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KREIN'S RESOLVENT FORMULA AND PERTURBATION THEORY

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Abstract. The difference between the resolvents of two selfadjoint extensions of a certain symmetric operator A is described by Krein's resolvent formula. We will prove an analog of Krein's formula in a general framework, apply it to extensions theory, and give a straightforward proof of Krein's formula including the case that A is not necessarily densely defined. We will also present a modification of Krein's formula adjusted to perturbation theory and prove the corresponding resolvent estimate.

Keywords: *Krein's formula, resolvent analysis, perturbation theory.*

MSC (2000): Primary 47A55, 47B15, 81Q15; Secondary 47A10, 81Q10.

1. INTRODUCTION

In the theory of selfadjoint extensions of symmetric operators Krein's resolvent formula describes the difference between the resolvents of two selfadjoint extensions H_0 and H of a closed symmetric (or Hermitian) operator A. The original Krein's formula was derived by M. Krein and M. Naimark for the case where A has deficiency indices $(1,1)$ ([8], [9], [11], [15]). It was then generalized to the case of general finite deficiency indices ([1], [10]) and finally to the case of arbitrary (including infinite) deficiency indices by S.N. Saakjan ([17]). F. Gesztesy, K.A. Makarov, and E. Tsekanovskii ([5]) stressed the importance of Saakjan's result, studied its consequences and, among other things, supplied a detailed proof and an extensive list of references on this problem ([5]). We remark, however, that [17] and [5] deal only with the case that A is densely defined. In the present paper, among other results, we present a proof of Krein's formula which includes the case that A is not densely defined. (The main difference between the densely defined and non densely defined cases concerns with the additional admissibility condition on the (operator) parameter which can be neglected in the former case.) Some of the results from the present paper have already appeared as a preprint ([12]).

The proofs are improved considerably and several new results are presented, in particular Theorems 3.2 and 4.2. We would like to emphasize that our research in this area was inspired by studies of singular perturbations of selfadjoint operators which attracted much attention in the recent years ([2], [3], [4], [6], [16], [18]).

We will prove an analog of Krein's formula in a framework more general than the extension theory. More specifically, we fix a selfadjoint operator H_0 in a Hilbert space $\mathcal{H}$ and consider pairs $\{M, \gamma\}$ of a closed subspace M of $\mathcal{H}$ and a selfadjoint operator γ in M. Then, we will give a simple bijective correspondence between all selfadjoint operators H in $\mathcal{H}$ and pairs $\{M, \gamma\}$ which satisfy a condition we call an admissibility condition. The correspondence is given by the difference between the resolvents at point i (Theorem 2.1). We will then derive a general formula connecting the resolvents for arbitrary nonreal values of the spectral parameter (see Theorem 2.3). This is our analog of Krein's formula. We will then apply it to the extension theory. In Krein's extension theory one considers a densely defined Hermitian operator A and one particular extension H_0 of A. Then, Krein's formula (in the form given in [17]) characterizes all other selfadjoint extensions H of A via Krein's resolvent formula. By taking H_0 to be our fixed operator we will show that our resolvent formula gives in this case exactly Krein's formula from [17]. We believe that our approach gives a rather straightforward and simple proof of original Krein's formula. We would like to emphasize once again that our method applies to the case that A is not necessarily densely defined. It is surprising that this non-densely defined case has not been dealt with in the literature. The necessity of the admissibility condition was discovered first by M. Krasnosel'skii ([7]) during the studies of selfadjoint extensions of Hermitian not densely defined operators. This condition is fulfilled automatically if the restricted operator is densely defined (see Theorem 2.11).

In the perturbation theory one describes selfadjoint operator H in relation to one fixed unperturbed, or free operator H_0. The difference between the two operators is described by a certain operator V. In many applications the operator V can be chosen as an additive perturbation of H_0 so that formula

$$H = H_0 + V \tag{1.1}$$

holds. In this way one cannot describe all selfadjoint operators H in relation to H_0, and natural generalization of (1.1) is a certain resolvent formula like it has been suggested in [13] and [14].

Another point discussed in the paper is a certain modification of Krein's resolvent formula adjusted to perturbation theory. The operator γ appearing in conventional Krein's formula cannot be considered as a perturbation parameter, since in general the difference between the resolvents of H and H_0 increases as the norm of γ decreases, the operators H and H_0 do not coincide if $\gamma = 0$. Trying to obtain the described generalization of Krein's formula we discovered a one-parameter family of resolvent formulas which contains Krein's formula as a special case (described in Section 4). Among these formulas one new correspondence appears for us to be important. Like in the conventional case H is described by a closed subspace N and a selfadjoint operator β in N, but in a different way (see (3.2) and (3.3)). With β we can estimate the difference of the resolvents (see (3.9)). It is natural to believe that this type of resolvent formulas can have important applications. As a corollary of (3.9) we found an interesting estimate

(3.10). With 2 on the right hand side it is well known, but it is curious that (3.10) has not been widely recognized.

The two correspondences $H \leftrightarrow \{M,\gamma\}$ and $H \leftrightarrow \{N,\beta\}$ (and corresponding resolvent formulas) are in fact particular cases of the family of correspondences parameterized by $\theta \in [0,2\pi)$, so that the cases $\theta = 0$ and $\theta = \pi$ coincide with the case of $H \leftrightarrow \{M,\gamma\}$ and $H \leftrightarrow \{N,\beta\}$, respectively. In Section 4 we write down the general case. The proof is similar and will be omitted. Logically, we could have started with general θ and applied the results to the case of $\theta = 0$ and π. Nevertheless, fearing that it would mar the simplicity of our proof of original Krein's formula, we preferred to deal with the case of $\theta = 0$ first.

