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NORMS OF ITERATES OF VOLTERRA OPERATORS ON L^2

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ABSTRACT. It has recently been established that if V is the classical Volterra (indefinite integration) operator acting on the Hilbert space $L^2([0,1])$, then the operator and Hilbert-Schmidt norms of V^n are both asymptotically $1/(2n!)$. We extend this in two ways: firstly, we give a generalisation which applies to Volterra convolution operators with kernels satisfying a mild smoothness condition, and secondly we show that in the constant-kernel case the same asymptotic behaviour is shared by the trace norm, and hence by a wide class of operator norms.

KEYWORDS: *Volterra operators.*

MSC (2000): 47G10.

1. INTRODUCTION

Consider the Volterra operator V acting on the Hilbert space $L^2([0,1],\mathbb{R})$ defined by

$$(Vu)(t) = \int_0^t u(s)\,\mathrm{d}s.$$

Lao and Whitley (Equation (17), [5]), prompted by a question raised by Halmos, showed that

$$\frac{1}{2\sqrt{\mathrm{e}}} \leqslant \liminf_{n\to\infty} n!\|V^n\| \leqslant \limsup_{n\to\infty} n!\|V^n\| \leqslant \frac{1}{2}$$

where $\|\cdot\|$ denotes the operator norm, and conjectured based on numerical evidence (Equation (16), [5]) that $n!\|V^n\| \to 1/2$ as $n\to\infty$. This conjecture was proved by Kershaw ([4]), who showed that

$$\frac{1}{2n!} \leqslant \|V^n\| \leqslant \frac{1}{2n!}\Big(1-\frac{1}{2n}\Big)^{-1/2} = \|V^n\|_2$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. In particular, both $\|V^n\|$ and $\|V^n\|_2$ are asymptotically equal to $1/(2n!)$ as $n \to \infty$. This result has been reproduced by Little and Reade ([6]) using slightly more elementary techniques. Independently, Thorpe (Remark 3, [10]) has shown that $\|V^n\|$ and $\|V^n\|_2$ are both asymptotically equal to $1/(2\Gamma(n+1))$ as $n \to \infty$ through $\mathbb{R}^+$, not just $\mathbb{N}$, where V^n is the Riemann-Liouville fractional integration operator defined for any real $n > 0$ by

$$(V^n u)(t) = \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} u(s)\,\mathrm{d}s, \tag{1.1}$$

(which agrees with the definition of V^n as the nth iterate of V for $n \in \mathbb{N}$).

Concerning the class of Volterra convolution operators V_k defined on $L^2([0,1],\mathbb{R})$ by

$$(V_k u)(t) = \int_0^t k(t-s)u(s)\,\mathrm{d}s$$

for $k \in L^1([0,1],\mathbb{R})$, Lao and Whitley also asked "for which convolution operators V_k does $\lim\limits_{n\to\infty} n!\|V_k^n\|$ exist and how can this limit be determined from the kernel function k?"

This paper addresses this and more general questions. To answer exactly this question, if k is any real L^1 function such that $k'(0)$ exists and $k(0) = \pm 1$ then

$$n!\|V_k^n\| \to \frac{\mathrm{e}^{k'(0)/k(0)}}{2}$$

as $n \to \infty$, and the same is true for the Hilbert-Schmidt norm. More generally, if $k(t) = f(t)t^r$ for some $r > -1$, $f(0) \neq 0$ and $f'(0)$ exists then

$$\|V_k^n\| \sim \|V_k^n\|_2 \sim \frac{\big(\Gamma(r+1)|f(0)|\big)^n \mathrm{e}^{f'(0)/f(0)}}{2\Gamma((r+1)n+1)}$$

as $n \to \infty$.

Returning to the fractional integration operators defined in (1.1), we also show that

$$\|V^n\|_1 \sim \frac{1}{2\Gamma(n+1)}$$

as $n \to \infty$ through $\mathbb{R}^+$, where $\|\cdot\|_1$ denotes the trace norm. It is then immediate that

$$\Phi(V^n) \sim \frac{1}{2\Gamma(n+1)}$$

where Φ is any Schatten-von Neumann norm corresponding to a symmetric sequence norm which is normalised so that the singular value sequence $(1,0,0,\ldots)$ has norm 1.

2. NOTATION

The singular values of a bounded operator T on a separable Hilbert space H are denoted by $(\sigma_n(T))_{n\geqslant 0}$, defined by

$$\sigma_n(T) = \inf\{\|T - R\| : \operatorname{rank}(R) \leqslant n\}.$$

It follows immediately from this definition that

$$\sigma_{m+n}(S + T) \leqslant \sigma_m(S) + \sigma_n(T). \tag{2.1}$$

As mentioned above, we denote by $\|\cdot\|$ the operator norm and by $\|\cdot\|_2$ the Hilbert-Schmidt norm. More generally, we shall use $\|\cdot\|_p$ for the Schatten p-norm defined by

$$\|T\|_p^p = \sum_{n=0}^{\infty} \sigma_n^p(T)$$

and $\mathcal{I}_p(H)$ for the ideal of operators on H for which this sum converges. For functions on the real line, we use $\|\cdot\|_p$ to denote the L^p norm; it should always be clear from context which norm is meant.

If $k \in L^1([0,1],\mathbb{R})$ we define the Volterra operator V_k on $L^2([0,1],\mathbb{R})$ by

$$(V_k u)(t) = \int_0^t k(t-s)u(s)\,\mathrm{d}s.$$

The Hilbert-Schmidt norm of V_h is evidently given by

$$\|V_k\|_2^2 = \int_0^1 (1-x)k(x)^2\,\mathrm{d}x.$$

For $n \in \mathbb{N}$, the nth convolution power k^{*n} of k is defined for $t \in [0,1]$ by

$$k^{*1}(t) = k(t), \quad k^{*(n+1)}(t) = \int_0^t k^{*n}(s)k(t-s)\,\mathrm{d}s,$$

so $V_{k^{*n}} = (V_k)^n$.

If (a_n) and (b_n) are positive real sequences indexed by either $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, we write $a_n \sim b_n$ as $n \to \infty$ to mean that $a_n/b_n \to 1$ as $n \to \infty$, $a_n \lesssim b_n$ as $n \to \infty$ to mean that $\limsup_{n\to\infty} a_n/b_n \leqslant 1$ and $a_n \gtrsim b_n$ as $n \to \infty$ to mean that $\liminf_{n\to\infty} a_n/b_n \geqslant 1$.

3. APPROXIMATION BY RANK 1 OPERATORS

A striking feature of Kershaw's and Thorpe's results is that the operator and Hilbert-Schmidt norms of V^n are asymptotically equal to each other. These norms only coincide exactly for operators of rank 0 or 1, so the fact that $\|V^n\|/\|V^n\|_2 \to 1$ as $n \to \infty$ should mean that V^n is tending towards the set of rank 1 operators in some sense. This idea, generalised to any Schatten p-class with $p < \infty$, is made precise in the following result (in fact, we shall only use $p = 1$ and $p = 2$ below).

