J. OPERATOR THEORY
51(2004), 89–104

ORTHOGONALITY IN $\mathfrak{S}_1$ AND $\mathfrak{S}_\infty$ SPACES AND NORMAL DERIVATIONS

DRAGOLJUB J. KEČKIĆ

Communicated by Florian-Horia Vasilescu

ABSTRACT. We introduce φ-Gateaux derivative, and use it to give the necessary and sufficient conditions for the operator Y to be orthogonal (in the sense of James) to the operator X, in both spaces $\mathfrak{S}_1$ and $\mathfrak{S}_\infty$ (nuclear and compact operators on a Hilbert space). Further, we apply these results to prove that there exists a normal derivation Δ_A such that $\overline{\operatorname{ran}\,\Delta_A} \oplus \ker \Delta_A \neq \mathfrak{S}_1$, and a related result concerning $\mathfrak{S}_\infty$.

KEYWORDS: *Gateaux derivative, orthogonality in Banach spaces, Schatten ideals, derivation, elementary operator.*

MSC (2000): Primary 46G05, 47B10, 47B47; Secondary 47A30, 46B99.

0. INTRODUCTION

Let H denote a separable Hilbert space, and let $\mathfrak{S}_p$ denote the Schatten ideal of those compact operators X acting on H such that $\|X\|_p = \Big(\sum\limits_{j=1}^{+\infty} s_j(X)^p\Big)^{1/p} < +\infty$, where $s_j(X) = \lambda_j(X^*X)^{1/2}$. Also, let $\mathfrak{S}_\infty$ denote the ideal of all compact operators equipped with the usual norm. Let us recall that these (Schatten) norms are special cases of so called unitarily invariant norms, associated with some two-sided ideal of compact operators. For further details the reader is referred to [6]. It is well known that $\mathfrak{S}_2$ has a Hilbert space structure, with the inner product $\langle X, Y\rangle = \operatorname{tr}(XY^*)$, and that this is not true in other $\mathfrak{S}_p$. Nevertheless, in all Banach spaces we can define the orthogonality in the following way (orthogonality in the sense of R.C. James).

DEFINITION 0.1. Let X be a Banach space. We say that $y \in X$ *is orthogonal to* $x \in X$ if for all complex numbers λ there holds

$$\|x + \lambda y\| \geqslant \|x\|. \tag{0.1}$$

REMARK 0.2. If X is a Hilbert space, then from (0.1) we can easily derive $\langle x, y \rangle = 0$.

REMARK 0.3. In Banach spaces, orthogonality from the previous definition is not symmetrical, i.e. y orthogonal to x does not imply x orthogonal to y.

REMARK 0.4. Definition 0.1 has a natural geometric interpretation. Namely, $y \perp x$ if and only if the complex line $\{x + \lambda y : \lambda \in \mathbb{C}\}$ is disjoint with the open ball $K(0, \|x\|)$, i.e. if and only if this complex line is a tangent one.

Such an orthogonality relation is closely related with Gateaux derivative of the norm and the smoothness of the sphere of radius $\|x\|$.

DEFINITION 0.5. The vector x is a smooth point of the sphere $S(0, \|x\|)$ if there exists a unique support functional $F_x \in X^*$, such that $F_x(x) = \|x\|$ and $\|F_x\| = 1$.

PROPOSITION 0.6. *If there exists the Gateaux derivative of the norm at the point x, i.e. if there exists the limit $\lim\limits_{\mathbb{R} \ni t \to 0} \frac{\|x+ty\|-\|x\|}{t} = 0$, then it is equal to* $\operatorname{Re} F_x(y)$, *where F_x is the functional from the previous definition. Moreover, in this case y is orthogonal to x if and only if $F_x(y) = 0$.*

It is also well known that if Banach space X has a strictly convex dual space then every nonzero point is a smooth point of the corresponding sphere. For details see [1] and references therein.

Orthogonality in the sense of James were used in investigation of so called elementary operators, introduced by Lumer and Rosenblum ([11]).

DEFINITION 0.7. Let $(A_1, A_2, \dots, A_n)$ and $(B_1, B_2, \dots, B_n)$ be the n-tuples of bounded Hilbert space operators. The mapping $X \mapsto \sum\limits_{j=1}^{n} A_j X B_j$ from $B(H)$ to $B(H)$ is called the *elementary operator* or *elementary mapping*.

REMARK 0.8. The same name "elementary operators" is used for operators of the same form, which maps $\mathfrak{J}$ to $\mathfrak{J}$, where $\mathfrak{J}$ is some two sided ideal equipped with a unitarily invariant norm.

The first result concerning the orthogonality in the sense of James and elementary operators was given by Anderson ([2]).

PROPOSITION 0.9. *If A is a normal operator on a separable Hilbert space H, then $AS = SA$ implies that for all bounded X there holds*

$$\|AX - XA + S\| \geqslant \|S\|.$$

In view of Definition 0.1, it means that the range of the mappings $\Delta_A : B(H) \to B(H)$, $\Delta_A(X) = AX - XA$ is orthogonal to its kernel. This result has been generalized in two directions, by extending the class of elementary mappings, and by extending this inequality to the other unitarily invariant norms; see for instance [4], [5], [8], [9].

In [2], Anderson also proved that equality $\overline{\operatorname{ran} \Delta_A} \oplus \ker \Delta_A = B(H)$ is true in very special cases, for example if and only if the spectrum of the normal operator A in Proposition 0.9 is finite. In [8] there was conjectured that it might be $\mathfrak{J} = \overline{\operatorname{ran} \Delta_A|\mathfrak{J}} \oplus \ker \Delta_A|\mathfrak{J}$ if the ideal $\mathfrak{J}$ is separable. In Section 3, we shall give the negative answer to this hypothesis.

The Gateaux derivative technique was used in [3], [10] and [12], in order to characterize those operators to which the range of a derivation is orthogonal. In these papers, the attention was directed to $\mathfrak{S}_p$ ideals for some $p > 1$, and to smooth points in $\mathfrak{S}_1$ and $\mathfrak{S}_\infty$, like in the following proposition, taken from [10].

PROPOSITION 0.10. *Let A be a bounded Hilbert space operator. The range of a derivation Δ_A is orthogonal to an operator S in $\mathfrak{S}_p$ if and only if $A\widetilde{S} = \widetilde{S}A$, where $\widetilde{S} = U|S|^{p-1}$, and $S = U|S|$.*

Smooth points in $\mathfrak{S}_1$ and $\mathfrak{S}_\infty$, were characterized by Holub ([7]).

PROPOSITION 0.11. *The operator X is a smooth point of the corresponding sphere in $\mathfrak{S}_1$ if and only if either X is injective or X^* is injective. The operator X is a smooth point of the corresponding sphere in $\mathfrak{S}_\infty$ if and only if it attains its norm at the unique vector (up to a complex scalar).*

The main purpose of this note is to characterize the orthogonality in the sense of James in $\mathfrak{S}_1$ and $\mathfrak{S}_\infty$ at the points which are not smooth, and to apply these characterizations to elementary operators. Namely, among other things, we prove that for a normal derivation $\Delta_A : \mathfrak{S}_p \to \mathfrak{S}_p$ there holds $\overline{\operatorname{ran} \Delta_A} \oplus \ker \Delta_A = \mathfrak{S}_p$ for $1 < p < +\infty$, and that such completeness result fails for $p = 1$. In the case $p = +\infty$ the situation is more complicated. Some results concerning more general elementary operators are also given.

