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# A MAPPING THEOREM FOR THE BOUNDARY SET $X_T$ OF AN ABSOLUTELY CONTINUOUS CONTRACTION $T$

G. CASSIER, I. CHALENDAR and B. CHEVREAU

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Abstract. Let $T$ be an absolutely continuous contraction acting on a Hilbert space. Its boundary set $X_T$ can be seen as a localization, on a Borel subset of the unit circle $\mathbb{T}$, of a sequence condition whose validity on all of $\mathbb{T}$ is equivalent to membership of $T$ in the class$\mathbb{A}_{\aleph_0}$. The main result is the following: if $b$ is a Blaschke product of degree $d$ for which there exist $d$ distinct Möbius transforms $u$ such that $b \circ u = b$, then $b(X_T) = X_{b(T)}$.

Keywords: *Absolutely continuous contraction, dual algebra theory, classes $\mathbb{A}_{n,m}$, boundary sets.*

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## 1. INTRODUCTION

For an absolutely continuous contraction $T$, its boundary set $X_T$, introduced in [12], can be seen as a localization, on a Borel subset of the unit circle $\mathbb{T}$, of a sequence condition whose validity on all of $\mathbb{T}$ is equivalent to membership of $T$ in the class $\mathbb{A}_{\aleph_0}$. In other words (cf. Corollary 3.4 in [12] and [1], with a different formulation):

$$T \in \mathbb{A}_{\aleph_0} \Leftrightarrow X_T = \mathbb{T}.$$

This class is the smallest of the subclasses $\mathbb{A}_{m,n}$ of the class $\mathbb{A}$ of absolutely continuous contractions with isometric functional calculus.

The set $X_T$, along with other subsets of $\mathbb{T}$ arising naturally from the minimal isometric dilation and minimal coisometric extension of $T$, has been useful in the study of the membership in the classes $\mathbb{A}_{m,n}$ (cf. [5], [10], [12]). In [5], starting from the observation that inner functions operate on $\mathbb{A}$ (i.e. if $u$ is inner and $T \in \mathbb{A}$, then $u(T) \in \mathbb{A}$), we studied how the subclasses $\mathbb{A}_{m,n}$ are transformed under this operation. In this study we were naturally led to investigate possible mapping theorems for boundary sets as initiated in [12]. In particular, we proved that for any non constant inner function $u$ and for any absolutely continuous contraction $T$, we have (Proposition 3.1 in [6]):

$$u(X_T) \subset X_{u(T)}. \tag{1.1}$$

In the particular case where $u(z) = \phi(z)^n$, $n \geqslant 1$ where $\phi$ is any conformal automorphism of $\mathbb{D}$ the inclusion (1.1) is an equality (Theorem 5.1 and Proposition 3.9 in [12]). Nevertheless the equality in (1.1) may drastically fail to hold as shown by the example of the unilateral shift and any inner function $u$ of infinite degree (recall that the degree of an inner function is the number of zeros, counting multiplicity, in the case of a finite Blaschke product and $\aleph_0$ elsewhere). Indeed, by Proposition 3.2 in [5], $u(S) \in \mathbb{A}_{\aleph_0}$ and thus $X_{u(S)} = \mathbb{T}$, while $X_S = \emptyset$ (Proposition 3.5 in [12]).

The main result of this paper (Theorem 4.1 below) says that equality holds in (1.1) whenever $u$ is a finite Blaschke product of degree $d$ with $d$ distinct analytic invariants. The invariants of a Blaschke product have been recently studied in [4]. The needed facts about the domain of analyticity of these invariants are recalled at the beginning of Section 3.

The paper is organized as follows. In Section 2, we recall some notations and terminology as well as some facts about the boundary set $X_T$. Then in Section 3, we state and prove an intermediate result. We prove that if $b$ is a finite Blaschke product of degree $d$ with $d$ distinct analytic invariants $\widetilde{u}_1, \dots, \widetilde{u}_d$, the membership of $b(T)$ in the class $\mathbb{A}_{\aleph_0}$ is equivalent to the membership of $\widetilde{u}_1(T) \oplus \cdots \oplus \widetilde{u}_d(T)$ in the class $\mathbb{A}_{\aleph_0}$. In Section 4 we prove the main result and develop additional mapping theorems for various boundary sets.

## 2. NOTATIONS AND TERMINOLOGY

Let $\mathcal{H}$ be a separable, infinite-dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A contraction $T \in \mathcal{L}(\mathcal{H})$ is absolutely continuous if $T$ is completely non unitary or if the spectral measure of its unitary part is absolutely continuous with respect to Lebesgue measure. If $T \in \mathcal{L}(\mathcal{H})$ is an absolutely continuous contraction, then, for any $x, y \in \mathcal{H}$, there exists a function $x \overset{T}{\cdot} y \in L^1$ such that the Fourier coefficients of $x \overset{T}{\cdot} y$ satisfy:

$$(x \overset{T}{\cdot} y)^\wedge(n) = \begin{cases} (T^{*n}x, y), & n \geqslant 0, \\ (T^{-n}x, y), & n < 0. \end{cases}$$

We write $\mathbb{D}$ for the open unit disc in the complex plane $\mathbb{C}$, and $\mathbb{T}$ for the unit circle. The spaces $L^p = L^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$ are the usual Lebesgue function spaces relative to Lebesgue measure $m$ on $\mathbb{T}$. The spaces $H^p = H^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$ are the usual Hardy spaces. It is well-known ([15]) that the dual space of $L^1/H_0^1$, where $H_0^1 = \left\{ f \in L^1 \int\limits_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\mathrm{i}nt}\,\mathrm{d}t = 0 : n = 0, 1, \dots \right\}$, can be identified with $H^\infty$. If we denote by $[g]$ the class of $g \in L^1$ in $L^1/H_0^1$, the duality is given by the pairing:

$$\langle f, [g] \rangle = \int\limits_{\mathbb{T}} fg\,\mathrm{d}m, \quad f \in H^\infty,\ g \in L^1.$$