Notation and terminology. Throughout the present paper we shall use the following notation. We shall work in a fixed Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}(\mathcal{H})$ the set of all bounded linear operators in $\mathcal{H}$. (The domain of $T \in \mathcal{L}(\mathcal{H})$ is equal to $\mathcal{H}$.) For brevity of the exposition we put

$$\mathcal{C}_{\mathrm{sa}}(\mathcal{H}) = \{\text{the set of all selfadjoint operators in } \mathcal{H}\}, \tag{1.2}$$

$$\mathcal{M} = \{\text{the set of all closed subspaces of } \mathcal{H}\}. \tag{1.3}$$

For $M \in \mathcal{M}$ we denote by $M^\perp$ the orthogonal complement of M and by P_M the orthogonal projection onto M.

For a closed operator A in $\mathcal{H}$ the resolvent set of A is denoted by $\rho(A)$. We express resolvents and related operators by fractions: $(H-z)^{-1} = \frac{1}{H-z}$, $(H-w)(H-z)^{-1} = \frac{H-w}{H-z}$. The following simple relations are rather useful in our discussion:

$$\frac{H-z'}{H-z} = 1 + \frac{z-z'}{H-z}, \qquad z \in \rho(H),\ z' \in \mathbb{C}, \tag{1.4}$$

$$\frac{1}{H-z} = \frac{1}{z-z'}\left\{\frac{H-z'}{H-z} - 1\right\}, \quad z \in \rho(H),\ z' \in \mathbb{C}. \tag{1.5}$$

We list now up some terminology we use in this paper.

(a) For two selfadjoint operators H and H_0 we denote by $H \wedge H_0$ the *maximal common restriction* of H and H_0:

$$\mathrm{Dom}\,(H \wedge H_0) = \{u \in \mathrm{Dom}\,(H) \cap \mathrm{Dom}\,(H_0) : Hu = H_0 u\} \tag{1.6}$$

and $(H \wedge H_0)u = Hu = H_0 u$, $u \in \mathrm{Dom}\,(H \wedge H_0)$.

(b) Following F. Riesz and M. Krasnosel'skii we call an operator A in $\mathcal{H}$ *Hermitian* if

$$\langle Af, g\rangle = \langle f, Ag\rangle$$

for all $f, g \in \mathrm{Dom}\,(A)$ without assuming that $\mathrm{Dom}(A)$ is dense in $\mathcal{H}$.

(c) Let A be a closed Hermitian operator. The closed subspace $M_A = [(A - \mathrm{i})\mathrm{Dom}\,(A)]^\perp$ is called the *deficiency subspace* of A (at point i).

(d) Let A be a closed Hermitian operator. Two selfadjoint extensions H and H_0 of A are called *relatively prime* if A coincides with the maximal common restriction of H and H_0:

$$H \wedge H_0 = A. \tag{1.7}$$

We also say that H is relatively prime to H_0.

2. Representation of selfadjoint operators and Krein's formula

2.1. The resolvent formula. In this subsection we are going to prove that an analog of Krein's resolvent formula, which with the complex parameter z being at one particular value, say $z = \mathrm{i}$, will describe all selfadjoint operators H in comparison with the (fixed) selfadjoint operator H_0 and in terms of a closed subspace M and a selfadjoint operator γ acting in M (Theorem 2.1). In Theorem 2.3 we prove that analog of Krein's formula for general z.

For $M \in \mathcal{M}$ and $\gamma \in \mathcal{C}_{\mathrm{sa}}(M)$ we introduce the condition

$$\mathrm{Ker}\, \left\{ \frac{1}{H_0 + \mathrm{i}} - \frac{1}{\gamma + \mathrm{i}} P_M \right\} = \{0\}, \tag{2.1}$$

and call it *admissibility condition*. A pair $\{M, \gamma\}$ satisfying this condition will be called *admissible*, for H_0. When we use the admissibility condition, H_0 is always fixed so that we do not refer to H_0 in the notation.

Theorem 2.1. *Let H_0 be fixed. Then for any selfadjoint operator H there exists a unique admissible pair $\{M, \gamma\}$, such that*

$$\frac{1}{H - \mathrm{i}} = \frac{1}{H_0 - \mathrm{i}} - \frac{H_0 + \mathrm{i}}{H_0 - \mathrm{i}} \frac{1}{\gamma + \mathrm{i}} P_M. \tag{2.2}$$

The correspondence $H \leftrightarrow \{M, \gamma\}$ is a bijection between $\mathcal{C}_{\mathrm{sa}}(\mathcal{H})$ and the set of all admissible pairs $\{M, \gamma\}$.

Definition 2.2. The operator $H = H(M, \gamma)$ is the unique selfadjoint operator determined by an admissible pair $\{M, \gamma\}$ via (2.2).

The next theorem will give a resolvent formula satisfied by $H(M, \gamma)$ and H_0. We put

$$Q(z) = P_M \frac{1 + zH_0}{H_0 - z}\Big|_M \in \mathcal{L}(M), \quad z \in \rho(H_0), \tag{2.3}$$

where $A|_M$ denotes the restriction of A to M.

