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A UNIFIED APPROACH TO EXEL-LACA ALGEBRAS AND C^*-ALGEBRAS ASSOCIATED TO GRAPHS

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Abstract. We define an ultragraph, which is a generalization of a directed graph, and describe how to associate a C^*-algebra to it. We show that the class of ultragraph algebras contains the C^*-algebras of graphs as well as the Exel-Laca algebras. We also show that many of the techniques used for graph algebras can be applied to ultragraph algebras and that the ultragraph provides a useful tool for analyzing Exel-Laca algebras. Our results include versions of the Cuntz-Krieger Uniqueness Theorem and the Gauge-Invariant Uniqueness Theorem for ultragraph algebras.

Keywords: *Ultragraph, graph C^*-algebra, Exel-Laca algebra, Cuntz-Krieger algebra, desingularization.*

MSC (2000): 46L55.

1. INTRODUCTION

In the early 1980's Cuntz and Krieger considered a class of C^*-algebras that arose in the study of topological Markov chains ([3]). These Cuntz-Krieger algebras $\mathcal{O}_A$ are generated by partial isometries whose relations are determined by a finite matrix A with entries in $\{0, 1\}$. In order for their C^*-algebras to be unique, Cuntz and Krieger assumed that the matrix A also satisfied a nondegeneracy condition called Condition (I). Since their introduction Cuntz-Krieger algebras have been generalized in a myriad of ways. Two important generalizations are the Exel-Laca algebras of [7] and the C^*-algebras of directed graphs [12], [13], [1], [9].

The Exel-Laca algebras are in some sense the most direct generalization of Cuntz-Krieger algebras. In 1999 Exel and Laca extended the definition of $\mathcal{O}_A$ to allow for infinite matrices ([7]). Furthermore, the only restriction placed on these matrices was that they had no zero rows. Motivated by the C^*-algebras associated to graphs, Exel and Laca avoided the need for Condition (I) by instead requiring the generating partial isometries to be universal for their defining relations.

Since Condition (I) was imposed by Cuntz and Krieger to insure uniqueness, this universal definition agreed with Cuntz and Krieger's for finite matrices satisfying Condition (I).

The generalization of Cuntz-Krieger algebras to C^*-algebras of directed graphs is slightly less direct. In 1982 Watatani noted that one could view $\mathcal{O}_A$ as the C^*-algebra of a finite directed graph with vertex adjacency matrix A ([19]). The fact that A satisfied Condition (I) implied, among other things, that this graph had no sinks or sources. It was to be approximately 15 years, however, before these graph ideas were explored more fully. In the late 1990's generalizations of these C^*-algebras were considered for possibly infinite graphs that were allowed to contain sinks and sources. Originally, a definition was given only for graphs that are row-finite; that is, each vertex is the source of only finitely many edges ([12], [13], [1]). However, in 2000 this definition was extended to obtain an appropriate notion of C^*-algebras for arbitrary graphs ([9]).

The relationship between Exel-Laca algebras and graph algebras is somewhat subtle. As mentioned before, the class of Exel-Laca algebras of finite matrices satisfying Condition (I) and the class of C^*-algebras of certain finite graphs both coincide with the Cuntz-Krieger algebras. However, the infinite case is more complicated.

Let G be a graph with no sinks or sources. The edge matrix A for the graph G is the matrix indexed by the edges of G with $A(e,f) = 1$ if $r(e) = s(f)$, and $A(e,f) = 0$ otherwise. It was shown in [9] that $C^*(G)$ is canonically isomorphic to $\mathcal{O}_A$. Thus C^*-algebras of graphs without sinks or sources are Exel-Laca algebras.

Unfortunately, the reverse inclusion is not true. It was shown in [15] that there exist Exel-Laca algebras that are not graph algebras. However, one can obtain a partial converse in the row-finite case. If A is a matrix, then one may form a graph $\mathrm{Gr}(A)$ by defining the vertex set of $\mathrm{Gr}(A)$ to be the index set of A, and defining the number of edges from v to w to be $A(v,w)$. If A is a $\{0,1\}$-matrix that is row-finite (i.e., each row of A is eventually zero), then $\mathrm{Gr}(A)$ is a row-finite graph and $\mathcal{O}_A \cong C^*(\mathrm{Gr}(A))$. This isomorphism is obtained through the use of the dual graph of $\mathrm{Gr}(A)$ and it is not canonical. It was shown in [15] that $C^*(\mathrm{Gr}(A))$ is always a C^*-subalgebra of $\mathcal{O}_A$, but when A is not row-finite this subalgebra may be very different from $\mathcal{O}_A$ (see Remark 15 in [17] and [4]).

These relationships are summarized in the following diagram:

$$\begin{array}{ccccccccc} \mathrm{CK} & \subset & \mathrm{F}' & \subset & \mathrm{RF}' & \subset & \mathrm{G}' & \subset & \mathrm{EL} \\ & & \cap & & \cap & & \cap & & \\ & & \mathrm{F} & \subset & \mathrm{RF} & \subset & \mathrm{G} & & \end{array}$$

$$
\begin{aligned}
\mathrm{CK} &= \text{Cuntz-Krieger algebras } \mathcal{O}_A \text{ with } A \text{ satisfying Condition (I)}\\
\mathrm{F}' &= C^*\text{-algebras of finite graphs with no sinks or sources}\\
\mathrm{RF}' &= C^*\text{-algebras of row-finite graphs with no sinks or sources}\\
&= \text{Exel-Laca algebras of row-finite matrices with no zero rows or columns}\\
\mathrm{G}' &= C^*\text{-algebras of graphs with no sinks or sources}\\
\mathrm{EL} &= \text{Exel-Laca algebras}\\
\mathrm{F} &= C^*\text{-algebras of finite graphs}\\
\mathrm{RF} &= C^*\text{-algebras of row-finite graphs}\\
\mathrm{G} &= C^*\text{-algebras of graphs.}
\end{aligned}
$$

In some sense, it is unfortunate that there are Exel-Laca algebras that are not graph algebras. The graph G is an extremely useful tool for analyzing $C^*(G)$, and many results take a more elegant form when stated in terms of graphs rather than matrices. In fact, when studying the $\mathcal{O}_A$'s Exel and Laca found it convenient to state many of their hypotheses and results in terms of the graph $\mathrm{Gr}(A)$.

In this paper we describe a generalized notion of graph, which we call an *ultragraph*, and describe a way to associate a C^*-algebra to it. We shall see that the C^*-algebras of ultragraphs with no sinks and in which every vertex emits finitely many edges are precisely the Exel-Laca algebras. Thus the ultragraph algebras fit into our diagram as follows:

$$
\begin{array}{ccccccccc}
\mathrm{CK} & \subset & \mathrm{F}' & \subset & \mathrm{RF}' & \subset & \mathrm{G}' & \subset & \mathrm{EL} = \widetilde{\mathrm{U}}\\
 & & \cap & & \cap & & \cap & & \cap\\
 & & \mathrm{F} & \subset & \mathrm{RF} & \subset & \mathrm{G} & \subset & \mathrm{U}
\end{array}
$$

$$
\begin{aligned}
\widetilde{\mathrm{U}} &= C^*\text{-algebras of ultragraphs with no sinks and in which every vertex emits}\\
&\qquad \text{which every vertex emits finitely many edges.}\\
\mathrm{U} &= C^*\text{-algebras of ultragraphs.}
\end{aligned}
$$

Therefore ultragraph algebras give an alternative (and in the author's opinion, more convenient) way to view Exel-Laca algebras. In addition, they provide a reasonable notion of "Exel-Laca algebras with sinks". We shall see that for a countably indexed matrix A, one may create an ultragraph $\mathcal{G}_A$ with edge matrix A and for which $C^*(\mathcal{G}_A)$ is canonically isomorphic to $\mathcal{O}_A$. Furthermore, the ultragraph $\mathcal{G}$ provides a useful tool for analyzing the structure of $C^*(\mathcal{G})$, just as the graph does for graph algebras.

We shall prove in Section 4 that the C^*-algebras of ultragraphs with no sinks and no infinite emitters are precisely the Exel-Laca algebras. However, when deducing results about ultragraph algebras we will almost always prove them from first principals rather than simply translating the corresponding result from $\mathcal{O}_A$ to the ultragraph setting. This keeps our treatment more self-contained, and more importantly it shows that ultragraph algebras are tractable objects of study. In addition, since many of our techniques generalize those used for graph algebras, this approach shows that by viewing Exel-Laca algebras as ultragraph algebras one can forget about matrix techniques entirely and instead apply the (often more straightforward) graph techniques.

This paper is organized as follows. In Section 2 we define an ultragraph $\mathcal{G}$ and describe a way to associate a C^*-algebra $C^*(\mathcal{G})$ to it. In Section 3 we discuss a natural way to view graph algebras as ultragraph algebras and discuss loops in ultragraphs. In Section 4 we show that the C^*-algebras of ultragraphs with no singular vertices are precisely the Exel-Laca algebras. In Section 5 we describe a method for realizing certain subalgebras of $C^*(\mathcal{G})$ as graph algebras. In Section 6 we discuss methods to remove singular vertices from ultragraphs by adding tails, and we obtain versions of the Gauge-Invariant Uniqueness Theorem and the Cuntz-Krieger Uniqueness Theorem for ultragraph algebras.

2. Ultragraphs and their C^*-algebras

To provide motivation for the definition of an ultragraph, recall that when G is a graph with no sinks or sourcesand with edge matrix A, then $C^*(G)$ is canonically isomorphic to $\mathcal{O}_A$. Thus, in some sense, what is preventing all Exel-Laca algebras from being graph algebras is the fact that not all matrices arise as the edge matrix of a graph. For example, the finite matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is not the edge matrix of any graph. This is because any such graph would have to have two edges, e_1 and e_2, and the relations coming from the matrix would imply that $r(e_1) = s(e_1)$ and $r(e_1) = s(e_2)$, but $s(e_1) \neq s(e_2)$.

The way that we will overcome this problem is to allow the range of each edge to be a set of vertices, rather than just a single vertex. For example, if we let v_1 and v_2 be two vertices, e_1 and e_2 be two edges, and define $s(e_1) = v_1$, $s(e_2) = v_2$, $r(e_1) = \{v_1, v_2\}$, and $r(e_2) = \{v_1\}$, then we see that the "edge matrix" of such an object would be $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ because the edge e_1 may be followed by either e_1 or e_2, and the edge e_2 may only be followed by e_1. Thus by allowing the edges to have a set of vertices as their range, we can view the matrix as an edge matrix.

Recall that a *graph* $G = (G^0, G^1, r, s)$ consists of a countable set of vertices G^0, a countable set of edges G^1, and maps $r, s : G^1 \to G^0$ identifying the range and source of each edge. For a set X let $\mathcal{P}(X)$ denote the collection of all subsets of X and let $P(X)$ denote the collection of all nonempty subsets of X.

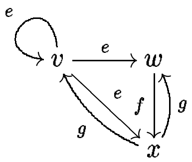
Definition 2.1. An *ultragraph* $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of a countable set of vertices G^0, a countable set of edges $\mathcal{G}^1$, and functions $s : \mathcal{G}^1 \to G^0$ and $r : \mathcal{G}^1 \to P(G^0)$.

Remark 2.2. Note that an ultragraph is a more general object than a graph. A graph may be viewed as a special type of ultragraph in which $r(e)$ is a singleton set for each edge e.