1. φ-GATEAUX DERIVATIVES

In this section we introduce φ-Gateaux derivative and, in Theorem 1.4, we give the necessary and sufficient condition for a vector y from an arbitrary Banach space to be orthogonal (in the sense of James) to a vector x, in terms of introduced φ-Gateaux derivative.

Definition 1.1. Let $(X, \|\cdot\|)$ be an arbitrary Banach space. φ-Gateaux derivative of the norm at the point x, and in y-direction is

$$D_{\varphi,x}(y) = \lim_{t\to 0^+} \frac{\|x + te^{i\varphi}y\| - \|x\|}{t}.$$

Proposition 1.2. (i) *The function* $\alpha_{x,y}(t) = \|x + te^{i\varphi}y\|$ *is convex.*
(ii) $D_{\varphi,x}(y)$ *is the right derivative of the function* $\alpha_{x,y}$ *at the point* 0, *and taking into account* (i) $D_{\varphi,x}(y)$ *always exists.*

Proof. Obvious. ∎

Proposition 1.3. (i) $D_{\varphi,x}$ *is subadditive, positively homogeneous functional on* X*;*
(ii) $D_{\varphi,x}(e^{i\theta}y) = D_{\varphi+\theta,x}(y)$*;*
(iii) $|D_{\varphi,x}(y)| \leqslant \|y\|$.

Proof. (i) We have $\|x + te^{i\varphi}(y_1 + y_2)\| \leqslant \left\|\frac{x}{2} + te^{i\varphi}y_1\right\| + \left\|\frac{x}{2} + te^{i\varphi}y_2\right\|$, and, by taking a limit we obtain

$$\begin{aligned} D_{\varphi,x}(y_1 + y_2) &= \lim_{t\to 0^+} \frac{\|x + te^{i\varphi}(y_1 + y_2)\| - \|x\|}{t} \\ &\leqslant \lim_{t\to 0^+} \frac{\|x + 2te^{i\varphi}y_1\| + \|x + 2te^{i\varphi}y_2\| - 2\|x\|}{2t} \\ &= D_{\varphi,x}(y_1) + D_{\varphi,x}(y_2), \end{aligned}$$

which proves the subadditivity. Positive homogeneity is obvious.
(ii) Obvious.
(iii) It is enough to see that $\big|\, \|x + te^{i\varphi}y\| - \|x\| \,\big| \leqslant \|x + te^{i\varphi}y - x\| = t\|y\|$. ∎

The previous simple construction allows us to characterize the orthogonality in the sense of James, in all Banach spaces (without care of smoothness) via φ-Gateaux derivative.

Theorem 1.4. *The vector* y *is orthogonal to* x *in the sense of James if and only if* $\inf_{\varphi} D_{\varphi,x}(y) \geqslant 0$.

Proof. Let us first prove the only if part of the statement. Indeed, let y be orthogonal to x in the sense of James, i.e. let for all $\lambda \in \mathbb{C}$ there holds $\|x + \lambda y\| \geqslant \|x\|$. Then $\frac{\|x+te^{i\varphi}y\|-\|x\|}{t} \geqslant 0$ for all $t > 0$, and passing to the limit we get $D_{\varphi,x}(y) \geqslant 0$ for an arbitrary φ.

Let us, now, prove the other, if, part of the statement. We have

$$D_{\varphi,x}(\mathrm{e}^{\mathrm{i}(\pi-\varphi)}x)=\lim_{t\to 0^+}\frac{\|x+t\mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{\mathrm{i}(\pi-\varphi)}x\|-\|x\|}{t}=\|x\|\lim_{t\to 0^+}\frac{|1-t|-1}{t}=-\|x\|.$$

From this, and from subadditivity we get

$$\begin{aligned}\|x\|=-D_{\varphi,x}(\mathrm{e}^{\mathrm{i}(\pi-\varphi)}x)&\leqslant D_{\varphi,x}(\mu y)-D_{\varphi,x}(\mathrm{e}^{\mathrm{i}(\pi-\varphi)}x)\leqslant D_{\varphi,x}(\mu y-\mathrm{e}^{\mathrm{i}(\pi-\varphi)}x)\\ &\leqslant\|\mu y-\mathrm{e}^{\mathrm{i}(\pi-\varphi)}x\|=\|x+\mu(-\mathrm{e}^{\mathrm{i}(\varphi-\pi)})y\|=\|x+\lambda y\|,\end{aligned}$$

if we take $\mu=-\mathrm{e}^{\mathrm{i}(\pi-\varphi)}\lambda$. ∎

REMARK 1.5. We can see that the previous theorem is reasonable if we look at it from an other aspect. Namely, y is orthogonal to x if and only if the convex function $\alpha_{x,y}(t)$ attains its minimum at the origin.

We conclude this section with two examples concerning two classical Banach spaces.

EXAMPLE 1.6. In the space $L^1(X,\mu)$ the function g is orthogonal to f, in the sense of James if and only if

$$\Big|\int\limits_{\{f\neq 0\}}\mathrm{e}^{-\mathrm{i}\theta(t)}g(t)\,\mathrm{d}\mu(t)\Big|\leqslant\int\limits_{\{f=0\}}|g(t)|\,\mathrm{d}\mu(t),$$

where $f(t)=|f(t)|\mathrm{e}^{\mathrm{i}\theta(t)}$.

Indeed, in the L^1 space there holds

$$D_{\varphi,f}(g)=\mathrm{Re}\Big\{\int\limits_{\{f\neq 0\}}\mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{-\mathrm{i}\theta(t)}g(t)\,\mathrm{d}\mu(t)\Big\}+\int\limits_{\{f=0\}}|g(t)|\,\mathrm{d}\mu(t),$$

since

$$\lim_{\rho\to 0^+}\frac{|f(t)+\rho\mathrm{e}^{\mathrm{i}\varphi}g(t)|-|f(t)|}{\rho}=\begin{cases}\cos(\varphi-\theta(t)+\psi(t))|g(t)|, & f(t)\neq 0,\\ |g(t)|, & f(t)=0.\end{cases}$$

(here $g(t)=|g(t)|\mathrm{e}^{\mathrm{i}\psi(t)}$), and also $\Big|\frac{|f(t)+\rho\mathrm{e}^{\mathrm{i}\varphi}g(t)|-|f(t)|}{\rho}\Big|\leqslant|g(t)|$. Thus, we get $g\perp f$ if and only if

$$\inf_{\varphi}\mathrm{Re}\Big\{\int\limits_{\{f\neq 0\}}\mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{-\mathrm{i}\theta(t)}g(t)\,\mathrm{d}\mu(t)\Big\}+\int\limits_{\{f=0\}}|g(t)|\,\mathrm{d}\mu(t)\geqslant 0.$$

However, the infimum will be attained for that φ, for which

$$\mathrm{e}^{\mathrm{i}\varphi}\int\limits_{\{f\neq 0\}}\mathrm{e}^{-\mathrm{i}\theta(t)}g(t)\,\mathrm{d}\mu(t)=-\Big|\int\limits_{\{f\neq 0\}}\mathrm{e}^{-\mathrm{i}\theta(t)}g(t)\,\mathrm{d}\mu(t)\Big|,$$

and the result follows.