Lemma 3.1. *Suppose that H is a Hilbert space and that (S_n) is a sequence of non-zero operators in $\mathcal{I}_p(H)$, where $1 \leqslant p < \infty$. Then the following are equivalent:*

(i) $\|S_n\|/\|S_n\|_p \to 1$ *as* $n \to \infty$*;*

(ii) *there exists a sequence (T_n) of rank 1 operators on H such that $\|S_n - T_n\|_p/\|S_n\|_p \to 0$ as $n \to \infty$.*

Proof. Let $r_n = \|S_n\|/\|S_n\|_p$ and $d_n = \inf\{\|S_n - S\|_p : \operatorname{rank}(S) \leqslant 1\}/\|S_n\|_p$. We claim that $r_n^p + d_n^p = 1$, from which the equivalence of the two statements immediately follows.

We have $\|S_n\| = \sigma_0(S_n)$, $\|S_n\|_p^p = \sum\limits_{j=0}^{\infty} \sigma_j^p(S_n)$ and the $\|\cdot\|_p$ distance between S_n and the set of rank 1 operators is $\Big(\sum\limits_{j=1}^{\infty} \sigma_j^p(S_n)\Big)^{1/p}$ (the finite-dimensional version of this is due to Mirsky ([7]), or see Example 7.4.52 in [3], and the extension to infinitely many dimensions is immediate). We now have

$$r_n^p = \frac{\sigma_0^p(S_n)}{\sum\limits_{j=0}^{\infty} \sigma_j^p(S_n)}, \quad d_n^p = \frac{\sum\limits_{j=1}^{\infty} \sigma_j^p(S_n)}{\sum\limits_{j=0}^{\infty} \sigma_j^p(S_n)}$$

so $r_n^p + d_n^p = 1$, as claimed. ∎

The well-known theorem of Adamjan, Arov and Krein, or "AAK Theorem" (see for example Chapter 6 in [8]), states that exactly one of the operator-norm optimal rank n approximants to a compact Hankel operator Γ, is itself a Hankel operator. This is not in general true for norms other than the operator norm, but there are various weaker results known; see Section 6 in [2]. We shall need only the following very special case which, for the sake of completeness, we prove.

Lemma 3.2. *Suppose that H is a Hilbert space and that $\Gamma \in \mathcal{I}_p(H)$ is a Hankel operator. Let Γ' be the rank 1 Hankel operator with $\|\Gamma - \Gamma'\| = \sigma_1(\Gamma)$. Then*

$$\|\Gamma - \Gamma'\|_2 \leqslant \sqrt{3} \inf\{\|\Gamma - T\|_2 : \operatorname{rank}(T) \leqslant 1\}.$$

Proof. Applying equation (2.1), we see that

$$\sigma_{j+1}(\Gamma - \Gamma') \leqslant \sigma_j(\Gamma) + \sigma_1(\Gamma') = \sigma_j(\Gamma)$$

since $\operatorname{rank}(\Gamma')=1$. Now,

$$\begin{aligned}\|\Gamma-\Gamma'\|_2^2 &= \sum_{j=0}^{\infty}\sigma_j^2(\Gamma-\Gamma') = \sigma_0^2(\Gamma-\Gamma')+\sigma_1^2(\Gamma-\Gamma')+\sum_{j=1}^{\infty}\sigma_{j+1}^2(\Gamma-\Gamma')\\ &\leqslant 2\sigma_1^2(\Gamma)+\sum_{j=1}^{\infty}\sigma_j^2(\Gamma)\leqslant 3\sum_{j=1}^{\infty}\sigma_j^2(\Gamma)=3\inf\{\|\Gamma-T\|_2^2:\operatorname{rank}(T)\leqslant 1\}\end{aligned}$$

again using the result of Mirsky cited in the proof of Lemma 3.1. ∎

For convenience, we also record the trivial results:

LEMMA 3.3. *Let (p_n) and (q_n) be sequences in a normed space. Then:*

(i) $\|p_n-q_n\|/\|q_n\|\to 0$ *as* $n\to\infty$ *if and only if* $\|p_n-q_n\|/\|p_n\|\to 0$ *as* $n\to\infty$*;*

(ii) *if* $\|p_n-q_n\|/\|q_n\|\to 0$ *as* $n\to\infty$ *then* $\|p_n\|/\|q_n\|\to 1$ *as* $n\to\infty$*;*

(iii) *in an inner product space,* $|\langle p_n,q_n\rangle|/(\|p_n\|\,\|q_n\|)\to 1$ *as* $n\to\infty$ *if and only if there exists a sequence* (α_n) *of non-zero scalars such that* $\|p_n-\alpha_n q_n\|/\|p_n\|\to 0$ *as* $n\to\infty$.

For the time being, we shall only consider the Hilbert-Schmidt and operator norms, returning to other norms in Section 8. We show first that when $S_n=V_h^n$ in Lemma 3.1, we need only consider rank 1 approximants T of the form

$$(Tu)(t)=\int_0^1 \alpha e^{\beta(t-s)}u(s)\,ds.$$

THEOREM 3.4. *Suppose that $k\in L^1([0,1],\mathbb{R})$ is such that V_k^n is Hilbert-Schmidt for some $n\in\mathbb{N}$. Then the following are equivalent:*

(i) $\|V_k^n\|/\|V_k^n\|_2\to 1$ *as* $n\to\infty$*;*

(ii) *there exist real sequences (α_n) and (β_n) with $\beta_n\to\infty$ as $n\to\infty$ such that*

$$\frac{\beta_n^2}{\alpha_n^2 e^{2\beta_n}}\int_0^1(1-x)(k^{*n}(x)-\alpha_n e^{\beta_n x})^2\,dx\to 0\quad \textit{as } n\to\infty;$$

(iii) *there exists a real sequence (β_n) tending to ∞ such that*

$$\frac{4\beta_n^2\Big(\int_0^1(1-x)k^{*n}(x)e^{\beta_n x}\,dx\Big)^2}{e^{2\beta_n}\int_0^1(1-x)k^{*n}(x)^2\,dx}\to 1\quad \textit{as } n\to\infty.$$

Proof. (i) ⇒ (ii) If $\|V_k^n\|\sim\|V_k^n\|_2$ as $n\to\infty$ then by Lemma 3.1 there exists a sequence (T_n) of rank 1 operators such that $\|V_k^n-T_n\|_2/\|V_k^n\|_2\to 0$ as $n\to\infty$.

