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$\mathbb{Z}$-ANALYTIC TAF ALGEBRAS AND PARTIAL DYNAMICAL SYSTEMS

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ABSTRACT. Connections between partial dynamical systems, a generalized notion of partial dynamical systems defined by nested sequences of partial hoeomorphisms, and triangular AF algebras which admit an integer-valued cocycle are established.

KEYWORDS: *Partial dynamical system, triangular* AF *algebra, $\mathbb{Z}$-analytic.*

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1. INTRODUCTION

Among the analytic triangular AF (TAF) algebras, the $\mathbb{Z}$-analytic algebras are those admitting an integer-valued cocycle. The purpose of this paper is to explore the connection between these algebras and dynamical systems.

The work of this paper may be, in a sense, considered a parallel version for nonselfadjoint algebras to the program of [7] and [5] studying the connections between zero-dimensional dynamical systems, crossed products, and their K-theoretic invariants. In some respects, the situation here is simpler: isomorphism of standard $\mathbb{Z}$-analytic dynamical systems is equivalent to conjugacy of the dynamical systems, so notions such as strong orbit equivalence are not needed. Also, since the K-theory for nonselfadjoint algebras is not well developed, we do not attempt to discuss K-theoretic invariants. On the other hand, there are complications in the nonselfadjoint situation not encountered in the selfadjoint case. First of all, the dynamical systems are only partially defined; worse yet, the "dynamical systems" may require an infinite sequence of partial homeomorphisms whose graphs are nested. The algebras to which the dynamical systems are associated are nonselfadjoint subalgebras of groupoid C^*-algebras (cf. [8]), or may also be viewed as nonselfadjoint subalgebras of generalized crossed products in the sense of Exel ([3], [4]).

By a *spectral triple* $(X, \mathcal{P}, \mathcal{R})$ we mean X is a compact space, which in our setting will always be zero-dimensional, $\mathcal{R}$ is a groupoid which is an equivalence relation on X having unit space X, and $\mathcal{P}$ an open subset of $\mathcal{R}$ with $\mathcal{P} \cup \mathcal{P}^{-1} = \mathcal{R}$, $\mathcal{P} \circ \mathcal{P} \subseteq \mathcal{P}$, and $\mathcal{P} \cap \mathcal{P}^{-1} = X$ ([17]). $\mathcal{P}$ determines a nonselfadjoint algebra $\mathfrak{A} = \mathfrak{A}(\mathcal{P}) \subset C^*(\mathcal{R})$ ([8]), which is "triangular" in the sense that $\mathfrak{A} \cap \mathfrak{A}^* \cong C(X) \subset C^*(\mathcal{R})$. The spectral triples we will study have the property that for all $(x, y) \in \mathcal{P}$ the set

$$\{z : (x, z), (z, y) \in \mathcal{P}\}$$

has finite cardinality. In other words, regarding $\mathcal{P}$ as giving an ordering to the equivalence classes of X, the order type is that of a subset of the integers.

For example, if X is a compact metrix space, $\varphi : X \to X$ a homeomorphism, set

$$\mathcal{R} = \{(x, \varphi^n(x)) : x \in X, n \in \mathbb{Z}\}, \quad \text{and} \quad \mathcal{P} = \{(x, \varphi^n(x)) : x \in X, n \geqslant 0\},$$

then $C^*(\mathcal{R})$ is the crossed product algebra $C(X) \times_\varphi \mathbb{Z}$, $\mathfrak{A}(\mathcal{P})$ the semicrossed product algebra $C(X) \times_\varphi \mathbb{Z}^+$. Suppose X is zero-dimensional, and φ a partial homeomorphism on X. That is, φ is a homeomorphism from $\operatorname{dom}(\varphi)$ to $\operatorname{ran}(\varphi)$, where both $\operatorname{dom}(\varphi), \operatorname{ran}(\varphi)$ are open subsets of X. Then, under certain conditions, $C^*(\mathcal{R})$ is an AF algebra, and $\mathfrak{A}(\mathcal{P})$ a standard $\mathbb{Z}$-analytic subalgebra. The partial homomorphism may arise as the restriction of a homeomorphism of X, (the case studied in [7]) or it may not admit an extension to a homeomorphism.

Returning to our general setting, suppose $(X, \mathcal{P}, \mathcal{R})$ is a spectral triple such that each equivalence classes of X has the order type of a subset of the integers. In that case it is possible to define a partial mapping Φ on X which maps each point to its immediate successor in the ordering. $\operatorname{dom}(\Phi)$ is the set of all points which have a successor. In general, such a map need not be continuous. Spectral triples of this kind can arise from nested sequences of partial homoemorphisms, $\{\varphi_n\}_{n \in \mathbb{Z}}$. By *nested* sequence we mean that $\varphi_{-n} = \varphi_n^{-1}$, $n \in \mathbb{Z}$, φ_0 is the identity map, and that the graphs are nested in the sense

$$\Gamma(\varphi_n \circ \varphi_m) \subset \Gamma(\varphi_{n+m}), \quad n, m \in \mathbb{Z}.$$

Given a partial homeomorphism φ of a zero dimensional space X, under what conditions is the groupoid $\mathcal{R}(X, \varphi) := \{(x, \varphi^n(x)) : x \in X,\ n \in \mathbb{Z}\}$ an AF groupoid? We prove that this happens exactly when (X, φ) is conjugate to (B, ψ), where B is an ordered Bratteli compactum and ψ the Veršik map on B. And this is the case if and only if for any clopen set U containing the complement of the range of φ, each $x \in X$ belongs to the forward orbit of some point $u \in U$ (Corollary 4.13).

The more general case in which $\mathcal{R}$ is the union of the graphs $\Gamma(\varphi_n)$ of a nested sequence of partial homeomorphims is precisely the case in which $\mathcal{R}$ admits a (continuous) integer valued cocycle. Here, too, we have necessary and sufficient conditions, given in terms of dynamical systems, for $\mathcal{R}$ to be an AF groupoid.

Though our goal initially was to study nonselfadjoint algebras, our approach leads to some new results in AF algebras and dynamical systems. Our approach, however, comes naturally from looking at nonselfadjoint algebras. The nonselfadjoint algebras we study are the strongly maximal subalgebras of AF algebras which admit an integer-valued cocycle, i.e., the $\mathbb{Z}$-analytic algebras. Theorem 3.1 in Section 3 gives various characterizations of $\mathbb{Z}$-analytic algebras (and spectral triples) assuming the enveloping groupoid is AF, and an analogous result for standard $\mathbb{Z}$-analytic algebras is presented in Theorem 3.2. In Section 4 two standard $\mathbb{Z}$-analytic algebras are shown to be isomorphic if and only if their partial dynamical systems are conjugate. (There is a corresponding result for semicrossed products and dynamical systems; cf. [14].) We return to this theme again in the context of nested sequences of partial homeomorphisms in Section 5 in which the dynamics may not be continuous (Theorem 5.3). An example of a $\mathbb{Z}$-analytic semigroupoid which is not standard is given in Theorem 5.1. This example admits a semi-saturation, and in the final of Section 5 we consider necessary and sufficient conditions for a nested sequence of partial homeomorphisms to admit a semi-saturation. Example 5.2 is one that does not admit a semi-saturation.

2. PRELIMINARIES

We begin this section with a review of terminology for AF groupoids and semigroupoids ([15]).

Let $\mathfrak{A}$ be a triangular AF algebra, or simply TAF algebra, with canonical masa $\mathfrak{D} = \mathfrak{A} \cap \mathfrak{A}^*$, and C^*-envelope $\mathfrak{B}$. In the spectral triple for $\mathfrak{A}$, denoted $(X, \mathcal{P}, \mathcal{R})$, X is the Gelfand space of the commutative C^*-algebra $\mathfrak{D}, \mathcal{R}$ is the AF groupoid of $\mathfrak{B}$, and $\mathcal{P}$ is the semigroupoid corresponding to the subalgebra $\mathfrak{A}$. If v is a matrix unit of $\mathfrak{A}$, or more generally if v is a $\mathfrak{D}$-normalizing partial isometry in $\mathfrak{A}$, let $\widehat{v}$ denote its support: i.e., $\widehat{v}$ is the support set of v, viewed as a function on the groupoid $\mathcal{R}$. Thus,

$$\mathcal{P} = \bigcup \{\widehat{v} : v \text{ is a matrix unit of } \mathfrak{A}\}.$$

The sets $\widehat{v}$, as v ranges over the matrix units of $\mathfrak{A}$, form a basis for the topology of $\mathcal{P}$. (And the sets $\widehat{v}$, form a basis for the topology of $\mathcal{R}$, as v ranges over the matrix units of $\mathfrak{B}$.)

A *cocycle* (or, more precisely, a real-valued 1-cocycle) on $\mathcal{R}$ is a map $c : \mathcal{R} \to \mathbb{R}$ satisfying:

- the cocycle condition: for all $(x, y), (y, z) \in \mathcal{R}$, $c(x, y) + c(y, z) = c(x, z)$;
- continuity: c is continuous from $\mathcal{R}$ to $\mathbb{R}$.

A TAF algebra $\mathfrak{A}$ with spectral triple $(X, \mathcal{P}, \mathcal{R})$ is said to be *analytic* if there is a (real-valued 1-) cocycle c such that $c^{-1}([0, \infty)) = \mathcal{P}$. We say in this case that $\mathfrak{A}$ is the analytic TAF algebra defined by c, and write $\mathfrak{A} = \mathfrak{A}_c$. $\mathfrak{A}$ is called $\mathbb{Z}$-*analytic* if the cocycle c can be chosen to be integer valued.

There is a proper subclass of the $\mathbb{Z}$-analytic algebras, called the *standard* $\mathbb{Z}$-analytic algebras. If $\mathfrak{A}$ is any strongly maximal TAF subalgebra of an AF algebra $\mathfrak{B}$, then by Lemma 1.1, [11], there is a sequence $\{\mathfrak{B}_n\}$ of finite dimensional C^*-algebras of $\mathfrak{B}$, say $\mathfrak{B}_n = \bigoplus_{k=1}^{r(n)} M_{m(n,k)}$, and a set of matrix units $\{e_{ij}^{(nk)}\}$ for $\bigcup_{n=1}^{\infty} \mathfrak{B}_n$ such that

$$\mathfrak{A}_n := \mathfrak{A} \cap \mathfrak{B}_n = \bigoplus_{k=1}^{r(n)} T_{m(n,k)},$$

where $T_{m(n,k)}$ is the upper triangular subalgebra of $M_{m(n,k)}$, and $\mathfrak{A} = \varinjlim \mathfrak{A}_n$ and $\mathfrak{B} = \varinjlim \mathfrak{B}_n$. Then $\mathfrak{A}$ is called *standard* if the embeddings $\mathfrak{A}_n \to \mathfrak{A}_{n+1}$ can be chosen to be standard ([13]); i.e., standard embeddings in the sense of Effros ([2]). Since $\mathbb{Z}$-analytic algebras are defined by means of the existence of an integer-valued cocycle, it is not obvious from the definition that standard $\mathbb{Z}$-analytic algebras form a subclass of the $\mathbb{Z}$-analytic. If $\mathfrak{A}$ is $\mathbb{Z}$-analytic with spectral triple $(X, \mathcal{P}, \mathcal{R})$, and hence admits an integer-valued cocycle d, it follows that if $x < y$ are two points in X belonging to the same equivalence class, then there are at most finitely many points in the equivalence class between x and y: indeed, the number of such points can be at most $d(x,y)$. Thus one can define an integer-valued function on $\widehat{d}$, on $\mathcal{P}$, called the *counting cocycle*:

$$\widehat{d}(x,y) = \text{ Cardinality of } \{z \in X : d(x,z) \geqslant 0 \text{ and } d(z,y) > 0\}.$$

It is clear that $\widehat{d}$ satisfies the additivity property for cocycles; however in general it is not continuous. From [13] it is known that the standard $\mathbb{Z}$-analytic algebras are precisely the ones for which the counting cocycle is continuous (cf. Theorem 3.2).