EXAMPLE 1.7. In the c_0 space, y is orthogonal to x if and only if there does not exist the open angle $D=\{z:\alpha<\arg z<\beta\}$, with $\beta-\alpha<\pi$, such that $\overline{\xi_\nu}\eta_\nu\in D$ for all those ν for which $|\xi_\nu|=\|x\|$ holds.

Let $k_1, k_2, \dots, k_n$ be all of indices, for which $|\xi_{k_j}| = \|x\|$ holds, and let $\delta > 0$ be a real number such that $\sup\limits_{\nu \neq k_j} |\xi_\nu| = \|x\| - \delta$, and let $\xi_\nu = |\xi_\nu| \mathrm{e}^{\mathrm{i}\theta_\nu}$. Now, for $t < \frac{\delta}{2\|y\|}$ we have:

$$\begin{cases} |\xi_\nu + t\mathrm{e}^{\mathrm{i}\varphi}\eta_\nu| \geqslant \|x\| - \frac{\delta}{2} & \text{for } \nu = k_j \\ |\xi_\nu + t\mathrm{e}^{\mathrm{i}\varphi}\eta_\nu| \leqslant \|x\| - \frac{\delta}{2} & \text{for } \nu \neq k_j \ . \end{cases}$$

Thus $\|x + t\mathrm{e}^{\mathrm{i}\varphi} y\| = \max\limits_{1 \leqslant j \leqslant n} |\xi_{k_j} + t\mathrm{e}^{\mathrm{i}\varphi}\eta_{k_j}|$, implying

$$\begin{aligned} D_{\varphi,x}(y) &= \lim_{t \to 0^+} \frac{\|x + t\mathrm{e}^{\mathrm{i}\varphi} y\| - \|x\|}{t} = \|x\| \lim_{t \to 0^+} \frac{\max\limits_{1 \leqslant j \leqslant n} \left|1 + \frac{t\mathrm{e}^{\mathrm{i}\varphi}\eta_{k_j}}{\xi_{k_j}}\right| - 1}{t} \\ &= \max_{1 \leqslant j \leqslant n} \operatorname{Re}\{\mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{-\mathrm{i}\theta_{k_j}}\eta_{k_j}\}, \end{aligned}$$

taking into account that for all n-tuples of complex numbers there holds

$$\lim_{t \to 0^+} \frac{\max\{|1 + tz_1|, |1 + tz_2|, \dots, |1 + tz_n|\} - 1}{t} = \max\{\operatorname{Re} z_1, \operatorname{Re} z_2, \dots, \operatorname{Re} z_n\}.$$

So, we have: y is orthogonal to x if and only if $\inf\limits_{\varphi} \max\limits_{\nu = k_j} \operatorname{Re} \mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{-\mathrm{i}\theta_\nu}\eta_\nu \geqslant 0$.

2. ORTHOGONALITY IN $\mathfrak{S}_1$ AND $\mathfrak{S}_\infty$

THEOREM 2.1. *Let $X, Y \in \mathfrak{S}_1$. Then, there holds*

$$\lim_{t \to 0^+} \frac{\|X + tY\|_{\mathfrak{S}_1} - \|X\|_{\mathfrak{S}_1}}{t} = \operatorname{Re}\{\operatorname{tr}(U^*Y)\} + \|QYP\|_{\mathfrak{S}_1},$$

where $X = U|X|$ is the polar decomposition of the operator X, $P = P_{\ker X}$, $Q = P_{\ker X^*}$.

For the proof of this theorem we need three technical lemmas.

LEMMA 2.2. *Let $X = U|X|$ and $X + tY = V_t|X + tY|$ be the polar decompositions of the operators X and $X + tY$, let $Q(P)$ be the projector to the kernel of $X^*(X)$, and let $\{\chi_j\}$ be some complete orthonormal system in* $\ker X$. *Then:*

(i) $V_{t_n}x \to Ux$, *strongly, for all $x \in \overline{\operatorname{ran} X^*}$, and for some sequence $t_n \to 0^+$. Also, $V_{t_n}^* x \to U^*x$, strongly, for all $x \in \overline{\operatorname{ran} X}$, and for some (or same) sequence $t_n \to 0^+$.*

(ii) $\lim\limits_{n \to +\infty} \sum\limits_j \langle V_{t_n}^*(I - Q)Y\chi_j, \chi_j\rangle = 0$, *provided that Y is nuclear.*

Proof. (i) Let e_j be some complete orthonormal system in H. For each j, the family $\{V_t e_j : t > 0\}$ is a bounded family, and hence there exists a sequence $t_n \to 0$, such that $V_{t_n}e_j$ converges weakly. Moreover, using Cantor's diagonal process, we conclude that there exists a sequence $t_n \to 0$ such that $V_{t_n}e_j$ converges weakly for all j, and therefore V_{t_n} converges weakly. Let V_0 denote the weak limit of the sequence V_{t_n}. Now, for all $y, z \in H$ we have $\langle V_{t_n}|X + t_nY|z, y\rangle =$

$\langle (X + t_n Y)z, y\rangle$. However, $X + t_n Y$ converges strongly (even uniformly) to X, as well as $|X + t_n Y|$ converges strongly to $|X|$, and passing to the limit we get $\langle V_0|X|z, y\rangle = \langle Xz, y\rangle = \langle U|X|z, y\rangle$ for all $z, y \in H$. Thus $V_0 x = Ux$ for all $x \in \operatorname{ran}|X|$. Since $\operatorname{ran}|X|$ is dense in $\overline{\operatorname{ran} X^*}$, we obtain that V_{t_n} converges weakly to Ux for all $x \in \overline{\operatorname{ran} X^*}$. However, this convergence is moreover strong. Indeed, let x be an arbitrary vector from $\operatorname{ran} X^*$. Then there exists $z \in H$ such that $x = |X|z$. We have that $V_{t_n}|X + t_n Y|z$ tends weakly to Ux. But, $V_{t_n}|X + t_n Y|z = (X + t_n Y)z$ which tends strongly to $Xz = Ux$. Thus $V_{t_n}|X + t_n Y|z$ tends strongly to Ux. Now we have $\|V_{t_n}x - Ux\| \leqslant \|V_{t_n}(|X|z - |X + t_n Y|z)\| + \|V_{t_n}|X + t_n Y|z - Xz\|$, which tends to zero as n tends to infinity.

In a similar way, we can obtain that $V_{t_n}^* x$ tends to $U^* x$ for all $x \in \overline{\operatorname{ran} X}$ and for some (same) sequence t_n.