Consider the unitary operator U on $L^2([0,1])$ defined by $(Uu)(t) = u(1-t)$, and extend the operators UV_k^n and $S_n := UT_n$ to map from $L^2([0,\infty))$ to itself by making them zero on $L^2([1,\infty))$. We have

$$(UV_k^n u)(t) = \int_0^{1-t} k^{*n}(1-t-s)u(s)\,\mathrm{d}s$$

for $0 \leqslant t \leqslant 1$, so if we define

$$k_n(y) = \begin{cases} k^{*n}(1-y) & \text{if } 0 \leqslant y \leqslant 1, \\ 0 & \text{if } y > 1, \end{cases}$$

then we have

$$(\Gamma_n u)(t) = \int_0^{\infty} k_n(t+s)u(s)\,\mathrm{d}s$$

for $u \in L^2([0,\infty)$ and $t \geqslant 0$, which shows that Γ_n is a Hankel operator on $L^2([0,\infty))$. Moroever, S_n has rank 1 and

$$\frac{\|\Gamma_n - S_n\|_2}{\|\Gamma_n\|_2} = \frac{\|UV_k^n - UT_n\|_2}{\|UV_k^n\|_2} = \frac{\|V_k^n - T_n\|_2}{\|V_k^n\|_2} \to 0$$

as $n \to \infty$. Now, Lemma 3.2 shows that there exists a sequence of rank 1 Hankel operators Γ'_n such that

$$\frac{\|\Gamma_n - \Gamma'_n\|_2}{\|\Gamma_n\|_2} \leqslant \frac{\sqrt{3}\|\Gamma_n - S_n\|_2}{\|\Gamma_n\|_2} \to 0$$

as $n \to \infty$ which implies (Lemma 3.3 part (i)) that

$$\frac{\|\Gamma_n - \Gamma'_n\|_2}{\|\Gamma'_n\|_2} \to 0$$

as $n \to \infty$. Rank 1 Hankel operators have a very simple structure: we have

$$(\Gamma'_n u)(t) = \int_0^{\infty} \gamma_n \mathrm{e}^{-\beta_n(t+s)}u(s)\,\mathrm{d}s$$

for some $\gamma_n, \beta_n \in \mathbb{R}$ with $\beta_n > 0$, and

$$\|\Gamma'_n\|_2^2 = \int_0^{\infty} \gamma_n^2 y \mathrm{e}^{-2\beta_n y}\,\mathrm{d}y = \frac{\gamma_n^2}{4\beta_n^2}.$$

This gives

$$\frac{4\beta_n^2}{\gamma_n^2} \int_0^{\infty} y(k_n(y) - \gamma_n \mathrm{e}^{-\beta_n y})^2\,\mathrm{d}y \to 0$$

as $n \to \infty$. Since $k_n(y) = 0$ if $y > 1$, we have

$$\begin{aligned}\frac{\|\Gamma_n - \Gamma'_n\|_2}{\|\Gamma'_n\|_2} &= \frac{4\beta_n^2}{\gamma_n^2}\int_0^1 y(k_n(y) - \gamma_n \mathrm{e}^{-\beta_n y})^2\,\mathrm{d}y + \frac{4\beta_n^2}{\gamma_n^2}\int_1^\infty \gamma_n^2 y \mathrm{e}^{-2\beta_n y}\,\mathrm{d}y \\ &= \frac{4\beta_n^2}{\gamma_n^2}\int_0^1 y(k_n(y) - \gamma_n \mathrm{e}^{-\beta_n y})^2\,\mathrm{d}y + \mathrm{e}^{-2\beta_n}(\beta_n + 1).\end{aligned}$$

But this sum tends to 0 as $n \to \infty$ and both terms are positive, so both terms must tend to zero; this shows that $\beta_n \to \infty$ and that

$$\frac{\beta_n^2}{\gamma_n^2}\int_0^1 y(k_n(y) - \gamma_n \mathrm{e}^{-\beta_n y})^2\,\mathrm{d}y \to 0$$

as $n \to \infty$. Now, $k_n(y) = k^{*n}(1-y)$ for $0 \leqslant y \leqslant 1$; using this and making the substitution $y = 1 - x$ in the above integral gives

$$\frac{\beta_n^2}{\gamma_n^2}\int_0^1 (1-x)(k^{*n}(x) - \gamma_n \mathrm{e}^{-\beta_n}\mathrm{e}^{\beta_n x})^2\,\mathrm{d}x \to 0$$

as $n \to \infty$. Finally, write $\alpha_n = \gamma_n \mathrm{e}^{-\beta_n}$ to give

$$\frac{\beta_n^2}{\alpha_n^2 \mathrm{e}^{2\beta_n}}\int_0^1 (1-x)(k^{*n}(x) - \alpha_n \mathrm{e}^{\beta_n x})^2\,\mathrm{d}x \to 0$$

which is statement (ii).

(ii) $\Rightarrow$ (i) By Lemma 3.1 and Lemma 3.3 part (i), $\|V_k^n\|/\|V_k^n\|_2 \to 1$ if and only if there exists a sequence (T_n) of rank 1 operators such that $\|V_k^n - T_n\|_2^2/\|T_n\|_2^2 \to 0$. It is therefore sufficient for this that sequences (α_n) and (β_n) exist with β_n tending to ∞ such that the operators (T_n) defined by

$$(T_n u)(t) = \int_0^1 \alpha_n \mathrm{e}^{\beta_n(t-s)} u(s)\,\mathrm{d}s$$

satisfy $\|V_k^n - T_n\|_2^2/\|T_n\|_2^2 \to 0$. We can evaluate these Hilbert-Schmidt norms by breaking the square into an upper and a lower triangle and making the obvious substitutions to give:

$$\|V_k^n - T_n\|_2^2 = \underbrace{\int_0^1 (1-x)(k^{*n}(x) - \alpha_n \mathrm{e}^{\beta_n x})^2\,\mathrm{d}x}_{=:a_n} + \underbrace{\frac{\alpha_n^2}{4\beta_n^2}(\mathrm{e}^{-2\beta_n} + 2\beta_n - 1)}_{=:b_n}$$

where the first term, a_n, is from the upper triangle and the second term, b_n, is from the lower triangle, and that

$$\|T_n\|_2^2 = \underbrace{\int_0^1 (1-x)(\alpha_n e^{\beta_n x})^2 \, dx}_{=:c_n} + \underbrace{\frac{\alpha_n^2}{4\beta_n^2}(e^{-2\beta_n} + 2\beta_n - 1)}_{=b_n}$$

$$= \underbrace{\frac{\alpha_n^2}{4\beta_n^2}(e^{2\beta_n} - 2\beta_n - 1)}_{=c_n} + \underbrace{\frac{\alpha_n^2}{4\beta_n^2}(e^{-2\beta_n} + 2\beta_n - 1)}_{=b_n}$$

where again the first term, c_n, is from the upper triangle and the second term, again b_n, is from the lower triangle. Now,

$$\frac{\|V_k^n - T_n\|_2^2}{\|T_n\|_2^2} = \frac{a_n + b_n}{c_n + b_n} = \frac{\frac{a_n}{c_n} + \frac{b_n}{c_n}}{1 + \frac{b_n}{c_n}}$$

and $b_n/c_n \to 0$ as $n \to \infty$ since $\beta_n \to \infty$ as $n \to \infty$. We therefore have $(\|V_k^n - T_n\|_2)/\|T_n\| \to 0$ as $n \to \infty$ if and only if $a_n/c_n \to 0$ as $n \to \infty$; explicitly

$$\frac{4\beta_n^2 \int_0^1 (1-x)(k^{*n}(x) - \alpha_n e^{\beta_n x})^2 \, dx}{\alpha_n^2(e^{2\beta_n} - 2\beta_n - 1)} \to 0$$

as $n \to \infty$. But since $\beta_n \to \infty$, this is equivalent to

$$\frac{\beta_n^2}{\alpha_n^2 e^{2\beta_n}} \int_0^1 (1-x)(k^{*n}(x) - \alpha_n e^{\beta_n x})^2 \, dx \to 0 \quad \text{as } n \to \infty$$

which is exactly statement (ii).