Theorem 2.3. *Let H_0 be fixed and let $H = H(M, \gamma)$, where $\{M, \gamma\}$ is an admissible pair. Then the resolvent of H is given by*

$$\frac{1}{H - z} = \frac{1}{H_0 - z} - \frac{H_0 + \mathrm{i}}{H_0 - z} \frac{1}{1 - \frac{\mathrm{i} - z}{\gamma + \mathrm{i}} P_M \frac{H_0 + \mathrm{i}}{H_0 - z}} \frac{1}{\gamma + \mathrm{i}} P_M \frac{H_0 - \mathrm{i}}{H_0 - z} \tag{2.4}$$

$$= \frac{1}{H_0 - z} - \frac{H_0 + \mathrm{i}}{H_0 - z} \frac{1}{\gamma + Q(z)} P_M \frac{H_0 - \mathrm{i}}{H_0 - z}, \quad z \in \rho(H) \cap \rho(H_0). \tag{2.5}$$

In (2.4) the operator in the denominator of the big fraction is boundedly invertible in $\mathcal{H}$, and in (2.5) $\gamma + Q(z)$ is boundedly invertible in M if and only if $z \in \rho(H) \cap \rho(H_0)$. (We say that a bounded operator A in $\mathcal{H}$ (or M) is boundedly invertible in $\mathcal{H}$ (or M) if A has an everywhere defined bounded inverse in $\mathcal{H}$ (or M).)

Proof of Theorem 2.1. (i) We prove first the existence of an admissible pair $\{M, \gamma\}$ satisfying (2.2). By (1.5) with $z = \mathrm{i}$ and $z' = -\mathrm{i}$ we see that formula (2.2) is equivalent to

$$\frac{H+\mathrm{i}}{H-\mathrm{i}} = \frac{H_0+\mathrm{i}}{H_0-\mathrm{i}}\Big(1 - \frac{2\mathrm{i}}{\gamma+\mathrm{i}}P_M\Big) = \frac{H_0+\mathrm{i}}{H_0-\mathrm{i}}\Big(P_{M^\perp} + \frac{\gamma-\mathrm{i}}{\gamma+\mathrm{i}}P_M\Big). \tag{2.6}$$

Given $H \in \mathcal{C}_{\mathrm{sa}}(\mathcal{H})$, we put

$$U = \frac{H_0-\mathrm{i}}{H_0+\mathrm{i}}\frac{H+\mathrm{i}}{H-\mathrm{i}}, \tag{2.7}$$

which is the ratio of the Cayley transforms of H_0 and H. U is a unitary operator in $\mathcal{H}$. Let $K = \{\psi \in \mathcal{H} : U\psi = \psi\}$ be the eigenspace of U corresponding to the eigenvalue 1 and put $M = K^\perp$. Then, K and M reduce U and the part $U|_M$ of U in M is a unitary operator in M which does not have 1 as an eigenvalue. Hence, by the theory of Cayley transform there exists a unique $\gamma \in \mathcal{C}_{\mathrm{sa}}(M)$ such that $U|_M = \frac{\gamma-\mathrm{i}}{\gamma+\mathrm{i}}$. Equation (2.6) follows from this observation and (2.7).

It is not difficult to see that M and γ given above satisfy (2.1). Indeed, the operator on the left hand side of (2.6) does not have 1 as an eigenvalue. By the first equality of (2.6) this means that

$$\mathrm{Ker}\Big\{1 - \frac{2\mathrm{i}}{\gamma+\mathrm{i}}P_M - \frac{H_0-\mathrm{i}}{H_0+\mathrm{i}}\Big\} = \{0\}. \tag{2.8}$$

This is equivalent to (2.1).

(ii) The uniqueness of $\{M, \gamma\}$ is an immediate consequence of (2.2). In fact, from (2.2) it follows that M is the orthogonal complement of $\mathrm{Ker}\Big\{\frac{1}{H-\mathrm{i}} - \frac{1}{H_0-\mathrm{i}}\Big\}$ and hence is uniquely determined. Then, for any $u \in M$, $\frac{1}{\gamma+\mathrm{i}}u$ is determined by (2.2), so that γ is unique.

(iii) Finally, we prove that the correspondence $H \mapsto \{M, \gamma\}$ is onto. Given a pair $\{M, \gamma\}$, the right hand side of (2.6) defines a unitary operator in $\mathcal{H}$. As is mentioned above, the admissibility condition is equivalent to (2.8), which in turn implies that the operator on the middle member of (2.6) does not have 1 as an eigenvalue. Hence, there exists $H \in \mathcal{C}_{\mathrm{sa}}(\mathcal{H})$ such that (2.2) holds. This completes the proof of Theorem 2.1. ∎

For the proof of Theorem 2.3 we shall use some simple relations satisfied by the difference of resolvents. For brevity of the exposition we use the following notation:

$$\Delta(z) = \frac{1}{H-z} - \frac{1}{H_0-z}, \quad z \in \rho(H_0) \cap \rho(H). \tag{2.9}$$

PROPOSITION 2.4. *$\Delta(z)$ satisfies the following relations:*

$$\Delta(z) = \frac{1}{z-z'}\Big(\frac{H-z'}{H-z} - \frac{H_0-z'}{H_0-z}\Big), \tag{2.10}$$

$$\frac{H-z}{H-z'}\Delta(z) = \Delta(z')\frac{H_0-z'}{H_0-z}, \tag{2.11}$$

$$(z-z')\Delta(z)\Delta(z') = \Delta(z)\frac{H_0-z}{H_0-z'} - \frac{H_0-z'}{H_0-z}\Delta(z'), \tag{2.12}$$

where, in (2.10), $z \in \rho(H_0) \cap \rho(H)$, *and* $z' \in \mathbb{C}$, $z' \neq z$, *while in* (2.11) *and* (2.12) z, $z' \in \rho(H_0) \cap \rho(H)$.