Example 2.3. A convenient way to draw ultragraphs is to first draw the set G^0 of vertices, and then for each edge $e \in \mathcal{G}^1$ draw an arrow labeled e from $s(e)$ to each vertex in $r(e)$. For instance, the ultragraph given by

$$\begin{array}{llll} G^0 = \{v, w, x\}, & s(e) = v, & s(f) = w, & s(g) = x, \\ \mathcal{G}^1 = \{e, f, g\}, & r(e) = \{v, w, x\}, & r(f) = \{x\}, & r(g) = \{v, w\} \end{array}$$

may be drawn as



We then identify any arrows with the same label, thinking of them as being a single edge. Thus in the above example there are only three edges, e, f, and g, despite the fact that there are six arrows drawn.

DEFINITION 2.4. If $\mathcal{G}$ is an ultragraph, the *edge matrix* of $\mathcal{G}$ is the $\mathcal{G}^1 \times \mathcal{G}^1$ matrix $A_{\mathcal{G}}$ given by $A_{\mathcal{G}}(e,f) = \begin{cases} 1 & \text{if } s(f) \in r(e), \\ 0 & \text{otherwise.} \end{cases}$

Although not every $\{0,1\}$-matrix is the edge matrix of a graph, every $\{0,1\}$-matrix is the edge matrix of an ultragraph.

DEFINITION 2.5. If I is a countable set and A is an $I \times I$ matrix with entries in $\{0,1\}$, then we may form the ultragraph $\mathcal{G}_A := (G_A^0, \mathcal{G}_A^1, r, s)$ defined by $G_A^0 := \{v_i : i \in I\}$, $\mathcal{G}_A^1 := I$, $s(i) = v_i$ for all $i \in I$, and $r(i) = \{v_j : A_{\mathcal{G}}(i,j) = 1\}$.

Note that the edge matrix of $\mathcal{G}_A$ is A. Also note that for each $v_i \in G_A^0$ there is exactly one edge i with source v_i.

If $\mathcal{G}$ is an ultragraph, then a vertex $v \in G^0$ is called a *sink* if $|s^{-1}(v)| = 0$ and an *infinite emitter* if $|s^{-1}(v)| = \infty$. We call a vertex a *singular vertex* if it is either a sink or an infinite emitter.

REMARK 2.6. If G is a graph, then G is said to be row-finite if it has no infinite emitters. This terminology comes from the fact that the edge matrix A_G is row-finite; that is, the rows of A_G are eventually zero. However, this is not the case for ultragraphs: If $\mathcal{G}$ has no infinite emitters, then it is not necessarily true that $A_{\mathcal{G}}$ is row-finite. In fact, for any matrix A we see that the ultragraph $\mathcal{G}_A$ will always have no infinite emitters. Thus we refrain from using the term row-finite when speaking of ultragraphs. Instead we shall always say that an ultragraph has no infinite emitters or (in the case that there are also no sinks) that the ultragraph has no singular vertices.

For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ we let $\mathcal{G}^0$ denote the smallest subcollection of $\mathcal{P}(G^0)$ that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite intersections and finite unions. Roughly speaking, the elements of $\{v : v \in G^0\} \cup \{r(e) : e \in \mathcal{G}^1\}$ play the role of "generalized vertices" and $\mathcal{G}^0$ plays the role of "subsets of generalized vertices".

DEFINITION 2.7. If $\mathcal{G}$ is an ultragraph, a *Cuntz-Krieger $\mathcal{G}$-family* is a collection of partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal ranges and a collection of projections $\{p_A : A \in \mathcal{G}^0\}$ that satisfy

(i) $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$, and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ for all $A, B \in \mathcal{G}^0$;
(ii) $s_e^* s_e = p_{r(e)}$ for all $e \in \mathcal{G}^1$;
(iii) $s_e s_e^* \leqslant p_{s(e)}$ for all $e \in \mathcal{G}^1$;

(iv) $p_v = \sum\limits_{s(e)=v} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$.

When A is a singleton set $\{v\}$, we write p_v in place of $p_{\{v\}}$.

For $n \geqslant 2$ we define $\mathcal{G}^n := \{\alpha = \alpha_1 \cdots \alpha_n : \alpha_i \in \mathcal{G}^1 \text{ and } s(\alpha_{i+1}) \in r(\alpha_i)\}$ and $\mathcal{G}^* := \bigcup\limits_{n=0}^{\infty} \mathcal{G}^n$. The map r extends naturally to $\mathcal{G}^*$, and we say that α has length $|\alpha| = n$ when $\alpha \in \mathcal{G}^n$. Note that the paths of length zero are the elements of $\mathcal{G}^0$, and when $A \in \mathcal{G}^0$ we define $s(A) = r(A) = A$.

LEMMA 2.8. *If $A \in \mathcal{G}^0$ and $e \in \mathcal{G}^1$, then*

$$p_A s_e = \begin{cases} s_e & \text{if } s(e) \in A, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad s_e^* p_A = \begin{cases} s_e^* & \text{if } s(e) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$p_A s_e = p_A s_e s_e^* s_e = p_A = p_A p_{s(e)} s_e s_e^* s_e = p_{A \cap s(e)} s_e = \begin{cases} s_e & \text{if } s(e) \in A, \\ 0 & \text{otherwise,} \end{cases}$$

and the second claim follows by taking adjoints. ∎

For a path $\alpha := \alpha_1 \cdots \alpha_n \in \mathcal{G}^*$ we define s_α to be $s_{\alpha_1} \cdots s_{\alpha_n}$ if $|\alpha| \geqslant 1$ and p_A if $\alpha = A \in \mathcal{G}^0$.

LEMMA 2.9. *Let $\{s_e, p_A\}$ be a Cuntz-Krieger $\mathcal{G}$-family, and let $\beta, \gamma \in \mathcal{G}^*$ with $|\beta|, |\gamma| \geqslant 1$. Then*

$$s_\beta^* s_\gamma = \begin{cases} s_{\gamma'} & \text{if } \gamma = \beta\gamma',\ \gamma' \notin \mathcal{G}^0, \\ p_{r(\gamma)} & \text{if } \gamma = \beta, \\ s_{\beta'}^* & \text{if } \beta = \gamma\beta',\ \beta' \notin \mathcal{G}^0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $e, f \in \mathcal{G}^1$ we have $s_e^* s_f = 0$ unless $e = f$, so

$$s_e^* s_\gamma = \delta_{e,\gamma_1} s_{\gamma_1}^* s_{\gamma_1} \cdots s_{\gamma_{|\gamma|}} = \delta_{e,\gamma_1} p_{r(\gamma_1)} s_{\gamma_2} \cdots s_{\gamma_{|\gamma|}}.$$

Because $s(\gamma_2) \in r(\gamma_1)$, this gives $s_e^* s_\gamma = \delta_{e,\gamma_1} s_{\gamma_2} \cdots s_{\gamma_{|\gamma|}}$. Repeated calculations of this form show that $s_\beta s_\gamma = 0$ unless either γ extends β or β extends γ. Suppose for the sake of argument that $\gamma = \beta\gamma'$ extends β. Then calculations as above show that $s_\beta^* s_\beta s_{\gamma'} = s_{\beta_{|\beta|}}^* s_{\beta_{|\beta|}} s_{\gamma'} = p_{r(\beta)} s_{\gamma'} = s_{\gamma'}$. ∎

REMARK 2.10. We see from Lemma 2.8 and Lemma 2.9 that any word in s_e, p_A, and s_f^* may be written in the form $s_\alpha p_A s_\beta^*$ for some $A \in \mathcal{G}^0$ and some $\alpha, \beta \in \mathcal{G}^*$ with$r(\alpha) \cap r(\beta) \cap A \neq \emptyset$.

THEOREM 2.11. *Let $\mathcal{G}$ be an ultragraph. There exists a C^*-algebra B generated by a universal Cuntz-Krieger $\mathcal{G}$-family $\{s_e, p_A\}$. Furthermore, the s_e's are nonzero and every p_A with $A \neq \emptyset$ is nonzero.*

Proof. We only give an outline here as the argument closely follows that of Theorem 2.1 from [12] and Theorem 2.1 from [10]. Let $S_\mathcal{G} := \{(\alpha, A, \beta) : \alpha, \beta \in \mathcal{G}^*,\ A \in \mathcal{G}^0, \text{ and } r(\alpha) \cap r(\beta) \cap A \neq \emptyset\}$ and let $k_\mathcal{G}$ be the space of functions of finite support on $S_{\mathcal{G}}$. The set of point masses $\{e_\lambda : \lambda \in S_{\mathcal{G}}\}$ forms a basis for $k_{\mathcal{G}}$. By thinking of $e_{(\alpha,A,\beta)}$ as $s_\alpha p_A s_\beta^*$ we may use Lemma 2.8, Lemma 2.9, and the relation $p_A p_B = p_{A\cap B}$ to define an associative multiplication and involution on $k_{\mathcal{G}}$ such that $k_{\mathcal{G}}$ is a $*$-algebra.

As a $*$-algebra $k_{\mathcal{G}}$ is generated by $q_A := e_{(A,A,A)}$ and $t_e := e_{(e,r(e),r(e))}$. From the way we have defined multiplication, the elements q_A have the property that $q_A q_B = e_{(A,A,A)} e_{(B,B,B)} = e_{(A\cap B, A\cap B, A\cap B)} = q_{A\cap B}$ and $q_v \geqslant t_e t_e^*$ for all $e \in \mathcal{G}^1$ with $s(e) = v$. Let us mod out by the ideal J generated by the elements $q_v - \sum_{e:s(e)=v} t_e t_e^*$ for all v with $0 < |s^{-1}(v)| < \infty$ and the elements $q_A + q_B - q_{A\cap B} - q_{A\cup B}$ for all $A, B \in \mathcal{G}^0$. Then the images r_A of q_A and u_e of t_e in $k_{\mathcal{G}}/J$ form a Cuntz-Krieger $\mathcal{G}$-family that generates $k_{\mathcal{G}}/J$. The triple $(k_{\mathcal{G}}, r_A, u_e)$ has the required universal property, though $k_{\mathcal{G}}/J$ is not a C^*-algebra. A standard argument shows that

$$\|a\|_0 := \sup \{ \; \|\pi(a)\| : \pi \text{ is a nondegenerate } *\text{-representation of } k_{\mathcal{G}}/J\}$$

is a well-defined, bounded seminorm on $k_{\mathcal{G}}/J$. The completion B of

$$(k_{\mathcal{G}}/J)/\{b \in k_{\mathcal{G}}/J : \|b\|_0 = 0\}$$

is a C^*-algebra with the same representation theory as $k_{\mathcal{G}}/J$. Thus if p_A and s_e are the images of r_A and u_e in B, then (B, s_e, p_A) has all the required properties.

Now for each $e \in \mathcal{G}^1$ let $\mathcal{H}_e$ be an infinite-dimensional Hilbert space. Also for each $v \in \mathcal{G}^0$ let $\mathcal{H}_v := \bigoplus_{s(e)=v} \mathcal{H}_e$ if v is not a sink, and let $\mathcal{H}_v$ be an infinite-dimensional Hilbert space if v is a sink. Let $\mathcal{H} := \bigoplus_{v\in G^0} \mathcal{H}_v$ and for each $e \in \mathcal{G}^1$ let S_e be the partial isometry with initial space $\bigoplus_{v\in r(e)} \mathcal{H}_v$ and final space $\mathcal{H}_e$. Finally, for $A \in \mathcal{G}^0$ define P_A to be the projection onto $\bigoplus_{v\in A} \mathcal{H}_v$, where this is interpreted as the zero projection when $A = \emptyset$. Then $\{S_e, P_A\}$ is a Cuntz-Krieger $\mathcal{G}$-family. By the universal property there exists a homomorphism $h : B \to C^*(\{S_e, P_A\})$. Since the S_e's and P_A's are nonzero, it follows that the s_e's and p_A's are also nonzero. ∎

The triple (B, s_e, p_A) is unique up to isomorphism and we write $C^*(\mathcal{G})$ for B. From Remark 2.10 we see that $C^*(\mathcal{G}) = \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{G}^* \text{ and } A \in \mathcal{G}^0\}$. The following lemma allows us to say slightly more.