(ii) By part (i) we have that $PV_{t_n}^*(I - Q)YP$ converges strongly to $PU^*(I - Q)YP$, and by Theorem III.6.3. from [6] $PV_{t_n}^*(I - Q)YP$ tends to $PU^*(I - Q)YP$ in nuclear norm, since Y is nuclear operator. However, $\sum\limits_j \langle V_{t_n}^*(I - Q)Y\chi_j, \chi_j\rangle$ is precisely the trace of the nuclear operator $PV_{t_n}^*(I - Q)YP$ and therefore it tends to the trace of the operator $PU^*(I - Q)YP$. But $PU^* = 0$ and the proof is complete. ∎

LEMMA 2.3. *Let A be a bounded operator, whose (usual) norm is at most one, and let $\{\varphi_j\}$ be an arbitrary orthonormal system. Then, we have* $\|X\|_1 \geqslant \Big|\sum\limits_j \langle AX\varphi_j, \varphi_j\rangle\Big|$.

Proof. Indeed $\Big|\sum\limits_j \langle AX\varphi_j, \varphi_j\rangle\Big| \leqslant |\operatorname{tr}(AX)| \leqslant \|A\|_\infty \|X\|_1 \leqslant \|X\|_1$. ∎

LEMMA 2.4. *Let φ_j be some orthonormal system (not necessarily complete) in H.*

(i) *for any vector $f \in H$, and for all $\varepsilon > 0$, there exists a vector f', such that $\|f - f'\| < \varepsilon$ and $\sum\limits_j |\langle f', \varphi_j\rangle| < +\infty$.*

(ii) *the set* $F = \Big\{A \in \mathfrak{S}_1 : \sum\limits_j \|A\varphi_j\| < +\infty\Big\}$ *is dense in $\mathfrak{S}_1$.*

Proof. (i) Let $f = f_1 + f_2$, $f_1 \in \mathcal{L}\{\varphi_j\}$, $f_2 \perp \mathcal{L}\{\varphi_j\}$. Since there holds $\|f_1\|^2 = \sum\limits_j |\langle f_1, \varphi_j\rangle|^2$, there exists n_0, such that $\sum\limits_{j > n_0} |\langle f_1, \varphi_j\rangle|^2 < \varepsilon^2$. We define f' as $f' = \sum\limits_{j \leqslant n_0} \langle f_1, \varphi_j\rangle\varphi_j + f_2$. We have $\sum\limits_j |\langle f', \varphi_j\rangle| = \sum\limits_{j \leqslant n_0} |\langle f_1, \varphi_j\rangle| < +\infty$, and also $\|f - f'\|^2 = \Big\|\sum\limits_{j > n_0} \langle f_1, \varphi_j\rangle\varphi_j\Big\|^2 = \sum\limits_{j > n_0} |\langle f_1, \varphi_j\rangle|^2 < \varepsilon^2$.

(ii) Let $Y \in \mathfrak{S}_1$, and let $Z = \sum\limits_{k=1}^N \sigma_k\langle\,\cdot\,, f_k\rangle g_k$ ($0 < \sigma_{k+1} \leqslant \sigma_k$, f_k, g_k orthonormal systems) be a finite rank operator such that $\|Y - Z\|_1 < \frac{\varepsilon}{2}$. By the previous part of the statement, there exist vectors f_k', such that $\sum\limits_j |\langle f_k', \varphi_j\rangle| < +\infty$, and $\|f_k - f_k'\| < \frac{\varepsilon}{2^k\sigma_k}$. Let $A = \sum\limits_{k=1}^N \sigma_k\langle\,\cdot\,, f_k'\rangle g_k$. We have $\|A - Z\|_1 = \Big\|\sum\limits_{k=1}^N \sigma_k\langle\,\cdot\,, f_k -$

$f'_k\rangle g_k\Big\|_1 \leqslant \sum\limits_{k=1}^{N} \sigma_k \|f_k - f'_k\| \leqslant \frac{\varepsilon}{2}$, and hence $\|A - Y\|_1 < \varepsilon$. On the other hand $\sum\limits_j \|A\varphi_j\| \leqslant \sum\limits_{k=1}^{N} \sum\limits_j \|\sigma_k \langle \varphi_j, f'_k\rangle g_k\| = \sum\limits_{k=1}^{N} \sigma_k \sum\limits_j |\langle \varphi_j, f'_k\rangle| < +\infty$. ∎

Now, we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let $X = \sum\limits_j s_j \langle \,\cdot\,, \varphi_j\rangle \psi_j$ be the Schmidt expansion of the operator X, and let χ_j be a complete orthonormal system in $\ker X$. Then, taking into account Lemma 2.3, we have:

$$
\begin{aligned}
&\frac{1}{t}\{\|X + tY\|_1 - \|X\|_1\} \\
&\quad = \frac{1}{t}\Big\{\|X + tY\|_1 - \sum_j s_j\Big\} \\
&\quad \geqslant \frac{1}{t}\Big(\Big|\sum_j \langle U^*(X + tY)\varphi_j, \varphi_j\rangle + \sum_j \langle V^*(X + tY)\chi_j, \chi_j\rangle\Big| - \sum_j s_j\Big),
\end{aligned}
$$

where $V : \ker X \to \ker X^*$ is given by $QYP = V|QYP|$. Further

$$
\begin{aligned}
&\frac{1}{t}\{\|X + tY\|_1 - \|X\|_1\} \\
&\quad \geqslant \frac{1}{t}\Big\{\Big|\sum_j \langle |X|\varphi_j, \varphi_j\rangle + t\sum_j \langle U^*Y\varphi_j, \varphi_j\rangle + \sum_j \langle V^*(X + tY)\chi_j, \chi_j\rangle\Big| - \sum_j s_j\Big\} \\
&\quad = \frac{1}{t}\Big\{\Big|\sum_j s_j + t\Big(\sum_j \langle U^*Y\varphi_j, \varphi_j\rangle + \sum_j \langle V^*Y\chi_j, \chi_j\rangle\Big)\Big| - \sum_j s_j\Big\}.
\end{aligned}
$$

But, since $V^*Q = V^*$ and $P\chi_j = \chi_j$ the last expression is equal to

$$
\begin{aligned}
&\frac{1}{t}\{\|X + tY\|_1 - \|X\|_1\} \\
&\quad = \frac{1}{t}\Big\{\Big|\sum_j s_j + t\Big(\sum_j \langle U^*Y\varphi_j, \varphi_j\rangle + \sum_j \langle V^*QYP\chi_j, \chi_j\rangle\Big)\Big| - \sum_j s_j\Big\} \\
&\quad = \frac{\Big|\sum\limits_j s_j + t(\operatorname{tr}(U^*Y) + \|QYP\|_1)\Big| - \sum\limits_j s_j}{t} \to \operatorname{Re}(\operatorname{tr}(U^*Y) + \|QYP\|_1),
\end{aligned}
$$

and thus $\lim\limits_{t\to 0^+} \frac{\|X+tY\|_1 - \|X\|_1}{t} \geqslant \operatorname{Re}(\operatorname{tr}(U^*Y)) + \|QYP\|_1$.