Proof. (2.10) follows from (2.9) and (1.4) at once. Using (2.10) twice, we obtain

$$\begin{aligned}\frac{H-z}{H-z'}\Delta(z) &= \frac{1}{z-z'}\frac{H-z}{H-z'}\left(\frac{H-z'}{H-z} - \frac{H_0-z'}{H_0-z}\right)\\ &= \frac{1}{z-z'}\left(\frac{H_0-z}{H_0-z'} - \frac{H-z}{H-z'}\right)\frac{H_0-z'}{H_0-z} = \Delta(z')\frac{H_0-z'}{H_0-z},\end{aligned}$$

which is (2.11). Multiplying (2.10) by $\Delta(z')$ from the right and using (2.11) with z and z' interchanged, we readily obtain (2.12). The proof is complete. ∎

REMARK 2.5. We could have used the notation Δ in (2.2), (2.4), and (2.5), but we preferred the explicit form of expressing the resolvent of H in terms of the resolvent of H_0 and other relevant quantities.

Proof of Theorem 2.3. Suppose $z \in \rho(H) \cap \rho(H_0)$. We start from (2.11) with $z' = \mathrm{i}$:

$$\frac{H-z}{H-\mathrm{i}}\Delta(z) = \Delta(\mathrm{i})\frac{H_0-\mathrm{i}}{H_0-z}. \tag{2.13}$$

Use (1.4) on the left hand side, express $\frac{1}{H-\mathrm{i}}$ and $\Delta(\mathrm{i})$ by (2.2), and apply $\frac{H_0-\mathrm{i}}{H_0+\mathrm{i}}$ from the left. Then, we obtain

$$\Big(\frac{H_0-z}{H_0+\mathrm{i}} - \frac{\mathrm{i}-z}{\gamma+\mathrm{i}}P_M\Big)\Delta(z) = -\frac{1}{\gamma+\mathrm{i}}P_M\frac{H_0-\mathrm{i}}{H_0-z}. \tag{2.14}$$

We modify this relation as

$$\left(1 - \frac{\mathrm{i}-z}{\gamma+\mathrm{i}}P_M\frac{H_0+\mathrm{i}}{H_0-z}\right)\frac{H_0-z}{H_0+\mathrm{i}}\Delta(z) = -\frac{1}{\gamma+\mathrm{i}}P_M\frac{H_0-\mathrm{i}}{H_0-z}. \tag{2.15}$$

We note that the process from (2.13) to (2.15) guarantees that the operator in parentheses on the left hand side of (2.15) is boundedly invertible in $\mathcal{H}$. Hence we obtain (2.4).

In order to prove the converse, we denote $S(z) = 1 - \frac{\mathrm{i}-z}{\gamma+\mathrm{i}}P_M\frac{H_0+\mathrm{i}}{H_0-z}$. As is mentioned above, $S(z)$ is boundedly invertible in $\mathcal{H}$ for all $z \in \rho(H) \cap \rho(H_0)$ and in particular for all non-real z. Suppose that $S(z_0)$ is boundedly invertible in $\mathcal{H}$ for a real $z_0 \in \rho(H_0)$. Then, so is $S(z)$ in a neighborhood of z_0 and $S(z)^{-1}$ is continuous there. This and (2.4) shows that $\frac{1}{H-z}$ is bounded in a complex neighborhood of z_0. This shows that $z_0 \in \rho(H)$.

Next, we prove (2.5). Let $S_0(z) = S(z)|_M$ be the restriction of $S(z)$ to M. It is obvious that $S_0(z)$ maps the domain of γ onto itself. Then, we see easily that the operator $S_1(z) \equiv (\gamma+\mathrm{i})S_0(z)$ (with the same domain as γ) is boundedly invertible in M and $\frac{1}{S_1(z)} = \frac{1}{S_0(z)}\frac{1}{\gamma+\mathrm{i}}$. Now, it is a simple computation to show that $S_1(z) = \gamma + Q(z)$. Thus, (2.5) is derived from (2.4), and the proof of Theorem 2.3 is complete. ∎

REMARK 2.6. In the first preprint version ([12]) of this paper, the resolvent relation (2.5) was proved by showing that:

(i) the operator appearing on the right handside of (2.5), call it $R(z)$, is a pseudo-resolvent, and

(ii) $R(z)$ is a resolvent if and only if admissibility condition (2.1) is met.

The method based of (2.11), which we found recently, has the advantage that it can serve to discover what expression should be on the right hand side of resolvent relation for general z, once the relation was found for $z = \mathrm{i}$.

2.2. APPLICATION TO EXTENSION THEORY. KREIN'S FORMULA. In this subsection we describe the relations between the correspondence established in Theorem 2.1 and the extension theory of symmetric operators. In particular, Krein's resolvent formula connecting the resolvents of two (different) selfadjoint extensions of one symmetric operator will be reconsidered in relation to resolvent formulas (2.4) and (2.5).