LEMMA 2.12. *If $\mathcal{G} := (G^0, \mathcal{G}^1, r, s)$ is an ultragraph, then*

$$\mathcal{G}^0 = \Big\{ \bigcap_{e\in X_1} r(e) \cup \cdots \cup \bigcap_{e\in X_n} r(e) \cup F : X_1, \ldots, X_n \textit{ are finite subsets of } \mathcal{G}^1 \textit{ and } F \textit{ is a finite subset of } G^0 \Big\}.$$

Furthermore, F may be chosen to be disjoint from $\bigcap_{e\in X_1} r(e) \cup \cdots \cup \bigcap_{e\in X_n} r(e)$.

Proof. Recall that $\mathcal{G}^0$ contains $\{v\}$ for all $v \in G^0$ and $r(e)$ for all $e \in \mathcal{G}^1$ and is closed under finite intersections and finite unions. Hence the right hand side of the above equation is contained in $\mathcal{G}^0$. To see the converse note that the right hand side contains $\{v\}$ for all $v \in G^0$ and $r(e)$ for all $e \in \mathcal{G}^1$ and is closed under finite intersections and finite unions. Furthermore, since F is a finite subset we may always choose it to be disjoint from $\bigcap_{e \in X_1} r(e) \cup \cdots \cup \bigcap_{e \in X_n} r(e)$ simply by discarding any unwanted vertices in F. ∎

Remark 2.13. This lemma combined with the comment preceding it shows that

$$C^*(\mathcal{G}) = \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{G}^* \text{ and either } A = r(e_1) \cap \cdots \cap r(e_n) \text{ for } e_1 \cdots e_n \in \mathcal{G}^1 \text{ or } A \text{ is a finite subset of } \mathcal{G}^0\}.$$

Furthermore, one can see that $C^*(\mathcal{G})$ is generated by $\{s_e : e \in \mathcal{G}^1\} \cup \{p_v : v \text{ is a singular vertex}\}$.

We conclude with a discussion of the gauge action for ultragraph algebras. If $\mathcal{G}$ is an ultragraph and $\{s_e, p_A\}$ is a Cuntz-Krieger $\mathcal{G}$-family, then for any $z \in \mathbb{T}$, the family $\{zs_e, p_A\}$ will be another Cuntz-Krieger $\mathcal{G}$-family that generates $C^*(\mathcal{G})$. Thus the universal property gives a homomorphism $\gamma_z : C^*(\mathcal{G}) \to C^*(\mathcal{G})$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_A) = p_A$. The homomorphism $\gamma_{\overline{z}}$ is an inverse for γ_z, so $\gamma_z \in \operatorname{Aut} C^*(\mathcal{G})$. Furthermore, a routine $\varepsilon/3$ argument shows that γ is a strongly continuous action of $\mathbb{T}$ on $C^*(\mathcal{G})$. We call this action the *gauge action* for $C^*(\mathcal{G})$.

3. VIEWING GRAPH ALGEBRAS AS ULTRAGRAPH ALGEBRAS

The construction of $C^*(\mathcal{G})$ generalizes the C^*-algebra $C^*(G)$ associated to a directed graph G as described in [12] for row-finite graphs and in [9] for arbitrary graphs. If $G = (G^0, G^1, r, s)$ is a directed graph, then we may view G as an ultragraph $\mathcal{G}$ in a natural way; that is, let $\mathcal{G}^1 := G^1$, define $\widetilde{r} : \mathcal{G}^1 \to P(G^0)$ by $\widetilde{r}(e) = \{r(e)\}$, and then set $\mathcal{G} := (G^0, \mathcal{G}^1, \widetilde{r}, s)$.

Proposition 3.1. *If G is a graph and $\mathcal{G}$ is the ultragraph associated to G, then $C^*(G)$ is naturally isomorphic to $C^*(\mathcal{G})$.*

Proof. Since $\widetilde{r}(e)$ is a singleton set for all $e \in \mathcal{G}^1$, we see that $\mathcal{G}^0$ equals the collection of all finite subsets of G^0. If $\{s_e, p_v\}$ is a Cuntz-Krieger G-family ([9]) in $C^*(G)$, then we may define $p_A := \sum_{v \in A} p_v$ for all $A \in \mathcal{G}^0$. (Note that this will be a finite sum.) Then $\{s_e, p_A\}$ is a Cuntz-Krieger $\mathcal{G}$-family. Conversely, if $\{t_e, q_A\}$ is a Cuntz-Krieger $\mathcal{G}$-family, then it restricts to a Cuntz-Krieger G-family $\{t_e, q_{\{v\}}\}$. The result follows by applying the universal properties. ∎

Lemma 3.2. *If $\mathcal{G}$ is an ultragraph, then $C^*(\mathcal{G})$ is unital if and only if $G^0 \in \mathcal{G}^0$, and in this case $1 = p_{G^0}$.*

Proof. Let $C^*(\mathcal{G}) = C^*(\{s_e, p_A\})$. If $G^0 \in \mathcal{G}^0$, then consider the projection p_{G^0}. For any $\alpha, \beta \in \mathcal{G}^*$ and $A \in \mathcal{G}^0$ we have $(s_\alpha p_A s_\beta^*)p_{G^0} = s_\alpha p_A s_\beta^* p_{s(\beta)} p_{G^0} = s_\alpha p_A s_\beta^* p_{s(\beta)} = s_\alpha p_A s_\beta^*$. Similarly, $p_{G^0}(s_\alpha p_A s_\beta^*) = (s_\alpha p_A s_\beta^*)$. Since $\{s_\alpha p_A s_\beta^*\}$ is dense in $C^*(\mathcal{G})$, it follows that p_{G^0} is a unit for $C^*(\mathcal{G})$.

Conversely, suppose that $C^*(\mathcal{G})$ is unital. List the elements of $\mathcal{G}^1 = \{e_1, e_2, \ldots\}$ and $G^0 = \{v_1, v_2, \ldots\}$. Note that these sets are either finite or countably infinite.

For $n \geqslant 1$ define $A_n := \bigcup_{i=1}^{n} \{v_i\} \cup \bigcup_{i=1}^{n} r(e_i)$. Then $A_n \in \mathcal{G}^0$ for all n and $A_1 \subseteq A_2 \subseteq \cdots$. Also $\{p_{A_i}\}$ is an approximate unit since for any $s_\alpha p_A s_\beta^*$ we may choose n large enough so that p_{A_n} acts as the identity on $s_\alpha p_A s_\beta^*$. Now if $A_n \subsetneqq A_m$ for some $m > n$, then there exists $v \in A_m \backslash A_n$ and $(p_{A_m} - p_{A_n})p_v = p_{A_m \cap \{v\}} - p_{A_n \cap \{v\}} = p_v$ so $p_{A_m} - p_{A_n} \geqslant p_v$ and $\|p_{A_m} - p_{A_n}\| = 1$. Since $C^*(\mathcal{G})$ is unital, we must have $p_{A_n} \to 1$ in norm. But the only way that this could happen is if p_{A_n} is eventually constant. Thus $p_{A_k} = 1$ for some k. Furthermore, A_k must be all of G^0 for if there were $v \in G^0 \setminus A_k$, then $p_{A_k} p_v = 0$ contradicting the fact that $p_{A_k} = 1$. Hence $A_k = G^0$ and since $A_k \in \mathcal{G}^0$ we are done. ∎

REMARK 3.3. Note that when a graph G is viewed as an ultragraph as in Proposition 3.1, then the above lemma produces the familiar result that $C^*(G)$ is unital if and only if G has a finite number of vertices.

Recall that a loop in a graph G is a path $\alpha \in G^*$ with $|\alpha| \geqslant 1$ and $s(\alpha) = r(\alpha)$. If $\alpha = \alpha_1 \cdots \alpha_n$ is a loop, then an exit for α is defined to be an edge $e \in G^1$ with $s(e) = s(\alpha_i)$ for some $1 \leqslant i \leqslant n$ but $e \neq \alpha_i$. A graph G is said to satisfy Condition (L) if all loops have exits. Roughly speaking, an exit for a loop $\alpha := \alpha_1 \cdots \alpha_n$ is something that allows you to "get out" of the loop α; that is, it allows you to follow a path other than $\alpha_1 \cdots \alpha_n \alpha_1 \cdots \alpha_n \alpha_1 \cdots$.

DEFINITION 3.4. If $\mathcal{G}$ is an ultragraph, then a *loop* is a path $\alpha \in \mathcal{G}^*$ with $|\alpha| \geqslant 1$ and $s(\alpha) \in r(\alpha)$. An *exit* for a loop is either of the following:

(i) an edge $e \in \mathcal{G}^1$ such that there exists an i for which $s(e) \in r(\alpha_i)$ but $e \neq \alpha_{i+1}$;

(ii) a sink w such that $w \in r(\alpha_i)$ for some i.

We now extend Condition (L) to ultragraphs.

CONDITION (L). *Every loop in $\mathcal{G}$ has an exit; that is, for any loop $\alpha := \alpha_1 \cdots \alpha_n$ there is either an edge $e \in \mathcal{G}^1$ such that $s(e) \in r(\alpha_i)$ and $e \neq \alpha_{i+1}$ for some i, or there is a sink w with $w \in r(\alpha_i)$ for some i.*

Note that if $\alpha := \alpha_1 \cdots \alpha_n$ is a loop in $\mathcal{G}$ without an exit, then for all i we must have $r(\alpha_i) = \{s(\alpha_{i+1})\}$ and $s^{-1}(s(\alpha_i)) = \{\alpha_i\}$.

4. VIEWING EXEL-LACA ALGEBRAS AS ULTRAGRAPH ALGEBRAS

In this section we shall see that the C^*-algebras of ultragraphs with no singular vertices are precisely the Exel-Laca algebras.

DEFINITION 4.1. (Exel-Laca) Let I be any set and let $A = \{A(i,j)_{i,j \in I}\}$ be a $\{0,1\}$-matrix over I with no identically zero rows. The Exel-Laca algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by partial isometries $\{s_i : i \in I\}$ with commuting initial projections and mutually orthogonal range projections satisfying $s_i^* s_i s_j s_j^* = A(i,j) s_j s_j^*$ and

$$\prod_{x \in X} s_x^* s_x \prod_{y \in Y} (1 - s_y^* s_y) = \sum_{j \in I} A(X,Y,j) s_j s_j^* \tag{4.1}$$

whenever X and Y are finite subsets of I such that the function

$$j \in I \mapsto A(X,Y,j) := \prod_{x \in X} A(x,j) \prod_{y \in Y} (1 - A(y,j))$$

is finitely supported.

Although there is reference to a unit in (4.1), this relation applies to algebras that are not necessarily unital, with the convention that if a 1 still appears after expanding the product in (4.1), then the relation implicitly states that $\mathcal{O}_A$ is unital. It is also important to realize that the relation (4.1) also applies when the function $j \mapsto A(X,Y,j)$ is identically zero. This particular instance of (4.1) is interesting in itself so we emphasize it by stating the associated relation separately:

$$\prod_{x \in X} s_x^* s_x \prod_{y \in Y} (1 - s_y^* s_y) = 0 \tag{4.2}$$

whenever X and Y are finite subsets of I such that $A(X,Y,j) = 0$ for every $j \in I$.

