|\alpha P_+\|_{L^2(W)} = \lim_{n\to\infty} \|(|\alpha| + \varepsilon_n)P_+\|_{L^2(W)} = \lim_{n\to\infty} \inf_{k\in H^\infty} \left\| \frac{|\alpha|+\varepsilon_n}{\sqrt{1-|\varphi-k|^2}} \right\|_\infty ;$$

(ii) *if* $\inf_{k\in H^\infty} \|\varphi - k\|_\infty < 1$, *then we can take* $\varepsilon_n = \frac{1}{n}$.

Proof. We prove (i). Since $|\alpha| + \varepsilon_n > 0$, it follows from Corollary 3.4 that

$$\|(|\alpha| + \varepsilon_n)P_+\|_{L^2(W)} = \left\| \frac{|\alpha|+\varepsilon_n}{\sqrt{1-|\varphi-k|^2}} \right\|_\infty .$$

By the Koosis theorem ([9]), if $W^{-1} \in L^1$, then there exists $U \in L^1, U > 0$ such that

$$\|P_+ f\|_{L^2(U)} \leqslant \|f\|_{L^2(W)}, \quad f \in L^2(W).$$

Let $\varepsilon_n = \frac{1}{n}\sqrt{\frac{U}{W}}$. Since $U \leqslant W$, it follows that $\varepsilon_n \leqslant \frac{1}{n}$ and

$$\|\varepsilon_n P_+ f\|_{L^2(W)} \leqslant \frac{1}{n}\|f\|_{L^2(W)}, \quad f \in L^2(W).$$

Hence,

$$\lim_{n\to\infty} \big| \|(|\alpha| + \varepsilon_n)P_+\|_{L^2(W)} - \|\alpha P_+\|_{L^2(W)} \big| \leqslant \lim_{n\to\infty} \|\varepsilon_n P_+\|_{L^2(W)} \leqslant \lim_{n\to\infty} \frac{1}{n} = 0.$$

We prove (ii).

By the Helson-Szegö theorem ([7]), if $\inf_{k\in H^\infty} \|\varphi - k\|_\infty < 1$, then $\|P_+\|_{L^2(W)} < \infty$. Hence,

$$\lim_{n\to\infty} \left| \left\| \Big(|\alpha| + \frac{1}{n}\Big) P_+ \right\|_{L^2(W)} - \|\alpha P_+\|_{L^2(W)} \right| \leqslant \lim_{n\to\infty} \Big(\frac{1}{n} \|P_+\|_{L^2(W)} \Big) = 0.$$

This completes the proof. ∎

EXAMPLE 3.6. Suppose $W(\zeta) = |\zeta + 1|^{1/2}$, $h(\zeta) = (\zeta + 1)^{1/2}$, $\zeta = \mathrm{e}^{\mathrm{i}\theta}$, $\varphi(\zeta) = \overline{h(\zeta)}/h(\zeta) = \mathrm{e}^{-\mathrm{i}\theta/4}$ and $E \subset T$. By Corollaries 3.7 and 4.3,

$$\|\chi_E P_+\|_{L^2(W)} = \lim_{n\to\infty} \left\| \Big(\chi_E + \frac{1}{n}\Big) P_+ \right\|_{L^2(W)} = \lim_{n\to\infty} \inf_{k\in H^\infty} \left\| \frac{\chi_E + \frac{1}{n}}{\sqrt{1-|\varphi-k|^2}} \right\|_\infty$$

$$\leqslant \left\| \frac{\chi_E + 0.1}{\sqrt{1 - |\mathrm{e}^{-\mathrm{i}\theta/4} - \frac{1}{\sqrt{2}}|^2}} \right\|_\infty = \left\| \frac{\chi_E + 0.1}{\sqrt{\sqrt{2}\cos\frac{\theta}{4} - \frac{1}{2}}} \right\|_\infty .$$