(ii) $\Leftrightarrow$ (iii) In the notation from the previous part, write a_n/c_n in the form

$$\frac{\int_0^1 (1-x)(k^{*n}(x) - \alpha_n e^{\beta_n x})^2 \, dx}{\int_0^1 (1-x)(\alpha_n e^{\beta_n x})^2 \, dx}$$

which is of the form $\|p_n - \alpha_n q_n\|^2/\|\alpha_n q_n\|^2$ where (p_n) and (q_n) are sequences in a weighted L^2 space. By Lemma 3.3 (iii), this is equivalent to $|\langle p_n, q_n\rangle|/(\|p_n\|\,\|q_n\|) \to 1$ as $n \to \infty$, that is

$$\frac{\left(\int_0^1 (1-x)k^{*n}(x)e^{\beta_n x} \, dx\right)^2}{\int_0^1 (1-x)k^{*n}(x)^2 \, dx \int_0^1 (1-x)e^{2\beta_n x} \, dx} \to 1$$

as $n \to \infty$. Now,

$$\int_0^1 (1-x)e^{2\beta_n x} \, dx = \frac{e^{2\beta_n} - \beta_n - 1}{4\beta_n^2} \sim \frac{e^{2\beta_n}}{4\beta_n^2}$$

as $n \to \infty$, so this is equivalent to

$$\frac{4\beta_n^2 \Big(\int_0^1 (1-x)k^{*n}(x)\mathrm{e}^{\beta_n x}\,\mathrm{d}x \Big)^2}{\mathrm{e}^{2\beta_n} \int_0^1 (1-x)k^{*n}(x)^2\,\mathrm{d}x} \to 1$$

as $n \to \infty$, which is exactly the second condition above. ∎

4. SOME ASYMPTOTIC EXPANSIONS OF INTEGRALS

In order to apply Theorem 3.4 to some concrete examples, we need to find the asymptotic behaviour of some integrals. Consider the Laplace integral

$$I(\lambda) = \int_a^b f(t)\mathrm{e}^{-\lambda\varphi(t)}\,\mathrm{d}t$$

where φ and f are C^2 in a neighbourhood of b, the absolute minimum of φ over $[a,b]$ is attained at b, and $\varphi'(b) < 0$. It is well known that the first term in the asymptotic expansion of $I(\lambda)$ is

$$I(\lambda) \sim -\mathrm{e}^{-\lambda\varphi(b)} \frac{f(b)}{\varphi'(b)} \frac{1}{\lambda}$$

as $\lambda \to \infty$. For the applications below, we shall also need the second term in this expansion. This seems to be less easy to locate in the literature, so we sketch a proof.

Under the stated hypotheses, φ is certainly invertible in a neighbourhood of b, and since φ has an absolute minimum at b, standard considerations show that its values outside any neighbourhood of b have no effect on the aymptotic behaviour of $I(\lambda)$ as $\lambda \to \infty$. We may therefore assume without loss of generality that φ is invertible on $[a,b]$.

Using the same approach and notation as Bleistein and Handelsman (Section 5.1, [1]) use for the case of a stationary extremum, we write

$$I(\lambda) = -\mathrm{e}^{-\lambda\varphi(b)}(\mathcal{L}G)(\lambda), \qquad \text{where } G(\tau) = \frac{f(t)}{\varphi'(t)}\Big|_{t=\varphi^{-1}(\varphi(b)+\tau)}$$

and $\mathcal{L}$ denotes the Laplace transform. To obtain the asymptotics of $\mathcal{L}G$, we need to expand G as a Taylor series about the origin and apply Watson's Lemma (see for example Section 4.1 in [1]).

Plainly, $G(0) = f(b)/\varphi'(b)$. Next,

$$G'(\tau) = \frac{\varphi'(t)f'(t) - f(t)\varphi''(t)}{\varphi'(t)^2}\Big|_{t=\varphi^{-1}(\varphi(b)+\tau)} \frac{1}{\varphi'(\varphi^{-1}(\varphi(b)+\tau))}$$

so

$$G'(0) = \frac{\varphi'(b)f'(b) - f(b)\varphi''(b)}{\varphi'(b)^3}.$$

Now, Watson's Lemma shows that

$$(\mathcal{L}G)(\lambda) \sim \frac{f(b)}{\varphi'(b)}\frac{1}{\lambda} + \frac{\varphi'(b)f'(b) - f(b)\varphi''(b)}{\varphi'(b)^3}\frac{1}{\lambda^2}$$

as $\lambda \to \infty$. It follows that

$$I(\lambda) \sim -\mathrm{e}^{-\lambda\varphi(b)}\Big(\frac{f(b)}{\varphi'(b)}\frac{1}{\lambda} + \frac{\varphi'(b)f'(b) - f(b)\varphi''(b)}{\varphi'(b)^3}\frac{1}{\lambda^2}\Big)$$

as $\lambda \to \infty$.

We now record for future reference the asymptotic behaviour of some Laplace type integrals. All of these results can be obtained from the formula given above.

$$\int_0^1 (1-x)x^m \mathrm{e}^{(\xi+m)x}\,\mathrm{d}x \sim \frac{\mathrm{e}^{m+\xi}}{4m^2} \tag{4.1}$$

$$\int_0^1 (1-x)\mathrm{e}^{\xi x}x^m\,\mathrm{d}x \sim \frac{\mathrm{e}^{\xi}}{m^2} \tag{4.2}$$

$$\int_0^{1-\delta} \mathrm{e}^{\xi x}x^m \sim \mathrm{e}^{\xi(1-\delta)}\frac{(1-\delta)^{m-1}}{m} \tag{4.3}$$

as $m \to \infty$ for any $\xi \in \mathbb{R}$ and $\delta \in (0,1)$. Finally, it is an immediate consequence of Stirling's formula that for any $a, b \in \mathbb{R}$ with $a > 0$,

$$\frac{\Gamma(am+b)}{\Gamma(am)} \sim (am)^b \tag{4.4}$$

as $m \to \infty$.