We denote by $\mathbb{A} = \mathbb{A}(\mathcal{H})$ the class of all absolutely continuous contractions $T \in \mathcal{L}(\mathcal{H})$ for which the Sz.-Nagy–Foias functional calculus $\Phi_T : H^\infty \to \mathcal{L}(\mathcal{H})$ is an isometry. For a given $(m,n) \in \{1, \dots, \aleph_0\}^2$, the subclass $\mathbb{A}_{m,n}$ of $\mathbb{A}$, consists of those contractions $T \in \mathbb{A}$ for which given any family $(f_{i,j})_{i,j}$, $0 \leqslant i < m, 0 \leqslant j < n$ of elements of $L^1$, there exist two sequences $(x_i)_{0 \leqslant i < m}$ and $(y_j)_{0 \leqslant j < n}$ of elements of $\mathcal{H}$ such that $[f_{i,j}] = [x_i \overset{T}{\cdot} y_j]$, $i,j \geqslant 1$. The class $\mathbb{A}_{\aleph_0,\aleph_0} = \mathbb{A}_{\aleph_0}$ is well-known for being a class of contractions whose lattice of invariant subspaces is extremely rich since it contains a sub-lattice which is isomorphic to the lattice of all the closed subspaces of $\mathcal{H}$ (for further informations on the invariant subspaces of any element of the class $\mathbb{A}_{\aleph_0}$ see Chapter IX in [1]).

The boundary set $X_T$ introduced in [12], Proposition 3.1, is the unique maximal Borel subset $X_T$ of $\mathbb{T}$ such that, for any $f \in L^1(X_T), \|f\|_1 \leqslant 1$, there exist two sequences $(x_n)_n$ and $(y_n)_n$ in the unit ball of $\mathcal{H}$ such that:

$$\begin{cases} \lim\limits_{n\to\infty} \|[f] - [x_n \overset{T}{\cdot} y_n]\| = 0, & \\ \lim\limits_{n\to\infty} \|[x_n \overset{T}{\cdot} w]\| = 0, & w \in \mathcal{H}, \\ \lim\limits_{n\to\infty} \|[w \overset{T}{\cdot} y_n]\| = 0, & w \in \mathcal{H}. \end{cases}$$

In other words, $X_T$ is the largest Borel subset of the unit circle on which the "classical" Scott Brown approximation procedure works for $T$. (Recall that throughout the paper, expressions such as maximality, uniqueness, and equality of Borel subsets of $\mathbb{T}$ are to be interpreted as pertaining to the equivalence classes arising from the relation "equal a.e. (m)".) As was already the case in previous works (e.g. Theorem 6.2 in [12]), we will use the set $X_T$ in connection with membership of sufficiently many classes $[P_\lambda]$ of Poisson kernels in the approximation set $\chi_0(T)$ (cf. Proposition 3.6 below).

It has been known for some time that the doubly-infinite equation-solving procedure defining the class $\mathbb{A}_{\aleph_0}$ can be lifted from the quotient $L^1/H^1_0$ to $L^1$ itself. In fact this was established, via the functional model, for $T$ in the subclasses $(\mathrm{BCP})_\theta$ of $\mathbb{A}_{\aleph_0}$, in Corollary 6.9 of [2] (the extension to $\mathbb{A}_{\aleph_0}$, though not explicitly stated there, proceeds from standard dilation properties of the class $\mathbb{A}_{\aleph_0}$). More recently, the same result was obtained in Section 4 of [9] via a new more direct approach. In other words, we can see the class $\mathbb{A}_{\aleph_0}$ as the class of contractions $T \in \mathcal{L}(\mathcal{H})$ such that for any infinite array $(f_{i,j})_{i,j \geqslant 1}$ of functions in $L^1$, there exist some sequences $(x_i)_{i \geqslant 1}$ and $(y_j)_{j \geqslant 1}$ of elements of $\mathcal{H}$ such that $f_{i,j} = (x_i \overset{T}{\cdot} y_j)$, $i \geqslant 1$, $j \geqslant 1$. Equivalently, for $T \in \mathbb{A}_{\aleph_0}$, one can recover all the Fourier coefficients (and not simply the negative ones) of a system of functions in $L^1$. Corollary 6.9 of [2] also provides some localization of the above factorizations. The localization is generalized to the entire $X_T$ in [8], Theorem 2.3. Indeed, for any infinite array $(f_{i,j})_{i,j \geqslant 1}$ of functions in $L^1$, there exist some sequences $(x_i)_{i \geqslant 1}$ and $(y_j)_{j \geqslant 1}$ of elements of $\mathcal{H}$ such that $f_{i,j|X_T} = (x_i \overset{T}{\cdot} y_j)_{|X_T}$, $i \geqslant 1$, $j \geqslant 1$.

Sufficient conditions for factorizing lower semi-continuous positive $L^1$-functions in the form $x \overset{T}{\cdot} x$ have been obtained in [9] in the case $T \in \mathbb{A}_{\aleph_0}$ and in [7] in the case $T \in \mathbb{A}_{1,\aleph_0}$ via very different techniques.

3. INTERMEDIATE RESULTS

Let $b$ be a finite Blaschke product of degree $d$. By Theorem 3.1 in [4] we know that the set of continuous functions $u : \mathbb{T} \to \mathbb{T}$ such that $b \circ u = b$ on $\mathbb{T}$ is a cyclic group $G$ (under the composition) of order $d$. Moreover, by Proposition 3.1 in [4], each element $u$ of $G$ has an analytic extension $\widetilde{u}$ in the annulus $C_M = \{z \in \mathbb{C} : M < |z| < 1/M\}$ (where $M = \max\{|\alpha| : \alpha \in b(\alpha) = 0\}$), which still satisfies $b \circ \widetilde{u}_i = b$ in $C_M$.

Nevertheless, for general finite Blaschke products of degree at least equal to three, the elements of $G$ are not necessarily the restriction of functions in the disc algebra $A(\mathbb{D})$, the algebra of analytic functions on $\mathbb{D}$ which are continuous on $\mathbb{T}$ (see Section 4 in [4] for more informations about the finite Blaschke products whose invariants are analytic).