DEFINITION 2.1. A partial dynamical system $(X, X_{\max}, X_{\min}, \varphi)$ is a quadruple where X is a compact metric space, $X_{\max}, X_{\min}$ are closed subsets, and $\varphi : X \setminus X_{\max} \to X \setminus X_{\min}$ is a homeomorphism. Then φ is called a *partial homeomorphism on* X. The graph of φ will be denoted Γ_φ.

In this paper X will always be assumed to be zero dimensional. For n a positive integer, φ^n will denote the n-fold composite of φ, and for n a negative integer, φ^n will denote the n-fold composite of φ^{-1}. Also we adhere to the convention that $\varphi^0 = \mathrm{id}_X$.

For $U \subset X$, we write $\varphi(U)$ to mean $\varphi(U \setminus X_{\max})$, and similarly $\varphi^n(U) = \varphi^n(U \cap \mathrm{dom}(\varphi^n))$. Since $X_{\max}$ is closed it follows that if U is open, $\varphi(U)$ is open; and $\varphi^n(U)$ is open, $n \in \mathbb{Z}$.

3. CHARACTERIZATIONS OF $\mathbb{Z}$-ANALYTIC AND STANDARD $\mathbb{Z}$-ANALYTIC ALGEBRAS

We begin with several characterizations of $\mathbb{Z}$-analytic and standard $\mathbb{Z}$-analytic algebras. For subsets A, B of an operator algebra $\mathfrak{A}$, we will write $[A \cdot B]$ to denote the closed linear span of the set $\{ab : a \in A,\ b \in B\}$. Let $\mathbb{T}$ be the circle formed by identifying the end points of $[0,1]$.

THEOREM 3.1. *Let $\mathfrak{A}$ be a strongly maximal* TAF *algebra with spectral triple $(X, \mathcal{P}, \mathcal{R})$. The following conditions are equivalent:*

(i) *$C^*(\mathfrak{A})$ admits a continuous action α of $\mathbb{T}$, with fixed point algebra the diagonal $\mathfrak{D}$, and such that*

$$\mathfrak{A} = \Big\{ b \in C^*(\mathfrak{A}) : \int_{\mathbb{T}} \alpha_t(b) \mathrm{e}^{2\pi \mathrm{i} n t} \, \mathrm{d}t = 0 \text{ for all } n \geqslant 1 \Big\}.$$

(ii) *There is a sequence $\{\mathfrak{A}(n)\}$, $n \geqslant 0$, of closed linear subspaces of $\mathfrak{A}$ satisfying:*

(a) $\mathfrak{A}(0)$ *is the diagonal* $\mathfrak{D}$;
(b) $\mathfrak{A}(n) \cap \mathfrak{A}(m) = (0)$, $n \neq m$;
(c) $[\mathfrak{A}(n) \cdot \mathfrak{A}(m)] \subset \mathfrak{A}(n+m)$, *for all* $n, m \geqslant 0$;
(d) $\bigoplus_{n=0}^{\infty} \mathfrak{A}(n)$ *is dense in* $\mathfrak{A}$.

(iii) $\mathfrak{A}$ *is* $\mathbb{Z}$*-analytic.*

(iv) *There is a nested sequence* $\{\varphi_n\}$ *of partial homeomorphisms of* X*, with* $\varphi_0 = \mathrm{id}_X$*, the graph* Γ_{φ_n} *open in* $\mathcal{P}$*,* $n \geqslant 0$*, and such that*

$$\mathcal{P} = \bigsqcup_{n=0}^{\infty} \Gamma_{\varphi_n} \quad \textit{(disjoint union).}$$

Proof. (i) $\Rightarrow$ (ii) Note that α acts on $\mathfrak{A}$. Set $\mathfrak{A}(n) := \{a \in \mathfrak{A} : a = \int_0^1 \alpha_t(b) \mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t$, for some $b \in \mathfrak{A}\}$. It follows from the definition of $\mathfrak{A}$ that $\mathfrak{A}(n) = (0)$ for all $n < 0$. First we show (ii) (a). Since the fixed point algebra of α is the diagonal $\mathfrak{D}$, clearly $\mathfrak{A}(0) \supset \mathfrak{D}$. Now let $a \in \mathfrak{A}(0)$; then, if $a = \int_0^1 \alpha_t(b)\,\mathrm{d}t$

$$\alpha_s(a) = \int_0^1 \alpha_s(\alpha_t(b))\,\mathrm{d}t = \int_0^1 \alpha_{s+t}(b)\,\mathrm{d}t = \int_0^1 \alpha_t(b)\,\mathrm{d}t = a,$$

where the last equality uses the translation invariance of Lebesgue measure on $\mathbb{T}$. Thus, the fixed point algebra of α, which by assumption is $\mathfrak{D}$, contains $\mathfrak{A}(0)$. Hence, $\mathfrak{A}(0) = \mathfrak{D}$.

Next, observe that $\mathfrak{A}(n) = \{a \in \mathfrak{A} : \alpha_t(a) = \mathrm{e}^{2\pi\mathrm{i}nt}a, t \in \mathbb{T}\}$. Indeed, let $a = \int_0^1 \alpha_t(b)\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t$, $b \in \mathfrak{A}$. Then, for $s \in \mathbb{T}$,

$$\alpha_s(a) = \int_0^1 \alpha_{s+t}(b)\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t = \int_0^1 \alpha_u(b)\mathrm{e}^{-2\pi\mathrm{i}n(u-s)}\,\mathrm{d}u = \mathrm{e}^{2\pi\mathrm{i}ns}a.$$

Conversely, if $\alpha_s(a) = \mathrm{e}^{2\pi\mathrm{i}ns}a$, a similar calculation shows that $a \in \mathfrak{A}(n)$. Thus, (ii) (b) is clear, as is (ii) (c).

To show (ii) (d), let ω be a continuous linear functional on $\mathfrak{A}$, and suppose

$$\langle a, \omega \rangle = 0 \quad \text{for all } a \in \mathfrak{A}(n),\ n \in \mathbb{Z}.$$

Fix $b \in \mathfrak{A}$, and consider the continuous function f on $\mathbb{T}$, $f(t) = \langle \alpha_t(b), \omega \rangle$. By the characterization of $\mathfrak{A}(n)$ above, we have $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Thus, $f = 0$, and hence $\langle b, \omega \rangle = 0$. Since $b \in \mathfrak{A}$ was arbitrary, it follows that $\omega = 0$. Thus, $\bigoplus_{n\in\mathbb{Z}} \mathfrak{A}(n)$ is dense in $\mathfrak{A}$. Since by assumption the spaces $\mathfrak{A}(-n)$, $n > 0$, are (0), we have that $\bigoplus_{n=0}^{\infty} \mathfrak{A}(n)$ is dense in $\mathfrak{A}$.

(ii) $\Rightarrow$ (iii) Condition (c) of (ii) implies that each $\mathfrak{A}(n)$ is a closed $\mathfrak{D}$-bimodule. Hence, by Theorem 2.2, [11], $\mathfrak{A}(n)$ is spanned by the matrix units it contains. Write

$$\widehat{\mathfrak{A}}(n) = \bigcup\{\widehat{v} : v \text{ a matrix unit in } \mathfrak{A}(n)\}.$$

We claim: $\bigcup_{n=0}^{\infty} \widehat{\mathfrak{A}}(n) = \mathcal{P}$.

Let $(x_0, y_0) \in \mathcal{P}$, and suppose $(x_0, y_0) \notin \widehat{\mathfrak{A}}(n), n \geqslant 0$. Define a representation π of $\mathfrak{A}$ as follows: let $\mathcal{H}_\pi$ be a Hilbert space with orthonormal basis $\{\xi_x\}_{x\in\mathcal{O}(x_0)}$, and define π on matrix units by

$$\pi(v)\xi_x = \begin{cases} \xi_y & \text{if } (x,y) \in \widehat{v},\ x \in \mathcal{O}(x_0), \\ 0 & \text{otherwise.} \end{cases}$$

By [9], π extends to a representation of $\mathfrak{A}$. By supposition, $(\pi(a)\xi_{x_0}, \xi_{y_0}) = 0$ for $a \in \mathfrak{A}(n)$, hence for $a \in \bigoplus_{n=0}^{\infty} \mathfrak{A}(n)$. Since $\bigoplus_{n=0}^{\infty} \mathfrak{A}(n)$ is dense in $\mathfrak{A}$, we have that $(\pi(a)\xi_{x_0}, \xi_{y_0}) = 0$ for $a \in \mathfrak{A}$. That is impossible, since there is a matrix unit $v \in \mathfrak{A}$ with $(x_0, y_0) \in \widehat{v}$, hence $(\pi(v)\xi_{x_0}, \xi_{y_0}) = 1$. This proves the claim.

Next, $\widehat{\mathfrak{A}}(n) \cap \widehat{\mathfrak{A}}(m) = \emptyset$, for $n \neq m$. Indeed, the intersection $\widehat{\mathfrak{A}}(n) \cap \widehat{\mathfrak{A}}(m)$ is open, so if it is nonempty there is a matrix unit v such that $\widehat{v}$ lies in the intersection. But then $v \in \mathfrak{A}(n) \cap \mathfrak{A}(m)$, contradicting (ii) (b).

Define a cocycle $d : \mathcal{P} \to \mathbb{Z}$, by $d(x,y) = n$ if $(x,y) \in \widehat{\mathfrak{A}}(n)$. Then d is well-defined, and continuous since $d^{-1}(n) = \widehat{\mathfrak{A}}(n)$ is open in $\mathcal{P}$. Also, if $(x,y) \in \widehat{\mathfrak{A}}(n), (y,z) \in \widehat{\mathfrak{A}}(m)$, then by (ii) (c), $(x,z) \in \widehat{\mathfrak{A}}(n+m)$. So $d(x,y) + d(y,z) = d(x,z)$, i.e., the cocycle condition is satisfied on $\mathcal{P}$. Then d can be extended to a cocycle on $\mathcal{R}$ by setting $d(y,x) = -n$ if $d(x,y) = n$. One easily checks that the cocycle condition is satisfied on the groupoid $\mathcal{R}$.

(iii) $\Rightarrow$ (iv) Let d be a cocycle for $\mathfrak{A}$ and φ_n be the partial homeomorphism on X whose graph Γ_{φ_n} is $d^{-1}(n)$. It is clear from the cocycle condition that $\{\varphi_n\}$ form a nested sequence and that the graphs Γ_{φ_n} have the required properties.

(iv) $\Rightarrow$ (i) Write $\Gamma_n = \Gamma_{\varphi_n}$. We will define a $\mathbb{T}$ action on $\mathfrak{A}$ first by defining it on matrix units.