5. KERNELS OF THE FORM $k(t) = e^{\mu t}t^r$

We now consider kernels of the form $k(t) = \mathrm{e}^{\mu t}t^r$ for $r, \mu \in \mathbb{R}$ with $r > -1$. It is easy to check, either by induction or by using a Laplace transform, that the nth convolution power of k is given by

$$k^{*n}(t) = \frac{(\Gamma(r+1))^n}{\Gamma((r+1)n)}\mathrm{e}^{\mu t}t^{(r+1)n-1}.$$

Moreover, we can use this as a definition of k^{*n}, and hence of V_k^n, for any real $n > 0$. With this definition, we can give our first extension of Kershaw's and Thorpe's results.

THEOREM 5.1. *If $k(t) = c\mathrm{e}^{\mu t}t^r$ where $\mu, r \in \mathbb{R}$ with $r > -1$ then*

$$\|V_k^n\| \sim \|V_k^n\|_2 \sim \frac{(c\Gamma(r+1))^n \mathrm{e}^{\mu}}{2\Gamma((r+1)n+1)}$$

as $n \to \infty$ through $\mathbb{R}^+$.

Proof. Firstly, it is apparent from the explicit formula above that V_k^n is a Hilbert-Schmidt operator for sufficiently large n; precisely, if $2(r+1)n > 1$.

We claim that the condition in Theorem 3.4, item (iii), is satisfied with $\beta_n = (r+1)n - 1$. We need to show that

$$\frac{4\beta_n^2\Big(\int\limits_0^1 (1-x)(k^{*n}(x)\mathrm{e}^{\beta_n x}\,\mathrm{d}x\Big)^2}{\mathrm{e}^{2\beta_n}\int\limits_0^1 (1-x)k^{*n}(x)^2\,\mathrm{d}x} \to 1$$

as $n \to \infty$. Substituting the explicit formula for k^{*n} in the left-hand side gives

$$\frac{4\beta_n^2\Big(\int\limits_0^1 (1-x)\mathrm{e}^{(\mu+\beta_n)x}x^{\beta_n}\,\mathrm{d}x\Big)^2}{\mathrm{e}^{2\beta_n}\int\limits_0^1 (1-x)\mathrm{e}^{2\mu x}x^{2\beta_n}\,\mathrm{d}x}.$$

Now, the integral in the numerator is asymptotically $\mathrm{e}^{\mu+\beta_n}/(4\beta_n^2)$ as $n \to \infty$ by equation (4.1) with $\xi = \mu$ and $m = \beta_n$ and the integral in the denominator is asymptotically $\mathrm{e}^{2\mu}/(4\beta_n^2)$ as $n \to \infty$ by equation (4.2) with $\xi = 2\mu$ and $m = 2\beta_n$. Thus, the whole expression is asymptotically 1 as $n \to \infty$, as required.

It now follows from Theorem 3.4 that $\|V_k^n\| \sim \|V_k^n\|_2$ as $n \to \infty$. But

$$\|V_k^n\|_2^2 = \Big(\frac{(c\Gamma(r+1))^n}{\Gamma((r+1)n)}\Big)^2 \int\limits_0^1 (1-x)\mathrm{e}^{2\mu x}x^{2\beta_n}\,\mathrm{d}t \sim \Big(\frac{(c\Gamma(r+1))^n}{\Gamma((r+1)n)}\Big)^2 \frac{\mathrm{e}^{2\mu}}{4\beta_n^2}$$

by equation (4.2). We thus have that

$$\|V_k^n\| \sim \|V_k^n\|_2 \sim \frac{(c\Gamma(r+1))^n\mathrm{e}^{\mu}}{((r+1)n-1)\Gamma((r+1)n)} \sim \frac{(c\Gamma(r+1))^n\mathrm{e}^{\mu}}{\Gamma((r+1)n+1)}$$

as $n \to \infty$. ∎

6. ESTIMATES FOR MORE GENERAL KERNELS

If a kernel can be bounded above or below by a function of the form considered in the previous section, then we can make obvious estimates on the behaviour of its iterates. We need only observe that if $0 \leqslant k \leqslant h$ then for all $n \in \mathbb{N}$ we have $0 \leqslant k^{*n} \leqslant h^{*n}$ and hence $\|V_k^n\| \leqslant \|V_h^n\|$ and $\|V_k^n\|_2 \leqslant \|V_h^n\|_2$ to see that:

LEMMA 6.1. *Suppose that $k \in L^1([0,1],\mathbb{R})$, that $c, r, \mu, \nu \in \mathbb{R}$ with $r > -1$ and $c > 0$ and that $ct^r\mathrm{e}^{\mu t} \leqslant k(t) \leqslant ct^r\mathrm{e}^{\nu t}$ for $t \in [0,1]$. Then*

$$\frac{(c\Gamma(r+1))^n\mathrm{e}^{\mu}}{\Gamma((r+1)n+1)} \lesssim \|V_k^n\| \leqslant \|V_k^n\|_2 \lesssim \frac{(c\Gamma(r+1))^n\mathrm{e}^{\nu}}{\Gamma((r+1)n+1)}.$$

In particular,

$$\mathrm{e}^{\mu-\nu} \lesssim \frac{\|V_k^n\|}{\|V_k^n\|_2} \leqslant 1$$

as $n \to \infty$.

We shall need another result concerning kernels of this type. The significance of this will become apparent later, but it captures in a particular technical sense the idea that convolution powers take on much larger values near 1 than they do near 0.

Lemma 6.2. *Suppose that* $k \in L^1([0,1],\mathbb{R})$, *that* $c, r, \mu, \nu \in \mathbb{R}$ *with* $r > -1$ *and* $c > 0$ *and that* $ct^r e^{\mu t} \leqslant k(t) \leqslant ct^r e^{\nu t}$ *for* $t \in [0,1]$. *Then for any* $\delta \in (0,1)$, *any* $j \in \mathbb{Z}$ *and any polynomial* P,

$$P(n)\frac{\int_0^{1-\delta} k^{*(n-j)}(x)^2\,\mathrm{d}x}{\int_0^1 (1-x)k^{*n}(x)^2\,\mathrm{d}x} \to 0$$

as $n \to \infty$.