Definition 3.1. Let $b$ be a finite Blaschke product. We say that $b$ is of type (AI) if each element $u$ of $G$, the cyclic group of the continuous invariants of $b$, is the restriction of a function $\widetilde{u} \in A(\mathbb{D})$ which satisfies $b \circ \widetilde{u} = b$ on $\mathbb{D}$.

Remark 3.2. It follows from [14] that if $b$ is a finite Blaschke product and if $\widetilde{u} \in A(\mathbb{D})$ satisfies $b \circ \widetilde{u} = b$ on $\mathbb{D}$, then necessarily $\widetilde{u}$ is an elliptic Möbius transformation (i.e. an automorphism of the unit disc with a fixed point in $\mathbb{D}$) and thus, in particular, a Blaschke product of degree one.

The main result of this section is the following theorem:

Theorem 3.3. *Let $b$ be a finite Blaschke product of type* (AI) *and denote by $\widetilde{u}_1, \dots, \widetilde{u}_d$ its pairwise distinct analytic invariants. If $T$ is an absolutely continuous contraction, then the following assertions are equivalent:*

(i) *the operator $b(T)$ belongs to the class $\mathbb{A}_{\aleph_0}$;*

(ii) *the operator $\widetilde{u}_1(T) \oplus \cdots \oplus \widetilde{u}_d(T)$ belongs to the class $\mathbb{A}_{\aleph_0}$.*

Throughout the remainder of this section we will denote by $\widetilde{T}$ the operator $\widetilde{u}_1(T) \oplus \cdots \oplus \widetilde{u}_d(T)$.

3.1. Proof of Theorem 3.3. The next lemma is the "easy part" in the proof of the equivalence announced in Theorem 3.3.

Lemma 3.4. *Let $b$ be a finite Blaschke product of type* (AI) *and denote by $\widetilde{u}_1, \dots, \widetilde{u}_d$ its pairwise distinct analytic invariants. Let $T$ be an absolutely continuous contraction. If the operator $\widetilde{T}$ belongs to the class $\mathbb{A}_{\aleph_0}$, then the operator $b(T)$ belongs to the class $\mathbb{A}_{\aleph_0}$.*

*Proof.* By Theorem 2.2 in [6], if $\widetilde{T} \in \mathbb{A}_{\aleph_0}$, then $b(\widetilde{T}) \in \mathbb{A}_{\aleph_0}$. Since $b(\widetilde{T}) = b(T) \oplus \cdots \oplus b(T)$, using Theorem 3.8 in [1], we get $b(T) \in \mathbb{A}_{\aleph_0}$. ∎

For the proof of Theorem 3.3, it remains to check that if $b(T) \in \mathbb{A}_{\aleph_0}$ then $\widetilde{T} \in \mathbb{A}_{\aleph_0}$. In order to make clearer our strategy we review further classical definitions and results in dual algebra theory.

DEFINITION 3.5. (Definition 2.3 and 3.7 in [12]) If $T$ is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ we denote by $\chi_0(T)$ the subset of $(L^1/H_0^1)(\mathbb{T})$ consisting of those cosets $[f]$ for which there exist sequences $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ in the (closed) unit ball of $\mathcal{H}$ satisfying:

(a) $\lim \|[f] - [x_n \overset{T}{\cdot} y_n]\| = 0$, and

(b) $\lim\limits_{n\to\infty} (\|[x_n \overset{T}{\cdot} w]\| + \|[w \overset{T}{\cdot} y_n]\|) = 0$, $w \in \mathcal{H}$.

Thus, the definition of $X_T$ can be reformulated in the following way: $X_T$ is the maximal Borel subset of $\mathbb{T}$ such that, for any $f$ in $L^1(X_T)$, $[f]$ belongs to $\chi_0(T)$.

If $\lambda \in \mathbb{D}$, recall that the Poisson kernel $P_\lambda$ is defined by:

$$P_\lambda(z) = 1 + \sum_{n=1}^{\infty} \lambda^n z^n + \sum_{n=1}^{\infty} \lambda^n \overline{z}^n = \frac{1-|\lambda|^2}{|1-\lambda\overline{z}|^2}.$$

If $\Lambda \subset \mathbb{D}$ we write $\mathrm{NTL}(\Lambda)$ for the subset of $\mathbb{T}$ consisting of all the non-tangential limits of sequences from $\Lambda$. We will say that $\Lambda$ is dominating for $\mathbb{T}$ if $\mathbb{T} \setminus \mathrm{NTL}(\Lambda)$ has Lebesgue measure zero (this notion originated in [3]). The next result is just a reformulation of Proposition 6.1 in [1] using the Hahn–Banach theorem.

PROPOSITION 3.6. *Let $T$ be an absolutely continuous contraction. Suppose that $\Lambda \subset \mathbb{D}$ is dominating for $\mathbb{T}$ and suppose that there exists a positive constant $c$ such that, for every $\lambda \in \Lambda$, $c[P_\lambda] \in \chi_0(T)$. Then $T$ belongs to the class $\mathbb{A}_{\aleph_0}$.*

From now on, by means of several intermediate results, our aim is to check that if $b(T) \in \mathbb{A}_{\aleph_0}$ then there exist a dominating set $\Lambda$ for $\mathbb{T}$ and a positive constant $c$ such that for every $\lambda \in \Lambda$, $c[P_\lambda] \in \chi_0(\widetilde{T})$.

The next proposition and its corollary show how to transfer "left and right vanishing conditions" from the operator $b(T)$ to the operator $\widetilde{T}$.

PROPOSITION 3.7. *Let $b$ a finite Blaschke product of degree $d \geqslant 1$ and of type* (AI). *Take $f_1, \dots, f_d \in H^\infty$ and $g_1, \dots, g_d \in H^\infty$ such that $b \circ g_i = l_i \circ b$ with $l_i \in H^\infty$, $1 \leqslant i \leqslant d$. Set $\widetilde{\widetilde{T}} = g_1(T) \oplus \cdots \oplus g_d(T)$ and for $x_n \in \mathbb{D}$, set $\widetilde{x}_n = f_1(T)x_n \oplus \cdots \oplus f_d(T)x_n$. Then we have:*

(i) $\lim\limits_{n\to\infty} \left\| \left[ x_n \overset{b(T)}{\cdot} w \right] \right\| = 0$, $w \in \mathcal{H} \Rightarrow \lim\limits_{n\to\infty} \left\| \left[ \widetilde{x}_n \overset{\widetilde{\widetilde{T}}}{\cdot} \widetilde{w} \right] \right\| = 0$, $\widetilde{w} = w_1 \oplus \cdots \oplus w_d \in \mathcal{H}^{(d)}$.