Let v be a matrix unit in $\mathfrak{A}$; then (by compactness) $\widehat{v} = \bigcup_{k=0}^{N} \Gamma_k \cap \widehat{v}$, for some $N \in \mathbb{Z}$. Observe that Γ_k is both open and closed in $\mathcal{P}$, so that, setting $\widehat{v}_k := \Gamma_k \cap \widehat{v}$, each v_k is a matrix unit or sum of matrix units in $\mathfrak{A}$. Now set

$$\alpha_t(v) = \sum_{k=0}^{N} \mathrm{e}^{2\pi \mathrm{i} k t} v_k, \quad t \in \mathbb{T}.$$

Then α_t extends by linearity to (the dense subalgebra of) all finite linear combinations of matrix units. A short calculation shows that α_t is isometric, so it extends to an isometric map of $\mathfrak{A}$. By the nested property of the Γ_n, the automorphism property $\alpha_t(ab) = \alpha_t(a)\alpha_t(b)$ holds for a, b matrix units, hence for linear combinations of matrix units, and finally for arbitrary $a, b \in \mathfrak{A}$. One can verify directly, or use [11] or [15] to get that α_t is the restriction of a star automorphism of $C^*(\mathfrak{A})$. Finally, one notes that the action $t \to \alpha_t$ is continous in the pointwise-norm topology; i.e., for each $a \in \mathfrak{A}$, the map $t \to \alpha_t(a)$ is norm continuous. ∎

Recall from [3] that a continuous action α of $\mathbb{T}$ on a C^*-algebra $\mathfrak{B}$ is semi-saturated, if $\mathfrak{B}$ is generated as a C^*-algebra by the fixed point algebra and the first spectral subspace $\mathfrak{B}(1)$. (i.e., $\mathfrak{B}(1) = \{b \in \mathfrak{B} : \alpha_t(b) = \mathrm{e}^{2\pi \mathrm{i} t} b\}$.)

THEOREM 3.2. *Let* $\mathfrak{A}$ *be a strongly maximal* TAF *algebra with spectral triple* $(X, \mathcal{P}, \mathcal{R})$. *The following conditions are equivalent:*

(i) *There is a semi-saturated action* α *of* $\mathbb{T}$ *on* $C^*(\mathfrak{A})$ *with fixed point algebra the diagonal* $\mathfrak{D}$, *such that*

$$\mathfrak{A} = \Big\{ b \in C^*(\mathfrak{A}) : \int_{\mathbb{T}} \alpha_t(b) \mathrm{e}^{2\pi \mathrm{i} n t}\, \mathrm{d}t = 0 \text{ for all } n \geqslant 1 \Big\}.$$

(ii) *There is a sequence* $\mathfrak{A}(n)$, $n \geqslant 0$, *of closed linear subspaces of* $\mathfrak{A}$ *satisfying*

(a) $\mathfrak{A}(0)$ *is the diagonal* $\mathfrak{D}$;
(b) $\mathfrak{A}(n) \cap \mathfrak{A}(m) = (0)$, $n \neq m$;
(c) $[\mathfrak{A}(n) \cdot \mathfrak{A}(m)] = \mathfrak{A}(n+m)$, *for all* $n, m \geqslant 0$;
(d) $\bigoplus_{n=0}^{\infty} \mathfrak{A}(n)$ *is dense in* $\mathfrak{A}$.

(iii) *The counting cocycle* $\widehat{d}$ *is finite-valued and continuous on* $\mathcal{R}$, *and* $\widehat{d}^{-1}([0,\infty)) = \mathcal{P}$;

(iv) $\mathfrak{A}$ *is a standard* TAF *algebra;*

(v) *There is a partial homeomorphism* φ *of* X *such that the graph* Γ_φ *is open in* $\mathcal{P} \setminus \Delta(X)$, *and*

$$\mathcal{P} \setminus \Delta(X) = \bigcup_{n=1}^{\infty} \Gamma_{\varphi^n},$$

where φ^n *is the n-fold composite of* φ.

Proof. (i) $\Rightarrow$ (ii) With the spectral subspaces $\mathfrak{A}(n)$ defined as in the proof of Theorem 3.1, the condition that α be semi-saturated implies that $\mathfrak{A}(n) =$ closed span$\{a_1 \cdots a_n : a_j \in \mathfrak{A}(1)\}$, $n = 2, 3, \ldots$. In particular, this implies (ii) (c). Otherwise the proof is the same as (i) $\Rightarrow$ (ii) of Theorem 3.1.

(ii) $\Rightarrow$ (i) Define an action of $\mathbb{T}$ on the algebraic direct sum of the $\mathfrak{A}(n)$, $n \geqslant 0$, by

$$\alpha_t\Big(\sum a_j\Big) = \sum \mathrm{e}^{2\pi \mathrm{i} n_j t} a_j, \quad \text{where } a_j \in \mathfrak{A}(n_j).$$

Then α extends to an action of $\mathbb{T}$ on $\bigoplus_{n\in\mathbb{Z}} \mathfrak{A}(n)$, which is dense $\mathfrak{A}$, hence to an action of $\mathfrak{A}$. Define α_t on $\mathfrak{A}(n)^*$ by $\alpha_t(a) = \mathrm{e}^{-2\pi \mathrm{i} n t} a$, $a \in \mathfrak{A}(n)^*$, $n > 0$. This gives rise to an isometric action on $\mathfrak{A}^*$, and hence a star-action of $\mathbb{T}$ on the norm-closure of $\mathfrak{A} + \mathfrak{A}^*$, which is $C^*(\mathfrak{A})$.

(ii) (c) implies that the action is semi-saturated.

(ii) $\Rightarrow$ (v) We use the same notation as in Theorem 3.1. Thus, $\widehat{\mathfrak{A}}(n) = \bigcup\{\widehat{v} : v \text{ a matrix unit in } \mathfrak{A}(n)\}$. By assumption, $\mathfrak{A}(1) \cap \mathfrak{A}(0) = \mathfrak{A}(1) \cap \mathfrak{D} = (0)$, so that $\widehat{\mathfrak{A}}(1) \cap \Delta(X) = \emptyset$. So $\widehat{\mathfrak{A}}(1)$ is the graph Γ_φ of a partial homeomorphism φ of X, and the graph Γ_φ is disjoint from the diagonal set $\Delta(X)$. Also, repeated application of (ii) (c) implies that $\Gamma_{\varphi^n} = \widehat{\mathfrak{A}}(n)$, $n \geqslant 1$. Now as in the proof of

(ii) $\Rightarrow$ (iii) of Theorem 3.1, $\bigcup_{n=0}^{\infty} \widehat{\mathfrak{A}}(n) = \mathcal{P}$, so that $\bigcup_{n=1}^{\infty} \widehat{\mathfrak{A}}(n) = \mathcal{P} \setminus \Delta(X)$. Hence, $\bigcup_{n=1}^{\infty} \Gamma_{\varphi^n} = \mathcal{P} \setminus \Delta(X)$.

(v) $\Rightarrow$ (ii) Let $\mathfrak{A}(n)$ be the closed linear span of the set $\{v \in \mathfrak{A} : v \text{ a matrix unit}, \widehat{v} \subset \Gamma_{\varphi^n}\}$. We claim that

$$\Gamma_{\varphi^n} \cap \Gamma_{\varphi^m} = \emptyset \quad \text{for } n \neq m.$$

Case 1. $n = 0 < m$. Of course if $m = 1$, it is true by assumption. If $\Gamma_{\varphi^m} \cap \Delta(x) \neq \emptyset$ for some $m > 1$, let $(x,x) \in \Gamma_{\varphi^m} \cap \Delta(x)$. Then there is a $y \in X$ with $(x,y) \in \Gamma_{\varphi^{m-1}}$ and $(y,x) \in \Gamma_{\varphi}$. But then (x,y) and (y,x) are both in $\mathcal{P}$, or $(x,y) \in \mathcal{P} \cap \mathcal{P}^{-1} = \Delta(X)$, so that $y = x$ and $(x,x) \in \Gamma_{\varphi}$, contrary to hypothesis.

Case 2. $0 < n < m$. Suppose $(x,y) \in \Gamma_{\varphi^n} \cap \Gamma_{\varphi^m}$. Then $\varphi^n(x) = y$ and $\varphi^m(x) = y$. But $\varphi^m(x) = \varphi^{m-n} \circ \varphi^n(x)$, so that $\varphi^{m-n}(y) = y$. Since by Case 1 the graph of φ^{m-n} is disjoint from the diagonal, this is impossible. Thus the claim is established.

From the claim we have $\mathfrak{A}(n) \cap \mathfrak{A}(m) = (0)$ for $n \neq m$. Then (ii) (d) follows from the fact that $\bigoplus_{n=0}^{\infty} \mathfrak{A}(n)$ contains the algebra spanned by the matrix units of $\mathfrak{A}$.

(iii) $\Leftrightarrow$ (iv) Follows from Proposition 2.8 and Theorem 2.9 of [13].

(iii) $\Rightarrow$ (v) Let $\Gamma_{\varphi} = \widehat{d}^{-1}(1)$. This is open in $\mathcal{P}$ and disjoint from $\Delta(X)$. It follows that $\Gamma_{\varphi^n} = \widehat{d}^{-1}(n)$, and hence

$$\mathcal{P} = \widehat{d}^{-1}([0,\infty)) = \bigcup_{n=0}^{\infty} \Gamma_{\varphi^n},$$

so (with $\varphi^0 = \mathrm{id}_X$), $\mathcal{P} \setminus \Delta(X) = \bigcup_{n=1}^{\infty} \Gamma_{\varphi^n}$.

(v) $\Rightarrow$ (iii) Observe that the condition that Γ_{φ} is open in $\mathcal{P} \setminus \Delta(X)$ implies that the sets Γ_{φ^n} are disjoint and open. Furthermore, the orbit (or equivalence class) of a point $x \in X$ is given by $\mathcal{O}(x) = \{\varphi^n(x) : x \in \mathrm{dom}(\varphi^n), n \in \mathbb{Z}\}$ (where for n negative, $\varphi^n(x)$ denotes the n-fold composite of φ^{-1} at x). In particular, each orbit has the order type of a subset of $\mathbb{Z}$, so that $\widehat{d}$ is finite on the groupoid $\mathcal{R} : \widehat{d}(x,y) = n$ if and only if $y = \varphi^n(x)$. Thus the counting cocycle $\widehat{d}$ is finite and continuous on $\mathcal{R}$, and $\mathcal{P} = \widehat{d}^{-1}([0,\infty))$. ∎

Let X be a zero-dimensional compact space and $(X, \mathcal{P}, \mathcal{R})$ a spectral triple defined by a partial homeomorphism (respectively, a nested sequence of partial homeomorphisms) on X. Then Theorem 3.2 (respectively, 3.1) shows that $\mathcal{A}(\mathcal{P})$ is standard $\mathbb{Z}$-analytic (respectively, $\mathbb{Z}$-analytic) if and only if $\mathcal{R}$ is an AF groupoid. In the next two sections we will give necessary and sufficient conditions for $\mathcal{R}$ to be AF.

4. ORDERED BRATTELI DIAGRAMS AND PARTIAL DYNAMICAL SYSTEMS

Let V and W be two non-empty finite sets. An *ordered diagram* from V to W consists of a partially ordered set E and surjective maps $r : E \to W$ and $s : E \to V$ such that e and e' are comparable if and only if $r(e) = r(e')$. Sometimes we just write E for (E, r, s). The elements of V and W are the *vertices* and the elements of E are the *edges* of the diagram.

An *ordered Bratteli diagram* $(\mathcal{V}, \mathcal{E})$ consists of a vertex set

$$\mathcal{V} = V_0 \cup V_1 \cup \cdots \quad \text{(disjoint union of finite sets)},$$

where V_0 is a singleton, and

$$\mathcal{E} = \{(E_n, r_n, s_n) : n \geqslant 1\},$$

where (E_n, r_n, s_n) is an ordered diagram from V_{n-1} to V_n. If $e_i \in E_i$ with $r_i(e_i) = s_{i+1}(e_{i+1})$ for all i, $m < i < n$, then $(e_{m+1}, e_{m+2}, \ldots, e_n)$ is a *path* from V_m to V_n. Let $X = X(\mathcal{V}, \mathcal{E})$ consist of all infinite sequences $(e_1, e_2, \ldots)$ of edges with $e_i \in E_i$ and $r_{i-1}(e_{i-1}) = s_i(e_i)$ for all i. For $x = (e_n)_{n=1}^{\infty} \in X$ and $n \geqslant 1$, we will write $x(n) = e_n$.