If $-\pi \leqslant \theta < \pi$, then $\frac{1}{\sqrt{2}} \leqslant \cos\frac{\theta}{4} \leqslant 1$. Hence, $\sqrt{\frac{2}{2\sqrt{2}-1}} \leqslant \frac{1}{\sqrt{\sqrt{2}\cos\frac{\theta}{4}-\frac{1}{2}}} \leqslant \sqrt{2}$. Hence, $\frac{\chi_E+0.1}{\sqrt{\sqrt{2}\cos\frac{\theta}{4}-\frac{1}{2}}} < 0.15$ on E^{c}. If $\theta = 0$, then

$$\frac{1}{\sqrt{\sqrt{2}\cos\frac{\theta}{4}-\frac{1}{2}}} = \sqrt{\frac{2}{2\sqrt{2}-1}} = 1.04\ldots < 1.05.$$

Hence, for sufficiently small $\varepsilon > 0$, $E = (-\varepsilon, \varepsilon)$ satisfies $\|\chi_E P_+\|_{L^2(W)} < 1.05 \cdot 1.1 = 1.155$. Since $\|P_+\|_{L^2(W)} = \frac{1}{\cos(\pi/4)} = \sqrt{2}$ (cf. [12]), it follows that

$$\|\chi_E P_+\|_{L^2(W)} < \|\chi_E\|_\infty \|P_+\|_{L^2(W)}.$$

Hence, $\|\alpha P_+\|_{L^2(W)} = \|\alpha\|_\infty \|P_+\|_{L^2(W)}$ does not hold in general.

Next, we compare $\|S_{\alpha,\beta}\|_{L^2(W)}$ and $\|S\|_{L^2(W)}$. Let $\alpha, \beta \in L^\infty$. Since $S_{\alpha,\beta} = \frac{\alpha+\beta}{2} I + \frac{\alpha-\beta}{2} S$, it follows that

$$\|S_{\alpha,\beta}\|_{L^2(W)} \leqslant \Big\|\frac{\alpha+\beta}{2}\Big\|_\infty + \Big\|\frac{\alpha-\beta}{2}\Big\|_\infty \|S\|_{L^2(W)}.$$

By Theorem 3.1, we have

Corollary 3.7. *If $\alpha\overline{\beta} \in H^\infty$ and $|\alpha-\beta| > 0$, then*

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \leqslant \|S_{\alpha,\beta}\|_{L^2(W)} \leqslant \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \|S\|_{L^2(W)}.$$

Proof. Without loss of generality, we assume that $\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} = 1$ and $\|S\|_{L^2(W)} < \infty$. Let $c = \inf_{k \in H^\infty} \|\varphi - k\|_\infty$. By the Helson-Szegö theorem ([7]), $c < 1$. It follows from Theorem 3.1 that

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \inf_{\substack{k \in H^\infty \\ |\varphi-k|<1}} \left\| \sqrt{\gamma_k + \Big(\frac{|\alpha|+|\beta|}{2}\Big)^2} + \sqrt{\gamma_k + \Big(\frac{|\alpha|-|\beta|}{2}\Big)^2} \right\|_\infty,$$

where $\gamma_k := \left|\frac{\alpha-\beta}{2}\right|^2 \Big(\frac{|\varphi-k|^2}{1-|\varphi-k|^2}\Big)$. Since $\max\{|\alpha|, |\beta|\} \leqslant 1$, it follows that $\gamma_k \leqslant \frac{|\varphi-k|^2}{1-|\varphi-k|^2} \leqslant \frac{c^2}{1-c^2}$. Hence,

$$\|S_{\alpha,\beta}\|_{L^2(W)} \leqslant \left\| \sqrt{\frac{c^2}{1-c^2} + \Big(\frac{|\alpha|+|\beta|}{2}\Big)^2} + \sqrt{\frac{c^2}{1-c^2} + \Big(\frac{|\alpha|-|\beta|}{2}\Big)^2} \right\|_\infty.$$

It follows from Theorem 3.1 that

$$\begin{aligned}\|S\|_{L^2(W)} &= \inf_{\substack{k \in H^\infty \\ |\varphi-k|<1}} \left\| \sqrt{\frac{|\varphi-k|^2}{1-|\varphi-k|^2} + 1} + \sqrt{\frac{|\varphi-k|^2}{1-|\varphi-k|^2}} \right\|_\infty \\ &= \sqrt{\frac{c^2}{1-c^2} + 1} + \sqrt{\frac{c^2}{1-c^2}}.\end{aligned}$$