LEMMA 4.2. *Let $\mathcal{G}$ be an ultragraph. If $A \subseteq G^0$ is a finite set, then*

$$p_A = \sum_{v \in A} p_v.$$

Proof. Simply use the fact that A is the disjoint union of its singleton sets. ∎

LEMMA 4.3. *Let $\mathcal{G}$ be an ultragraph. If $Y \subseteq \mathcal{G}^1$ is a finite set, then for any $A \in \mathcal{G}^0$*

$$\prod_{y \in Y} (p_A - p_A p_{r(y)}) = p_A - p_A p_B, \quad \textit{where } B = \bigcup_{y \in Y} r(y).$$

Proof. We shall induct on the number of elements in Y. If $|Y| = 1$, then the claim holds trivially. Assume the claim is true for sets containing $n-1$ elements. Let $|Y| = n$ and choose $e \in Y$. If we let $B' := \bigcup_{y \in Y \setminus \{e\}} r(y)$, then

$$\begin{aligned}
\prod_{y \in Y} (p_A - p_A p_{r(y)}) &= \Big(\prod_{y \in Y \setminus \{e\}} (p_A - p_A p_{r(y)}) \Big) (p_A - p_A p_{r(e)}) \\
&= (p_A - p_A p_{B'})(p_A - p_A p_{r(e)}) \\
&= p_A - p_A p_{B'} - p_A p_{r(e)} + p_A p_{B' \cap r(e)} \\
&= p_A - p_A (p_{B'} + p_{r(e)} - p_{B' \cap r(e)}) \\
&= p_A - p_A p_{B' \cup r(e)} \\
&= p_A - p_A p_B,
\end{aligned}$$

where $B := \bigcup_{y \in Y} r(y)$. ∎

PROPOSITION 4.4. *Let $\mathcal{G}$ be an ultragraph with no sinks, and let $A_{\mathcal{G}}$ be the edge matrix of $\mathcal{G}$. If $\{s_e, p_A\}$ is a Cuntz-Krieger $\mathcal{G}$-family, then $\{s_e : e \in \mathcal{G}^1\}$ is a collection of partial isometries satisfying the relations defining $\mathcal{O}_{A_{\mathcal{G}}}$.*

Proof. By definition the s_e's have mutually orthogonal range projections. Furthermore, $s_e^* s_e s_f^* s_f = p_{r(e)} p_{r(f)} = p_{r(e)\cap r(f)} = p_{r(f)} p_{r(e)} = s_f^* s_f s_e^* s_e$ so the initial projections commute. Furthermore, if $e, f \in \mathcal{G}^1$, then by Lemma 2.8

$$s_e^* s_e s_f s_f^* = p_{r(e)} s_f s_f^* = \begin{cases} s_f s_f^* & \text{if } s(f) \in r(e), \\ 0 & \text{otherwise}, \end{cases} = A_{\mathcal{G}}(e,f) s_f s_f^*.$$

Now we shall show that condition (4.1) of Exel-Laca algebras holds. Suppose that X and Y are finite subsets of $\mathcal{G}^1$ such that the function $j \mapsto A_{\mathcal{G}}(X, Y, j)$ has finite (or empty) support. We divide the proof of (4.1) into two cases.

CASE 1. $X = \emptyset$.

We claim that $G^0 \in \mathcal{G}^0$. The support of $j \mapsto \prod_{y\in Y} (1 - A_{\mathcal{G}}(y, j))$, which is finite (or empty) by assumption, is given by

$$F := \{j \in \mathcal{G}^1 : A_{\mathcal{G}}(y, j) = 0 \text{ for all } y \in Y\} = \Big\{j \in \mathcal{G}^1 : s(j) \notin \bigcup_{y\in Y} r(y)\Big\}.$$

Because there are no sinks, the source map s is surjective and $G^0 = s(F) \sqcup \bigcup_{y\in Y} r(y)$. Since F and Y are finite this implies that $G^0 \in \mathcal{G}^0$ and by Lemma 3.2 $C^*(\mathcal{G})$ is unital with $1 = p_{G^0}$. Furthermore, since $G^0 = s(F) \sqcup \bigcup_{y\in Y} r(y)$ we have

$$1 = p_{s(F)} + p_B, \quad \text{where } B := \bigcup_{y\in Y} r(y)$$

and using Lemma 4.2

$$1 - p_B = \sum_{v\in s(F)} p_v.$$

Now applying Lemma 4.3 with $p_A = 1$ gives

$$\prod_{y\in Y} (1 - p_{r(y)}) = \sum_{v\in s(F)} p_v \quad \text{or} \quad \prod_{y\in Y} (1 - s_y^* s_y) = \sum_{v\in s(F)} p_v.$$

If F is empty the right hand side vanishes and (4.1) holds. If F is nonempty, then $s(F) \neq \emptyset$ and for each $v \in s(F)$ we have $\{j : s(j) = v\} \subseteq \Big\{j : j \notin \bigcup_{y\in Y} r(y)\Big\} = F$. Hence $0 < |\{j : s(j) = v\}| < \infty$ and we may use the definition of a Cuntz-Krieger $\mathcal{G}$-family to write $p_v = \sum_{\{j:s(j)=v\}} s_j s_j^*$. Summing over all vertices in $s(F)$ and substituting above gives

$$\prod_{y\in Y} (1 - s_y^* s_y) = \sum_{j\in F} s_j s_j^*$$

which is (4.1).

CASE 2. $X \neq \emptyset$.

Once again let $F = \{j \in \mathcal{G}^1 : s(j) \in r(x) \text{ for all } x \in X \text{ and } s(j) \notin r(y) \text{ for all } y \in Y\}$ be the support of $j \mapsto A(X, Y, j)$.

SUBCASE 2a. Assume F is empty. Let $x_0 \in X$. Since there are no sinks, for each $v \in r(x_0)$ there exists $j_v \in \mathcal{G}^1$ such that $s(j_v) = v$. Because F is empty, either $v = s(j_v) \notin r(x_v)$ for some $x_v \in X$ or $v = s(j_v) \in r(y_v)$ for some $y_v \in Y$. Thus

$$r(x_0) \cap \bigcap_{x_v} r(x_v) \subseteq \bigcup_{y_v} r(y_v)$$

and the left hand side of (4.2) contains

$$\Big(\prod_{x_v} s^*_{x_v} s_{x_v}\Big)\Big(s^*_{x_0} s_{x_0}\Big) \prod_{y_v} (1 - s^*_{y_v} s_{y_v}) = \prod_{x_v} p_{r(x_v)} \prod_{y_v} \Big(p_{r(x_0)} - p_{r(x_0)} p_{r(y_v)}\Big).$$

If we let $A := \bigcap_{x_v} r(x_v)$ and $B := \bigcup_{y_v} r(y_v)$, then Lemma 4.3 shows that

$$\begin{aligned}\Big(\prod_{x_v} s^*_{x_v} s_{x_v}\Big)\Big(s^*_{x_0} s_{x_0}\Big) \prod_{y_v} (1 - s^*_{y_v} s_{y_v}) &= p_A(p_{r(x_0)} - p_{r(x_0)} p_B) \\ &= p_{r(x_0)\cap A} - p_{r(x_0)\cap A \cap B} = p_{r(x_0)\cap A} - p_{r(x_0)\cap A} = 0\end{aligned}$$

so (4.2) holds.

SUBCASE 2b. Suppose F is nonempty. Since $F = \Big\{j : s(j) \in \bigcap_{x\in X} r(x) \cap \Big(\bigcup_{y\in Y} r(y)\Big)^{\mathrm{c}}\Big\}$ is nonempty and finite and since $\mathcal{G}$ has no sinks, it follows that $\bigcap_{x\in X} r(x) \cap \Big(\bigcup_{y\in Y} r(y)\Big)^{\mathrm{c}}$ is nonempty and finite. Thus $\bigcap_{x\in X} r(x) \cap \Big(\bigcup_{y\in Y} r(y)\Big)^{\mathrm{c}} \in \mathcal{G}^0$. For convenience of notation, let $A := \bigcap_{x\in X} r(x)$ and $B := \bigcup_{y\in Y} r(y)$. Then

$$\sum_{j\in F} s_j s_j^* = \sum_{\{j : s(j) \in A \cap B^{\mathrm{c}}\}} s_j s_j^*.$$

Furthermore, since $F = \{j : s(j) \in A \cap B^{\mathrm{c}}\}$ is finite, we know that any vertex in $A \cap B^{\mathrm{c}}$ can emit only finitely many vertices. Thus $p_v = \sum_{s(e)=v} s_e s_e^*$ for all $v \in A \cap B^{\mathrm{c}}$. Applying this to the above equation and using Lemma 4.2 gives

$$\sum_{j\in F} s_j s_j^* = \sum_{v \in A\cap B^{\mathrm{c}}} p_v = p_{A\cap B^{\mathrm{c}}}.$$

Now we also have

$$p_{A\cap B^{\mathrm{c}}} + p_{A\cap B} = p_A - p_\emptyset = p_A$$

and combining this with Lemma 4.3 gives

$$\begin{aligned}\sum_{j\in F} s_j s_j^* = p_A - p_{A\cap B} &= \prod_{y\in Y} (p_A - p_A p_{r(y)}) = \prod_{x\in X} p_{r(x)} \prod_{y\in Y} (1 - p_{r(y)}) \\ &= \prod_{x\in X} s_x^* s_x \prod_{y\in Y} (1 - s_y^* s_y)\end{aligned}$$

so (4.1) holds and we are done. ∎

THEOREM 4.5. *Let $\mathcal{G}$ be an ultragraph with no singular vertices (i.e. no sinks and no infinite emitters). If $A_{\mathcal{G}}$ is the edge matrix of $\mathcal{G}$, then $\mathcal{O}_{A_{\mathcal{G}}}$ is canonically isomorphic to $C^*(\mathcal{G})$.*

Proof. By Proposition 4.4 and the universal property of $\mathcal{O}_{A_{\mathcal{G}}}$, there exists a homomorphism $\varphi : \mathcal{O}_{A_{\mathcal{G}}} \to C^*(\mathcal{G})$ with the property that $\varphi(S_e) = s_e$. Since this homomorphism is equivariant for the gauge actions, it follows from the Gauge-Invariant Uniqueness Theorem for Exel-Laca algebras (Theorem 2.7 in [15]) that φ is injective. Furthermore, since $\mathcal{G}$ has no singular vertices, the s_e's generate $C^*(\mathcal{G})$. Thus φ is also surjective. ∎

REMARK 4.6. Since any matrix A with entries in $\{0,1\}$ is the edge matrix of the ultragraph $\mathcal{G}_A$, the above shows that the C^*-algebras of ultragraphs with no singular vertices are precisely the Exel-Laca algebras.

COROLLARY 4.7. *Let $\mathcal{G}$ be an ultragraph with no singular vertices. If A is the edge matrix of $\mathcal{G}$, then $C^*(\mathcal{G})$ is canonically isomorphic to $C^*(\mathcal{G}_A)$.*

Proof. From Theorem 4.5 we have $C^*(\mathcal{G}) \cong \mathcal{O}_A \cong C^*(\mathcal{G}_A)$. ∎

REMARK 4.8. This corollary shows that if one wishes to study ultragraphs with no singular vertices, then there is no loss of generality in considering only ultragraphs of the form $\mathcal{G}_A$ for a matrix A with entries in $\{0,1\}$. This is somewhat surprising since any $\mathcal{G}_A$ has the property that $|s^{-1}(v)| = 1$ for all $v \in G^0$.

In conclusion, we have seen that if A is a $\{0,1\}$-matrix with no zero rows, then $\mathcal{O}_A \cong C^*(\mathcal{G}_A)$, and hence we may view any Exel-Laca algebra as an ultragraph algebra. While this has the advantage that one no longer needs to deal with the complicated condition (4.1) of Exel-Laca algebras, one must now deal with the collection $\mathcal{G}^0_A$. At first it may seem as though we are simply trading one set of troubles for another. However, we contend that the ultragraph approach is often easier. Despite the somewhat complicated definition of $\mathcal{G}^0$, we shall see in the next sections that techniques much like those used for graph algebras can be applied to ultragraph algebras.