(ii) $\lim\limits_{n\to\infty} \|[w \overset{b(T)}{\cdot} x_n]\| = 0$, $w \in \mathcal{H} \Rightarrow \lim\limits_{n\to\infty} \left\| \left[ \widetilde{w} \overset{\widetilde{\widetilde{T}}}{\cdot} \widetilde{x}_n \right] \right\| = 0$, $\widetilde{w} = w_1 \oplus \cdots \oplus w_d \in \mathcal{H}^{(d)}$.

*Proof.* For $h \in H^\infty$, $\widetilde{w} = w_1 \oplus \cdots \oplus w_d \in \mathcal{H}^{(d)}$ and $x_n \in \mathcal{H}$ we compute

$$J_n = \left\langle \left[ \widetilde{x}_n \overset{\widetilde{\widetilde{T}}}{\cdot} \widetilde{w} \right], h \right\rangle = \left( h(\widetilde{\widetilde{T}})\widetilde{x}_n, \widetilde{w} \right) = \sum_{i=1}^{d} (h \circ g_i(T) f_i(T) x_n, w_i).$$

By Theorem 2.1 in [6] we know that there exist some functions $v_1, \dots, v_d \in H^\infty \cap (H^2 \ominus bH^2)$ such that for any function $h \in H^\infty$:

$$h(z) = \sum_{k=1}^{d} h_k(b(z)) v_k(z)$$

with $h_k \in H^\infty$, $\|f_k\|_\infty \leqslant C\|h\|_\infty$ and where $C$ is a positive numerical constant. We thus obtain

$$J_n = \sum_{i=1}^{d} \sum_{k=1}^{d} ((h_k \circ b \circ g_i)(T)(v_k \circ g_i)(T) f_i(T)x, w_i).$$

Since $b \circ g_i = l_i \circ b$ for $1 \leqslant i \leqslant d$, we get:

$$J_n = \sum_{i=1}^{d} \sum_{k=1}^{d} ((h_k \circ l_i \circ b)(T)(v_k \circ g_i)(T) f_i(T)x, w_i) = \sum_{i=1}^{d} \sum_{k=1}^{d} \Big\langle \Big[x \overset{b(T)}{\cdot} M_{k,i} w_i\Big], h_k \circ l_i \Big\rangle,$$

with $M_{k,i} = ((v_k \circ g_i)(T) f_i(T))^*$. From this computation we get the transfer of the right vanishing condition. Indeed suppose that $\lim_{n\to\infty} \|[x_n \overset{b(T)}{\cdot} w]\| = 0, w \in \mathcal{H}$. Then for a given $\widetilde{w} = w_1 \oplus \cdots \oplus w_d \in \mathcal{H}^{(d)}$ and each $n$ there exists a function $h_n \in H^\infty$ of norm 1 such that

$$\Big\langle \Big[\widetilde{x}_n \overset{\widetilde{\widetilde{T}}}{\cdot} \widetilde{w}\Big], h_n \Big\rangle = \Big\| \Big[\widetilde{x}_n \overset{\widetilde{\widetilde{T}}}{\cdot} \widetilde{w}\Big] \Big\|.$$

By the previous computation we get

$$\Big\| \Big[\widetilde{x}_n \overset{\widetilde{\widetilde{T}}}{\cdot} \widetilde{w}\Big] \Big\| \leqslant K \sum_{i=1}^{d} \sum_{k=1}^{d} \Big\| \Big[x \overset{b(T)}{\cdot} M_{k,i} w_i\Big] \Big\|,$$

where $K$ is a positive numerical constant equal to $\max\{\|h_{n,k} \circ l_i\|_\infty : 1 \leqslant k, i \leqslant d\}$ with $\|h_{n,k}\|_\infty \leqslant C$ for every $n$. Therefore $\lim_{n\to\infty} \Big\| \Big[\widetilde{x}_n \overset{\widetilde{\widetilde{T}}}{\cdot} \widetilde{w}\Big] \Big\| = 0$.

The second assertion follows from similar arguments. ∎

Corollary 3.8. *Let $b$ be a finite Blaschke product of type* (AI) *and denote by $\widetilde{u}_1, \dots, \widetilde{u}_d$ its pairwise distinct analytic invariants. Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction and $\varphi \in H^\infty$. For $x_n \in \mathbb{D}$, set $\widetilde{x}_n = (\varphi \circ \widetilde{u}_1)(T)x_n \oplus \cdots \oplus (\varphi \circ \widetilde{u}_d)(T)x_n$, $\widehat{x}_n = x_n \oplus \cdots \oplus x_n$. Then we have:*

(i) $\lim_{n\to\infty} \Big\| \Big[x_n \overset{b(T)}{\cdot} w\Big] \Big\| = 0$ *for all* $w \in \mathcal{H} \Rightarrow \lim_{n\to\infty} v \Big\| \Big[\widetilde{x}_n \overset{\widetilde{T}}{\cdot} \widetilde{w}\Big] \Big\| = 0$ *for all* $\widetilde{w} = w_1 \oplus \cdots \oplus w_d \in \mathcal{H}^{(d)}$.

(ii) $\lim_{n\to\infty} \Big\| \Big[w \overset{b(T)}{\cdot} x_n\Big] \Big\| = 0$ *for all* $w \in \mathcal{H} \Rightarrow \lim_{n\to\infty} \Big\| \Big[\widetilde{w} \overset{\widetilde{T}}{\cdot} \widehat{x}_n\Big] \Big\| = 0$ *for all* $\widetilde{w} = w_1 \oplus \cdots \oplus w_d \in \mathcal{H}^{(d)}$.