Suppose $(\mathcal{V}, \mathcal{E})$ is an ordered Bratteli diagram. For each path $p = (e_1, e_2 \ldots, \ldots, e_n)$ from V_0 to V_n, let $C(p) = \{(f_1, f_2, \ldots) \in X : f_i = e_i \text{ for all } 1 \leqslant i \leqslant n\}$. We give X the smallest topology where each $C(p)$ is open. In this topology, each $C(p)$ is actually both closed and open (clopen).

PROPOSITION 4.1. *X is a separable compact metrizable space.*

Proof. Each E_n is a discrete space, hence metrizable. Let $Y = \prod_{n=1}^{\infty} E_n$, with the product topology. Therefore, Y is compact and metrizable and $X \subseteq Y$. The topology of X is equal to that inherited from Y. X is separable because it has a countable base $\{C(p) : n \geqslant 1,\, p = (e_1, e_2, \ldots, e_n)\}$. ∎

Given an ordered Bratteli diagram $(\mathcal{V}, \mathcal{E})$, let $X_{\max}$ to be the set of maximal paths, i.e., $X_{\max} = \{(e_i) \in X : e_i \text{ is maximal in } E_i \text{ for all } i\}$. Similarly, define $X_{\min} = \{(e_i) \in X : e_i \text{ is minimal in } E_i \text{ for all } i\}$. Since X is compact, it follows that these sets are always nonempty. Also it is clear that the sets $X_{\max}, X_{\min}$ are closed. Now for every $(e_i) \in X \setminus X_{\max}$, let $k = \min\{i : e_i \text{ is not maximal in } E_i\}$ and let f_k be the successor of e_k in E_k. For $1 \leqslant i < k$, define f_i so that $(f_1, \ldots, f_{k-1})$ is the unique minimal path (i.e., each f_i is minimal in E_i) from V_0 to V_{k-1} such that $r_{k-1}(f_{k-1}) = s_k(f_k)$. Finally, let $f_n = e_n$ for $n > k$. Define a partial mapping φ on X by $\varphi((e_i)) = (f_i)$. For $x, y \in X$, we will write $x \leqslant y$ if $\varphi^n(x) = y$ for some $n \geqslant 0$. Then $\leqslant$ is a partial ordering on X.

Let $x = (e_i) \in X$ and $n \geqslant 1$. Define $C_n(x) = C(p)$, where $p = (e_1, \ldots, e_n)$. If $x = (e_i) \in X \setminus X_{\max}$, then for every $n \geqslant 1$, there exists $m \geqslant n$ such that $(e_1, e_2, \ldots, e_m)$ is not maximal. We have $\varphi(C_k(x)) = C_k(\varphi(x))$ for all $k \geqslant m$. Since every open subset of $X \setminus X_{\max}$ is a union of clopen subsets $C_k(x)$, φ is a partial homeomorphism.

DEFINITION 4.2. Let $\mathfrak{B} = (\mathcal{V}, \mathcal{E}, \geqslant)$ be an ordered Bratteli diagram, and let $(X, X_{\max}, X_{\min}, \varphi)$ be the partial dynamical system constructed above. The partial homeomorphism φ is called a *Veršik transformation*, and $(X, X_{\max}, X_{\min}, \varphi)$ the *Veršik partial dynamical system associated with* $\mathfrak{B}$.

For $x \in X$, let

$$\begin{aligned} \mathcal{O}(x) &= \{\varphi^n(x) : n \in \mathbb{Z}, x \in \operatorname{dom} \varphi^n\}, \\ \mathcal{O}^+(x) &= \{\varphi^n(x) : n \geqslant 0, x \in \operatorname{dom} \varphi^n\} = \{y \in X : y \geqslant x\}, \\ \mathcal{O}^-(x) &= \{\varphi^n(x) : n \leqslant 0, x \in \operatorname{dom} \varphi^n\} = \{y \in X : y \leqslant x\}, \end{aligned}$$

and $\omega^+(x)$ and $\omega^-(x)$ be the accumulation points of $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$.

PROPOSITION 4.3. *Let $(\mathcal{V}, \mathcal{E})$ be an ordered Bratteli diagram and $X_{\max}$, $X_{\min}$, and φ as defined above. Suppose U (respectively, V) is a clopen subset of X containing $X_{\min}$ (respectively, $X_{\max}$). Then we have:*

(i) $\bigcup_{n=0}^{\infty} \varphi^n(U) = X$;

(ii) $\bigcup_{n=0}^{\infty} \varphi^{-n}(V) = X$.

Proof. We prove only (i), as the proof of (ii) is similar. Let $x \in X$. If $\mathcal{O}^-(x)$ is finite, then $\varphi^{-n}(x) \in X_{\min}$ for some $n \geqslant 0$. Therefore, $x \in \varphi^n(X_{\min}) \subseteq \varphi^n(U)$. So, we may assume that $\mathcal{O}^-(x)$ is infinite.

We are going to show that there exists a sequence $n_1 < n_2 < \cdots$ such that $\varphi^{-n_i}(x)$ converges to some $x_0 \in X_{\min}$. Choose m such that $C_m(x_0) \subseteq U$. Then we have $\varphi^{-n_i}(x) \in C_m(x_0) \subseteq U$ for sufficiently large i, and hence $x \in \varphi^{n_i}(C_m(x_0)) \subseteq \varphi^{n_i}(U)$.

For each $k \geqslant 1$, let $(e_1^k, \ldots, e_k^k)$ be the unique minimal path from V_0 to V_k such that $r(e_k^k) = r(x(k))$. Let $x^k \in X$ be the point satisfying $x^k(i) = e_i^k$ for $1 \leqslant i \leqslant k$ and $x^k(i) = x(i)$ for $i > k$. Then $x^k = \varphi^{-n(k)}(x)$ for some $n(k) > 0$. Since $\mathcal{O}^-(x)$ is infinite, $\{x^k : k \geqslant 1\}$ is also infinite. Let x_0 be an accumulation point of $\{x^k : k \geqslant 1\}$. Then $x_0 \in X_{\min}$ and there exists a subsequence $n_i = n(k_i)$ such that $\varphi^{-n_i}(x)$ converges to x_0. ∎

We will be studying the relationship of ordered Bratteli diagrams and partial dynamical systems. For the rest of this section, we will mainly consider partial dynamical system satisfying the following conditions:

DEFINITION 4.4. A Bratteli system is a quadruple $(X, X_{\max}, X_{\min}, \varphi)$ where $X_{\max}$ and $X_{\min}$ are closed subsets of the zero dimensional compact space X, and $\varphi : X \setminus X_{\max} \to X \setminus X_{\min}$ is a homeomorphism such that if U (respectively, V) is a clopen subset of X containing $X_{\min}$ (respectively, $X_{\max}$), then

(i) $\bigcup_{n=0}^{\infty} \varphi^n(U) = X$;

(ii) $\bigcup_{n=0}^{\infty} \varphi^{-n}(V) = X$.

Two Bratteli systems $(X^1, X^1_{\max}, X^1_{\min}, \varphi_1)$ and $(X^2, X^2_{\max}, X^2_{\min}, \varphi_2)$ are said to be *conjugate* to each other if there exists a homeomorphism $h : X^1 \to X^2$ such that $h(X^1_{\max}) = X^2_{\max}$, $h(X^1_{\min}) = X^2_{\min}$, and $h \circ \varphi_1 = \varphi_2 \circ h$.

REMARK 4.5. It will be shown that every Bratteli system is conjugate to one arising from an ordered Bratteli diagram. This, together with Proposition 4.3, justifies the above definition.

Since the clopen sets form a base for the topology of X, one could just as well substitute "open" for "clopen" in the Definition 4.4. However, mostly it will be convenient to work with clopen sets.

LEMMA 4.6. *Let Y be a clopen set containing $X_{\min}$. Set $Z = \varphi^{-1}(Y) \cup X_{\max}$. Then Z is clopen and $Y = \varphi(Z) \cup X_{\min}$.*

Proof. This follows from the fact that $\varphi|_{X\setminus Z}$ is a homeomorphism from $X \setminus Z$ to $\varphi(X \setminus Z) = X \setminus Y$. ∎

Similarly, we have

LEMMA 4.7. *Let Z be a clopen set containing $X_{\max}$. Set $Y = \varphi(Z) \cup X_{\min}$. Then Y is clopen and $Z = \varphi^{-1}(Y) \cup X_{\max}$.*

We note that in the proof of the last two lemmas, the conditions (i), (ii) in Definition 4.4 are not needed.

PROPOSITION 4.8. *Condition* (i) *in the definition of Bratteli system is equivalent to condition* (ii).

Proof. We show (i) $\Rightarrow$ (ii) By Lemma 4.7, we may assume that $V = \varphi^{-1}(Y) \cup X_{\max}$ for some clopen $Y \supseteq X_{\min}$. Suppose (ii) fails to hold. Then there exists $x_0 \in X$ such that for all $k \geqslant 0, \varphi^k(x_0) \notin V$. In particular, $\varphi^k(x_0) \notin X_{\max}$, so the forward orbit $\{\varphi^k(x_0) : k \geqslant 0\}$ is defined. Since V is open, the closed orbit $\mathrm{cl}\{\varphi^k(x_0) : k \geqslant 0\}$ does not intersect V, and hence $\omega^+(x_0) \cap V = \emptyset$.

Let $x \in \omega^+(x_0)$. Thus there is a sequence $0 \leqslant k_1 < k_2 < \cdots$ with $x = \lim \varphi^{k_n}(x_0)$. By assumption (i), $x = \varphi^m(y')$ for some $m \geqslant 0$, $y' \in Y$. Thus, $y' = \lim\limits_n \varphi^{k_n - m}(y)$. Since Y is open, there is an n with $k_n - m > 0$ for which $\varphi^{k_n - m}(x_0) \in Y$. But then $\varphi^{k_n - m - 1}(x_0) \in V$. This is a contradiction, and the proof is complete.

The implication (ii) $\Rightarrow$ (i) is analogous. ∎

REMARK 4.9. *Note that Bratteli systems do not admit periodic points; that is, there is no point $x \in X$ and positive integer n such that $x \in \operatorname{dom} \varphi^n$ and $\varphi^n(x) = x$.*

Proof. Assume to the contrary there is a periodic point x with period n. Then the orbit $\mathcal{O}(x) = \{\varphi^j(x) : 0 \leqslant j < n\}$ is finite and evidently disjoint from $X_{\max}$. Thus there is a clopen neighborhood Z of $X_{\max}$ which is disjoint from $\mathcal{O}(x)$. But by assumption (ii) of Bratteli systems there is a nonnegative integer j with $x \in \varphi^{-j}(Z)$; i.e., $\varphi^j(x) \in Z$, a contradiction. ∎

4.1. BRATTELI DIAGRAMS FROM PARTIAL DYNAMICAL SYSTEMS. In this section we will show that any partial dynamical system satisfying (i), (ii) (i.e., a Bratteli system) is given by a Bratteli diagram. The proof follows Putnam's construction in [16].