Proof. Since $0 < ct^r e^{\mu t} \leqslant k(t) \leqslant ct^r e^{\nu t}$ for all $t \in [0,1]$, it follows that for all $n \in \mathbb{N}$,

$$0 < \frac{(c\Gamma(r+1))^n}{\Gamma((r+1)n)} e^{\mu t} t^{(r+1)n-1} \leqslant k^{*n}(t) \leqslant \frac{(c\Gamma(r+1))^n}{\Gamma((r+1)n)} e^{\nu t} t^{(r+1)n-1}.$$

The left-hand inequality gives

$$\begin{aligned}\int_0^1 (1-x)k^{*n}(x)^2\,\mathrm{d}x &\geqslant \frac{(c\Gamma(r+1))^{2n}}{\Gamma((r+1)n)^2}\int_0^1 (1-x)e^{2\mu x}x^{2(r+1)n-2}\,\mathrm{d}x \\ &\sim \frac{(c\Gamma(r+1))^{2n}}{\Gamma((r+1)n)^2}\frac{e^{2\mu}}{(2(r+1)n)^2}\end{aligned}$$

by equation (4.2) with $\xi = 2\mu$ and $m = 2(r+1)n-2$. The right-hand inequality with $n-j$ in place of n gives

$$\begin{aligned}\int_0^{1-\delta} k^{*(n-j)}(x)^2\,\mathrm{d}x &\leqslant \frac{(c\Gamma(r+1))^{2(n-j)}}{\Gamma((r+1)(n-j))^2}\int_0^{1-\delta} e^{2\nu x}x^{2(r+1)(n-j)-2}\,\mathrm{d}x \\ &\sim \frac{(c\Gamma(r+1))^{2(n-j)}}{\Gamma((r+1)(n-j))^2} e^{2\nu(1-\delta)}\frac{(1-\delta)^{2(r+1)(n-j)-2}}{2(r+1)n}\end{aligned}$$

by equation (4.3) with $\xi = 2\nu$ and $m = 2(r-1)(n-j)-2$. We therefore have

$$\begin{aligned}&P(n)\frac{\int_0^{1-\delta} k^{*(n-j)}(x)^2\,\mathrm{d}x}{\int_0^1 (1-x)k^{*n}(x)^2\,\mathrm{d}x} \\ &\lesssim P(n)\frac{(c\Gamma(r+1))^{2(n-j)}}{\Gamma((r+1)(n-j))^2}\frac{(1-\delta)^{2(r+1)(n-j)-2}}{2(r+1)n}e^{2\nu(1-\delta)}\frac{\Gamma((r+1)n)^2}{(c\Gamma(r+1))^{2n}}\frac{(2(r+1)n)^2}{e^{2\mu}} \\ &= CnP(n)\frac{\Gamma((r+1)n)^2}{\Gamma((r+1)(n-j))^2}(1-\delta)^{2(r+1)n}\end{aligned}$$

where $C = 2(r+1)e^{2\nu(1-\delta)-2\mu}(c\Gamma(r+1))^{-2j}$, which does not depend on n

$$\sim CnP(n)((r+1)n)^{2(r+1)j}(1-\delta)^{2(r+1)(n-j)-2}$$

by equation (4.4). This tends to zero as $n \to \infty$ since $r > -1$ and $0 < \delta < 1$. ∎

7. MAIN RESULTS

We have now assembled the tools to analyse a reasonably wide class of kernels. The first result is a localisation lemma: if k and f agree in a neighbourhood of the origin and one of them can be bounded above and below in the way described in the previous section, then the Hilbert-Schmidt norms of V_k^n and V_f^n are asymptotically equal and the operator norms of V_k^n and V_f^n are asymptotically equal.

LEMMA 7.1. *Suppose that $k, h \in L^1([0,1], \mathbb{R})$, that there exists $\delta \in (0,1)$ such that for all $t \in [0,\delta]$, $k(t) = h(t)$, and that there exist $\mu, \nu, r, c \in \mathbb{R}$ with $r > -1$ and $c > 0$ such that for all $t \in [0,1]$, $ct^r e^{\mu t} \leqslant k(t) \leqslant ct^r e^{\nu t}$. Then $\|V_k^n\| \sim \|V_h^n\|$ as $n \to \infty$ and $\|V_k^n\|_2 \sim \|V_h^n\|_2$ as $n \to \infty$.*

Proof. Note that all of the calculations below take place in the convolution algebra $L^1([0,1], \mathbb{R})$.

Let $g = h - k$, so $h = k + g$ and g is zero on $[0,\delta]$. We have

$$h^{*n} = (k+g)^{*n} = k^{*n} + g^{*n} + \sum_{j=1}^{n-1} \binom{n}{j} k^{*(n-j)} * g^{*j}.$$

Since g is zero to the left of δ, g^{*n} is zero to the left of $n\delta$ and, since we are working only on $[0,1]$, is identically zero if $n > 1/\delta$. Similarly, $k^{*(n-j)} * g^{*j}$ is zero to the left of $j\delta$ and hence identically zero if $j > 1/\delta$. It follows that if we choose $N \in \mathbb{N}$ with $N > 1/\delta$ then for all $n > N$,

$$h^{*n} = k^{*n} + \sum_{j=1}^{N-1} \binom{n}{j} k^{*(n-j)} * g^{*j}.$$

Moreover, the fact that g^{*j} is zero to the left of $j\delta$ implies that

$$k^{*(n-j)} * g^{*j} = (k^{*(n-j)} \chi_{[0,1-j\delta]}) * g^{*j}$$

since values of $k^{*(n-j)}$ to the right of $1 - j\delta$ are multiplied by zero in the definition of $k^{*(n-j)} * g^{*j}$. We thus have

$$h^{*n} = k^{*n} + \sum_{j=1}^{N-1} \binom{n}{j} (k^{*(n-j)} \chi_{[0,1-j\delta]}) * g^{*j}.$$

Now,

$$\begin{aligned} \|V_h^n - V_k^n\|_2^2 &= \int_0^1 (1-x)(h^{*n}(x) - k^{*n}(x))^2 \,\mathrm{d}x \leqslant \int_0^1 (h^{*n}(x) - k^{*n}(x))^2 \,\mathrm{d}x \\ &= \|h^{*n} - k^{*n}\|_2^2. \end{aligned}$$

We therefore have

$$\begin{aligned}\frac{\|V_h^n - V_k^n\|_2}{\|V_k^n\|_2} &\leqslant \frac{1}{\|V_k^n\|_2}\sum_{j=1}^{N-1}\binom{n}{j}\|(k^{*(n-j)}\chi_{[0,1-j\delta]}) * g^{*j}\|_2\\ &\leqslant \sum_{j=1}^{N-1}\binom{n}{j}\frac{\|k^{*(n-j)}\chi_{[0,1-j\delta]}\|_2\|g^{*j}\|_1}{\|V_k^n\|_2}\\ &= \sum_{j=1}^{N-1}\binom{n}{j}\|g^{*j}\|_1\frac{\left(\int\limits_0^{1-j\delta}(k^{*(n-j)}(x)^2\,\mathrm{d}x\right)^{1/2}}{\left(\int\limits_0^1(1-x)k^{*n}(x)^2\,\mathrm{d}x\right)^{1/2}}.\end{aligned}$$

We can now apply Lemma 7.1: $\binom{n}{j}\|g^{*j}\|_1$ is a polynomial in n for each j, so each of these terms tends to zero as n tends to ∞. We thus have $\|V_h^n - V_k^n\|_2/\|V_k^n\|_2 \to 0$ as $n \to \infty$ and hence $\|V_h^n\|_2 \sim \|V_k^n\|_2$ as $n \to \infty$ (Lemma 3.3 (ii)). We also know from Lemma 6.1 that $\|V_k^n\|_2 \lesssim \mathrm{e}^{\mu-\nu}\|V_k^n\|$, so

$$\frac{\|V_h^n - V_k^n\|}{\|V_k^n\|} \leqslant \frac{\mathrm{e}^{\nu-\mu}\|V_h^n - V_k^n\|_2}{\|V_k^n\|_2} \to 0$$

as $n \to \infty$, and hence, again from Lemma 3.3 (ii), that $\|V_h^n\| \sim \|V_k^n\|$ as $n \to \infty$. ∎

We can now finally establish the results claimed in the introduction.