THEOREM 2.7. *Let A be a closed Hermitian operator in a Hilbert space $\mathcal{H}$ and let M_A be the deficiency subspace of A. Let H_0 be a (fixed) selfadjoint extension of A and let $H(M,\gamma)$ be the selfadjoint operator as defined by Definition* 2.2 *in reference to this H_0. Then:*

(i) *$H = H(M,\gamma)$ is a selfadjoint extension of A if and only if $M \subset M_A$;*

(ii) *$H = H(M,\gamma)$ is a selfadjoint extension of A which is relatively prime to H_0 if and only if $M = M_A$.*

By virtue of (ii) of the above theorem and the results of Subsection 2.1, we can restate the main results of the extension theory as follows.

THEOREM 2.8. *Let A be a closed symmetric operator and let H_0 be a selfadjoint extension of A. Let $M_A = [(A - \mathrm{i})\mathrm{Dom}(A)]^{\perp}$ be the deficiency subspace of A and put $Q(z) = P_{M_A} \frac{1+zH_0}{H_0 - z}\big|_{M_A}$. Then, selfadjoint extensions H of A which are relatively prime to H_0 are characterized by selfadjoint operators γ in M_A satisfying the admissibility condition* (2.1). *More precisely, the correspondence $\gamma \to H = H(M_A,\gamma)$ determined by* (2.2) *is a bijection from the set of all such γ to the set of all selfadjoint extensions of A which are relatively prime to H_0. Furthermore, for $H(M_A,\gamma)$ and H_0 the following resolvent formula holds:*

$$\frac{1}{H(M_A,\gamma) - z} = \frac{1}{H_0 - z} - \frac{H_0 + \mathrm{i}}{H_0 - z}\frac{1}{\gamma + Q(z)} P_{M_A} \frac{H_0 - \mathrm{i}}{H_0 - z}. \tag{2.16}$$

Formula (2.16) is the Krein's formula in the extension theory.

Proof of Theorems 2.7 *and* 2.8. It suffices to prove Theorem 2.7. We first note the following relation holds for any selfadjoint operators H and H_0:

$$\mathrm{Ker}\left(\frac{1}{H - \mathrm{i}} - \frac{1}{H_0 - \mathrm{i}}\right) = (H_0 - \mathrm{i})\mathrm{Dom}\,(H \wedge H_0) = (H - \mathrm{i})\mathrm{Dom}\,(H \wedge H_0). \tag{2.17}$$

From (2.2) and (2.17) it follows that

$$M = [(H_0 - \mathrm{i})\mathrm{Dom}\,(H(M,\gamma) \wedge H_0)]^{\perp}. \tag{2.18}$$

(This means that $M^\perp$ is the deficiency subspace of $H(M,\gamma)\wedge H_0$, or the latter is equal to $M^\perp$, irrespective of γ.)

On the other hand, since H_0 is an extension of A, we have

$$M_A = [(A-\mathrm{i})\mathrm{Dom}\,(A)]^\perp = [(H_0-\mathrm{i})\mathrm{Dom}\,(A)]^\perp. \tag{2.19}$$

(2.18) and (2.19) show that $M\subset M_A$ (or $M=M_A$) if and only if $\mathrm{Dom}\,(H\wedge H_0)\supset \mathrm{Dom}(A)$ (or $\mathrm{Dom}\,(H\wedge H_0)=\mathrm{Dom}\,(A)$). Since H_0 is an extension of A, this last relation is equivalent to $H\wedge H_0\supset A$ (or $H\wedge H_0 = A$.) Theorem 2.7 follows from this at once. ∎

The bounded operator $Q(z)$ defined by (2.3) depends analytically on $z\notin\mathbb{R}$ and has positive imaginary part in $\mathrm{Im}\, z>0$. Indeed,

$$\mathrm{Im}\, Q(z) = \mathrm{Im}\, z\, P_M \frac{H_0^2+1}{(H_0-\mathrm{Re}\, z)^2+\mathrm{Im}\, z^2} P_M \geqslant 0.$$

This operator is a generalization of Krein's Q-function. In the literature this function is usually defined up to a certain real constant (selfadjoint operator) ([1]). We find it more convenient to determine it uniquely using the normalization condition

$$Q(\mathrm{i}) = \mathrm{i} I_M.$$

2.3. Some remarks.

More on admissibility condition.

THEOREM 2.9. *Let H_0 be a selfadjoint operator and let $M\in\mathcal{M}$. Then the following alternative* (i) *and* (ii) *holds:*

(i) $M\cap \mathrm{Dom}\,(H_0)=\{0\}$. *In this case, $\{M,\gamma\}$ is admissible for all $\gamma\in\mathcal{C}_{\mathrm{sa}}(M)$ and the domain* $\mathrm{Dom}\,(H(M,\gamma)\wedge H_0)$ *is dense in $\mathcal{H}$.*

(ii) $M\cap \mathrm{Dom}\,(H_0)\neq\{0\}$. *In this case there exist both admissible $\{M,\gamma\}$ and non-admissible $\{M,\gamma\}$. The domain* $\mathrm{Dom}\,(H(M,\gamma)\wedge H_0)$ *is not dense for any admissible $\{M,\gamma\}$.*

In the proof of Theorem 2.9 we use the following simple lemma.