Suppose $(X, X_{\max}, X_{\min}, \varphi)$ is a Bratteli system. Let Y be a clopen subset containing $X_{\min}$ and $Z = \varphi^{-1}(Y) \cup X_{\max}$. Define $\lambda : Y \to \mathbb{Z}$ by

$$\lambda(y) = \min\{k \geqslant 0 : \varphi^k(y) \in Z\}.$$

By condition (ii) of a Bratteli system, $\lambda(y) < \infty$ for all $y \in Y$. By the compactness of X, there is an $N \in \mathbb{Z}^+$ such that $\bigcup_{n=0}^{N} \varphi^{-n}(Z) = X$. Therefore, $\lambda(Y)$ only takes a finite number of values, say, $J_1 < J_2 < \cdots < J_m$. Set

$$Y_k = \lambda^{-1}(J_k) \subset Y, \quad \text{and} \quad Y(k,j) = \varphi^j(Y_k) \quad \text{for } j = 0, \ldots, J_k,$$

and $k = 1, \ldots, m$. Then we have

(1) $\bigcup_{k=1}^{m} Y(k,1) = \varphi(Y)$;
(2) $\varphi(Y(k,j)) = Y(k,j+1)$, for $0 \leqslant j < J_k$;
(3) $\bigcup_{k=1}^{m} Y(k,J_k) = Z$;
(4) $\bigcup_{k=1}^{m} \bigcup_{j=0}^{J_k} Y(k,j) = X$.

It follows from the definition that the sets $Y(k,j)$, $1 \leqslant k \leqslant m$, $0 \leqslant j \leqslant J_k$ are disjoint.

(1) and (2) are clear. To show (3), let $z \in Z$. By assumption (i), $\bigcup_{n=0}^{\infty} \varphi^n(Y) = X$, so there is a smallest nonnegative integer j for which $z \in \varphi^j(Y)$. Set $y = \varphi^{-j}(z)$.

CLAIM. *For $0 \leqslant \ell < j$, $\varphi^\ell(y) \notin Z$.*

Suppose to the contrary that for some ℓ, $0 \leqslant \ell < j$, $\varphi^\ell(y) \in Z$. Note that $\varphi^\ell(y) \notin X_{\max}$, since $\varphi^\ell(y) \in \operatorname{dom} \varphi^{j-\ell}$. Thus, by definition of Z, $y_1 = \varphi(\varphi^\ell(y)) \in Y$, and $z = \varphi^{j-\ell-1}(y_1)$. Since $j - \ell - 1 \geqslant 0$, this contradicts our choice of j. This establishes the claim.

Thus, $\lambda(y) = j$, so $j \in \{J_1, \ldots, J_k\}$, and hence $z \in \varphi^{J_k}(y) \in Y(k, J_k)$ for some k.

The proof of (4) is similar. Let $x \in X$, and let j be the smalllest nonnegative integer with $x \in \varphi^j(Y)$. Set $y = \varphi^{-j}(x)$. Then, as in the proof of (3), $\varphi^\ell(y) \notin Z$ for $0 \leqslant \ell < j$. If $y \in Y_k$, then $x \in Y(k,j)$, since $0 \leqslant j \leqslant J_k$.

NOTE. The sets $Y(k,j)$, $1 \leqslant k \leqslant m$, $0 \leqslant j \leqslant J_k$ are open and form a partition of X. Hence, each of them is both closed and open (clopen).

DEFINITION 4.10. We will refer to such a partition as a Kakutani-Rohlin partition, and to the sets $\{Y(k,j) : 0 \leqslant j \leqslant J_k\}$ as a tower.

Let the partial dynamical system $(X, X_{\max}, X_{\min}, \varphi)$ be a Bratteli system, and let $\{Y_n\}_{n=1}^{\infty}$ be a nested sequence of clopen sets containing $X_{\min}$ such that

$\bigcap_{n=1}^{\infty} Y_n = X_{\min}$. Then $\bigcap_{n=1}^{\infty} (\varphi^{-1}(Y_n) \cup X_{\max}) = X_{\max}$. Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a nested sequence of finite clopen partitions of X such that each Y_n is a union of clopen sets in $\mathcal{P}_n$, $n = 1, 2, \ldots$, and $\bigvee \mathcal{P}_n$ is a base for the topology of X.

Inductively, construct sequences $\mathcal{Q}_n, \mathcal{P}'_n$ where $\mathcal{Q}_n$ is a finite clopen partition of Y_n, $\mathcal{P}'_n$ is a finite clopen partition of X as follows: set

$$\mathcal{Q}_1 = \bigvee_{j=0,\ldots,J(1,k)} \ \bigvee_{k=1,\ldots,m(1)} \{\varphi^{-j}[(Y_1(k,j) \cap P] : P \in \mathcal{P}_1\}.$$

So $\mathcal{Q}_1$ is a partition of Y_1, and each set $Y_1(k,0)$ is the union of sets $Y_1(k,0) \cap Q$ as Q runs through $\mathcal{Q}_1$. Index these sets $Y_1(k,0,i)$, $1 \leqslant i \leqslant r(1,k)$. The sets

$$Y_1(k,j,i) = \varphi^j(Y_1(k,0,i)),$$

$1 \leqslant k \leqslant m(1)$, $1 \leqslant i \leqslant r(1,k)$, $0 \leqslant j \leqslant J(1,k)$, partition X. Denote this partition $\mathcal{P}'_1$. Let $\mathcal{P}'_0 = \{X\}$.

Suppose now that $n > 1$ and $\mathcal{P}'_1, \ldots, \mathcal{P}'_{n-1}$ have been defined so that $\mathcal{P}'_l$ is a refinement of $\mathcal{P}_l$ and $\mathcal{P}'_{l-1}$, for $1 \leqslant l \leqslant n-1$. Set

$$\mathcal{Q}_n = \bigvee \{\varphi^{-j}(Y_n(k,j) \cap P) : P \in \mathcal{P}_n \vee \mathcal{P}'_{n-1},\ 1 \leqslant j \leqslant J(n,k),\ 1 \leqslant k \leqslant m(n)\}.$$

Thus the set $Y_n(k,0)$ is a union of sets $Y_n(k,0) \cap Q$, $Q \in \mathcal{Q}_n$. Index these sets by $Y_n(k,0,i), 1 \leqslant i \leqslant r(n,k)$. Then, let $\mathcal{P}'_n$ be the partition consisting of the sets

$$Y_n(k,j,i) = \varphi^j(Y_n(k,0,i)),$$

$1 \leqslant k \leqslant m(n)$, $1 \leqslant i \leqslant r(n,k)$, $1 \leqslant j \leqslant J(n,k)$.

Note that since $\mathcal{P}'_n$ is finer than $\mathcal{P}_n$, $\bigvee \mathcal{P}'_n$ generates the topology of X.

THEOREM 4.11. *Let the partial dynamical system $(X, X_{\max}, X_{\min}, \varphi)$ be a Bratteli system. Then there is a Bratteli diagram $\mathfrak{B} = \mathfrak{B}(\mathcal{V}, \mathcal{E}, \geqslant)$ such that the Veršik partial dynamical system $(X', X'_{\max}, X'_{\min}, \varphi')$ associated to $\mathfrak{B}$ is conjugate to $(X, X_{\min}, X_{\max}, \varphi)$.*

Proof. The Bratteli diagram $\mathfrak{B}$ to be constructed will have as its vertices the set

$$V_n = \{Y_n(k,0,i) : 1 \leqslant i \leqslant r(n,k), 1 \leqslant k \leqslant m(n)\} \quad \text{for } n \geqslant 1.$$

For convenience of notation, define $V_0 = \{Y_0(1,0,1)\} = \{X\}$. Let $n \geqslant 1$. For each $Y_n(k,0,i)$ and $0 \leqslant \ell \leqslant J(n,k)$, if $Y_n(k,\ell,i) \subseteq Y_{n-1}(k',0,i')$ for some $1 \leqslant k' \leqslant m(n-1)$, $1 \leqslant i' \leqslant r(n-1,k')$, we put an edge e from $Y_{n-1}(k',0,i')$ to $Y_n(k,0,i)$. We will indicate the correspondence between e and ℓ by $j(e) = \ell$. Define $s(e) = Y_{n-1}(k',0,i')$ and $r(e) = Y_n(k,0,i)$. For two edges e_1, e_2 with $r(e_1) = r(e_2)$, we put $e_1 \leqslant e_2$ if $j(e_1) \leqslant j(e_2)$. This defines an ordered Bratteli diagram $\mathfrak{B}(\mathcal{V}, \mathcal{E})$. Let $(X', X'_{\max}, X'_{\min}, \varphi')$ be the corresponding Veršik partial dynamical system. We are going to show that $(X', X'_{\max}, X'_{\min}, \varphi')$ is conjugate to $(X, X_{\min}, X_{\max}, \varphi)$.

Since $Y_n \subseteq Y_{n-1}$, we have $j(e) = 0$ for all minimal edges e. Suppose e is a maximal edge from $Y_{n-1}(k',0,i')$ to $Y_n(k,0,i)$. Let $\ell = j(e)$. Then we have $Y_n(k,\ell,i) \subseteq Y_{n-1}(k',0,i')$ but $Y_n(k,j,i) \cap Y_{n-1} = \emptyset$ for all $\ell < j \leqslant J(n,k)$. Since for each $0 \leqslant j \leqslant J(n-1,k')$, $Y_{n-1}(k',j,i') \subseteq Y_{n-1}$ or $Y_{n-1}(k',j,i') \cap Y_{n-1} = \emptyset$, we have $Y_{n-1}(k',j-\ell,i') \cap Y_{n-1} = \emptyset$ for $\ell < j \leqslant J(n,k)$. On the other hand,

$\varphi(Y_{n-1}(k', J(n,k)-\ell, i'))\cap Y_{n-1} \supset \varphi(Y_n(k, J(n,k), i))\cap Y_n = \varphi(Y_n(k, J(n,k), i)) \neq \emptyset$. Hence, $J(n,k) - \ell = J(n-1,k')$, which implies that $\ell = J(n,k) - J(n-1,k')$.

For $1 \leqslant m \leqslant n$, let e_m be an edge from $Y_{m-1}(k_{m-1}, 0, i_{m-1})$ to $Y_m(k_m, 0, i_m)$, then $p = (e_1, \ldots, e_n)$ is a finite path in $\mathfrak{B}$ from V_0 to V_n. For each $1 \leqslant m \leqslant n$, let $j_m = j(e_m)$. Define $\Psi(p) = Y_n(k_n, s_n, i_n)$ where $s_n = \sum_{m=1}^{n} j_m$. Suppose $n > 1$. Let $p' = (e_1, \ldots, e_{n-1})$ and $s_{n-1} = \sum_{m=1}^{n-1} j_m$. We have

$$\begin{aligned}\Psi(p) &= Y_n(k_n, s_{n-1} + j_n, i_n) = \varphi^{s_{n-1}}(Y_n(k_n, j_n, i_n)) \\ &\subseteq \varphi^{s_{n-1}}(Y_{n-1}(k_{n-1}, 0, i_{n-1})) = \Psi(p').\end{aligned}$$