We shall derive the opposite inequality for those Y which belong to $F = \Big\{A \in \mathfrak{S}_1 : \sum\limits_j \|A\varphi_j\| < +\infty\Big\}$. It will be enough, since we will get two sublinear bounded functionals $Y \mapsto \lim\limits_{t\to 0^+} \frac{\|X+tY\|_1 - \|X\|_1}{t}$ and $Y \mapsto \operatorname{Re}\operatorname{tr}(U^*Y) + \|QYP\|_1$, that coincide on the set F which is dense by Lemma 2.4. Also, in the following, wherever V_t as $t \to 0^+$, is written, it means V_{t_n} as $n \to +\infty$, where t_n is a sequence from Lemma 2.2. (We do not need to care about it, since by Proposition 1.2 (ii) always exists the limit that we consider.) At the first we have

$$\begin{aligned}\frac{1}{t}&\Big\{\|X+tY\|_1-\|X\|_1\Big\}\\ &=\frac{1}{t}\Big\{\sum_j\langle|X+tY|\varphi_j,\varphi_j\rangle+\sum_j\langle|X+tY|\chi_j,\chi_j\rangle-\sum_j s_j\Big\}.\end{aligned}\tag{2.1}$$

However,

$$\frac{1}{t}\sum_j\langle|X+tY|\chi_j,\chi_j\rangle=\frac{1}{t}\sum_j\langle V_t^*(X+tQY)\chi_j,\chi_j\rangle+\sum_j\langle V_t^*(I-Q)Y\chi_j,\chi_j\rangle,$$

and also,

$$\begin{aligned}\|QYP\|_1\geqslant\Big|\sum_j\langle V_t^*QYP\chi_j,\chi_j\rangle\Big|&=\Big|\sum_j\langle V_t^*QY\chi_j,\chi_j\rangle\Big|\\ &=\frac{1}{t}\Big|\sum_j\langle V_t^*(X+tQY)\chi_j,\chi_j\rangle\Big|,\end{aligned}$$

so that a real number $\frac{1}{t}\sum\limits_j\langle|X+tY|\chi_j,\chi_j\rangle$ is equal to a sum of a complex number whose modulus is less or equal to $\|QYP\|_1$, and an other complex number, whose modulus is, for t small enough, less or equal to ε (Lemma 2.2). Thus, for t small enough, we get $\frac{1}{t}\sum\limits_j\langle|X+tY|\chi_j,\chi_j\rangle\leqslant\|QYP\|_1+\varepsilon$. On the other hand, by Jensen's inequality applied to the integration with respect to the spectral measure we have:

$$\begin{aligned}\sum_j\langle|X+tY|\varphi_j,\varphi_j\rangle&\leqslant\sum_j\sqrt{\langle|X+tY|^2\varphi_j,\varphi_j\rangle}=\\ &=\sum_j\sqrt{s_j^2+2t\mathrm{Re}\,\langle Y\varphi_j,s_j\psi_j\rangle+t^2\|Y\varphi_j\|^2}\end{aligned}$$

and, applying (2.1)

$$\begin{aligned}\frac{1}{t}&\{\|X+tY\|_1-\|X\|_1\}\\ &\leqslant\frac{\sum\limits_j\sqrt{s_j^2+2t\mathrm{Re}\,\langle Y\varphi_j,s_j\psi_j\rangle+t^2\|Y\varphi_j\|^2}-\sum\limits_j s_j}{t}+\|QYP\|_1+\varepsilon\\ &=\sum_j\frac{\sqrt{s_j^2+2t\mathrm{Re}\,\langle Y\varphi_j,s_j\psi_j\rangle+t^2\|Y\varphi_j\|^2}-s_j}{t}+\|QYP\|_1+\varepsilon\\ &=\sum_j\frac{t2\mathrm{Re}\,\langle Y\varphi_j,s_j\psi_j\rangle+t^2\|Y\varphi_j\|^2}{t\Big(\sqrt{s_j^2+2t\mathrm{Re}\,\langle Y\varphi_j,s_j\psi_j\rangle+t^2\|Y\varphi_j\|^2}+s_j\Big)}+\|QYP\|_1+\varepsilon\\ &\to\sum_j\mathrm{Re}\,\langle Y\varphi_j,\psi_j\rangle+\|QYP\|_1+\varepsilon=\sum_j\mathrm{Re}\,\langle Y\varphi_j,U\varphi_j\rangle+\|QYP\|_1+\varepsilon\\ &=\mathrm{Re}\,(\mathrm{tr}\;U^*Y)+\|QYP\|_1+\varepsilon.\end{aligned}$$

The inequality

$$\left|\frac{t2\mathrm{Re}\,\langle Y\varphi_j, s_j\psi_j\rangle + t^2\|Y\varphi_j\|^2}{t\Big(\sqrt{s_j^2 + 2t\mathrm{Re}\,\langle Y\varphi_j, s_j\psi_j\rangle + t^2\|Y\varphi_j\|^2} + s_j\Big)}\right| \leqslant |2\mathrm{Re}\,\langle Y\varphi_j, \psi_j\rangle| + \|Y\varphi_j\|$$

allows us to take a limit as $t \to 0^+$ under the sum. The result now follows, since ε can be arbitrarily small. ∎

The following corollary characterizes orthogonality in the sense of James in the space $\mathfrak{S}_1$.

COROLLARY 2.5. *The operator Y is orthogonal to the operator X in the space $\mathfrak{S}_1$ if and only if $|\operatorname{tr}(U^*Y)| \leqslant \|QYP\|_{\mathfrak{S}_1}$, where $X = U|X|$, $P = P\ker X$, and $Q = P\ker X^*$.*

Proof. By Theorem 1.4, Y is orthogonal to X if and only if $\inf\limits_{\varphi} D_{\varphi,X}(Y) \geqslant 0$. However, by Theorem 2.1, there holds $\inf\limits_{\varphi} D_{\varphi,X}(Y) = \inf\limits_{\varphi} \mathrm{Re}\,(\mathrm{e}^{\mathrm{i}\varphi}\operatorname{tr}(U^*Y)) + \|QYP\|_1$, and we get the result, by choosing the most suitable φ. ∎

THEOREM 2.6. *Let X, Y be in $\mathfrak{S}_\infty$. Then we have*

$$\lim_{t\to 0^+} \frac{\|X + tY\|_\infty - \|X\|_\infty}{t} = \max_{\substack{f\in\Phi\\ \|f\|=1}} \mathrm{Re}\,\langle U^*Yf, f\rangle,$$

where $X = U|X|$, and Φ is the characteristic subspace of the operator $|X|$ with respect to its eigenvalue s_1.

For the proof of this theorem we need a technical lemma, as well.