5. UNIQUENESS THEOREMS FOR ULTRAGRAPH ALGEBRAS

In this section we prove versions of the Gauge-Invariant Uniqueness Theorem and the Cuntz-Krieger Uniqueness Theorem for C^*-algebras of ultragraphs with no singular vertices. In Section 6 we extend these results to allow for singular vertices. We obtain our results in this section by approximating ultragraph algebras with C^*-algebras of finite graphs as in [15]. We mention that Raeburn and Szymański gave one method for approximating C^*-algebras of infinite graphs by those of finite graphs (Definition 1.1 in [15]) and another for approximating Exel-Laca algebras by C^*-algebras of finite graphs (Definition 2.1 in [15]). We shall show that we are able to approximate C^*-algebras of ultragraphs by a method much like the one used for Exel-Laca algebras. We also remark that although our computations in Proposition 5.3 are done for ultragraphs rather than matrices, they are similar to those in Proposition 2.2 of [15].

Let $\mathcal{G}$ be an ultragraph. For any finite subsets $X, Y \subseteq \mathcal{G}^1$ define $V(X,Y) := \bigcap_{x\in X} r(x) \cap \Big(\bigcup_{y\in Y} r(y)\Big)^{c}$ and $E(X,Y) := \{e \in \mathcal{G}^1 : s(e) \in V(X,Y)\}$.

DEFINITION 5.1. For any finite subset $F \subseteq \mathcal{G}^1$ define the graph G_F by

$$G_F^0 := F \cup \{X : \emptyset \neq X \subseteq F \text{ satisfies } E(X, F \setminus X) \not\subseteq F\}, \text{ and}$$
$$G_F^1 := \{(e,f) \in F \times F : s(f) \in r(e)\} \cup \{(e,X) : e \in X\};$$

with

$$\begin{array}{lll} s((e,f)) = e, & s((e,v)) = e, & s((e,X)) = e, \\ r((e,f)) = f, & r((e,v)) = v, & r((e,X)) = X. \end{array}$$

LEMMA 5.2. *If $P_1, \ldots, P_n$ are commuting projections, then*

$$1 = \sum_{Y \subseteq \{1,\ldots,n\}} \Big(\prod_{i \in Y} P_i\Big)\Big(\prod_{i \notin Y}(1 - P_i)\Big).$$

Proof. Induct on n: multiply the formula for $n = k$ by $P_{k+1} + (1 - P_{k+1})$. ∎

PROPOSITION 5.3. *Let $\mathcal{G}$ be an ultragraph with no sinks, $\{s_e, p_A\}$ be a Cuntz-Krieger $\mathcal{G}$-family, and F a finite subset of $\mathcal{G}^1$. Define $A_X := \bigcap_{x \in X} r(x)$ and $B_X := \bigcup_{y \in F \setminus X} r(y)$. Then*

$$\begin{array}{ll} Q_e := s_e s_e^*, & Q_X := p_{A_X}(1 - p_{B_X})\Big(1 - \sum_{f \in F} s_f s_f^*\Big), \\ T_{(e,f)} := s_e Q_f, & T_{(e,X)} := s_e Q_X, \end{array}$$

forms a Cuntz-Krieger G_F-family that generates $C^*(\{s_e : e \in F\})$. If every s_e is nonzero, then every Q is nonzero.

Proof. First we shall show that this is a Cuntz-Krieger G_F-family. The projections Q_e are mutually orthogonal because the s_e's have mutually orthogonal ranges, and are orthogonal to the Q_X's because of the factor $1 - \sum_{f \in F} s_f s_f^*$. To see that the Q_X's are mutually orthogonal suppose that $X \neq Y$. Then, without loss of generality, we may assume that there exists $x \in X \setminus Y$, and because $X \subseteq F$ we see that $A_X \subseteq r(x)$ and $r(x) \subseteq B_Y$. Thus $p_{A_X}(1 - p_{B_Y}) = p_{A_X} p_{r(x)}(1 - p_{B_Y}) = p_{A_X}(p_{r(x)} - p_{r(x)} p_{B_Y}) = p_{A_X}(p_{r(x)} - p_{r(x)}) = 0$ and Q_X is orthogonal to Q_Y.

Furthermore,

$$\begin{aligned} T_{(e,f)}^* T_{(e,f)} &= Q_f^* s_e^* s_e Q_f = s_f s_f^* p_{r(e)} s_f s_f^* = s_f s_f^* p_{s(f)} p_{r(e)} s_f s_f^* \\ &= s_f s_f^* p_{s(f)} s_f s_f^* = s_f s_f^* = Q_f, \end{aligned}$$

and since $Q_X \leqslant s_e^* s_e$ whenever $e \in X$, we have

$$T_{(e,X)}^* T_{(e,X)} = Q_X^* s_e^* s_e Q_X = Q_X$$

so the first Cuntz-Krieger relation holds.

Note that the elements X in G_F^0 are all sinks in G_F. Thus to check the second Cuntz-Krieger relation, we need only consider edges whose source is some $e \in F$.

If X is a subset of G^1 and $E(X, F \setminus X) \subseteq F$, then $V(X, F \setminus X) := \bigcap_{x \in X} r(x) \cap \Big(\bigcup_{y \in (F \setminus X)} r(y)\Big)^{\mathrm{c}}$ is a finite set and $V(X,Y) \in \mathcal{G}^0$. Furthermore, since $E(X, F \setminus$

$X) \subseteq F$ we see that each vertex in $V(X, F \setminus X)$ is the source of finitely many edges. Hence

$$p_{V(X,F\setminus X)} = \sum_{v\in V(X,F\setminus X)} p_v = \sum_{e\in E(X,F\setminus X)} s_e s_e^* \leqslant \sum_{f\in F} s_f s_f^*$$

and

$$p_{A_X}(1-p_{B_X})\Big(1-\sum_{f\in F} s_f s_f^*\Big) = p_{V(X,Y)}\Big(1-\sum_{f\in F} s_f s_f^*\Big) = 0.$$

Thus if we fix $e \in G^1$, we have

$$\begin{aligned}\sum_{\{X:e\in X\}} Q_X &= \sum_{\{X:e\in X\}} p_{A_X}(1-p_{B_X})\Big(1-\sum_{f\in F} s_f s_f^*\Big)\\ &= p_{r(e)}\Big(\sum_{Y\subseteq F\setminus\{e\}} p_{A_Y}(1-p_{B_{Y\cup\{e\}}})\Big(1-\sum_{f\in F} s_f s_f^*\Big)\Big)\\ &\qquad\text{and by Lemma 4.3}\\ &= p_{r(e)}\sum_{Y\subseteq F\setminus\{e\}}\Big(\prod_{x\in Y} p_{r(x)}\Big)\Big(\prod_{y\in(F\setminus\{e\})\setminus Y}(1-p_{r(y)})\Big)\Big(1-\sum_{f\in F} s_f s_f^*\Big) \qquad (5.1)\\ &\qquad\text{and by Lemma 5.2}\\ &= p_{r(e)}\Big(1-\sum_{f\in F} s_f s_f^*\Big)\\ &= s_e^* s_e\Big(1-\sum_{\{f\in F:s(f)\in r(e)\}} s_f s_f^*\Big).\end{aligned}$$

Now we have

$$\sum_{s(f)\in r(e)} T_{(e,f)}T_{(e,f)}^* + \sum_{\{X:e\in X\}} T_{(e,X)}T_{(e,X)}^* = \sum_{s(f)\in r(e)} s_e s_f s_f^* s_e^* + \sum_{\{X:e\in X\}} s_e Q_X s_e^*$$

which equals $s_e s_e^* = Q_e$ by (5.1). Thus the T's and Q's form a Cuntz-Krieger G_F-family.

Equation (5.1) also implies that we can recover s_e as

$$s_e = \sum_{\{f\in F:s(f)\in r(e)\}} T_{(e,f)} + \sum_{\{X:e\in X\}} T_{(e,X)} = s_e\Big(\sum_{\{X:e\in X\}} Q_X + \sum_{\{f\in F:s(f)\in r(e)\}} s_f s_f^*\Big)$$

so the operators T_e and Q_v generate $C^*(\{s_e : e \in F\})$. For the last comment note that $E(X, F \setminus X) \not\subseteq F$ implies $Q_X \geqslant s_f s_f^*$ for some $f \notin F$, and hence $Q_X \neq 0$. ∎

COROLLARY 5.4. *Let $\mathcal{G}$ be an ultragraph with no sinks and let $F \subseteq \mathcal{G}^1$ be a finite set of edges. Then $C^*(G_F)$ is canonically isomorphic to the C^*-subalgebra of $C^*(\mathcal{G})$ generated by $\{s_e : e \in F\}$.*

Proof. Applying the proposition to the canonical family $\{s_e, p_A\}$ of $C^*(\mathcal{G})$ gives a Cuntz-Krieger G_F-family $\{T_e, Q_v\}$ that generates $C^*(\{s_e : e \in F\})$. Thus we have a homomorphism $\varphi : C^*(G_F) \to C^*(\mathcal{G})$ whose image is $C^*(\{s_e : e \in F\})$. If α is the gauge action on $C^*(G_F)$ and γ is the gauge action on $C^*(\mathcal{G})$, then we see that $\varphi\circ\alpha_z = \gamma_z\circ\varphi$ for all $z \in \mathbb{T}$. Since each projection Q_v is nonzero, it follows from the Gauge-Invariant Uniqueness Theorem for graph algebras (Theorem 2.1 in [1]) that φ is injective. ∎

The following is an analogue of the Gauge-Invariant Uniqueness Theorem for C^*-algebras of ultragraphs with no singular vertices. We shall extend this result to all ultragraph algebras in Section 6.

PROPOSITION 5.5. *Let $\mathcal{G}$ be an ultragraph with no singular vertices (i.e. no sinks or infinite emitters). Also let $\{S_e, P_A\}$ be a Cuntz-Krieger $\mathcal{G}$-family on Hilbert space and let π be the representation of $C^*(\mathcal{G})$ such that $\pi(s_e) = S_e$ and $\pi(p_A) = P_A$. Suppose that each P_A is nonzero for every nonempty A, and that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(S_e, P_A)$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}$. Then π is faithful.*

Proof. Let F be a finite subset of $\mathcal{G}^1$. Then $C^*(\{s_e : e \in F\})$ is isomorphic to the graph algebra $C^*(G_F)$ by Corollary 5.4, and this isomorphism is equivariant for the gauge actions. Furthermore, the projections in $\mathcal{B}(\mathcal{H})$ corresponding to the vertices of G_F are all nonzero: Q_e because T_e is, and Q_X because of the existence of an f such that $T_f T_f^* \leqslant Q_X$. Applying the Gauge-Invariant Uniqueness Theorem for graph algebras (Theorem 2.1 in [1]) to the corresponding representation of $C^*(G_F)$ shows that π is faithful on $C^*(\{s_e : e \in F\})$, and hence is isometric there. Thus π is isometric on the subalgebra generated by $\{s_e : e \in \mathcal{G}^1\}$. Since $\mathcal{G}$ has no singular vertices, this subalgebra is dense in $C^*(\mathcal{G})$. Hence π is isometric on all of $C^*(\mathcal{G})$. ∎

We can also use this method of approximating ultragraph algebras by graph algebras to prove a version of the Cuntz-Krieger Uniqueness Theorem. Again, we shall extend this result to all ultragraph algebras in Section 6.