Define a map $\psi : X' \to X$ as follows. Let $x' \in X'$ be the infinite path $x' = (e_1, e_2, \ldots)$. Then let $\{\psi(x')\} = \bigcap_{n=1}^{\infty} \Psi((e_1, \ldots, e_n))$. Clearly, ψ is a homeomorphism such that $\psi(X'_{\min}) = X_{\min}$ and $\psi(X'_{\max}) = X_{\max}$. Let $x' = (e_1, e_2, \ldots) \in X'$ such that e_ℓ is an edge from $Y_{\ell-1}(k_{\ell-1}, 0, i_{\ell-1})$ to $Y_\ell(k_\ell, 0, i_\ell)$ and $j_\ell = j(e_\ell)$ for all $1 \leqslant \ell \leqslant n$. If x' is not maximal, let m be the smallest integer such that e_m is not maximal. Let j'_m be the smallest integer $j > j_m$ such that $Y_m(k_m, j, i_m) \subseteq Y_{m-1}(k'_{m-1}, 0, i'_{m-1})$ for some k'_{m-1}, i'_{m-1}. Since $Y_m(k_m, j, i_m) \cap Y_{m-1} = \emptyset$ for all $j_m < j < j'_m$ and $Y_m(k_m, j'_m, i_m) \subseteq Y_{m-1}$, we have $j'_m - j_m = 1 + J(m-1, k_{m-1})$. For $1 \leqslant \ell < m$, $j_\ell = J(\ell, k_\ell) - J(\ell-1, k'_\ell)$ because e_ℓ is maximal. Therefore, $j'_m = 1 + J(m-1, k_{m-1}) + j_m = 1 + \sum_{\ell=1}^{m} j_\ell$. Let f_m be the edge from $Y_{m-1}(k'_{m-1}, 0, i'_{m-1})$ to $Y_m(k_m, 0, i_m)$ with $j(f_m) = j'_m$. Let $(f_1, \ldots, f_{m-1})$ be the unique minimal path from V_0 to $Y_{m-1}(k'_{m-1}, 0, i'_{m-1})$. Then $\varphi'(x') = (f_1, \ldots, f_m, e_{m+1}, e_{m+2}, \ldots)$. For $n > m$, let $s_n = \sum_{\ell=m+1}^{n} j_\ell$. We have

$$\begin{aligned}\{\psi(\varphi'(x'))\} &= \bigcap_{n=m+1}^{\infty} Y_n\Big(k_n, \sum_{\ell=1}^{m} j(f_\ell) + s_n, i_n\Big) = \bigcap_{n=m+1}^{\infty} Y_n(k_n, j'_m + s_n, i_n) \\ &= \varphi\Big(\bigcap_{n=m+1}^{\infty} Y_n\Big(k_n, \sum_{\ell=1}^{m} j_\ell + s_n, i_n\Big)\Big) = \{\varphi(\psi(x'))\}.\end{aligned}$$

It follows that $\psi \circ \varphi' = \varphi \circ \psi$. ∎

REMARK 4.12. It follows that for *any* choice of nested clopen sets Y_n with intersection $X_{\max}$ the ordered Bratteli diagram with corresponding Veršik transformation is conjugate to the given partial dynamical system. Thus, the Veršik transformation is independent of the nested sequence $\{Y_n\}$.

COROLLARY 4.13. *Let $(X, X_{\max}, X_{\min}, \varphi)$ be a partial dynamical system. Then the following are equivalent:*

(i) $\mathcal{R} = \{(x, \varphi^n(x)) : x \in \mathrm{dom}\,(\varphi^n),\ n \in \mathbb{Z}\}$ *is an* AF *groupoid;*

(ii) *the partial dynamical system $(X, X_{\max}, X_{\min}, \varphi)$ is conjugate to a Veršik map on a Bratteli compactum;*

(iii) *for any clopen subset U containing $X_{\min}$, $\bigcup_{n=0}^{\infty} \varphi^n(U) = X$;*

(iv) *for any clopen subset V containing $X_{\max}$, $\bigcup_{n=0}^{\infty} \varphi^{-n}(V) = X$.*

For the next result we will need the notion of equivalence for ordered Bratteli diagrams from [13] (Definitions 3.4, 3.5 and 3.6); for the reader's convenience, we recall the definition here.

DEFINITION 4.14. Let V, W be (finite) vertex sets. Two ordered diagrams $(V, E, r, s), (V, E', r', s')$ are *order equivalent* if there is an order-preserving bijection $\Phi : E \to E'$ such that

$$r(e) = r'(\Phi(e)) \quad \text{and} \quad s(e) = s'(\Phi(e)).$$

Now, let $(\mathcal{V}, \mathcal{E}), (\mathcal{W}, \mathcal{F})$ be two ordered Bratteli diagrams. The Bratteli diagrams are said to be order equivalent if there exist strictly increasing maps $g, h : \mathbb{N} \to \mathbb{N}$ with $g(0) = h(0)$ and ordered diagrams E'_n from V_n to $W_{g(n)}$ and F'_n from W_n to $V_{h(n)}$ such that

$$F'_{g(n)} \circ E'_n \text{ is order equivalent to } E_{h(g(n))} \circ \cdots \circ E_{n+1}$$

and

$$E'_{h(n)} \circ F'_n \text{ is order equivalent to } F_{g(h(n))} \circ \cdots \circ F_{n+1}.$$

REMARK 4.15. A contraction of an ordered Bratteli diagram $\mathfrak{B} = (\mathcal{V}, \mathcal{E})$ is another ordered Bratteli diagram $(\mathcal{V}', \mathcal{E}')$ together with a subsequence $\{n_k\}$ of the positive integers such that $V'_k = V_{n_k}$ and the edge set E'_k consists of all paths from $V_{n_{k-1}}$ to V_{n_k}, ordered lexicographically.

Another definition of order equivalence of ordered Bratteli diagrams is as follows: say $\mathfrak{B} = (\mathcal{V}, \mathcal{E})$, $\mathfrak{B}' = (\mathcal{V}', \mathcal{E}')$ are order equivalent if there is a third ordered Bratteli diagram $\mathfrak{B}'' = (\mathcal{V}'', \mathcal{E}'')$ such that the contraction of $\mathfrak{B}''$ to the even vertices is a contraction of $\mathfrak{B}$, and the contraction of $\mathfrak{B}''$ to the odd vertices is a contraction of $\mathfrak{B}'$.

It is easy to see that this definition is equivalent to the one we have given above.

While we have been viewing ordered Bratteli diagrams as a partial dynamical systems, one can just as well view them as defining a standard $\mathbb{Z}$-analytic algebra by viewing the edges as (ordered) standard embeddings, as in [13]. Now it is known that two semicrossed products are isomorphic if and only if their actions (i.e., homeomorphisms) are conjugate ([10], [6], [14]). Exel's generalized notion of crossed product by a partial action ([3], [4]) allows one to consider standard $\mathbb{Z}$-analytic algebras as semicrossed products. In this sense, the following theorem extends the earlier results on semicrossed products.

THEOREM 4.16. *Let* $(\mathcal{V},\mathcal{E})$ *(respectively* $(\mathcal{V}',\mathcal{E}')$*) be an ordered Bratteli diagram,* $(X,X_{\max},X_{\min},\varphi)$ *(respectively,* $(X',X'_{\max},X'_{\min},\varphi')$*) the Veršik partial dynamical system constructed from the diagram, and* $\mathfrak{A}=\mathfrak{A}(\mathcal{V},\mathcal{E})$ *(respectively,* $\mathfrak{A}'=\mathfrak{A}'(\mathcal{V}',\mathcal{E}')$*) the standard* $\mathbb{Z}$*-analytic* TAF *algebra defined by the diagram* $(\mathcal{V},\mathcal{E})$ *(respectively,* $(\mathcal{V}',\mathcal{E}')$*). Then the following are equivalent:*

(i) *the Bratteli diagrams* $(\mathcal{V},\mathcal{E}),(\mathcal{V}',\mathcal{E}')$ *are order equivalent;*

(ii) *the* $(X,X_{\max},X_{\min},\varphi)$, $(X',X'_{\max},X'_{\min},\varphi')$ *Veršik partial dynamical systems are conjugate;*

(iii) $\mathfrak{A}$ *is isometrically isomorphic to* $\mathfrak{A}'$;

(iv) $\mathfrak{A}$ *is algebrically isomorphic to* $\mathfrak{A}'$.

Proof. (i) $\Leftrightarrow$ (iii) was proved by Power ([13], Theorem 3.7).

(ii) $\Rightarrow$ (iii) To begin, we need the fact that if a strongly maximal TAF algebra is the inductive limit of a system $(\mathfrak{A}_n,\sigma_n)$, then the spectrum $\mathcal{P}$ of $\mathfrak{A}$ is the projective limit of the spectra of $\mathfrak{A}_n$. It follows in this case that the spectrum $\mathcal{P}$ of $\mathfrak{A}$ is $\{(x,y)\in X\times X : y=\varphi^n(x) \text{ for some } n\geqslant 0\}$.

Suppose (ii) is satisfied, and let $\psi : X\to X'$ be a homeomorphism which induces a conjugacy of the two partial dynamical systems. Define $\Psi:\mathcal{P}\to\mathcal{P}'$,

$$\Psi(x,\varphi^j(x)) = (\psi(x),\psi(\varphi^j(x)) = (\psi(x),{\varphi'}^j(\psi(x)),$$

$x\in X\setminus X_{\max}$. Then Ψ is a semigroupoid isomorphism, and so by Theorem 7.5 in [15], $\mathfrak{A},\mathfrak{A}'$ are isometrically isomorphic.

(iii) $\Rightarrow$ (ii) Conversely, suppose there is a semigroupoid isomorphism $\Psi : \mathcal{P}\to\mathcal{P}'$. Setting $\Delta(X)=\{(x,x) : x\in X\}$, note that $\Psi(\Delta(X))=\Delta(X')$. Indeed, if $\Psi(x,x)=(x',y')$, then

$$(x',y') = \Psi((x,x)\circ(x,x)) = \Psi(x,x)\circ\Psi(x,x) = (x',y')\circ(x',y'),$$

forcing $y'=x'$. Thus there is a homeomorphism $\psi : X\to X'$ so that $\Psi(x,x) = (\psi(x),\psi(x))$, $x\in X$. Also,

$$\begin{aligned}\Psi(x,\varphi(x)) &= \Psi(x,x)\circ\Psi(x,\varphi(x))\circ\Psi(\varphi(x),\varphi(x))\\ &= (\psi(x),\psi(x))\circ\Psi(x,\varphi(x))\circ(\psi(\varphi(x)),\psi(\varphi(x))),\end{aligned}$$

from which $\Psi(x,\varphi(x)) = (\psi(x),\psi(\varphi(x)))$. Since $(\psi(x),\psi(\varphi(x))\in\mathcal{P}'$, $\psi(\varphi(x)) = {\varphi'}^j(\psi(x))$ for some $j\geqslant 0$. Now $j=0$ is impossible, as Ψ^{-1} maps $\Delta(X')$ into $\Delta(X)$. Now if $j\geqslant 2$, then

$$(\psi(x),{\varphi'}^j(\psi(x))) = (\psi(x),\varphi'(\psi(x)))\circ(\varphi'(\psi(x)),{\varphi'}^j(\psi(x)))$$

is the composition of two elements of $\mathcal{P}'\setminus\Delta(X')$. Applying Ψ^{-1}, we have that $(x,\varphi(x))$ is the composition of two elements of $\mathcal{P}\setminus\Delta(x)$. But this is impossible, as $\varphi(x)$ is the immediate successor of x. Thus,

$$\Psi(x,\varphi(x)) = (\psi(x),\varphi'(\psi(x))), \quad \text{and hence } \varphi'(\psi(x)) = \psi(\varphi(x)).$$

In other words, ψ is a conjugacy of the two partial dynamical systems.

(iii) $\Leftrightarrow$ (iv) was proved by Donsig, Pitts and Power ([1]). ∎

5. NESTED SEQUENCES AND AF ALGEBRAS

In this section, we are going to characterize nested sequences of partial homeomorphism that will give rise to AF algebras.