LEMMA 2.7. *Let A be a positive compact operator, let Φ be the subspace where A attains its norm, let Φ_γ be the set of those vectors from the Hilbert space H which forms with Φ an angle less or equal to γ. Let, further, B be a selfadjoint compact operator, such that $\|B\| \leqslant \delta$, where δ is a real number such that $\frac{2\delta}{s_1(A)-s_2(A)-2\delta} \leqslant \tan\gamma$. Then we have*

$$\|A + B\| = \max_{f\in\Phi_\gamma, \|f\|=1} \langle (A + B)f, f\rangle.$$

Proof. If the unit vector x is represented as $x = f + g$, where $f \in \Phi$, $g \in \Phi^\perp$, then $x \in \Phi_\gamma$ if and only if $\frac{\|g\|}{\|f\|} \leqslant \tan\gamma$. Also, $s_2(A) = \max\limits_{\substack{f\in\Phi^\perp\\ \|f\|=1}} \langle Af, f\rangle$. Since the operator $A + B$ is compact and selfadjoint, there exists a unit vector x such that $\langle (A + B)x, x\rangle = \|A + B\|$. If we represent this vector as $x = f + g$, then it will be $\langle (A + B)x, x\rangle = \langle (A + B)f, f\rangle + 2\mathrm{Re}\,\langle Bf, g\rangle + \langle (A + B)g, g\rangle$. But, since $|\langle (A+B)f, f\rangle| \leqslant \|A+B\|\,\|f\|^2$, $|\langle Bf, g\rangle| \leqslant \delta\|f\|\,\|g\|$, $|\langle (A+B)g, g\rangle| \leqslant (s_2+\delta)\|g\|^2$, we have $\|A + B\| \leqslant \|A + B\|\,\|f\|^2 + 2\delta\|f\|\,\|g\| + (s_2 + \delta)\|g\|^2$, i.e. $(s_1 - \delta)\|g\| \leqslant \|A + B\|\,\|g\| \leqslant 2\delta\|f\| + (s_2 + \delta)\|g\|$, taking into account $\|A + B\| \geqslant s_1 - \delta$, respectively $(s_1 - s_2 - 2\delta)\|g\| \leqslant 2\delta\|f\|$, from which we conclude $x \in \Phi_\gamma$. ∎

Proof of Theorem 2.6. At first, we have

$$\lim_{t\to0^+}\frac{\|X+tY\|_\infty-\|X\|_\infty}{t}=\lim_{t\to0^+}\frac{\|(X^*+tY^*)(X+tY)\|^{1/2}-\|X\|}{t}$$
$$=\lim_{t\to0^+}\frac{\|X^*X+t(X^*Y+Y^*X)+t^2Y^*Y\|-\|X\|^2}{t(\|X^*X+t(X^*Y+Y^*X)+t^2Y^*Y\|^{1/2}+\|X\|)}.$$

It is obvious that the denominator $\|X^*X+t(X^*Y+Y^*X)+t^2Y^*Y\|^{1/2}+\|X\|$ tends to $2\|X\|$. Let us consider the limit of the numerator. The operators X^*X and $t(X^*Y+Y^*X)+t^2Y^*Y$ satisfy the assumptions of Lemma 2.7, for an arbitrary $\gamma>0$ and for the corresponding small enough t, and we get

$$\lim_{t\to0^+}\frac{\|X^*X+t(X^*Y+Y^*X)+t^2Y^*Y\|-s_1^2}{t}$$
$$=\lim_{t\to0^+}\frac{\max\limits_{\substack{f\in\Phi_\gamma\\ \|f\|=1}}\langle(X^*X+t(X^*Y+Y^*X)+t^2Y^*Y)f,f\rangle-s_1^2}{t}$$
$$\leqslant\lim_{t\to0^+}\max_{\substack{f\in\Phi_\gamma\\ \|f\|=1}}[\langle(X^*Y+Y^*X)f,f\rangle+t\langle Y^*Yf,f\rangle]=\max_{\substack{f\in\Phi_\gamma\\ \|f\|=1}}2\mathrm{Re}\,\langle Yf,Xf\rangle.$$

On the other hand

$$\lim_{t\to0^+}\frac{\|X^*X+t(X^*Y+Y^*X)+t^2Y^*Y\|-s_1^2}{t}$$
$$\geqslant\lim_{t\to0^+}\frac{\max\limits_{\substack{f\in\Phi\\ \|f\|=1}}\langle(X^*X+t(X^*Y+Y^*X)+t^2Y^*Y)f,f\rangle-s_1^2}{t}$$
$$=\lim_{t\to0^+}\max_{\substack{f\in\Phi\\ \|f\|=1}}[\langle(X^*Y+Y^*X)f,f\rangle+t\langle Y^*Yf,f\rangle]=\max_{\substack{f\in\Phi\\ \|f\|=1}}2\mathrm{Re}\,\langle Yf,Xf\rangle,$$

so that for all $\gamma>0$ we have

$$\frac{1}{\|X\|}\max_{\substack{f\in\Phi_\gamma\\ \|f\|=1}}\mathrm{Re}\,\langle Yf,Xf\rangle\leqslant\lim_{t\to0^+}\frac{\|X+tY\|_\infty-\|X\|_\infty}{t}\leqslant\frac{1}{\|X\|}\max_{\substack{f\in\Phi_\gamma\\ \|f\|=1}}\mathrm{Re}\,\langle Yf,Xf\rangle.$$

Note that

$$\inf_{\gamma>0}\max_{\substack{f\in\Phi_\gamma\\ \|f\|=1}}\mathrm{Re}\,\langle Yf,Xf\rangle=\max_{\substack{f\in\Phi\\ \|f\|=1}}\mathrm{Re}\,\langle Yf,Xf\rangle,$$

since Y and X are continuous in the sphere metric. So, we can get the result, by taking an infimum over all γ, since for $f\in\Phi$ there holds $Xf=\|X\|Uf$. ∎

COROLLARY 2.8. *In the space $\mathfrak{S}_\infty$ the following three conditions are mutually equivalent:*

(i) *Y is orthogonal to X in the sense of James.*

(ii) $\inf\limits_{0\leqslant\varphi<2\pi}\max\limits_{\substack{f\in\Phi\\ \|f\|=1}}\mathrm{Re}\,\mathrm{e}^{\mathrm{i}\varphi}\langle U^*Yf,f\rangle\geqslant0$, *where $X=U|X|$ and Φ is the subspace where the operator X attains its norm.*

(iii) *There exists the vector* $f \in \Phi$ *such that* $Yf \perp Xf$.

Proof. The equivalence between (i) and (ii) follows from Theorems 1.4 and 2.6. However, the condition (ii) tells us that the numerical range of the operator U^*Y (on the subspace Φ) has in the complex plane, such a position that it contains at least one value with positive real part, under all rotations around the zero, i.e. that is not contained in an open half-plane, whose boundary contains the origin. But by Toeplitz-Haussdorf Theorem the numerical range is a closed convex set, so the last condition is equivalent to the condition that the numerical range of the operator U^*Y contains the origin. Since the vectors Uf and Xf always have the same direction, we conclude that (iii) is equivalent to (ii). ∎

3. THE SUM OF THE RANGE AND THE KERNEL OF THE ELEMENTARY OPERATORS

Let us, first, recall some facts concerning ideals of compact operators.