LEMMA 2.10. *Let M be a Hilbert space.*

(i) *Let φ, $\psi\in M$, $\varphi\neq 0$, and assume that (φ,ψ) is real. Then there exists a selfadjoint operator γ such that $\varphi\in\mathrm{Dom}\,(\gamma)$ and $\gamma\,\varphi=\psi$.*

(ii) *Let B be a Hermitian operator in M. Then there exists a selfadjoint operator γ such that $u\in\mathrm{Dom}\,(\gamma)\cap\mathrm{Dom}\,(B)$ and $\gamma u = Bu$ imply $u=0$.*

Proof. (i) It suffices to construct such an operator in the two-dimensional space containing φ and ψ. Then the proof is elementary.

(ii) We may assume that B is closed. Then $\gamma = P_{\ker B}$ satisfies the requirement. ∎

Proof of Theorem 2.9. Suppose first that $\{M,\gamma\}$ is admissible and put $H = H(M,\gamma)$. Since $(H_0 - \mathrm{i})\mathrm{Dom}\,(H \wedge H_0)$ is a closed subspace, we see from (2.18) that $(H_0 - \mathrm{i})\mathrm{Dom}\,(H \wedge H_0) = M^{\perp}$. From this it follows that

$$u \in \mathrm{Dom}\,(H \wedge H_0)^{\perp} \Leftrightarrow \frac{1}{H_0 + \mathrm{i}} u \in M. \tag{2.20}$$

In the case (i) $\frac{1}{H_0+\mathrm{i}} u \in M$ is possible only for $u = 0$, because $\frac{1}{H_0+\mathrm{i}} u \in \mathrm{Dom}\,(H_0)$. Hence, $\mathrm{Dom}\,(H \wedge H_0)$ is dense. In the case (ii) take $0 \neq v \in \mathrm{Dom}\,(H_0) \cap M$ and put $u = (H_0 + \mathrm{i})v$. Then, $\frac{1}{H_0+\mathrm{i}} u \in M$ and $u \neq 0$. Hence, $\mathrm{Dom}\,(H \wedge H_0)$ is not dense by (2.20).

Next, we consider the admissibility condition. It is easy to see that (2.1) is equivalent to

$$[v \in \mathrm{Dom}\,(\gamma) \cap \mathrm{Dom}\,(H_0) \quad \text{and} \quad \gamma v = P_M H_0 v] \Rightarrow v = 0. \tag{2.21}$$

In the case (i) $v \in \mathrm{Dom}\,(\gamma) \cap \mathrm{Dom}\,(H_0)$ alone implies $v = 0$, so that all $\{M,\gamma\}$ are admissible. Let us proceed to the case (ii). To prove the existence of admissible pair, let $B = P_M H_0|_M$. Then B is a Hermitian operator in M. Hence, (ii) of Lemma 2.10 guarantees the existence of γ satisfying (2.21). To prove the existence of non-admissible γ, let $0 \neq \varphi \in M \cap \mathrm{Dom}(H_0)$ and put $\psi = P_M H_0 \varphi$. Then, $(\varphi, \psi) = (\varphi, H_0\varphi)$ is real. Then, by (i) of Lemma 2.10 there exists γ such that $\gamma\,\varphi = \psi$. It is clear that this γ does not satisfy (2.21). The proof is complete. ∎

Consider two extreme cases $M = \{0\}$ and $M = \mathcal{H}$. When $M = \{0\}$, the only selfadjoint operator in M is the zero operator and the admissibility condition (2.1) is satisfied. We have $H(\{0\}, 0) = H_0$. Next let $M = \mathcal{H}$. A pair $\{\mathcal{H},\gamma\}$ satisfies (2.1) if and only if $\mathrm{Dom}\,(H_0 \wedge \gamma) = \{0\}$ (see (2.17)). In particular, if $\mathrm{Dom}\,(H_0) \cap \mathrm{Dom}\,(\gamma) = \{0\}$, then $(\mathcal{H},\gamma)$ is admissible.

We next apply Theorem 2.9 to the extension theory.

Theorem 2.11. *Let A and H_0 be as in Theorem* 2.7.

(i) *If* $\mathrm{Dom}\,(A)$ *is dense in $\mathcal{H}$, then $M_A \cap \mathrm{Dom}\,(H_0) = \{0\}$ (alternative* (i) *in Theorem* 2.9*) and the admissibility condition can be neglected.*

(ii) *If* $\mathrm{Dom}\,(A)$ *is not dense, then $M_A \cap \mathrm{Dom}(H_0) \neq \{0\}$ (alternative* (ii) *in Theorem* 2.9*) and the admissibility condition is relevant.*

Proof. (i) is clear from $\mathrm{Dom}\,(H(M,\gamma) \wedge H_0) \supset \mathrm{Dom}\,(A)$.

(ii) By Theorem 2.9 an admissible pair $\{M_A,\gamma\}$ exists irrespective of (i) or (ii) of Theorem 2.9 holds for M_A. Then, $H(M_A,\gamma)$ is relatively prime to H_0 by Theorem 2.7 and hence $\mathrm{Dom}\,(H(M_A,\gamma) \wedge H_0) = \mathrm{Dom}\,(A)$ is not dense, that is alternative (ii) in Theorem 2.9. The proof is complete. ∎

Limit of large γ. The resolvent formula proven in Section 2.1 is not suitable for perturbation theory, since the operator γ appearing in (2.2) cannot be considered as a perturbation parameter: "the difference between the resolvents of H and H_0 decreases as the norm of γ increases". For example, we have the following theorem.