Let $x = \frac{c^2}{1-c^2}$. It is sufficient to show that

$$\sqrt{x + \Big(\frac{|\alpha|+|\beta|}{2}\Big)^2} + \sqrt{x + \Big(\frac{|\alpha|-|\beta|}{2}\Big)^2} \leqslant \sqrt{x+1} + \sqrt{x}.$$

Since $(1-|\alpha|^2)(1-|\beta|^2) \geqslant 0$, it follows that $\Big(\frac{|\alpha|^2-|\beta|^2}{2}\Big)^2 \leqslant \Big(1-\frac{|\alpha|^2+|\beta|^2}{2}\Big)^2$. Hence, $\Big(\frac{|\alpha|^2-|\beta|^2}{2}\Big)^2 + 2(|\alpha|^2+|\beta|^2)x \leqslant \Big(1-\frac{|\alpha|^2+|\beta|^2}{2}\Big)^2 + 4x$. Hence,

$$4\Big(x+\Big(\frac{|\alpha|+|\beta|}{2}\Big)^2\Big)\Big(x+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2\Big) \leqslant \Big(1-\frac{|\alpha|^2+|\beta|^2}{2}\Big)^2 + 4x(x+1)$$
$$\leqslant \Big(1-\frac{|\alpha|^2+|\beta|^2}{2}\Big)^2 + 4\Big(1-\frac{|\alpha|^2+|\beta|^2}{2}\Big)\sqrt{x(x+1)} + 4x(x+1).$$

Hence,

$$2x + \frac{|\alpha|^2+|\beta|^2}{2} + 2\sqrt{x+\Big(\frac{|\alpha|+|\beta|}{2}\Big)^2}\sqrt{x+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}$$
$$\leqslant 2x+1+2\sqrt{x(x+1)}.$$

Hence,

$$\left(\sqrt{x+\Big(\frac{|\alpha|+|\beta|}{2}\Big)^2} + \sqrt{x+\Big(\frac{|\alpha|-|\beta|}{2}\Big)^2}\right)^2 \leqslant \Big(\sqrt{x+1}+\sqrt{x}\Big)^2.$$

This completes the proof. ∎

4. THE THIRD FORMULA OF $S_{\alpha,\beta}$ ON $L^2(W)$

The following Theorem 4.2 is the third formula of $\|S_{\alpha,\beta}\|_{L^2(W)}$. Suppose $\alpha(\zeta) = 1, \beta(\zeta) = \zeta$. Then $\alpha\overline{\beta} \notin H^\infty$. By Corollary 2.9, $\|S_{\alpha,\beta}\|_{L^2} = \sqrt{2}$. Since $\|S\|_{L^2} = 1$, it follows that $\|S_{\alpha,\beta}\|_{L^2} > \|S\|_{L^2}$. Hence Corollary 3.7 does not hold in general. Hence Theorem 3.1 does not hold in general. But the following Theorem 4.2 holds even when $\alpha\overline{\beta} \notin H^\infty$. We give Lemma 4.1 to prove Theorem 4.2.

LEMMA 4.1. *If* $|g| < 1$, *then*

(i) $\frac{|\alpha|^2+|\beta|^2-2\mathrm{Re}(\overline{\alpha}\beta g)}{1-|g|^2} = |\alpha|^2 + \frac{|\overline{\beta}-\overline{\alpha}g|^2}{1-|g|^2} = |\beta|^2 + \frac{|\alpha-\beta g|^2}{1-|g|^2}$;

(ii) $\max\{|\alpha|^2, |\beta|^2\} \leqslant \frac{|\alpha|^2+|\beta|^2-2\mathrm{Re}(\overline{\alpha}\beta g)}{1-|g|^2}$;

(iii) $\frac{(|\alpha|-|\beta|)^2}{1-|g|^2} \leqslant \frac{\max\{|\overline{\beta}-\overline{\alpha}g|^2, |\alpha-\beta g|^2\}}{1-|g|^2} \leqslant \frac{|\alpha|^2+|\beta|^2-2\mathrm{Re}(\overline{\alpha}\beta g)}{1-|g|^2} \leqslant \frac{(|\alpha|+|\beta|)^2}{1-|g|^2}$.