Let $\mathfrak{A}$ be a $\mathbb{Z}$-analytic algebra and $\mathfrak{B} = C^*(\mathfrak{A})$. Choose a sequence $\{\mathfrak{B}_n\}$ of finite dimensional C^*-algebras of $\mathfrak{B}$, say $\mathfrak{B}_n = \bigoplus_{k=1}^{r(n)} M_{m(n,k)}$, and a set of matrix units $\{e_{ij}^{(Nk)}\}$ for $\bigcup_{n=1}^{\infty} \mathfrak{B}_n$ such that

$$\mathfrak{A}_n := \mathfrak{A} \cap \mathfrak{B}_n = \bigoplus_{k=1}^{r(n)} T_{m(n,k)},$$

where $T_{m(n,k)}$ is the upper triangular subalgebra of $M_{m(n,k)}$, and $\mathfrak{A} = \lim\limits_{\to} \mathfrak{A}_n$ and $\mathfrak{B} = \lim\limits_{\to} \mathfrak{B}_n$. Denote $e_{ii}^{(Nk)}$ by $e_i^{(Nk)}$. Let $(X, \mathcal{P}, \mathcal{R})$ be the associated spectral triple. Then there exists an integer valued cocycle d on $\mathcal{R}$ with $d^{-1}([0,\infty)) = \mathcal{P}$. For each n, define a partial homeomorphism φ_n on X by $\varphi_n(x) = y$ if $d(x,y) = n$. Then $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a nested sequence of partial homeomorphisms on X such that

$$\mathcal{P} = \bigsqcup_{n=0}^{\infty} \Gamma_{\varphi_n}.$$

For $n \in \mathbb{Z}$, let $X_{\max}^n = X_{\min}^{-n} = X \setminus \operatorname{dom} \varphi_n$, $X_{\max} = \bigcap_{n=1}^{\infty} X_{\max}^n$ and $X_{\min} = \bigcap_{n=1}^{\infty} X_{\min}^n$.

We have

$$X_{\min} = \bigcap_n \Big(\bigcup_k \widehat{e}_1^{(Nk)} \Big) \quad \text{and} \quad X_{\max} = \bigcap_n \Big(\bigcup_k \widehat{e}_{m(n,k)}^{(Nk)} \Big).$$

Suppose U and V are clopen subsets of X containing $X_{\min}$ and $X_{\max}$ respectively. Then there exists N such that $\bigcup_k \widehat{e}_1^{(Nk)} \subseteq U$ and $\bigcup_k \widehat{e}_{m(N,k)}^{(Nk)} \subseteq V$. For $n \in \mathbb{Z}$, let $\widetilde{\varphi}_n$ be the restriction of φ_n to $\mathcal{D}_n = \{x \in X : (x, \varphi_n(x)) \in \widehat{e}_{ij}^{(Nk)}$ for some $i, j\}$. Then each $\mathcal{D}_n$ is clopen. Let

$$M = \max\{d(x,y) : (x,y) \in \widehat{e}_{ij}^{(Nk)},\ 1 \leqslant i,j \leqslant m(N,k),\ 1 \leqslant k \leqslant r(N)\}.$$

If $q > M$ then the domain of $\widetilde{\varphi}_q$ is empty. This proves the necessity of the conditions in Theorem 5.2. We first prove the sufficiency of the conditions for a special case of the theorem.

LEMMA 5.1. *Let $\mathcal{N} = \{\varphi_n\}_{n\in\mathbb{Z}}$ be a nest of partial homeomorphisms on a compact zero-dimensional space X such that:*

(i) *$X_{\min}$ and $X_{\max}$ are clopen;*

(ii) *for each $n \geqslant 1$, $\operatorname{dom} \varphi_n$ and $\operatorname{dom} \varphi_{-n}$ are clopen;*

(iii) *there exists $M \geqslant 1$ such that $\operatorname{dom} \varphi_n = \emptyset$ for $|n| \geqslant M$.*

Then the groupoid defined by $\mathcal{N}$ is AF.

Proof. Define $\Phi : X \setminus X_{\max} \to X \setminus X_{\min}$ by

$$\Phi(x) = \varphi_k(x) \quad \text{where } k = \min\{n \geqslant 1 : x \in \operatorname{dom} \varphi_n\}.$$

We are going to prove that

(1) Φ is bijective and

$$\Phi^{-1}(y) = \varphi_{-k}(y) \quad \text{where } k = \min\{n \geqslant 1 : y \in \operatorname{dom} \varphi_{-n}\}.$$

(2) Φ is a homeomorphism.

(3) $\bigcup_{n=0}^{\infty} \Phi^n(X_{\min}) = X = \bigcup_{n=0}^{\infty} \Phi^{-n}(X_{\max})$.

Proof of (1). Suppose $x, y \in X \setminus X_{\max}$ and $\Phi(x) = \Phi(y)$. Let

$$r = \min\{n \geqslant 1 : x \in \operatorname{dom} \varphi_n\} \quad \text{and} \quad s = \min\{n \geqslant 1 : y \in \operatorname{dom} \varphi_n\}.$$

So $\Phi(x) = \varphi_r(x) = \varphi_s(y) = \Phi(y)$. If $r > s$, then we have $y = \varphi_{-s} \circ \varphi_r(x) = \varphi_{r-s}(x)$. Therefore, $x \in \operatorname{dom} \varphi_{r-s}$ and $1 \leqslant r - s < r$, a contradiction. Therefore, $r \leqslant s$. Similarly, $r \geqslant s$ and consequently, $r = s$. This proves that Φ is one to one.

Let $y \in X \setminus X_{\min}$ and $k = \min\{n \geqslant 1 : y \in \operatorname{dom} \varphi_{-n}\}$. Then $y = \varphi_k(x)$ for some $x \in X \setminus X_{\max}$. Suppose $n \geqslant 1$ and $x \in \operatorname{dom} \varphi_n$. Then

$$y = \varphi_k(x) = \varphi_k(\varphi_{-n} \circ \varphi_n(x)) = (\varphi_k \circ \varphi_{-n})\varphi_n(x) = \varphi_{k-n} \circ \varphi_n(x) \in \operatorname{dom} \varphi_{-(k-n)},$$

and $k - n < k$. Therefore, $k - n \leqslant 0$. Hence, $n \geqslant k$ and $y = \Phi(x)$.

Proof of (2). Let C be a closed subset of $X \setminus X_{\max}$. We will show that $\Phi(C)$ is closed. Suppose $x_n \in C$ and $y = \lim_{n\to\infty} \Phi(x_n)$. By choosing a subsequence if necessary, we may assume that for some fixed $k \geqslant 1$, $\Phi(x_n) = \varphi_k(x_n)$ for all n and $x_0 = \lim_{n\to\infty} x_n \in C \cap \operatorname{dom} \varphi_k$. Therefore, $y = \varphi_k(x_0)$. Suppose $m \geqslant 1$ and $x_0 \in \operatorname{dom} \varphi_m$. Then $x_n \in \operatorname{dom} \varphi_m$ for sufficiently large n. Hence, $k \leqslant m$ and $y = \varphi_k(x_0) = \Phi(x_0) \in \Phi(C)$. Therefore, $\Phi(C)$ is closed. Since Φ is bijective, $\Phi(O)$ is open for every open set O in $X \setminus X_{\max}$. The same proof shows that $\Phi^{-1}(O)$ is open for any open set O in $X \setminus X_{\min}$. Consequently, Φ is a homeomorphism.

Proof of (3). Since $\operatorname{dom} \Phi^M = \emptyset$, we have

$$X = X \setminus \operatorname{dom} \Phi^M = \bigcup_{n=0}^{M-1} \Phi^{-n}(X_{\max}).$$

Hence, $X = \bigcup_{n=0}^{\infty} \Phi^{-n}(X_{\max})$. Similarly, $X = \bigcup_{n=0}^{\infty} \Phi^n(X_{\min})$.

It follows that $(X, X_{\max}, X_{\min}, \Phi)$ is a Bratteli system. By construction, this system generates the same groupoid as that defined by the nest $\mathcal{N}$. ∎

THEOREM 5.2. *Let $\mathcal{N} = \{\varphi_n\}_{n\in\mathbb{Z}}$ be a nest of partial homeomorphisms on a compact zero-dimensional space X and $\mathcal{R}$ the groupoid defined by $\mathcal{N}$. Then $\mathcal{R}$ is* AF *if and only if the following is satisfied:*

Given clopen subsets U and V of X containing $X_{\min}$ and $X_{\max}$ respectively, there exist clopen subsets Y and Z with $X_{\min} \subseteq Y \subseteq U$ and $X_{\max} \subseteq Z \subseteq V$ such that for $n \geqslant 1$, if $\widetilde{\varphi}_n$ is the restriction of φ_n to $\mathcal{D}_n = \{x \in X \setminus Z : \varphi_n(x) \in X \setminus Y\}$, $\widetilde{\varphi}_{-n} = \widetilde{\varphi}_n^{-1}$, and $\varphi_0 = \mathrm{id}_X$, then the system $\{\widetilde{\varphi}_n\}$ satisfies the conditions of Lemma 5.1.

Proof. Suppose $\mathcal{N}$ is a nest satisfying the above conditions. Then we can choose a sequence of clopen subsets U^k and V^k such that $\bigcap\limits_k U^k = X_{\min}$ and $\bigcap\limits_k V^k = X_{\max}$. For each k, applying the condition to $U = U^k$ and $V = V^k$, we have clopen sets Y^k and Z^k with $\widetilde{\varphi}_n^{(k)}$ and $\operatorname{dom}\widetilde{\varphi}_n^{(k)}$ defined accordingly. Let $\mathcal{R}^k$ be the groupoid defined by the nest $\{\widetilde{\varphi}_n^{(k)}\}$. Since $\mathcal{R}_k \subseteq \mathcal{R}_{k+1}$ and $\mathcal{R} = \bigcup\limits_{k=1} \mathcal{R}_k$, it suffices to prove that each $\mathcal{R}_k$ is AF. Since the nest $\{\widetilde{\varphi}_n^{(k)}\}$ satisfies the conditions in Lemma 5.1, the result follows. ∎

EXAMPLE 5.3. This is an example of a nested sequence of partial homeomorphisms satisfying the conditions in Theorem 5.2 but the counting cocycle on the associated groupoid is not continuous. Therefore, the corresponding TAF algebra is $\mathbb{Z}$-analytic but not standard $\mathbb{Z}$-analytic. This TAF algebra is isomorphic to the example given by Donsig and Hopenwasser ([12]).

Suppose $X = \prod\limits_{n=1}^{\infty} \{0,1\}$. Define a homeomorphism $\varphi : X \to X$ by $\varphi((x_n)) = (y_n)$, where

$$y_n = \begin{cases} 0 & \text{if } x_i = 1 \text{ for all } 1 \leqslant i \leqslant n, \\ 1 & \text{if } x_i = 1 \text{ for all } 1 \leqslant i \leqslant n-1 \text{ and } x_n = 0, \\ x_n & \text{if } x_i = 0 \text{ for some } 1 \leqslant i \leqslant n-1. \end{cases}$$

(φ is usually referred to as the odometer map.)

For $x \in X$, let $X_{\max} = \{x\}$ and $X_{\min} = \{\varphi(x)\}$. Restricting φ to $X \setminus X_{\max}$, we have a partial dynamical system $(X, X_{\max}, X_{\min}, \varphi)$, which is a Bratteli system. The conjugacy class of this system is independent of the choice of x. Indeed, viewing X as a solenoidal group, given any two points x, x' there is a homeomorphism h of X mapping x to x' which commutes with φ, namely $h(y) = y + x' - x$. It follows that the system with x as the maximal point is conjugate to the system with x' as the maximal point. The corresponding TAF algebra is standard $\mathbb{Z}$-analytic with $\mathfrak{B}_n = M_{2^n}$ and $\mathfrak{A}_n = T_{2^n}$.