Theorem 7.2. *Suppose that $k \in L^1([0,1],\mathbb{R})$ and that we can write $k(t) = t^r f(t)$, where $r > -1$, $f(0) \neq 0$ and in some neighbourhood of the origin f does not change sign and $\log(f(t)/f(0))$ is bounded (which implies that f is continuous at 0). Let*

$$\mu = \liminf_{t\to 0+}\log\frac{f(t)}{f(0)}, \quad \nu = \limsup_{t\to 0+}\log\frac{f(t)}{f(0)}.$$

Then

$$\frac{(\Gamma(r+1)|f(0)|)^n\mathrm{e}^{\mu}}{2\Gamma((r+1)n+1)} \lesssim \|V_k^n\| \leqslant \|V_k^n\|_2 \lesssim \frac{(\Gamma(r+1)|f(0)|)^n\mathrm{e}^{\nu}}{2\Gamma((r+1)n+1)}.$$

If f is differentiable (from the right) at 0 then $\mu = \nu = f'(0)/f(0)$ and

$$\|V_k^n\| \sim \|V_k^n\|_2 \sim \frac{(\Gamma(r+1)|f(0)|)^n\mathrm{e}^{f'(0)/f(0)}}{2\Gamma((r+1)n+1)}.$$

Proof. Note that since k is bounded in a neighbourhood of 0, it can be written as the sum of a bounded function and a function nilpotent in the convolution algebra $L^1([0,1])$. It follows that all sufficiently large powers of V_k are Hilbert-Schmidt operators.

We can assume without loss of generality that f is positive in a neighbourhood of 0. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < t < \delta$ then

$$\mu - \varepsilon \leqslant \frac{1}{t}\log\frac{f(t)}{f(0)} \leqslant \nu + \varepsilon$$

and hence

$$f(0)t^r \mathrm{e}^{(\mu-\varepsilon)t} \leqslant k(t) \leqslant f(0)t^r \mathrm{e}^{(\nu+\varepsilon)t}.$$

Let

$$k_\varepsilon(t) = \begin{cases} k(t) & \text{if } 0 \leqslant t \leqslant \delta, \\ k(0)t^r \mathrm{e}^{(\mu-\varepsilon)t} & \text{if } \delta < t \leqslant 1, \end{cases}$$

so k agrees with k_ε on $[0,\delta]$ and for all $t \in [0,1]$ we have

$$f(0)t^r \mathrm{e}^{(\mu-\varepsilon)t} \leqslant k_\varepsilon(t) \leqslant f(0)t^r \mathrm{e}^{(\nu+\varepsilon)t}.$$

By Lemma 6.1,

$$\frac{(f(0)\Gamma(r+1))^n \mathrm{e}^{\mu-\varepsilon}}{\Gamma((r+1)n+1)} \lesssim \|V_{k_\varepsilon}^n\| \leqslant \|V_{k_\varepsilon}^n\|_2 \lesssim \frac{(f(0)\Gamma(r+1))^n \mathrm{e}^{\nu+\varepsilon}}{\Gamma((r+1)n+1)}$$

as $n \to \infty$. Now, Lemma 6.2 shows that $\|V_{k_\varepsilon}^n\| \sim \|V_k^n\|$ and that $\|V_{k_\varepsilon}^n\|_2 \sim \|V_k^n\|_2$ as $n \to \infty$, so we have

$$\frac{(f(0)\Gamma(r+1))^n \mathrm{e}^{\mu-\varepsilon}}{\Gamma((r+1)n+1)} \lesssim \|V_k^n\| \leqslant \|V_k^n\|_2 \lesssim \frac{(f(0)\Gamma(r+1))^n \mathrm{e}^{\nu+\varepsilon}}{\Gamma((r+1)n+1)}$$

as $n \to \infty$. Since ε was arbitrary, it follows that

$$\frac{(f(0)\Gamma(r+1))^n \mathrm{e}^{\mu}}{\Gamma((r+1)n+1)} \lesssim \|V_k^n\| \leqslant \|V_k^n\|_2 \lesssim \frac{(f(0)\Gamma(r+1))^n \mathrm{e}^{\nu}}{\Gamma((r+1)n+1)}.$$

If f is differentiable from the right at 0, then so is $\log f$ and

$$(\log f)'(0) = \lim_{t\to 0+} \frac{\log(f(t)) - \log(f(0))}{t} = \lim_{t\to 0+} \frac{\log(f(t)/f(0))}{t}$$

so we have $\mu = \nu = (\log f)'(0) = f'(0)/f(0)$ and the two asymptotic bounds coincide to give

$$\|V_k^n\| \sim \|V_k^n\|_2 \sim \frac{(\Gamma(r+1)|f(0)|)^n \mathrm{e}^{f'(0)/f(0)}}{2\Gamma((r+1)n+1)}. \quad ∎$$

8. THE TRACE NORM AND OTHER NORMS

We now have a class of kernels k for which $\|V_k^n\|$ and $\|V_k^n\|_2$ have the same asymptotic behaviour. One immediate consequence of this is that if $p \geqslant 2$ then $\|V_k^n\|_p$ also has the same asymptotic behaviour (because $\|\cdot\| \leqslant \|\cdot\|_p \leqslant \|\cdot\|_2$).

To make similar inferences about a wider class of operator norms, we need to consider the trace norm $\|\cdot\|_1$. The significance of the trace norm is that if H is a separable Hilbert space and Φ is an complete operator norm on a non-trivial ideal $\mathcal{I}$ in $B(H)$ such that for all $A, \in B(H)$ and $B \in \mathcal{I}$ we have $\Phi(ABC) \leqslant \|A\|\Phi(B)\|C\|$, then there is a constant $c > 0$ such that for all $T \in \mathcal{I}$,

$$c\|T\| \leqslant \Phi(T) \leqslant c\|T\|_1.$$

(See, for example, Simon (Chapter 2, [9]).)

If we could show that $\|V_k^n\|_1 \sim \|V_k^n\|$ then it would follow from this that $\Phi(V_k^n) \sim c\|V_k^n\|$ for any of the Schatten-von Neumann norms described in the previous paragraph. We shall now show that this is indeed true for the classical Volterra operator V and its iterates, the fractional integration operators defined in (1.1). Although V itself is not trace class (it is not difficult to show that its mth singular value is $((m+1/2)\pi)^{-1}$), V^n is trace class for all $n > 1$, because $V^n = V^{n/2}V^{n/2}$ and a simple calculation shows that $V^{n/2}$ is Hilbert-Schmidt if $n > 1$. It therefore makes sense to ask about the asymptotic behaviour of $\|V^n\|_1$ as $n \to \infty$ through $\mathbb{R}^+$.