Proof. Since

$$|\alpha|^2+|\beta|^2-2\mathrm{Re}(\overline{\alpha}\beta g) - |\alpha|^2(1-|g|^2) = |\overline{\beta}-\overline{\alpha}g|^2$$

and

$$|\alpha|^2+|\beta|^2-2\mathrm{Re}(\overline{\alpha}\beta g) - |\beta|^2(1-|g|^2) = |\alpha-\beta g|^2,$$

we have (i).

(i) implies (ii).

We prove (iii). Since $|g| < 1$, it follows that $|\alpha|^2+|\beta|^2-2\mathrm{Re}(\overline{\alpha}\beta g) \leqslant (|\alpha|+|\beta|)^2$. Then we have

$$|\overline{\beta}-\overline{\alpha}g|^2 \geqslant (|\beta|-|\alpha g|)^2 \geqslant (|\beta|-|\alpha|)^2, \quad \text{for } |\alpha| \leqslant |\beta|,$$
$$|\alpha-\beta g|^2 \geqslant (|\alpha|-|\beta g|)^2 \geqslant (|\alpha|-|\beta|)^2, \quad \text{for } |\alpha| \geqslant |\beta|.$$

This completes the proof. ∎

THEOREM 4.2. *Let $\alpha, \beta \in L^\infty$. Let φ and W be functions such that there exists an outer function $h \in H^2$ satisfying $\varphi = \overline{h}/h$ and $W = |h|^2$.*

(i) *If $|\,|\alpha| - |\beta|\,| > 0$, then*

$$\|S_{\alpha,\beta}\|^2_{L^2(W)} = \inf_{\substack{k\in H^\infty\\ |\varphi-k|<1}} \Big\| \frac{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\overline{\alpha}\beta(1-\overline{\varphi}k))}{1-|\varphi-k|^2} \Big\|_\infty$$
$$= \inf_{\substack{k\in H^\infty\\ |\varphi-k|<1}} \Big\| |\alpha|^2 + \frac{|\overline{\beta} - \overline{\alpha}(1-\overline{\varphi}k)|^2}{1-|\varphi-k|^2} \Big\|_\infty$$
$$= \inf_{\substack{k\in H^\infty\\ |\varphi-k|<1}} \Big\| |\beta|^2 + \frac{|\alpha - \beta(1-\overline{\varphi}k)|^2}{1-|\varphi-k|^2} \Big\|_\infty.$$

The infimum is attained.

(ii) *If $m\{\zeta \in T : |\alpha(\zeta)| = |\beta(\zeta)|\} > 0$ and $W^{-1} \in L^1$, then there exists $\varepsilon_n \in L^\infty$, $\varepsilon_n > 0$, $\|\varepsilon_n\|_\infty \to 0$, $n \to \infty$, such that $|\,|\alpha_n| - |\beta|\,| > 0$ and*

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \lim_{n\to 0} \inf_{\substack{k\in H^\infty\\ |\varphi-k|<1}} \|S_{\alpha_n,\beta}\|_{L^2(W)},$$

where

$$\alpha_n := \alpha + (\alpha\chi_{E_1} + \chi_{E_0})\varepsilon_n,$$
$$E_0 := \{\zeta \in T : \alpha(\zeta) = \beta(\zeta) = 0\},$$
$$E_1 := \{\zeta \in T : |\alpha(\zeta)| = |\beta(\zeta)| > 0\}.$$

Proof. We prove (i). Let $s = \|S_{\alpha,\beta}\|^2_{L^2(W)}$. It is well known that $\max\{|\alpha|^2, |\beta|^2\} \leqslant s$. Since $|\,|\alpha| - |\beta|\,| > 0$, it follows that $s > 0$. For any $k \in H^\infty$ satisfying $|\varphi - k| < 1$, we define the quantity M_k according to

$$M_k = \Big\| \frac{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\overline{\alpha}\beta(1-\overline{\varphi}k))}{1-|\varphi-k|^2} \Big\|_\infty.$$