Let $\mathbf{x}_0 = (0, 0, \ldots)$ and $\mathbf{x}_n = \varphi^n(\mathbf{x}_0)$ for $n \in \mathbb{Z}$. Define

$$\begin{aligned}
X^0_{\max} &= \emptyset, \\
X^1_{\max} &= \{\mathbf{x}_k : k = -2, -1, 0\}, \\
X^2_{\max} &= \{\mathbf{x}_k : k = -3, 0\}, \\
&\vdots \\
X^n_{\max} &= \{\mathbf{x}_k : -n-1 \leqslant k \leqslant 0, k \neq -n, -1\} \text{ for } n > 2, \\
X^{-n}_{\max} &= \varphi^n(X^n_{\max}) \text{ for } n \geqslant 1.
\end{aligned}$$

Let $\varphi_0 = \mathrm{id}_X$. For $n \neq 0$, let $\varphi_n = \varphi^n|_{X \setminus X^n_{\max}}$. Then $\mathcal{N} = \{\varphi_n\}_{n \in \mathbb{Z}}$ is a nested sequence of partial homeomorphism on X with $X_{\max} = \{\mathbf{x}_0\}$ and $X_{\min} = \{\mathbf{x}_{-1}\}$. We are going to show that:

(1) $\mathcal{N}$ satisfies the conditions in Theorem 5.2;
(2) the counting cocycle $\widehat{d}$ is not continuous.

Proof of (1). Given $k \geqslant 1$ and $c_i \in \{0, 1\}$ for $i = 1, \ldots, k$, define the k-cylinder sets

$$[c_1, \ldots, c_k]_k = \{\mathbf{x} = \{x_n\} \in X : x_i = c_i \text{ for } 1 \leqslant i \leqslant k\}.$$

The cylinder sets are clopen and form a basis of the topology of X. Given any clopen sets U and V with $X_{\min} \subseteq U$ and $X_{\max} \subseteq V$, there exists $k \geqslant 1$ such that $[1, \ldots, 1]_k \subseteq U$ and $[0, \ldots, 0]_k \subseteq V$. Let $\mathcal{C}_k$ be the collection of all k-cylinder sets. Then

$$\mathcal{D}_n = \begin{cases} \bigcup\{C \in \mathcal{C}_k : C \cap X^{(n)}_{\max} = \emptyset\} & \text{for } n \neq 0, \\ X & \text{for } n = 0. \end{cases}$$

It is straightforward to check that the condition in Theorem 5.2 is satisfied.

Proof of (2). Let $\mathbf{x}^n = \{x^n_i\}_{i \geqslant 1}$, where $x^n_i = 1$ for $1 \leqslant i \leqslant n$ and $x^n_i = 0$ otherwise. Then each $\mathbf{x}^n$ is in $\operatorname{dom} \varphi_1 \circ \varphi_1$ but $\mathbf{x}^n$ converges to $\mathbf{x}_{-1} \in \operatorname{dom} \varphi_2 \setminus (\operatorname{dom} \varphi_1 \circ \varphi_1)$. Therefore, $\widehat{d}(\mathbf{x}^n, \varphi_2(\mathbf{x}^n)) = 2$ but $\widehat{d}(\mathbf{x}_{-1}, \varphi_2(\mathbf{x}_{-1})) = 1$. Hence, $\widehat{d}$ is not continuous.

Let $\mathcal{N} = \{\varphi_n\}_{n \in \mathbb{Z}}$ be a nested sequence of partial homeomorphisms satisfying the condition in Theorem 5.2. Define $\Phi : X \setminus X_{\max} \to X \setminus X_{\min}$ by

$$\Phi(x) = \varphi_k(x) \quad \text{where } k = \min\{n \geqslant 1 : x \in \operatorname{dom} \varphi_n\}$$

(as defined in Lemma 5.1). In Example 5.3, we have $\mathbf{x}^n \to \mathbf{x}_{-1}$ but $\Phi(\mathbf{x}^n) \to \mathbf{x}_0 \neq \mathbf{x}_1 = \Phi(x_{-1})$. Therefore, Φ may not be continuous. However, Φ turns out to be very useful in the study of the isomorphism of the associated $\mathbb{Z}$-analytic algebra. The proof of the following theorem is similar to the proof for the equivalence of conditions (ii) and (iii) in Theorem 4.16.

THEOREM 5.4. *Suppose $\mathcal{N} = \{\varphi_n\}_{n\in\mathbb{Z}}$ and $\mathcal{N}' = \{\varphi'_n\}_{n\in\mathbb{Z}}$ are two nested sequences of partial homeomorphisms on X and X' satisfying the conditions in Theorem 5.2. Then the associated $\mathbb{Z}$-analytic algebras $\mathfrak{A}$ and $\mathfrak{A}'$ are isometrically isomorphic if and only if there exists a homeomorphism $\psi : X \to X'$ such that $\psi(X_{\max}) = X'_{\max}$, $\psi(X_{\min}) = X'_{\min}$ and $\psi \circ \Phi = \Phi' \circ \psi$.*

SEMI-SATURATING A NESTED SEQUENCE. In Example 5.3 there is a single partial homeomorphism φ of X (actually in this case a homeomorphism) such that each φ_n in the nested sequence arises as a restriction of the n-fold composition φ^n. A *semi-saturation* of a nested sequence $\{\varphi_n\}$ of partial homeomorphisms is a partial homeomorphism φ on X satisfying $\operatorname{dom}(\varphi_n) \subset \operatorname{dom}(\varphi^n)$, and for all $x \in \operatorname{dom}(\varphi_n)$, $\varphi_n(x) = \varphi^n(x)$, $n = 1, 2, \ldots$.

We note that in the above example, $\operatorname{dom}\varphi_1$ is dense in $\bigcup_{n=1}^{\infty} \operatorname{dom}\varphi_n$. For such systems we have the following result.

PROPOSITION 5.5. *Let $\{\varphi_n\}_{n=0}^{\infty}$ be a nested sequence of partial homeomorphism on X such that $\operatorname{dom}\varphi_1$ is dense in $\bigcup_{n=1}^{\infty} \operatorname{dom}\varphi_n$. Then the following conditions are equivalent:*

(i) *There exists a homeomorphism φ on $\bigcup_{n=1}^{\infty} \operatorname{dom}\varphi_n$ such that for all $n \geqslant 1$ $\varphi_n = \varphi^n|\operatorname{dom}\varphi_n$.*

(ii) (a) *there exists $\lim_{k\to\infty} \varphi_1(\mathbf{x}^{(k)})$ for every sequence $\{\mathbf{x}^{(k)}\}$ in $\operatorname{dom}\varphi_1$ such that $\lim_{k\to\infty} \mathbf{x}^{(k)} \in \bigcup_{n=1}^{\infty} \operatorname{dom}\varphi_n$; and*

(b) *there exists $\lim_{k\to\infty} \varphi_1^{-1}(\mathbf{y}^{(k)})$ for every sequence $\{\mathbf{y}^{(k)}\} \in \operatorname{range}\varphi_1$ such that $\lim_{k\to\infty} \mathbf{y}^{(k)} \in \bigcup_{n=1}^{\infty} \operatorname{range}\varphi_n$.*

Proof. Clearly, (ii) follows from (i). Suppose (ii) holds. Let $\mathbf{x} \in \bigcup_{n=2}^{\infty} \operatorname{dom}\varphi_n$ and $\mathbf{x}^{(k)} \in \operatorname{dom}\varphi_1$ such that $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{x}$. Then $\mathbf{y} = \lim_{k\to\infty} \varphi_1(\mathbf{x}^{(k)})$ exists and the limit is independent of the choice of $\{\mathbf{x}^{(k)}\}$. Therefore, we can extend φ_1 to a continuous map φ on $\bigcup_{n=1}^{\infty} \operatorname{dom}\varphi_n$. It follows from (b) that φ is a homeomorphism on $\bigcup_{n=1}^{\infty} \operatorname{dom}\varphi_n$. ∎

EXAMPLE 5.6. Let $X = \left\{\left(\frac{1}{n}, i\right) : i = 1, 2 \text{ and } n \geqslant 1\right\} \cup \{(0,1), (0,2)\} \subset \mathbb{R}^2$, $X_1 = X \setminus \{(0,1), (0,2)\}$. Define a nested sequence $\{\varphi_n\}_{n=0}^{\infty}$ of partial homeomorphism on X by: $\varphi_0 = \mathrm{id}_X$; $\varphi_1\left(\left(\frac{1}{2k-1}, i\right)\right) = \left(\frac{1}{2k}, i\right)$ for $i = 1, 2$, $\varphi_1\left(\left(\frac{1}{2k}, 1\right)\right) = \left(\frac{1}{2k-1}, 2\right)$, and $\varphi_1\left(\left(\frac{1}{2k}, 2\right)\right) = \left(\frac{1}{2k+1}, 1\right)$ for all $k \geqslant 1$; $\varphi_2((0,1)) = (0,2)$ and $\varphi_2(\mathbf{x}) = \varphi_1^2(\mathbf{x})$ for all $\mathbf{x} \in X_1$ and for $k \geqslant 2$, $\varphi_{2k-1} = \varphi_1^{2k-1}$ and $\varphi_{2k} = \varphi_2^k$.

Let $\mathbf{x}^k = \left(\frac{1}{2k-1}, 1\right)$ and $\mathbf{y}^k = \left(\frac{1}{2k}, 1\right)$. Then $\lim\limits_{k\to\infty} \mathbf{x}^{(k)} = \lim\limits_{k\to\infty} \mathbf{y}^{(k)} = (0,1)$ but $\lim\limits_{k\to\infty} \varphi_1(\mathbf{x}^{(k)}) = (0,1)$ and $\lim\limits_{k\to\infty} \varphi_1(\mathbf{y}^{(k)}) = (0,2)$. So there exists no partial homeomorphism φ on X such that $\varphi_n = \varphi^n|X_n$ for all n.

This shows that the nested sequence $\{\varphi_n\}_{n=1}^{\infty}$ does not admit a semi-saturation. Let $(X, \mathcal{P}_1, \mathcal{R}_1)$ be the spectral triple associated with $\{\varphi_n\}_{n=0}^{\infty}$. We want to show that $(X, \mathcal{P}_1, \mathcal{R}_1)$ is the spectral triple of a $\mathbb{Z}$-analytic algebra. For this it is enough to know that $\mathcal{R}_1$ is an AF groupoid.

Define $X_{\max} = \{(0,2)\}$, $X_{\min} = \{(1,1)\}$ and $\varphi : X \setminus X_{\max} \to X \setminus X_{\min}$ by $\varphi((0,1)) = (0,2)$, $\varphi\Big(\left(\frac{1}{n}, 1\right)\Big) = \left(\frac{1}{n}, 2\right)$ and $\varphi\Big(\left(\frac{1}{n}, 2\right)\Big) = \left(\frac{1}{n+1}, 1\right)$ for all $n \geqslant 1$. Then $(X, X_{\max}, X_{\min}, \varphi)$ is a Bratteli system with associated spectral triple $(X, \mathcal{P}_2, \mathcal{R}_2)$. It is easy to check that $\mathcal{R}_2 = \mathcal{R}_1$. By Theorem 4.11, the quadruple $(X, X_{\max}, X_{\min}, \varphi)$ is conjugate to a Veršik transformation of an ordered Bratteli diagram. The (unordered) Bratteli diagram defines the AF algebra $C^*(\mathcal{R}_2) = C^*(\mathcal{R}_1)$. Another way to see that $\mathcal{R}_1$ is an AF groupoid is to apply Theorem 5.2. Therefore, $\mathfrak{A}(\mathcal{P}_1)$ is a $\mathbb{Z}$-analytic subalgebra of $C^*(\mathcal{R}_1)$ but $\mathfrak{A}(\mathcal{P}_1)$ is not standard $\mathbb{Z}$-analytic.

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