*Proof.* It is an immediate application of Proposition 3.7 taking $f_i = \varphi \circ \widetilde{u}_i$ or $f_i(z) = z$ and $g_i = \widetilde{u}_i$, $1 \leqslant i \leqslant d$, so that $b \circ \widetilde{u}_i = b = l_i \circ b$ with $l_i(z) = z$. ∎

The next two results are devoted to show how to transfer a factorization in $L^1/H^1_0$ of a Poisson kernel $P_{b(\lambda)}$ linked to the operator $b(T)$ into a factorization of $[P_\lambda]$ linked to the operator $\widetilde{T}$. In order to state the next proposition we need the following lemma.

LEMMA 3.9. *Let $b$ be a finite Blaschke product. Suppose that $f$ analytic in $\mathbb{D}$ satisfies $f(z) = f(z')$ as soon as $b(z) = b(z')$. Then there exists $\widetilde{f}$ analytic in $\mathbb{D}$ such that $f = \widetilde{f} \circ b$. Moreover, if $f \in H^\infty(\mathbb{D})$ then $\widetilde{f} \in H^\infty(\mathbb{D})$.*

*Proof.* Define $\tilde{f}$ on $\mathbb{D}$ by $\tilde{f}(w) = f(z)$ for any $z \in \mathbb{D}$ such that $b(z) = w$ (note that $b$ is onto). The hypothesis $f(z) = f(z')$ as soon as $b(z) = b(z')$ guarantees that $\tilde{f}$ is well-defined. It remains to check that $\tilde{f}$ is holomorphic on $\mathbb{D}$. In fact, locally, except for finitely many points in $b(\{z \in \mathbb{D} : b'(z) = 0\})$, $b$ has an analytic inverse and thus $\tilde{f} = f \circ b^{-1}$ is holomorphic in a neighborhood of any $w \in \mathbb{D} \backslash b(\{z \in \mathbb{D} : b'(z) = 0\})$. Also, its definition implies that $\tilde{f}$ is bounded on every compact subset of $\mathbb{D}$; hence its singularities are removable and it has an analytic extension on $\mathbb{D}$. Obviously $\sup_{z \in \mathbb{D}} |f(z)| = \sup_{z \in \mathbb{D}} |\tilde{f}(z)|$, which proves the last assertion of the lemma. ∎

PROPOSITION 3.10. *Let $b$ be a finite Blaschke product of type* (AI) *and denote by $\widetilde{u}_1, \ldots, \widetilde{u}_d =$* Id *its pairwise distinct analytic invariants. Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction. Take $w \in \mathbb{D}$, $\lambda = b(w)$ and $\varphi_w(z) = \prod_{i=1}^{d-1} \frac{z - \widetilde{u}_i(w)}{w - \widetilde{u}_i(w)}$. For $x \in \mathbb{D}$, set $\widetilde{x} = (\varphi_w \circ \widetilde{u}_1)(T)x \oplus \cdots \oplus (\varphi_w \circ \widetilde{u}_d)(T)x$, $\widehat{x} = x \oplus \cdots \oplus x$. Then we have:*

$$[P_\lambda] = [x \overset{b(T)}{\cdot} x] \Rightarrow [P_w] = [\widetilde{x} \overset{\widetilde{T}}{\cdot} \widehat{x}].$$

*Proof.* For $h \in H^\infty$ we compute

$$\left\langle \left[\widetilde{x} \overset{\widetilde{T}}{\cdot} \widehat{x}\right], h \right\rangle = (h(\widetilde{T})\widetilde{x}, \widehat{x}) = \sum_{i=1}^{d} (h \circ \widetilde{u}_i)(T)(\varphi_w \circ \widetilde{u}_i)(T)x, x).$$

By Theorem 2.1 in [6] we know that there exist some functions $v_1, \ldots, v_d \in H^\infty \cap (H^2 \ominus bH^2)$ such that for any function $h \in H^\infty$:

$$h(z) = \sum_{k=1}^{d} h_k(b(z)) v_k(z)$$

with $h_k \in H^\infty$, $\|f_k\|_\infty \leqslant C\|h\|_\infty$ and where $C$ is a positive numerical constant. We thus obtain

$$\begin{aligned} \left\langle \left[\widetilde{x} \overset{\widetilde{T}}{\cdot} \widehat{x}\right], h \right\rangle &= \sum_{i=1}^{d} \sum_{k=1}^{d} ((h_k \circ b)(T)(v_k \circ \widetilde{u}_i)(T)(\varphi_w \circ (T)\widetilde{u}_i)(T)x, x) \\ &= \sum_{k=1}^{d} (h_k(b(T)) A_{k,\varphi_w}(T)x, x), \end{aligned}$$

where $A_{k,\varphi_w}$ is a function of $H^\infty$ equal to $\sum_{i=1}^{d}(v_k\varphi_w)\circ\widetilde{u}_i$. Obviously this function is invariant (by composition) under every $\widetilde{u}_i$, $1 \leqslant i \leqslant d$. It follows from Lemma 3.9 that $A_{k,\varphi_w}$ is of the form $\widetilde{A}_{k,\varphi_w}\circ b$. Thus we get:

$$\Big\langle \Big[\widetilde{x} \overset{\widetilde{T}}{\cdot} \widehat{x}\Big], h \Big\rangle = \Big\langle \Big[x \overset{b(T)}{\cdot} x\Big], \sum_{k=1}^{d} h_k \widetilde{A}_{k,\varphi_w} \Big\rangle.$$

Since $[x \overset{b(T)}{\cdot} x] = [P_\lambda]$, we obtain:

$$\Big\langle \Big[\widetilde{x} \overset{\widetilde{T}}{\cdot} \widehat{x}\Big], h \Big\rangle = \sum_{k=1}^{d} h_k(\lambda)\widetilde{A}_{k,\varphi_w}(\lambda).$$

Moreover, by definition of the function $\varphi_w$ we have:

$$\widetilde{A}_{k,\varphi_w}(\lambda) = \sum_{i=1}^{d}(v_k\varphi_w)\circ\widetilde{u}_i(w) = v_k(w).$$