LEMMA 5.6. *Let $\mathcal{G}$ be ultragraph with no sinks and let $F \subseteq \mathcal{G}^1$ be a finite set. If $L = x_1 \cdots x_n$ is a loop in G_F, then there exists a loop $L' = e_1 \cdots e_n$ in $\mathcal{G}$ such that $\{e_i\}_{i=1}^n \subseteq F$, $x_i = (e_i, e_{i+1})$ for $i = 1, \ldots, n-1$ and $x_n = (e_n, e_1)$. Furthermore, L has an exit if and only if L' does.*

Proof. If L has an exit, then either there is an edge of the form (e_i, f) with $f \neq e_{i+1}$, or there is an edge of the form (e_i, X). In the first case, $s(f) \in r(e_i)$ and $f \neq e_{i+1}$ so f is an exit for L'. In the second case $e_i \in X$ and $E(X, F \setminus X) \nsubseteq F$. Hence there exists $g \in \mathcal{G}^1$ such that $s(g) \in \mathcal{G}^1 \setminus F$ for which $s(g) \in V(X, F \setminus F)$. But then $s(g) \in r(e_i)$ and $g \neq e_{i+1}$ so g is an exit for L'.

Conversely, suppose L' has an exit. Since $\mathcal{G}$ has no sinks, there exists an edge $f \in \mathcal{G}^1$ with $s(f) \in r(e_i)$ and $f \neq e_{i+1}$. If $f \in F$, then (e_i, f) is an exit for L. If $f \notin F$, let $X := \{e \in F : s(f) \in r(e)\}$. Then $f \in E(X, F \setminus F)$, so $\emptyset \neq E(X, F \setminus X) \nsubseteq F$. Since $e_i \in X$ we see that (e_i, X) is an exit for L. ∎

PROPOSITION 5.7. *Suppose that $\mathcal{G}$ is an ultragraph with no singular vertices (i.e. no sinks or infinite emitters), and that $\mathcal{G}$ satisfies Condition* (L). *If $\{S_e, P_A\}$ and $\{T_e, Q_A\}$ are two Cuntz-Krieger $\mathcal{G}$-families in which all the projections P_A and Q_A are nonzero for nonempty A, then there is an isomorphism φ of $C^*(S_e, P_A)$ onto $C^*(T_e, Q_A)$ such that $\varphi(S_e) = T_e$ and $\varphi(P_A) = Q_A$.*

Proof. We shall prove the theorem by showing that the representations $\pi_{S,P}$ and $\pi_{T,Q}$ of $C^*(\mathcal{G})$ are faithful, and then $\varphi := \pi_{T,Q} \circ \pi_{S,P}^{-1}$ is the required isomorphism.

Write $\mathcal{G}^1 = \bigcup_{n=1}^{\infty} F_n$ as the increasing union of finite subsets F_n, and let B_n be the C^*-subalgebra of $C^*(\mathcal{G})$ generated by $\{s_e : e \in F_n\}$. By Lemma 5.4 there are isomorphisms $\varphi_n : C^*(G_{F_n}) \to B_n$ that respect the generators. Since all the loops in F_n have exits by Lemma 5.6, the Cuntz-Krieger Uniqueness Theorem for graph algebras (Theorem 3.1 in [1]) implies that $\pi_{S,P} \circ \varphi_n$ is an isomorphism, and hence is isometric. Thus $\pi_{S,P}$ is isometric on the $*$-subalgebra $\bigcup_n B_n$ of $C^*(\mathcal{G})$. But since $\mathcal{G}$ has no sinks or infinite emitters, $C^*(\mathcal{G})$ is generated by the s_e's and thus $\bigcup_n B_n$ is a dense $*$-subalgebra of $C^*(\mathcal{G})$. Hence $\pi_{S,P}$ is isometric on all of $C^*(\mathcal{G})$, and in particular, it is an isomorphism. ∎

Let I be a countable (or finite) set, and let A be an $I \times I$ matrix with entries in $\{0,1\}$ and no zero rows. In [7] Exel and Laca associated a graph $\mathrm{Gr}(A)$ to A whose vertex matrix is equal to A. Specifically, we define the vertices of $\mathrm{Gr}(A)$ to be I, and for each pair of vertices $i, j \in I$ we define there to be $A(i,j)$ edges from i to j.

In Section 13 of [7], Exel and Laca proved a uniqueness theorem for $\mathcal{O}_A$ when $\mathrm{Gr}(A)$ satisfies Condition (L) (or in their terminology, when $\mathrm{Gr}(A)$ has no terminal circuits). The following shows that their uniqueness theorem is equivalent to the one we proved in Proposition 5.7.

LEMMA 5.8. *Let $\mathcal{G}$ be an ultragraph with no sinks and with edge matrix A. Then $\mathrm{Gr}(A)$ satisfies Condition (L) if and only if $\mathcal{G}$ satisfies Condition (L).*

Proof. Suppose that $\mathcal{G}$ satisfies Condition (L). Let $\alpha := \alpha_1 \cdots \alpha_n$ be a loop in $\mathrm{Gr}(A)$ with $a_i := s(\alpha_i)$ for $1 \leqslant i \leqslant n$. Then $a_1 \cdots a_n$ is a loop in $\mathcal{G}$. Let b be an exit for this loop, and without loss of generality assume that $s(b) \in r(a_1)$ and $b \neq a_2$. Since $A(a_1, b) = 1$, there exists an edge f in $\mathrm{Gr}(A)$ from a_1 to b. Since $b \neq a_2$ we know that $f \neq \alpha_1$ and hence f is an exit for α.

Conversely, suppose that $\mathrm{Gr}(A)$ satisfies Condition (L). Let $a = a_1 \cdots a_n$ be a loop in $\mathcal{G}$. Then $A(a_i, a_{i+1}) = 1$ for all $1 \leqslant i \leqslant n-1$ and $A(a_n, a_1) = 1$. Hence there exists a loop $\alpha = \alpha_1 \cdots \alpha_n$ in $\mathrm{Gr}(A)$ with $s(\alpha_i) = a_i$ for all i. Let f be an exit for α in $\mathrm{Gr}(A)$, and without loss of generality assume $s(f) = s(\alpha_1)$ and $f \neq \alpha_1$. Let $b := r(f)$. Since A has entries in $\{0,1\}$ we know that $b \neq r(\alpha_1) = a_2$. Hence b is an exit for $a = a_1 \cdots a_n$. ∎

COROLLARY 5.9. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be ultragraphs with no sinks and with the same edge matrix A. Then $\mathcal{G}_1$ satisfies Condition (L) if and only if $\mathcal{G}_2$ satisfies Condition (L).*

Proof. $\mathcal{G}_1$ satisfies Condition (L) $\Leftrightarrow$ $\mathrm{Gr}(A)$ satisfies Condition (L) $\Leftrightarrow$ $\mathcal{G}_2$ satisfies Condition (L). ∎

6. SINGULAR VERTICES

In this section we deal with singular vertices in a manner similar to what was done in [1] for sinks in graphs and in [4] for infinite emitters in graphs.

LEMMA 6.1. *Let $\mathcal{G}$ be an ultragraph, let A be a C^*-algebra generated by a Cuntz-Krieger $\mathcal{G}$-family $\{s_e, p_A\}$, and let $\{q_n\}$ be a sequence of projections in A. If $q_n s_\alpha p_A s_\beta^*$ converges for all $\alpha, \beta \in \mathcal{G}^*, A \in \mathcal{G}^0$, then $\{q_n\}$ converges strictly to a projection $q \in \mathcal{M}(A)$.*

Proof. Since we can approximate any $a \in A := C^*(s_e, p_A)$ by a linear combination of $s_\alpha p_A s_\beta^*$, an $\varepsilon/3$ argument shows that $\{q_n a\}$ is Cauchy for every $a \in A$. We define $q : A \to A$ by $q(a) := \lim_{n\to\infty} q_n a$. Since

$$b^* q(a) = \lim_{n\to\infty} b^* q_n a = \lim_{n\to\infty} (q_n b)^* a = q(b)^* a,$$

the map q is an adjointable operator on the Hilbert C^*-module A_A, and hence defines (left multiplication by) a multiplier q of A (Theorem 2.47 in [16]). Taking adjoints shows that $aq_n \to aq$ for all $a \in A$ so $q_n \to q$ strictly. It is easy to check that $q^2 = q = q^*$. ∎

By adding a tail at a sink w we mean adding a graph of the form

$$w \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \cdots$$

to $\mathcal{G}$ to form a new ultragraph $\mathcal{F}$; thus $F^0 := G^0 \cup \{v_i : 1 \leqslant i < \infty\}$, $\mathcal{F}^1 := \mathcal{G}^1 \cup \{e_i : 1 \leqslant i < \infty\}$, and r and s are extended to $\mathcal{F}^1$ by $r(e_i) = \{v_i\}$ and $s(e_i) = v_{i-1}$ and $s(e_1) = w$. Just as with graphs, when we add tails to sinks in $\mathcal{G}$ any Cuntz-Krieger $\mathcal{F}$-family will restrict to a Cuntz-Krieger $\mathcal{G}$-family. This is because $\mathcal{F}^0$ is generated by $G^0 \cup \{v_i : 1 \leqslant i < \infty\} \cup \{r(e) : e \in \mathcal{G}^1\}$ and thus by Lemma 2.12 we see that $\mathcal{F}^0 = \{A \cup F : A \in \mathcal{G}^0$ and F is a finite subset of $\{v_i\}_{i=1}^\infty\}$.

PROPOSITION 6.2. *Let $\mathcal{G}$ be a directed graph and let $\mathcal{F}$ be the ultragraph formed by adding a tail to each sink of $\mathcal{G}$.*

(i) *For each Cuntz-Krieger $\mathcal{G}$-family $\{S_e, P_A\}$ on a Hilbert space $\mathcal{H}_\mathcal{G}$, there is a Hilbert space $\mathcal{H}_\mathcal{F} := \mathcal{H}_\mathcal{G} \oplus \mathcal{H}_T$ and a Cuntz-Krieger $\mathcal{F}$-family $\{T_e, Q_A\}$ such that $T_e = S_e$ for $e \in \mathcal{G}^1$, $Q_A = P_A$ for $A \in \mathcal{G}^0$, and $\sum_{v \notin G^0} Q_v$ is the projection on $\mathcal{H}_T$.*

(ii) *If $\{T_e, Q_A\}$ is a Cuntz-Krieger $\mathcal{F}$-family, then $\{T_e, Q_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\}$ is a Cuntz-Krieger $\mathcal{G}$-family. If w is a sink in E such that $Q_w \neq 0$, then $Q_v \neq 0$ for every vertex on the tail attached to w.*

(iii) *If $\{t_e, q_A\}$ are the canonical generators of $C^*(\mathcal{F})$, then the homomorphism $\pi_{t,q}$ corresponding to the Cuntz-Krieger $\mathcal{G}$-family $\{t_e, q_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\}$ is an isomorphism of $C^*(E)$ onto a full corner in $C^*(\mathcal{F})$.*

Proof. For the sake of simplicity we consider the case in which a single tail has been added to a sink w. As mentioned earlier, any element $B \in \mathcal{F}^0$ may be (uniquely) written as $B = A \cup F$ for some $A \in \mathcal{G}^0$ and some finite set $F \subseteq \{v_i\}_{i=1}^\infty$. To extend $\{S_e, P_A\}$, we let $\mathcal{H}_T$ be the direct sum of infinitely many copies of $P_w \mathcal{H}_\mathcal{G}$, define P_{v_i} to be the projection on the i^{th} summand, and let S_{e_i} be the partial isometry whose initial space is the i^{th} summand and whose final space is the $(i-1)^{\text{st}}$, with S_{e_1} taking the first summand of $\mathcal{H}_T$ onto $P_w\mathcal{H}_\mathcal{G} \subseteq \mathcal{H}_\mathcal{G}$. Now for any $B \in \mathcal{G}^0$ we write B (uniquely) as $A \cup F$ and define

$$P_B := P_A + \sum_{v \in F} P_v.$$

One can check that $\{S_e, P_B\}$ is a Cuntz-Krieger $\mathcal{F}$-family, and hence (i) holds.