PROPOSITION 3.1. *If* $\mathfrak{J}$ *is a separable ideal of compact operators, associated with some unitarily invariant norm, then its dual is isometrically isomorphic with another ideal of compact operators (not necessarily separable) and it admits the representation:* $\varphi_Y(X) = \operatorname{tr}(XY)$.

Proof. This is, in fact, Theorem III.12.2. from [6]. ∎

PROPOSITION 3.2. *Let* $\mathfrak{J}$ *be some separable ideal of compact operators, and let* $E : \mathfrak{J} \to \mathfrak{J}$ *be some elementary operator given by* $E(X) = \sum_{j=1}^{n} A_j X B_j$. *Then its conjugate operator* $E^* : \mathfrak{J}^* \to \mathfrak{J}^*$ *has the form* $E^*(Y) = \sum_{j=1}^{n} B_j Y A_j$.

Proof. We have

$$\begin{aligned} \varphi_Y(E(X)) &= \operatorname{tr}(E(X)Y) = \operatorname{tr}\Big(\sum_{j=1}^{n} A_j X B_j Y\Big) \\ &= \operatorname{tr}\Big(X \sum_{j=1}^{n} B_j Y A_j\Big) = \operatorname{tr}(XE^*(Y)) = \varphi_{E^*(Y)}(X). \quad ∎ \end{aligned}$$

Consider an arbitrary separable ideal of compact operators $\mathfrak{J}$, such that $\mathfrak{J}^*$ is strictly convex. According to Proposition 0.9, for all $X \in \mathfrak{J}$ there exists a unique operator $\widetilde{X} \in \mathfrak{J}^*$ such that $\widetilde{X}(X) = \|X\|$ and $\|\widetilde{X}\| = 1$. If, moreover, $\mathfrak{J}$ is reflexive then the mapping $X \to \widetilde{X}$, $\widetilde{X} = \omega(X)$ is a bijection (and also involution) of the unit spheres of the spaces $\mathfrak{J}$ and $\mathfrak{J}^*$. Moreover, Y is orthogonal to X in the space $\mathfrak{J}$ if and only if $\widetilde{X}(Y) = 0$.

THEOREM 3.3. *Let $\mathfrak{J}$ be a reflexive ideal in $B(H)$ such that $\mathfrak{J}^*$ is strictly convex, and let $E:\mathfrak{J}\to\mathfrak{J}$ be an elementary operator given by $E(X)=\sum_{j=1}^{n}A_jXB_j$. Then $\overline{\operatorname{ran} E}$ is orthogonal (in the sense of James) to the operator S if and only if $\omega(S)=\widetilde{S}\in\ker E^*$.*

Proof. Taking into account Propositions 0.6 and 3.2, we have that $\overline{\operatorname{ran} E}\perp S$ implies that for all $X\in\mathfrak{J}$, $\widetilde{S}(E(X))=0$ or $(E^*(\widetilde{S}))(X)=0$, for all X, and consequently $E^*(\widetilde{S})=0$. ∎

REMARK 3.4. Theorem 3.3 is the general result and it holds on an arbitrary Banach space.

LEMA 3.5. *Let X be a reflexive Banach space and let V be a closed subspace of X. If $V_\perp=\{x\in X:\forall v\in V\ \|v+x\|\geqslant\|x\|\}=\{0\}$ then $V=X$.*

Proof. This is Lemma 3.6. from [14]. ∎

THEOREM 3.6. *Let $\mathfrak{J}$ satisfy the assumptions of the previous theorem, and let $E:\mathfrak{J}\to\mathfrak{J}$ be an elementary operator given by $E(X)=AXB+CXD$, where A,B,C and D are normal operator such that $AC=CA$, $BD=DB$ and $A^*A+C^*C>0$, $B^*B+D^*D>0$. Then $\mathfrak{J}=\overline{\operatorname{ran} E}\oplus\ker E$.*

Proof. In [8], it is proved that for such elementary operators its range is orthogonal to its kernel, and by this and by previous Theorem we have the following implications:

$$E(S)=0\Rightarrow\forall X\in\mathfrak{J},\ \|E(X)+S\|\geqslant\|S\|\Rightarrow E^*(\widetilde{S})=0\Rightarrow$$
$$\Rightarrow\forall X\in\mathfrak{J}^*,\ \|E^*(X)+\widetilde{S}\|\geqslant\|\widetilde{S}\|\Rightarrow E^{**}(\widetilde{\widetilde{S}})=0\Leftrightarrow E(S)=0.$$

Thus we have $E(S)=0$ if and only if $E(X)\perp S$ for all $X\in\mathfrak{J}$.

From the orthogonality of the range and the kernel it follows that the sum $\overline{\operatorname{ran} E}+\ker E$ is closed. Indeed, if x_n+y_n for $x_n\in\overline{\operatorname{ran} E}$, $y_n\in\ker E$ tends to z, then, by inequality $\|y_n-y_m\|\leqslant\|x_n+y_n-x_m-y_m\|$ we conclude that y_n is a Cauchy sequence, and therefore $y_n\to y\in\ker E$. Further $x_n\to z-y\in\overline{\operatorname{ran} E}$, and thus $z\in\overline{\operatorname{ran} E}+\ker E$. Suppose that $\|E(X)+Y+Z\|_{\mathfrak{J}}\geqslant\|Z\|_{\mathfrak{J}}$, for all X, and for all $Y\in\ker E$. By choosing $Y=0$ we see that $Z\in\ker E$. Now, we can put $Y=-Z$ and $X=0$, implying $Z=0$. Hence $(\overline{\operatorname{ran} E}+\ker E)_\perp=\{0\}$. This, by Lemma 3.5, finishes the proof. ∎

COROLLARY 3.7. *Let $p>1$, and let $E:\mathfrak{S}_p\to\mathfrak{S}_p$, $E(X)=AXB+CXD$, where A,B,C and D are as in the previous theorem. Then $\mathfrak{S}_p=\overline{\operatorname{ran} E}\oplus\ker E$. Moreover, for any elementary operator on $\mathfrak{S}_p$ it is valid that $\overline{\operatorname{ran} E}$ is orthogonal to S if and only if $E^*(|S|^{p-1}U^*)=0$.*

Proof. It is well known that $\mathfrak{S}_p^*\cong\mathfrak{S}_q$, $q>1$, and that $\mathfrak{S}_q$ is strictly convex (Clarckson-McCarthy inequalities; see [13]). Further, we can easily check that in the case of $\mathfrak{S}_p$, $\widetilde{S}=\frac{1}{\|S\|_p^{p/q}}|S|^{p-1}U^*$, which concludes the proof. ∎

REMARK 3.8. The special case of this theorem is Proposition 3 from [10].

THEOREM 3.9. *There exists a normal derivation* $\Delta_A : \mathfrak{S}_1 \to \mathfrak{S}_1$, $\Delta_A(X) = AX - XA$, *with* $AA^* = A^*A$ *such that* $\mathfrak{S}_1 \neq \overline{\operatorname{ran} \Delta_A} \oplus \ker \Delta_A$.