We prove that $s \geqslant \inf\{M_k : k \in H^\infty, |\varphi - k| < 1\}$. It follows from Theorem 2.8 that $F(s) = s$. Hence there exists a function $k \in H^\infty$ such that $G(s - \alpha\overline{\beta} - \overline{\varphi}k) \leqslant s$. It follows from Lemma 2.5 that $G\big(\sqrt{s-|\alpha|^2}\sqrt{s-|\beta|^2}\big) = s$. Since

$$\Big|\alpha\overline{\beta} - s\Big(1 - \overline{\varphi}\frac{k}{s}\Big)\Big|^2 = |s - \alpha\overline{\beta} - \overline{\varphi}k|^2 \leqslant (s-|\alpha|^2)(s-|\beta|^2),$$

it follows that there exists a $k_0 \in H^\infty$ such that $|\alpha\overline{\beta} - s(1-\overline{\varphi}k_0)|^2 \leqslant (s - |\alpha|^2)(s-|\beta|^2)$. Hence, $|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\overline{\alpha}\beta(1-\overline{\varphi}k_0)) \leqslant s(1-|\varphi-k_0|^2)$. Since $s|\varphi - k_0| \leqslant \sqrt{s-|\alpha|^2}\sqrt{s-|\beta|^2} + |\alpha\beta| \leqslant s$, it follows that $|\varphi - k_0| \leqslant 1$. Since $|\,|\alpha| - |\beta|\,| > 0$, it follows that $|\varphi - k_0| < 1$. Hence, $\inf_{\substack{k\in H^\infty\\ |\varphi-k|<1}} M_k \leqslant M_{k_0} \leqslant s$. We prove that $s \leqslant \inf\{M_k : k \in H^\infty, |\varphi - k| < 1\}$. This is the easy direction of the theorem.

Since $|\alpha|^2+|\beta|^2-2\mathrm{Re}(\overline{\alpha}\beta(1-\overline{\varphi}k))\leqslant M_k(1-|\varphi-k|^2)$, it follows that

$$\begin{aligned}|M_k-\alpha\overline{\beta}-\overline{\varphi}M_kk|^2&=|M_k(1-\overline{\varphi}k)-\alpha\overline{\beta}|^2\\&=|M_k(1-\overline{\varphi}k)|^2-2M_k\mathrm{Re}(\overline{\alpha}\beta(1-\overline{\varphi}k))+|\alpha\beta|^2\\&\leqslant M_k^2-(|\alpha|^2+|\beta|^2)M_k+|\alpha\beta|^2\\&=(M_k-|\alpha|^2)(M_k-|\beta|^2).\end{aligned}$$

It follows from Lemma 4.1 that $\max\{|\alpha|^2,|\beta|^2\}\leqslant M_k$. It follows from Lemma 2.5 that $G\big(\sqrt{M_k-|\alpha|^2}\sqrt{M_k-|\beta|^2}\,\big)=M_k$. Hence, $G(M_k-\alpha\overline{\beta}-\overline{\varphi}M_kk)\leqslant M_k$. Hence $F(M_k)\leqslant M_k$. It follows from Lemma 2.6 that $s\leqslant M_k$. Therefore,

$$\inf_{\substack{k\in H^\infty\\|\varphi-k|<1}} M_k\leqslant M_{k_0}\leqslant s\leqslant \inf_{\substack{k\in H^\infty\\|\varphi-k|<1}} M_k.$$

Hence the equalities hold, and the infimum is attained by $k=k_0$.

We prove (ii). By Corollary 3.5, there exists $\varepsilon_n\in L^\infty$, $\varepsilon_n>0$, $\|\varepsilon_n\|_\infty\to 0$, $n\to\infty$, such that $|\,|\alpha_n|-|\beta|\,|>0$, and $\|\varepsilon_nP_+\|_{L^2(W)}\to 0$, $n\to\infty$. Since

$$\|(\alpha_n-\alpha)P_+\|_{L^2(W)}=\|(\alpha\chi_{E_1}+\chi_{E_0})\varepsilon_nP_+\|_{L^2(W)}\leqslant\|(\alpha\chi_{E_1}+\chi_{E_0})\|_\infty\|\varepsilon_nP_+\|_{L^2(W)},$$