So, finally, we get:

$$\Big\langle \Big[\widetilde{x} \overset{\widetilde{T}}{\cdot} \widehat{x}\Big], h \Big\rangle = \sum_{k=1}^{d} h_k(\lambda)v_k(w) = \sum_{k=1}^{d} h_k(b(w))v_k(w) = h(w),$$

that is, $\Big[\widetilde{x} \overset{\widetilde{T}}{\cdot} \widehat{x}\Big] = [P_w]$, which ends the proof of the proposition. ∎

We are now ready to conclude the proof of the main result of this section. We know from [4] that

$$\min\{|w - \widetilde{u}_i(w)| : w \in \mathbb{T},\ 1 \leqslant i \leqslant d-1\} > 0.$$

Thus, by continuity, there exists $\delta > 0$ and an annulus $\Omega$ of the form $\Omega = \{z \in \mathbb{C} : r < |z| < 1\}$ such that:

$$|w - \widetilde{u}_i(w)| \geqslant \delta \quad \text{for } w \in \Omega \text{ and } 1 \leqslant i \leqslant d-1.$$

For $w \in \Omega$, let $\lambda = b(w)$ and $\varphi_w$ the polynomial defined by:

$$\varphi_w(z) = \prod_{i=1}^{d-1} \frac{z - \widetilde{u}_i(w)}{w - \widetilde{u}_i(w)}.$$

Note that $\|\varphi_w\|_\infty \leqslant M$ where $M = (2\delta^{-1})^{d-1}$.

Since $b(T) \in \mathbb{A}_{\aleph_0}$, by Proposition 6.1 in [1], there exists a sequence of unit vectors $(x_n)_n$ in $\mathcal{H}$ such that:

$$[P_\lambda] = \Big[x_n \overset{b(T)}{\cdot} x_n\Big] \quad \text{and} \quad \lim_{n\to\infty}\Big(\Big\|\Big[w \overset{b(T)}{\cdot} x_n\Big]\Big\| + \Big\|\Big[x_n \overset{b(T)}{\cdot} w\Big]\Big\|\Big) = 0, \quad w \in \mathcal{H}.$$

It follows from Proposition 3.10 and Corollary 3.8 that if we take $\widetilde{x}_n = (\varphi_w \circ \widetilde{u}_1)(T)x_n \oplus \cdots \oplus (\varphi_w \circ \widetilde{u}_d)(T)x_n$ and $\widehat{x}_n = x_n \oplus \cdots \oplus x_n$, we obtain:

$$[P_w] = \Big[\widetilde{x}_n \overset{\widetilde{T}}{\cdot} \widehat{x}_n\Big] \quad \text{and} \quad \lim_{n\to\infty}\Big(\Big\|\Big[\widetilde{w} \overset{\widetilde{T}}{\cdot} \widehat{x}_n v\Big]\Big\| + \Big\|\Big[\widetilde{x}_n \overset{\widetilde{T}}{\cdot} \widetilde{w}\Big]\Big\|\Big) = 0, \quad \widetilde{w} \in \mathcal{H}^{(d)}.$$

Now, observing that $\|\widetilde{x}_n\| \leqslant M\sqrt{d}$ and $\|\widehat{x}_n\| \leqslant \sqrt{d}$, we see that $\frac{1}{Md}[P_w] \in \chi_0(\widetilde{T})$ for any $w \in \Omega$. Therefore, by Proposition 3.6, $\widetilde{T}$ belongs to $\mathbb{A}_{\aleph_0}$ as was to be shown.

## 4. PROOF OF THE MAIN RESULT AND ADDITIONAL MAPPING THEOREMS

It will be convenient in what follows to record some useful properties of the boundary set $X_T$ (see Proposition 3.5 and Proposition 3.9 in [12]).

PROPOSITION 4.1. *Let $T$ and $T'$ be absolutely continuous contractions in $\mathcal{L}(\mathcal{H})$. Then*

(i) $X_{T\oplus T'} = X_T \cup X_{T'}$;

(ii) *if $\phi$ is any conformal automorphism of $\mathbb{D}$ (i.e. Blaschke product of degree one), then $\phi(X_T) = X_{\phi(T)}$.*

PROPOSITION 4.2. (Proposition 3.1 in [6]) *For any non constant inner function $u$ and for any absolutely continuous contraction $T$, we have:*

$$u(X_T) \subset X_{u(T)}.$$

The main application of Theorem 3.3 is the following mapping result.

THEOREM 4.3. *Let $b$ be a finite Blaschke product of type* (AI) *and let $T$ be an absolutely continuous contraction. Then we have the following mapping result:*

$$b(X_T) = X_{b(T)}.$$

*Proof.* First, suppose that $X_{b(T)} = \mathbb{T}$, that is, suppose that $b(T) \in \mathbb{A}_{\aleph_0}$. Our aim is to prove that $b(X_T) = \mathbb{T}$. In order to prove that, let us introduce the operator $\widetilde{T}$ defined by $\widetilde{T} = \widetilde{u}_1(T) \oplus \cdots \oplus \widetilde{u}_d(T)$ where $d$ is the degree of $b$ and where $(\widetilde{u}_i)_{1\leqslant i\leqslant d}$ is the finite sequence of elliptic Möbius transformations satisfying $b \circ \widetilde{u}_i = b$ on $\mathbb{D}$ for every $i \in \{1, \dots, d\}$. Using Proposition 4.1, we get

$$X_{\widetilde{T}} = \widetilde{u}_1(X_T) \cup \cdots \cup \widetilde{u}_d(X_T),$$

and thus we obtain:

$$b(X_{\widetilde{T}}) = b \circ \widetilde{u}_1(X_T) \cup \cdots \cup b \circ \widetilde{u}_d(X_T) = b(X_T).$$

Since $b(T) \in \mathbb{A}_{\aleph_0}$, by Theorem 3.3, we get $\widetilde{T} \in \mathbb{A}_{\aleph_0}$. Therefore we have $X_{\widetilde{T}} = \mathbb{T}$ and $b(X_{\widetilde{T}}) = b(X_T) = \mathbb{T}$.