THEOREM 2.12. *Let $\gamma \in \mathcal{C}_{\rm sa}(\mathcal{H})$ be such that* $\mathrm{Dom}\,(\gamma) \cap \mathrm{Dom}\,(H_0) = \{0\}$ *and zero is not an eigenvalue of γ. Then, the pair $(\mathcal{H}, t\gamma)$ is admissible for any real $t \neq 0$. Furthermore, as $t \to \pm\infty$, $H(\mathcal{H}, t\gamma)$ converges to H_0 in the sense of strong resolvent convergence.*

Proof. Consider any element f of the Hilbert space. Then, by (2.2) with $M = \mathcal{H}$, the difference of the resolvents can be written using the spectral measure $\mu_f(\lambda)$ for the operator γ and the element $P_M f$ as follows

$$\left\| \Big(\frac{1}{H(\mathcal{H}, t\gamma) - \mathrm{i}} - \frac{1}{H_0 - \mathrm{i}} \Big) f \right\|^2 = \int_{\mathbb{R}} \left| \frac{1}{t\lambda + \mathrm{i}} \right|^2 \mathrm{d}\mu_f(\lambda) \underset{t \to \pm\infty}{\to} 0,$$

since zero is not an eigenvalue of the operator γ. The theorem is proven. ∎

3. RESOLVENT FORMULA FITTED TO PERTURBATION THEORY

3.1. THE RESOLVENT FORMULA. In view of Theorem 2.12, effective resolvent formula (useful in perturbation theory) can be obtained if one substitutes the operator γ by its inverse, for example $\beta = -\gamma^{-1}$. (Of course, the described transformation can be considered only formally, since the operator γ is not necessarily invertible.) In this subsection we are going to derive such resolvent formula. Our second resolvent formula describes any selfadjoint operator H in comparison with the (fixed) selfadjoint operator H_0 and in terms of a closed subspace N (different from M) and a selfadjoint operator β acting in N. The admissibility condition for the pair $\{N, \beta\}$, $N \in \mathcal{M}$, $\beta \in \mathcal{C}_{\rm sa}(N)$ is

$$\mathrm{Ker} \Big\{ \frac{H_0}{H_0 + \mathrm{i}} - \frac{\mathrm{i}}{\beta + \mathrm{i}} P_N \Big\} = \{0\}. \tag{3.1}$$

A *pair* $\{N, \beta\}$ satisfying this condition will be called *admissible* throughout this section.

THEOREM 3.1. *Let H_0 be fixed. For any $H \in \mathcal{C}_{\rm sa}(\mathcal{H})$ there exists a unique admissible pair $\{N, \beta\}$ (satisfying* (3.1)*) such that*

$$\frac{1}{H - \mathrm{i}} - \frac{1}{H_0 - \mathrm{i}} = \frac{H_0 + \mathrm{i}}{H_0 - \mathrm{i}} \Big(\mathrm{i} P_{N^\perp} + \frac{\mathrm{i}\beta}{\beta + \mathrm{i}} P_N \Big) \tag{3.2}$$

$$= \frac{H_0 + \mathrm{i}}{H_0 - \mathrm{i}} \Big(\mathrm{i} + \frac{1}{\beta + \mathrm{i}} P_N \Big). \tag{3.3}$$

The correspondence $H \leftrightarrow \{N, \beta\}$ is a bijection between $\mathcal{C}_{\rm sa}(\mathcal{H})$ and the set of all admissible pairs $\{N, \beta\}$.

THEOREM 3.2. *Let H_0 be fixed and let H be the selfadjoint operator determined by $\{N, \beta\}$ via* (3.2)*. Then the resolvent of H is given by*

$$\begin{aligned} \frac{1}{H - z} = {} & \frac{1}{H_0 - z} \\ & + \frac{H_0 + \mathrm{i}}{H_0 - z} \, \frac{1}{1 + (\mathrm{i} - z) \Big(\mathrm{i} + \frac{1}{\beta + \mathrm{i}} P_N \Big) \frac{H_0 + \mathrm{i}}{H_0 - z}} \Big(\mathrm{i} + \frac{1}{\beta + \mathrm{i}} P_N \Big) \frac{H_0 - \mathrm{i}}{H_0 - z}. \end{aligned} \tag{3.4}$$

In (3.4) *the operator in the denominator of the big fraction is boundedly invertible in $\mathcal{H}$ if and only if $z \in \rho(H) \cap \rho(H_0)$.*

Proof of Theorem 3.1. The proof is similar to that of Theorem 2.1. By (1.5) with $z = \mathrm{i}$ and $z' = -\mathrm{i}$ we see that the formula (3.2) is equivalent to

$$\frac{H+\mathrm{i}}{H-\mathrm{i}} = -\frac{H_0+\mathrm{i}}{H_0-\mathrm{i}}\Big(1 - \frac{2\mathrm{i}}{\beta+\mathrm{i}}P_N\Big) = -\frac{H_0+\mathrm{i}}{H_0-\mathrm{i}}\Big(P_{N^\perp} + \frac{\beta-\mathrm{i}}{\beta+\mathrm{i}}P_N\Big). \tag{3.5}$$

Given $H \in \mathcal{C}_{\mathrm{sa}}(\mathcal{H})$, we put

$$U = -\frac{H_0-\mathrm{i}}{H_0+\mathrm{i}}\,\frac{H+\mathrm{i}}{H-\mathrm{i}}. \tag{3.6}$$