For the same reasons, throwing away the extra elements of a Cuntz-Krieger $\mathcal{F}$-family gives a Cuntz-Krieger $\mathcal{G}$-family. The last statement in (i) holds because

$$S_{e_1}S_{e_1}^* = P_w \neq 0 \Rightarrow S_{e_2}S_{e_2}^* = P_{v_1} = S_{e_1}^*S_{e_1} \neq 0 \Rightarrow S_{e_3}S_{e_3}^* = P_{v_2} = S_{e_2}^*S_{e_2} \neq 0 \cdots.$$

For the first part of (iii), just use part (i) to see that every representation of $C^*(\mathcal{G})$ factors through a representation of $C^*(\mathcal{F})$.

We still have to show that the image of $C^*(\mathcal{G})$ is a full corner. List the elements of $G^0 = \{w_1, w_2, w_3, \dots\}$ and the elements $\mathcal{G}^1 = \{e_1, e_2, e_3, \dots\}$. Define $A_n := \{v_i : 1 \leqslant i \leqslant n\} \cup \bigcup_{i=1}^{n} r(e_i)$. Then given any $e \in \mathcal{G}^1$ and $A \in \mathcal{G}^0$ we see that for large enough n we have $p_{A_n}s_e = s_e$ and $p_{A_n}p_A = p_A$. Hence Lemma 6.1 applies and the sequence $\{p_{A_n}\}$ converges strictly to a projection p in $\mathcal{M}(C^*(\mathcal{F}))$ satisfying

$$ps_e := \begin{cases} s_e & \text{if } s(e) \in G^0, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \quad pp_A = p_{A \cap G^0}.$$

Thus the corner $pC^*(\mathcal{F})p$ is precisely $C^*(\mathcal{G})$.

To see that this corner is full suppose J is an ideal containing $pC^*(\mathcal{F})p$. Then J contains $\{q_{r(e)} : e \in \mathcal{G}^1\}$ and $\{q_v : v \in G^0\}$. Furthermore, if v is a vertex on the tail attached to w, then there is a unique path α with $s(\alpha) = w$ and $r(\alpha) = v$, and

$$q_w \in J \Rightarrow t_\alpha = q_w t_\alpha \in J \Rightarrow q_v = t_\alpha^* t_\alpha \in J.$$

Thus J contains $\{q_{r(e)} : e \in \mathcal{F}^1\} \cup \{q_v : v \in F^0\}$ and hence is all of $C^*(\mathcal{F})$. ∎

Now suppose that $\mathcal{G}$ is an ultragraph with an infinite emitter v_0. We *add a tail* at v_0 by performing the following procedure. List the edges $g_1, g_2, \dots$ of $s^{-1}(v_0)$. We begin by adding vertices and edges as we did with sinks:

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \cdots.$$

Then we remove the edges in $s^{-1}(v_0)$ from $\mathcal{F}$, and for each j we draw an edge f_j with source v_{j-1} and range $r(g_j)$.

To be precise, if $\mathcal{G}$ is an ultragraph with an infinite emitter v_0, we define $F^0 := G^0 \cup \{v_1, v_2, \dots\}$ and

$$\mathcal{F}^1 := \{e \in G^1 : s(e) \neq v_0\} \cup \{e_i\}_1^\infty \cup \{f_j : 1 \leqslant j < \infty\}.$$

We extend r and s to $\mathcal{F}^1$ as indicated above. In particular, $s(e_i) = v_{i-1}$, $r(e_i) = \{v_i\}$, $s(f_j) = v_{j-1}$, and $r(f_j) = r(g_j)$.

For any j we shall often have need to refer to the path $\alpha^j := e_1e_2\cdots e_{j-1}f_j$ in $\mathcal{F}$. Also note that if $\mathcal{G}$ is an ultragraph and $\mathcal{F}$ is the graph formed by adding a tail at an infinite emitter, then $\mathcal{F}^0$ is generated by $\{\{v\} : v \in F^0\} \cup \{r(e) : e \in \mathcal{F}^1\} = \{\{v\} : v \in G^0\} \cup \{r(e) : e \in \mathcal{G}^1\} \cup \{\{v_i\} : 1 \leqslant i < \infty\}$. Thus by Lemma 2.12 we see that $\mathcal{F}^0 = \{A \cup F : A \in \mathcal{G}^0$ and F is a finite subset of $\{v_i\}_{i=1}^\infty\}$.

DEFINITION 6.3. If $\mathcal{G}$ is an ultragraph, a *desingularization of* $\mathcal{G}$ is an ultragraph $\mathcal{F}$ obtained by adding a tail at every singular vertex of $\mathcal{G}$.

LEMMA 6.4. *Suppose $\mathcal{G}$ is a graph and let $\mathcal{F}$ be a desingularization of $\mathcal{G}$. If $\{T_e, Q_A\}$ is a Cuntz-Krieger $\mathcal{F}$-family, then there exists a Cuntz-Krieger $\mathcal{G}$-family in $C^*(\{T_e, Q_A\})$.*

Proof. For every $A \in \mathcal{G}^0$, define $P_A := Q_A$. For every edge $e \in \mathcal{G}^1$ with $s(e)$ not a singular vertex, define $S_e := T_e$. If $e \in \mathcal{G}^1$ with $s(e) = v_0$, a singular vertex, then $e = g_j$ for some j and we define $S_e := T_{\alpha^j}$. The fact that $\{S_e, P_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\}$ is a Cuntz-Krieger $\mathcal{G}$-family follows immediately from the fact that $\{T_e, Q_A : e \in \mathcal{F}^1, A \in \mathcal{F}^0\}$ is a Cuntz-Krieger $\mathcal{F}$-family. ∎

LEMMA 6.5. *Let $\mathcal{G}$ be an ultragraph and let $\mathcal{F}$ be a desingularization of $\mathcal{G}$. For every Cuntz-Krieger $\mathcal{G}$-family $\{S_e, P_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\}$ on a Hilbert space $\mathcal{H}_\mathcal{G}$, there exists a Hilbert space $\mathcal{H}_\mathcal{F} = \mathcal{H}_\mathcal{G} \oplus \mathcal{H}_T$ and a Cuntz-Krieger $\mathcal{F}$-family $\{T_e, Q_A : e \in \mathcal{F}^1, A \in \mathcal{F}^0\}$ on $\mathcal{H}_\mathcal{F}$ satisfying:*

(i) *$P_A = Q_A$ for every $A \in \mathcal{G}^0$;*
(ii) *$S_e = T_e$ for every $e \in \mathcal{G}^1$ such that $s(e)$ is not a singular vertex;*
(iii) *$S_e = T_{\alpha^j}$ for every $e = g_j \in \mathcal{G}^1$ such that $s(g_j)$ is a singular vertex;*
(iv) *$\sum_{v \notin G^0} Q_v$ is the projection onto $\mathcal{H}_T$.*

Proof. We prove the case where $\mathcal{G}$ has just one singular vertex v_0. If v_0 is a sink, then the result follows from Proposition 6.2. Thus we need only consider when v_0 is an infinite emitter. Given a Cuntz-Krieger $\mathcal{G}$-family $\{S_e, P_A\}$, and a nonnegative integer n we define $R_0 = 0$ and $R_n := \sum_{j=1}^{n} S_{g_j} S_{g_j}^*$. Note that the R_n's are projections because the S_{g_j}'s have orthogonal ranges. Furthermore, $R_n \leqslant R_{n+1} < P_{v_0}$ for every n.

Now, for every integer $n \geqslant 1$, define $\mathcal{H}_n := (P_{v_0} - R_n)\mathcal{H}_\mathcal{G}$ and set

$$\mathcal{H}_\mathcal{F} := \mathcal{H}_\mathcal{G} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n.$$

As mentioned previously any $B \in \mathcal{F}^0$ may be written (uniquely) as $A \cup F$ for some $A \in \mathcal{G}^0$ and some finite set F of vertices on the added tail.

For every $A \in \mathcal{G}^0$, define Q_A to equal P_A on the $\mathcal{H}_\mathcal{G}$ component of $\mathcal{H}_\mathcal{F}$ and zero elsewhere. That is, $Q_A(\xi_\mathcal{G}, \xi_1, \xi_2, \ldots) = (P_A \xi_\mathcal{G}, 0, 0, \ldots)$. Similarly, for every $e \in \mathcal{G}^1$ with $s(e) \neq v_0$, define $T_e = S_e$ on the $\mathcal{H}_\mathcal{G}$ component; $T_e(\xi_\mathcal{G}, \xi_i, \xi_2, \ldots) = (S_e \xi_\mathcal{G}, 0, 0, \ldots)$. For each vertex v_n on the added tail, define Q_{v_n} to be the projection onto $\mathcal{H}_n$; $Q_{v_n}(\xi_\mathcal{G}, \xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots) = (0, 0, \ldots, \xi_n, 0, \ldots)$. Note that, because the R_n's are non-decreasing, $\mathcal{H}_n \subseteq \mathcal{H}_{n-1}$ for each n. Thus, for each edge e_n of the tail, we may define T_{e_n} to be the inclusion of $\mathcal{H}_n$ into $\mathcal{H}_{n-1}$ (where $\mathcal{H}_0$ is taken to mean $P_{v_0}(\mathcal{H}_\mathcal{G})$). More precisely,

$$T_{e_n}(\xi_\mathcal{G}, \xi_1, \xi_2, \ldots) = (0, 0, \ldots, 0, \xi_n, 0, \ldots),$$

where the ξ_n is in the $\mathcal{H}_{n-1}$ component.

Finally, for each edge g_j and for each $\xi \in \mathcal{H}_\mathcal{G}$, we have that $S_{g_j}\xi \in \mathcal{H}_{j-1}$. Thus, we can define T_{f_j} as

$$T_{f_j}(\xi_\mathcal{G}, \xi_1, \xi_2, \ldots) = (0, \ldots, 0, S_{g_j}\xi_\mathcal{G}, 0, \ldots),$$

where the nonzero term appears in the $\mathcal{H}_{j-1}$ component.

Now for any $B \in \mathcal{F}^0$ we can (uniquely) write $B := A \cup F$ for some $A \in \mathcal{G}^0$ and some finite subset F of vertices on the added tail. Thus we may define $Q_B := Q_A + \sum_{v \in F} Q_v$. It then follows from calculations similar to those in Lemma 2.10 of [5] that $\{T_e, Q_A\}$ is a Cuntz-Krieger $\mathcal{F}$-family satisfying the bulleted points. ∎

PROPOSITION 6.6. *Let $\mathcal{G}$ be an ultragraph and let $\mathcal{F}$ be a desingularization of $\mathcal{G}$. Then $C^*(\mathcal{G})$ is isomorphic to a full corner of $C^*(\mathcal{F})$.*

Proof. Again, for simplicity we assume $\mathcal{G}$ has only one singular vertex v_0. If v_0 is a sink, then the claim follows from Proposition 6.2. Therefore, let us assume that this singular vertex is an infinite emitter. Let $\{t_e, q_A : e \in \mathcal{F}^1, A \in \mathcal{F}^0\}$ denote the canonical set of generators for $C^*(\mathcal{F})$ and let $\{s_e, p_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\}$ denote the Cuntz-Krieger $\mathcal{G}$-family in $C^*(\mathcal{F})$ constructed in Lemma 6.4. Define $B := C^*(\{s_e, p_A\})$. Also, list the elements of $G^0 = \{w_1, w_2, \ldots\}$ and the elements of $\mathcal{G}^1 = \{h_1, h_2, \ldots\}$. For each nonnegative integer n let $A_n := \{w_i : 1 \leqslant i \leqslant n\} \cup \bigcup_{i=1}^{n} r(h_i)$. It follows from Lemma 6.1 that the sequence $\{p_{A_n}\}$ converges to a projection $p \in \mathcal{M}(C^*(\mathcal{F}))$ satisfying

$$pt_e := \begin{cases} t_e & \text{if } s(e) \in G^0, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad pq_A = q_{A \cap G^0}.$$

From these relations one can see that $B \cong pC^*(\mathcal{F})p$.