Now suppose that $X_{b(T)} \neq \mathbb{T}$. If $\gamma$ is any Borel subset of $\mathbb{T}$ we denote by $M_\gamma$ the absolutely continuous unitary operator on $L^2(\gamma)$, defined by:

$$(M_\gamma x)(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}t} x(\mathrm{e}^{\mathrm{i}t}), \quad x \in L^2(\gamma),\ \mathrm{e}^{\mathrm{i}t} \in \mathbb{T},$$

and by $\widetilde{M_\gamma}$ the direct sum of $\aleph_0$ copies of $M_\gamma$ acting on the Hilbert space $\widetilde{L}^2(\Gamma) = \bigoplus\limits_{n\in\mathbb{N}} L^2(\Gamma)$. Note that $X_{\widetilde{M_\gamma}} = \gamma$. Take $\sigma = \mathbb{T} \setminus X_{b(T)}$ and consider the operator $\widehat{T} = T \oplus \widetilde{M_\gamma}$ where $\gamma$ is a Borel subset of $\mathbb{T}$ such that $b(\gamma) = \sigma$. It follows that

$$b(\widehat{T}) = b(T) \oplus \widetilde{M_\sigma},$$

and thus $X_{b(\widehat{T})} = X_{b(T)} \cup \sigma = \mathbb{T}$ by Proposition 4.1. Previously we have seen that necessarily $b(X_{\widehat{T}}) = \mathbb{T}$. Since we have:

$$X_{b(T)} \cup \sigma = \mathbb{T} = b(X_T \cup \gamma) = b(X_T) \cup \sigma,$$

the fact that $b(X_T) \subset X_{b(T)}$ (cf. Proposition 4.2) implies that $X_{b(T)} = b(X_T)$. ∎

The second assertion of the next corollary has already been proved in [12] but we would like to emphasize the fact that it is now an immediate consequence of Theorem 4.3.

Corollary 4.4. *Let $b$ be a finite Blaschke product satisfying at least one of the following conditions:*

(i) *the degree of $b$ is less than or equal to 2, or*

(ii) *the set of the zeros of $b$, $Z(b)$, is reduced to a single point of $\mathbb{D}$.*

*Then we have $X_{b(T)} = b(X_T)$ for any absolutely continuous contraction $T$.*

*Proof.* By Theorem 4.3, the only thing to do is to check that the finite Blaschke products satisfying one of the above conditions are in fact of type (AI). If $b$ is of degree one, it is clear since the identity map $\mathrm{Id} : z \mapsto z$ is always an analytic invariant for every Blaschke product. When $b$ is of degree two, that is, if $b$ is of the form $b(z) = \mathrm{e}^{\mathrm{i}\theta} \frac{z-\alpha}{1-\overline{\alpha}z} \frac{z-\beta}{1-\overline{\beta}z}$ with $\theta \in [0, 2\pi)$ and $\alpha, \beta$ in $\mathbb{D}$, then $b$ is also of type (AI) since the group $G$ of the continuous invariants of $b$ on $\mathbb{T}$ is generated by the involution $z \mapsto \frac{\gamma - z}{1-\overline{\gamma}z}$ where $\gamma$ is the solution of the equation

$$\gamma + \overline{\gamma}(\alpha\beta) = \alpha + \beta.$$

Finally, if $b$ is of the form $b(z) = \mathrm{e}^{\mathrm{i}\theta}\left(\frac{\alpha-z}{1-\overline{\alpha}z}\right)^n, n \geqslant 1$, with $\theta \in [0, 2\pi)$ and $\alpha \in \mathbb{D}$, then $b$ is of type (AI) since the group $G$ of the continuous invariants of $b$ on $\mathbb{T}$ is generated by the function $\varphi_\alpha \circ \xi_n \circ \varphi_\alpha$ where $\varphi_\alpha(z) = \frac{\alpha - z}{1-\overline{\alpha}z}$ and $\xi_n(z) = \mathrm{e}^{2\mathrm{i}\pi/n} z$. ∎

The next corollary is an obvious consequence of Theorem 4.3 and generalizes Corollary 4.4.

Corollary 4.5. *Let $b$ be a Blaschke product for which there exists a finite sequence $(b_k)_{1 \leqslant k \leqslant n}$ of finite Blaschke products of type* (AI) *such that*

$$b = b_1 \circ \cdots \circ b_n.$$

*Then we have $X_{b(T)} = b(X_T)$ for any absolutely continuous contraction $T$.*

*Proof.* Applying Theorem 4.3 $n$ times we obtain:

$$\begin{aligned} X_{b(T)} = X_{b_1 \circ \cdots \circ b_n(T)} &= b_1(X_{b_2 \circ \cdots \circ b_n(T)}) \\ &= b_1 \circ b_2(X_{b_3 \circ \cdots \circ b_n(T)}) \\ &\ \ \vdots \\ &= b_1 \circ \cdots \circ b_n(X_T) = b(X_T). \end{aligned}$$ ∎

Remark 4.6. Blaschke products for which there exists a finite sequence $(b_k)_{1 \leqslant k \leqslant n}$ of finite Blaschke products of type (AI) such that $b = b_1 \circ \cdots \circ b_n$ are not necessarily of type (AI). Indeed, take for example $b(z) = z^2\left(\frac{z-\alpha}{1-\overline{\alpha}z}\right)^2$ where $\alpha \in \mathbb{D} \setminus \{0\}$. Clearly we have $b = b_1 \circ b_2$ with $b_1(z) = z^2$ and $b_2(z) = z\frac{z-\alpha}{1-\overline{\alpha}z}$. Since the degree of $b_1$ and $b_2$ is two, $b_1$ and $b_2$ are of type (AI) but the only analytic invariants of $b$ are the identity map and the involution $z \mapsto \frac{\alpha - z}{1-\overline{\alpha}z}$, and thus $b$ is not of type (AI).