Proof. Let $H = l^2(\mathbb{Z})$, and let A be the bilateral shift operator, i.e. for all $n \in \mathbb{Z}$ let $Ae_n = e_{n-1}$.

Let us, first, perceive that the kernel of the derivation Δ_A is trivial. Indeed, let $X \in \ker \Delta_A$. Then $AX = XA$, implies $\langle Xe_i, e_j\rangle = \langle Xe_i, A^*e_{j-1}\rangle = \langle AXe_i, e_{j-1}\rangle = \langle XAe_i, e_{j-1}\rangle = \langle Xe_{i-1}, e_{j-1}\rangle$. However, taking into account the compactness of the operator X we get $0 = \lim_{n\to+\infty} \langle Xe_{i+n}, e_{j+n}\rangle = \langle Xe_i, e_j\rangle$, for all $i, j \in \mathbb{Z}$, and therefore $X = 0$. Thus $\overline{\operatorname{ran} \Delta_A} \oplus \ker \Delta_A = \overline{\operatorname{ran} \Delta_A}$.

Let us now construct the operator $S \in \mathfrak{S}_1$ in the following way:

$$\begin{cases} Se_j = 0 & \text{for } j \leqslant 0, \\ Se_j = \frac{1}{2^j} e_{j-1} & \text{for } j > 0 \end{cases}.$$

If $S = U|S|$ then clearly $U^*e_j = 0$ for $j < 0$, and $U^*e_j = e_{j+1}$ for $j \geqslant 0$. We shall prove that $\overline{\operatorname{ran} \Delta_A}$ is orthogonal to S. Indeed, this orthogonality is, by Corollary 2.5, equivalent to $|\operatorname{tr}(U^*(AX - XA))| \leqslant \|Q(AX - XA)P\|_1$, where $P = P_{\ker S}$ and $Q = P_{\ker S^*}$. However $|\operatorname{tr}(U^*(AX-XA))| = |\operatorname{tr}((AU^*-U^*A)X)|$, whereas $AU^* - U^*A = \langle \cdot, e_0\rangle e_0$, and, in fact $|\operatorname{tr}(U^*(AX - XA))| = |\langle Xe_0, e_0\rangle|$. On the other hand it is easy to check that $P = P_{\overline{\mathcal{L}(\dots,e_{-2},e_{-1},e_0)}}$, $Q = P_{\overline{\mathcal{L}(\dots,e_{-2},e_{-1})}}$, and we get (taking in Lemma 2.3 the bounded operator A^*)

$$\begin{aligned}
&\|Q(AX - XB)P\|_{\mathfrak{S}_1} \\
&\quad \geqslant \Big|\sum_{j=-\infty}^{+\infty} \langle Q(AX - XA)Pe_{j+1}, e_j\rangle\Big| = \Big|\sum_{j=-\infty}^{+\infty} \langle (AX - XA)Pe_{j+1}, Qe_j\rangle\Big| \\
&\quad = \Big|\sum_{j=-\infty}^{-1} (\langle AXe_{j+1}, e_j\rangle - \langle XAe_{j+1}, e_j\rangle)\Big| = \Big|\sum_{j=-\infty}^{-1} (\langle Xe_{j+1}, e_{j+1}\rangle - \langle Xe_j, e_j\rangle)\Big| \\
&\quad = |\langle Xe_0, e_0\rangle|,
\end{aligned}$$

finishing the proof. ∎

REMARK 3.10. The trivial kernel of the derivation from the previous theorem is a completely unessential detail. Indeed, considering Hilbert space $H \oplus H$ and the operator $A \oplus I$ on it, we can construct a normal derivation which has the same properties as that in Theorem 3.9, and whose kernel is nontrivial.

THEOREM 3.11. *There exists a normal derivation* $\Delta_B : \mathfrak{S}_\infty \to \mathfrak{S}_\infty$, $\Delta_B(X) = BX - XB$, *with* $BB^* = B^*B$, *and an operator* $S \in \mathfrak{S}_\infty$ *such that* $\operatorname{ran} \Delta_B$ *is orthogonal to* S, *and* $S \notin \ker \Delta_B$.

Proof. Let A be the operator from the proof of Theorem 3.9, and let $B = A \oplus I$ acting on $H = l^2(\mathbb{Z}) \oplus \mathbb{C}$. Further, let $S = S_1 \oplus 2I$, where $S_1 : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is any operator of norm at most one. The operator S attains its norm at the unique (up to a scalar) vector $\varphi = 0 \oplus 1 \in l^2(\mathbb{Z}) \oplus \mathbb{C}$. It is obvious that $S\varphi = 2\varphi$, $B\varphi = B^*\varphi = \varphi$, and therefore

$$\langle (BX - XB)\varphi, S\varphi\rangle = 2\langle X\varphi, B^*\varphi\rangle - 2\langle XB\varphi, \varphi\rangle = 2\langle X\varphi, \varphi\rangle - 2\langle X\varphi, \varphi\rangle = 0.$$

Thus ran $\Delta_B \perp S$. On the other hand $BS = SB$ implies $AS_1 = S_1A$, and $S_1 = 0$. So if we take $S_1 \neq 0$ we are done. ∎

REMARK 3.12. It is not possible to prove $\mathfrak{S}_\infty \neq \overline{\operatorname{ran} \Delta_B} \oplus \ker \Delta_B$. On the contrary, the sum $\overline{\operatorname{ran} \Delta_B} \oplus \ker \Delta_B$ is always equal to $\mathfrak{S}_\infty$. Namely, let $f_Y \in \mathfrak{S}_\infty^*$ be a functional of the form $f_Y(X) = \operatorname{tr}(XY)$ for some $Y \in \mathfrak{S}_1 \cong \mathfrak{S}_\infty^*$ which annihilates ran Δ_B. It immediately follows $BY^* - Y^*B = 0$, i.e. $Y^* \in \ker \Delta_B$. Since $f_Y(Y^*) \neq 0$, f_Y can not annihilate $\ker \Delta_B$. Thus ran $\Delta_B + \ker \Delta_B$ is always dense in $\mathfrak{S}_\infty$.

REMARK 3.13. One can find that Remark 3.12 is in a confusion with Theorem 3.11. However, this is a consequence of the fact that both $V^\perp = \{X \in \mathfrak{S}_\infty : \forall U \in V, \|X + U\| \geqslant \|U\|\}$ and $V_\perp = \{X \in \mathfrak{S}_\infty : \forall U \in V, \|X + U\| \geqslant \|X\|\}$, in general, does not make a subspace, but a cone!

Acknowledgements. The author is deeply grateful to Professor V.S. Shulman for suggestions in formulating Theorem 3.11, and to Professor D.R. Jocić for suggestions in proving Theorem 2.1.

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DRAGOLJUB J. KEČKIĆ
Faculty of Mathematics
University of Belgrade
Studentski trg 16–18
11000 Beograd
YUGOSLAVIA
E-mail: keckic@poincare.matf.bg.ac.yu

Received November 13, 2001; revised September 15, 2002 and February 22, 2003.