We shall now show that $B \cong C^*(\mathcal{G})$. Since B is generated by a Cuntz-Krieger $\mathcal{G}$-family, it suffices to show that B satisfies the universal property for $C^*(\mathcal{G})$. Let $\{S_e, P_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\}$ be a Cuntz-Krieger $\mathcal{G}$-family on a Hilbert space $\mathcal{H}_\mathcal{G}$. Then by Lemma 6.5 we can construct a Hilbert space $\mathcal{H}_\mathcal{F}$ and a Cuntz-Krieger $\mathcal{F}$-family $\{T_e, Q_A : e \in \mathcal{F}^1, A \in \mathcal{F}^0\}$ on $\mathcal{H}_\mathcal{F}$ such that $Q_A = P_A$ for every $A \in \mathcal{G}^0$, $T_e = S_e$ for every $e \in \mathcal{F}^1$ with $s(e) \neq v_0$, and $S_{g_j} = T_{\alpha^j}$ for every edge $g_j \in \mathcal{G}^1$ whose source is v_0. By the universal property of $C^*(\mathcal{F})$ we have a homomorphism π from $C^*(\mathcal{F})$ onto $C^*(\{T_e, Q_A : e \in \mathcal{F}^1, A \in \mathcal{F}^0\})$ that takes t_e to T_e and q_A to Q_A. Now $p_A = q_A$ for any $A \in \mathcal{G}^0$, so $\pi(p_A) = Q_A = P_A$. Let $e \in \mathcal{G}^1$ and $s(e) \neq v_0$. Then $s_e = t_e$ and $\pi(s_e) = T_e = S_e$. Finally, if $s(e) = v_0$, then $e = g_j$ for some j, $s_e = t_{\alpha^j}$, and $\pi(s_{g_j}) = T_{\alpha^j} = S_{g_j}$. Thus $\pi|B$ is a representation of B on $\mathcal{H}_\mathcal{G}$ that takes generators of B to the corresponding elements of the given Cuntz-Krieger $\mathcal{G}$-family. Therefore B satisfies the universal property of $C^*(\mathcal{G})$ and $C^*(\mathcal{G}) \cong B$.

Finally, we note that the corner $C^*(\mathcal{G}) \cong B \cong pC^*(F)p$ is full by an argument similar to the one given in Proposition 6.2. ∎

These results give us a way to extend the uniqueness theorems of Section 5 to C^*-algebras of ultragraphs that contain singular vertices. Also note that an ultragraph satisfies Condition (L) if and only if its desingularization satisfies Condition (L).

THEOREM 6.7. (Uniqueness) *Suppose that $\mathcal{G}$ is an ultragraph and that every loop in $\mathcal{G}$ has an exit. If $\{S_e, P_A\}$ and $\{T_e, Q_A\}$ are two Cuntz-Krieger $\mathcal{G}$-families in which all the projections P_A and Q_A are nonzero, then there is an isomorphism φ of $C^*(S_e, P_A)$ onto $C^*(T_e, Q_v)$ such that $\varphi(S_e) = T_e$ and $\varphi(P_A) = Q_A$.*

Proof. Let $\mathcal{F}$ be a desingularization of $\mathcal{G}$. Then we may use Lemma 6.5 to extend the $\mathcal{G}$-families to $\mathcal{F}$-families. Since every loop in $\mathcal{G}$ has an exit, it follows that every loop in $\mathcal{F}$ has an exit, and thus we may apply Proposition 5.7 to get an isomorphism φ that restricts to our desired isomorphism from $C^*(S_e, P_A)$ onto $C^*(T_e, Q_v)$. ∎

THEOREM 6.8. (Gauge-Invariant Uniqueness) *Let $\mathcal{G}$ be an ultragraph, $\{s_e, p_A\}$ the canonical generators in $C^*(\mathcal{G})$, and γ the gauge action on $C^*(\mathcal{G})$. Also let B be a C^*-algebra, and $\varphi : C^*(\mathcal{G}) \to B$ be a homomorphism for which $\varphi(p_A) \neq 0$ for all nonempty A. If there exists a strongly continuous action β of $\mathbb{T}$ on B such that $\beta_z \circ \varphi = \varphi \circ \gamma_z$ for all $z \in \mathbb{T}$, then φ is faithful.*

Proof. Since $\beta : \mathbb{T} \to \operatorname{Aut} B$ is an action of $\mathbb{T}$ on B, there exists a Hilbert space $\mathcal{H}_\mathcal{G}$, a faithful representation $\pi : B \to \mathcal{B}(\mathcal{H}_\mathcal{G})$, and a unitary representation $U : \mathbb{T} \to U(\mathcal{H}_\mathcal{G})$ such that

$$\pi(\beta_z(x)) = U_z \pi(x) U_z^* \quad \text{for all } x \in B \text{ and } z \in \mathbb{T}.$$

Let $S_e := \pi \circ \varphi(s_e)$ and $P_A := \pi \circ \varphi(p_A)$. Also let $\mathcal{F}$ be a desingularization of $\mathcal{G}$. For simplicity, we shall assume that $\mathcal{G}$ has only one singular vertex v_0.

If v_0 is a sink, then it follows from Proposition 6.2 that there exists a Hilbert space $\mathcal{H}_\mathcal{F} := \mathcal{H}_\mathcal{G} \oplus \mathcal{H}_T = \mathcal{H}_\mathcal{G} \oplus \bigoplus_{i=1}^{\infty} P_{v_0}\mathcal{H}_\mathcal{G}$, and a Cuntz-Krieger $\mathcal{F}$-family $\{T_e, Q_A\}$ in $\mathcal{B}(\mathcal{H}_\mathcal{F})$ that restricts to $\{S_e, P_A\}$. We shall define a unitary representation $V : \mathbb{T} \to U(\mathcal{H}_\mathcal{F})$ as follows: If $h \in \mathcal{H}_\mathcal{G}$, then we define $V_z h := U_z h \in \mathcal{H}_\mathcal{G}$. If $h \in Q_{v_i}\mathcal{H}_\mathcal{G} = P_{v_0}\mathcal{H}_\mathcal{G}$ is in the i^{th} component of $\mathcal{H}_T$, then we define $V_z h := z^{-i} U_z h \in Q_{v_i}\mathcal{H}_\mathcal{G}$ also in the i^{th} component of $\mathcal{H}_T$. We define V_z on all of $\mathcal{H}_\mathcal{F}$ by extending it linearly.

Now let $\pi_{t,q} : C^*(\mathcal{F}) \to \mathcal{B}(\mathcal{H}_\mathcal{F})$, be the representation for which $\pi_{t,q}(t_e) = T_e$ and $\pi_{t,q}(q_A) = Q_A$, and let $\gamma^\mathcal{F}$ be the gauge action on $C^*(\mathcal{F})$. Then one can check that

$$\pi_{t,q}(\gamma_z^\mathcal{F}(x)) = V_z \pi_{t,q}(x) V_z^* \quad \text{for all } x \in C^*(\mathcal{F}) \text{ and all } z \in \mathbb{T}.$$

(To see this simply check the relation on the generators $\{t_e\}$ and use the fact that $\pi \circ \varphi(s_e) = \pi_{t,q}(t_e)$ for all $e \in \mathcal{G}^1$.) Now if we define $\widetilde{\beta} : \mathbb{T} \to \operatorname{Aut} C^*(T_e, Q_A)$ by $\widetilde{\beta}_z(X) := V_z X V_z^*$, then we see that $\widetilde{\beta}_z \circ \pi_{t,q} = \pi_{t,q} \circ \gamma_z^F$ for all $z \in \mathbb{T}$. Since $\pi_{t,q}(q_A) = Q_A \neq 0$ and since $\mathcal{F}$ has no sinks or infinite emitters, it follows from Proposition 5.5 that $\pi_{t,q}$ is faithful. Now, if $\iota : C^*(\mathcal{G}) \to C^*(\mathcal{F})$ denotes the canonical inclusion of $C^*(\mathcal{G})$ into $C^*(\mathcal{F})$, then we see that $\pi_{t,q} \circ \iota = \pi \circ \varphi$ (since each map agrees on the generators $\{s_e, p_A\}$). Because ι and $\pi_{t,q}$ are both injective, it follows that φ is injective.

If v_0 is an infinite emitter, then an argument almost identical to the one above works. We obtain a faithful representation $\pi : C^*(\mathcal{G}) \to \mathcal{H}_\mathcal{G}$ and a unitary representation $U : \mathbb{T} \to U(\mathcal{H}_\mathcal{G})$ as before, and we then extend this Hilbert space to $\mathcal{H}_{\mathcal{F}} := \mathcal{H}_{\mathcal{G}} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_n := \Big(P_{v_0} - \sum_{j=1}^{n-1} S_{g_j} S_{g_j}^*\Big)(\mathcal{H}_{\mathcal{G}})$ as in Lemma 6.5. Similarly, we define $V : \mathbb{T} \to \mathcal{B}(\mathcal{H}_{\mathcal{F}})$ as follows: If $h \in \mathcal{H}_{\mathcal{G}}$, then $V_z h := U_z h \in \mathcal{H}_{\mathcal{G}}$. If $h \in \mathcal{H}_n \subseteq \mathcal{H}_{\mathcal{G}}$, then $V_z h := z^{-n} U_z h \in \mathcal{H}_n$. The rest of the argument follows much like the one above.

If $\mathcal{G}$ has more than one sink or more than one infinite emitter, then we simply account for multiple tails. The previous argument will still work, we need only keep track of the multiple pieces added on when extending $\mathcal{H}_{\mathcal{G}}$ to $\mathcal{H}_{\mathcal{F}}$. ∎

7. CONCLUDING REMARKS

We have seen in this paper that ultragraph algebras contain both the Exel-Laca algebras and the graph algebras. Furthermore, it is shown in a forthcoming article ([18]) that there exist ultragraph algebras that are neither Exel-Laca algebras nor graph algebras. Throughout this paper we have seen that many of the techniques of graph algebras can be applied to ultragraph algebras and that analogues of the results for graph algebras and Exel-Laca algebras hold for ultragraph algebras.

These observations are important for many reasons. First of all, ultragraphs give a context in which many results concerning graph algebras and Exel-Laca algebras may be proven simultaneously. In the past, many similar results (e.g. the Cuntz-Krieger Uniqueness Theorem, the Gauge-Invariant Uniqueness Theorem) were proven separately for graph algebras and for Exel-Laca algebras. Since ultragraph algebras contain both the graph algebras and the Exel-Laca algebras, it suffices to prove these results once for ultragraph algebras. Hence these classes are in some sense unified under the umbrella of ultragraph algebras. Second, since many of the graph techniques may be used for ultragraphs, we see that we may study Exel-Laca algebras in this context and the (often complicated) matrix techniques may be avoided in favor of graph techniques. Finally, ultragraph algebras are a larger class of C^*-algebras than the Exel-Laca algebras and the graph algebras. Thus with only slightly more work, we are able to extend these results to a larger class of C^*-algebras.

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