We recall that if $T$ is an arbitrary absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and if $\sigma$ is a Borel subset of $\mathbb{T}$, then we say that $\sigma$ is essential for $T$ and we write $\sigma \subset \mathrm{ess}(T)$ (cf. Definition 3.1 in [11]) if:

$$\|f(T)\| \geqslant \|f_{|\sigma}\|_{\infty}, \quad f \in H^{\infty}(\mathbb{T}).$$

We will denote by $\mathrm{Ess}(T)$ the maximal essential Borel subset for $T$ (see Proposition 3.3 of [11]). Using this terminology we have $T \in \mathbb{A}$ if and only if $\mathrm{Ess}(T) = \mathbb{T}$. Also denote by $\Sigma_T$ (respectively $\Sigma_{*T}$) the support of the spectral measure of the unitary part $R$ (respectively $R_*$) of the minimal isometric dilation (respectively minimal coisometric extension) of $T$. The link between all the boundary sets $\mathrm{Ess}(T), \Sigma_T, \Sigma_{*T}$ and $X_T$ is given by Corollary 4.4 in [12]:

$$\mathrm{Ess}(T) = X_T \cup \Sigma_T \cup \Sigma_{*T}.$$

In [12] the authors introduced the boundary set $E_T^{\mathrm{l}}$ equal to $X_T \cup \Sigma_T$ (respectively $E_T^{\mathrm{r}}$ equal to $X_T \cup \Sigma_{*T}$) whose equality to the unit circle is equivalent to the membership of $T$ in the class $\mathbb{A}_{\aleph_0,1}$ (respectively $\mathbb{A}_{1,\aleph_0}$). Since, by Corollary 3.1 in [6], we have $u(\Sigma_T) = \Sigma_{u(T)}$ and $u(\Sigma_{*T}) = \Sigma_{*u(T)}$ for every inner functions $u$, we obtain immediately the following corollary. Note that the last two equivalences show that the sufficient conditions for membership of $b(T)$ in $\mathbb{A}_{\aleph_0,1}$ or $\mathbb{A}_{1,\aleph_0}$ given in [6], Corollary 3.2, are indeed necessary.

COROLLARY 4.7. *Let $b$ be a Blaschke product for which there exists a finite sequence $(b_k)_{1 \leqslant k \leqslant n}$ of finite Blaschke products of type* (AI) *such that $b = b_1 \circ \cdots \circ b_n$. Then we have:*

$$\mathrm{Ess}(b(T)) = b(\mathrm{Ess}(T)), \quad E^{\mathrm{l}}_{b(T)} = b(E^{\mathrm{l}}_T)] \quad \textit{and} \quad E^{\mathrm{r}}_{b(T)} = b(E^{\mathrm{r}}_T)$$

*for any absolutely continuous contraction $T$. In particular we get the following equivalences:*

$$\begin{cases} b(T) \in \mathbb{A} \Leftrightarrow b(X_T) \cup b(\Sigma_T) \cup b(\Sigma_{*T}) = \mathbb{T} \\ b(T) \in \mathbb{A}_{\aleph_0,1} \Leftrightarrow b(X_T) \cup b(\Sigma_T) = \mathbb{T} \\ b(T) \in \mathbb{A}_{1,\aleph_0} \Leftrightarrow b(X_T) \cup b(\Sigma_{*T}) = \mathbb{T}. \end{cases}$$

REMARK 4.8. Let $b$ be an arbitrary Blaschke product of finite degree; its invariants have analytic extension $\widetilde{u}_i$ in the annulus $C_M = \left\{ z \in \mathbb{C} : M < |z| < \frac{1}{M} \right\}$ (where $M = \max\{|\alpha| : \alpha \in b(\alpha)\}$) which still satisfy $b \circ \widetilde{u}_i = b$ in $C_M$ (cf. beginning of Section 3. Suppose that

$$\sigma(T) \subset \{z \in \mathbb{C} : M < |z| \leqslant 1\}. \tag{4.1}$$

The operators $\widetilde{u}_i(T)$ are well-defined (for instance via the Riesz-Dunford-Schwarz functional calculus). Provided they are absolutely continuous contractions, the operator $\widetilde{T} = \widetilde{u}_1(T) \oplus \cdots \oplus \widetilde{u}_d(T)$ is also an absolutely continuous contraction and the previous argument goes through. In fact, we will show shortly that under the above spectral inclusion the absolute continuity is automatic. Therefore we have also

$$b(X_T) = X_{b(T)}$$

when (4.1) holds and $\|\widetilde{u}_i(T)\| \leqslant 1$, $1 \leqslant i \leqslant d$.

To prove the absolute continuity we note that under (4.1), with $\Omega = \{z \in \mathbb{C} : M < |z| < 1\}$, we have an $H^\infty(\Omega)$ w*-continuous functional calculus $h \mapsto h(T)$ extending the Sz.-Nagy–Foias functional calculus of $T$ and the rational functional one (this is standard and can be proved, for example, along the lines of Section 7 in [13]). The representation $h \mapsto h \circ \widetilde{u}_i$, $h \in H^\infty(\mathbb{D})$, extends the polynomial functional calculus associated with $\widetilde{u}_i(T)$ and is clearly w*-continuous. Hence, when $\|\widetilde{u}_i(T)\| \leqslant 1$, this representation is the Sz.-Nagy–Foias functional calculus for $\widetilde{u}_i(T)$ which is therefore absolutely continuous.

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*G. CASSIER*
*Institut Girard Desargues*
*UFR de Mathématiques*
*Université Claude Bernard Lyon 1*
*69622 Villeurbanne Cedex*
*FRANCE*
*E-mail:* cassier@igd.univ-lyon1.fr

*I. CHALENDAR*
*Institut Girard Desargues*
*UFR de Mathématiques*
*Université Claude Bernard Lyon 1*
*69622 Villeurbanne Cedex*
*FRANCE*
*E-mail:* chalenda@igd.univ-lyon1.fr

*B. CHEVREAU*
*Université Bordeaux I*
*351 Cours de la Libération*
*33405 Talence Cedex*
*FRANCE*
*E-mail:* chevreau@math.u-bordeaux.fr

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