it follows that

$$\|(\alpha_n-\alpha)P_+\|_{L^2(W)}\to 0,\quad n\to\infty.$$

Since

$$\big|\,\|S_{\alpha_n,\beta}\|_{L^2(W)}-\|S_{\alpha,\beta}\|_{L^2(W)}\big|\leqslant\|S_{\alpha_n,\beta}-S_{\alpha,\beta}\|_{L^2(W)}=\|(\alpha_n-\alpha)P_+\|_{L^2(W)},$$

it follows that

$$|\,\|S_{\alpha_n,\beta}\|_{L^2(W)}-\|S_{\alpha,\beta}\|_{L^2(W)}|\to 0,\quad n\to\infty.$$

This completes the proof. ∎

If W is a constant function, then φ becomes a constant function. In this case, the formula of $\|S_{\alpha,\beta}\|_{L^2}$ was given by Corollary 2.9. The another formula of $\|S_{\alpha,\beta}\|_{L^2}$ is given by Corollary 4.3.

COROLLARY 4.3. *Let $\alpha,\beta\in L^\infty$. If $|\,|\alpha|-|\beta|\,|>0$, then*

$$\begin{aligned}\|S_{\alpha,\beta}\|_{L^2}^2&=\inf_{\substack{g\in H^\infty\\|g|<1}}\left\|\frac{|\alpha|^2+|\beta|^2-2\mathrm{Re}(\overline{\alpha}\beta g))}{1-|g|^2}\right\|_\infty\\&=\inf_{\substack{g\in H^\infty\\|g|<1}}\left\||\alpha|^2+\frac{|\overline{\beta}-\overline{\alpha}g|^2}{1-|g|^2}\right\|_\infty=\inf_{\substack{g\in H^\infty\\|g|<1}}\left\||\beta|^2+\frac{|\alpha-\beta g|^2}{1-|g|^2}\right\|_\infty.\end{aligned}$$

COROLLARY 4.4. *Let $\alpha,\beta\in L^\infty$. Let φ and W be functions such that there exists an outer function $h\in H^2$ satisfying $\varphi=\overline{h}/h$ and $W=|h|^2$. If $|\,|\alpha|-|\beta|\,|>0$, then*

$$\text{(i)}\ \inf_{k\in H^\infty}\left\|\frac{|\alpha|-|\beta|}{\sqrt{1-|\varphi-k|^2}}\right\|_\infty\leqslant\inf_{k\in H^\infty}\left\|\frac{\max\{|\overline{\beta}-\overline{\alpha}(1-\overline{\varphi}k)|,|\alpha-\beta(1-\overline{\varphi}k)|\}}{\sqrt{1-|\varphi-k|^2}}\right\|_\infty$$
$$\leqslant\|S_{\alpha,\beta}\|_{L^2(W)}\leqslant\inf_{k\in H^\infty}\left\|\frac{|\alpha|+|\beta|}{\sqrt{1-|\varphi-k|^2}}\right\|_\infty;$$

(ii) $$\max\left\{\|\alpha\|_\infty, \|\beta\|_\infty, \inf_{k\in H^\infty}\left\|\frac{|\alpha|-|\beta|}{\sqrt{1-|\varphi-k|^2}}\right\|_\infty\right\}$$
$$\leqslant \|S_{\alpha,\beta}\|_{L^2(W)} \leqslant \inf_{k\in H^\infty}\left\|\frac{\alpha-\beta}{\sqrt{1-|\varphi-k|^2}}\right\|_\infty + \min\{\|\alpha\|_\infty, \|\beta\|_\infty\}.$$

Proof. (i) follows from Theorem 4.2 (i) and Lemma 4.1.
(ii) follows from Corollary 3.4 and

$$\|S_{\alpha,\beta}\|_{L^2(W)} \leqslant \|(\alpha-\beta)P_+\|_{L^2(W)} + \|\beta\|_\infty,$$
$$\|S_{\alpha,\beta}\|_{L^2(W)} \leqslant \|(\beta-\alpha)P_-\|_{L^2(W)} + \|\alpha\|_\infty.$$

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