Theorem 8.1.

$$\|V^n\|_1 \sim \frac{1}{2\Gamma(n+1)}$$

as $n \to \infty$ *through* $\mathbb{R}^+$.

Proof. We have for any $n > 0$

$$(V^n u)(t) = \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} u(s)\,\mathrm{d}s.$$

For $n > k > 0$, define rank 1 operators $T_{k,n}$ and $S_{k,n}$ by

$$\begin{aligned}
(T_{k,n}u)(t) &= \frac{1}{\Gamma(n)\mathrm{e}^{n-k}} \int_0^1 \mathrm{e}^{(n-k)(t-s)} u(s)\,\mathrm{d}s \\
&= \frac{1}{\Gamma(n)} \Big(\int_0^1 \mathrm{e}^{-(n-k)s} u(s)\,\mathrm{d}s \Big) \mathrm{e}^{-(n-k)} \mathrm{e}^{(n-k)t} \\
(S_{k,n}u)(t) &= \frac{1}{\Gamma(n)} \Big(\int_0^1 \mathrm{e}^{-(n-k)s} u(s)\,\mathrm{d}s \Big) t^{n-1}.
\end{aligned}$$

If we introduce the notation $p_n(t) = t^n$ and $e_n(t) = \mathrm{e}^{nt}$ then we have

$$\begin{aligned}
T_{k,n}u &= \frac{1}{\Gamma(n)} \langle u, e_{-(n-k)} \rangle \mathrm{e}^{-(n-k)} e_{n-k} \\
S_{k,n}u &= \frac{1}{\Gamma(n)} \langle u, e_{-(n-k)} \rangle p_{n-1}.
\end{aligned}$$

Since $S_{k,n}$ has rank 1, we can write its trace norm as

$$\|S_{k,n}\|_1 = \frac{1}{\Gamma(n)} \|e_{-(n-k)}\|_2 \|p_{n-1}\|_2.$$

The two norms are easily evaluated, and we have $\|S_{k,n}\|_1 \sim 1/(2n\Gamma(n))$ as $n \to \infty$. Similarly, $\operatorname{rank}(S_{k,n} - T_{k,n}) = 1$, so in the same way

$$\|S_{k,n} - T_{k,n}\|_2 = \frac{1}{\Gamma(n)} \|e_{-(n-k)}\|_2 \|\mathrm{e}^{-n(n-k)} e_{-(n-k)} - p_{n-1}\|_2.$$

The first of these norms is again easy to evaluate, but the second is not so easy. Expanding the square in the integral defining $\|e_{-(n-k)} - p_{n-1}\|_2^2$ gives three integrals, two straightforward and one which can be expanded to third order using the methods described in Section 4. A tedious calculation leads to

$$\|S_{k,n} - T_{k,n}\|_2 \sim \frac{C_k}{n^2\Gamma(n)}$$

where C_k is a constant depending only on k.

We also have

$$\|V^n - S_{k,n}\|_2 = \frac{1}{\Gamma(n)}\Big(\int_0^1 (1-x)(\mathrm{e}^{-(n-k)}\mathrm{e}^{(n-k)x} - x^{n-1})^2\,\mathrm{d}x\Big)^{1/2} + \frac{\mathrm{O}(\mathrm{e}^{-n})}{\Gamma(n)}$$

and a similiar calculation involving a fourth-order expansion gives

$$\|V^n - S_{k,n}\|_2 \sim \frac{C'_k}{n^2\Gamma(n)}$$

as $n \to \infty$, where C'_k is another constant depending only on k. Combining these last two results gives us

$$\|V^n - S_{k,n}\|_2 \leqslant \frac{C''_k}{n^2\Gamma(n)}.$$

Now, we shall use this Hilbert-Schmidt norm estimate to obtain a trace norm estimate. Abbreviating $S_{1,n}$ to S_n, we have

$$\begin{aligned}(V^n - S_n)u &= V^n u - \frac{1}{\Gamma(n)}\langle u, e_{-(n-1)}\rangle p_{n-1} = V^n u - \frac{1}{\Gamma(n)}\langle u, e_{-(n-1)}\rangle \Gamma(n) V^{n-1}1\\ &= V^n u - \langle u, e_{-(n-1)}\rangle V^{n-1}1 = V^\alpha(V^{n-\alpha}u - \langle u, e_{-(n-1)}\rangle V^{n-1-\alpha}1)\\ &= V^\alpha\bigg(V^{n-\alpha}u - \langle u, e_{-(n-1)}\rangle \frac{p_{n-\alpha-1}}{\Gamma(n-\alpha)}\bigg) = V^\alpha(V^{n-\alpha}u - S_{1-\alpha,n-\alpha}u).\end{aligned}$$

We therefore have that if $\alpha > 1/2$, so V^α is Hilbert-Schmidt,

$$\|V^n - S_n\|_1 \leqslant \|V^\alpha\|_2 \|V^{n-\alpha} - S_{1-\alpha,n-\alpha}\|_2 \leqslant \|V^\alpha\|_2 \frac{C''_{1-\alpha}}{(n-\alpha)^2\Gamma(n-\alpha)}$$

so

$$\frac{\|V^n - S_n\|_1}{\|S_n\|_1} \leqslant \frac{2n\Gamma(n)\|V^\alpha\|_2 C''_{1-\alpha}}{(n-\alpha)^2\Gamma(n-\alpha)} \sim 2\|V^\alpha\|_2 C''_{1-\alpha} n^{\alpha-1}$$

using (4.4). This estimate tends to zero as $n \to \infty$ if $\alpha < 1$ so, by the Lemma 3.1, $\|V^n\|_1 \sim \|V^n\|$ as $n \to \infty$. But we know from Theorem 5.1 that $\|V_n\| \sim 1/(2\Gamma(n+1))$ as $n \to \infty$, so $\|V^n\|_1 \sim 1/(2\Gamma(n+1))$ as $n \to \infty$, as claimed. ∎

9. FURTHER QUESTIONS

These results raise some further questions:

Firstly, what happens if a kernel is not of the form needed for Theorem 7.2? Perhaps the simplest example of this is $k(t) = 1 + t^{1/2}$. If we write $\|V_k^n\| = c_n/n!$, then we have $c_n \to \infty$ and $a^n c_n \to 0$ whenever $|a| < 1$, but the exact rate of growth is unclear.

Secondly, the calculations in the last section are rather tedious, and the final estimate for $\|V^n - T_n\|_1/\|T_n\|_1$ is, up to a constant factor, $n^{-1/2+\varepsilon}$. Limited numerical evidence suggests that the decay is closer to $1/n$ than $1/n^{1/2}$. An alternative approach might be able to clarify the speed of decay and avoid the unpleasant computations.

Finally, for what other kernels do the trace and operator norms have the same asymptotic behaviour? The methods in Section 8, based on fractional powers, clearly have very limited applicability to more general kernels.

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