Let N be the orthogonal complement of the eigenspace of the unitary operator U corresponding to the eigenvalue 1. Since $U|_N$ is unitary in N and does not have 1 as an eigenvalue, there exists a unique selfadjoint operator β in N such that

$$U = P_{N^\perp} \oplus \frac{\beta-\mathrm{i}}{\beta+\mathrm{i}}P_N. \tag{3.7}$$

It is straightforward to see that this pair satisfies (3.5) and the admissibility condition (3.1). The uniqueness follows from (3.7) at once. The proof of onto property is similar and we omit the details. ∎

Proof of Theorem 3.2. The proof proceeds in the same way as in the proof of (2.4) of Theorem 2.3. We use (3.3) instead of (2.2). Then, instead of (2.14) we obtain

$$\Big(\frac{H_0-z}{H_0+\mathrm{i}} + (\mathrm{i}-z)\Big(\mathrm{i} + \frac{1}{\beta+\mathrm{i}}P_N\Big)\Big)\Delta(z) = \Big(i + \frac{1}{\beta+i}P_N\Big)\frac{H_0-i}{H_0-z}. \tag{3.8}$$

The rest of the proof is the same and we do not repeat it. ∎

REMARK 3.3. In Theorem 3.2 we have derived only (3.4) which corresponds to (2.4) in Theorem 2.3. We have not been able to derive a formula corresponding to (2.5) in the case of Theorem 3.2.

3.2. RESOLVENT ESTIMATES.

Parameterization of the selfadjoint operators as perturbations of a given selfadjoint operator H_0 using the pair $\{N, \beta\}$ leads to efficient estimates of the difference between the resolvents of the perturbed and unperturbed operators. Let us denote by $H'(N, \beta)$ the unique selfadjoint operator determined by an admissible pair $\{N, \beta\}$ via (3.2) or (3.3).

THEOREM 3.4. *The difference between the resolvents of the unperturbed operator H_0 and the perturbed operator $H = H'(N, \beta)$ determined by the admissible pair $\{N, \beta\}$ can be estimated as follows*

$$\Big\|\frac{1}{H-\mathrm{i}} - \frac{1}{H_0-\mathrm{i}}\Big\|_{\mathcal{H}} = \begin{cases} 1 & \textit{if } N \neq \mathcal{H}, \\ \frac{\|\beta\|}{\sqrt{\|\beta\|^2+1}} \leqslant \min\{\|\beta\|, 1\} & \textit{if } N = \mathcal{H}; \end{cases} \tag{3.9}$$

where the right hand side is understood to be 1 *when β is not bounded.*

Proof. The difference between the resolvents at point i is given by (3.2). To estimate the norm of the operator $\mathrm{i}P_{N^\perp} + \mathrm{i}\frac{\beta}{\beta+\mathrm{i}}P_N$ we note first that this sum is orthogonal. Therefore the norm of this operator is equal to the maximum of the norms of the summands. The norm of the operator $\mathrm{i}\frac{\beta}{\beta-\mathrm{i}}P_N$ is equal to $\frac{\|\beta\|}{\sqrt{\|\beta\|^2+1}}$. The orthogonal projector $P_{N^\perp}$ has norm 1 if $N \neq \mathcal{H}$. The norm of the Cayley transform $\frac{H_0+\mathrm{i}}{H_0-\mathrm{i}}$ is equal to 1 and formula (3.9) is proven. ∎

In the course of our investigation we found the following estimate (3.10) as an easy corollary of the last theorem. We decided to include this estimate in the paper since we could not trace it in the literature and believe that it could have interesting applications. The proof given below, however, is more direct.

PROPOSITION 3.5. *Let H and H_0 be two arbitrary selfadjoint operators in the Hilbert space $\mathcal{H}$. Then the difference between the resolvents at point* i *satisfies the estimate*

$$\left\| \frac{1}{H-\mathrm{i}} - \frac{1}{H_0-\mathrm{i}} \right\| \leqslant 1. \tag{3.10}$$

Proof. Using formula (1.4), we argue as follows:

$$\left\| \frac{1}{H-\mathrm{i}} - \frac{1}{H_0-\mathrm{i}} \right\| = \frac{1}{2} \left\| \frac{H+\mathrm{i}}{H-\mathrm{i}} - \frac{H_0+\mathrm{i}}{H_0-\mathrm{i}} \right\| \leqslant 1. \quad ∎$$

REMARK 3.6. Equations (3.9) and (3.10) show that $N \neq \mathcal{H}$ (or $N = \mathcal{H}$) if and only if the equality (or the inequality) holds in (3.10). In the case $N = \mathcal{H}$ the operator β serves as a perturbation parameter.

EXAMPLE 3.7. Let the original operator H_0 be equal to zero $H_0 = 0$ with the domain $\mathrm{Dom}\,(H_0) = \mathcal{H}$. Then the pair $\{N, \beta\}$ is admissible only if the subspace N coincides with $\mathcal{H}$. Then any operator β is admissible. The formula (3.2) reads as follows

$$\frac{1}{H-\mathrm{i}} = \frac{1}{-\beta-\mathrm{i}}.$$

The perturbation operator β coincides with the operator $-H$ in this case.

4. GENERAL CORRESPONDENCE

The two resolvent formulas proven in the previous sections are particular cases of the family of correspondences described here. This family can be parameterized by $\theta \in [0, 2\pi)$, so that the cases $\theta = 0$ and $\theta = \pi$ coincide with the correspondences described in Sections 2 and 3 respectively. For $M \in \mathcal{M}$ and $\gamma \in \mathcal{C}_{\mathrm{sa}}(M)$ we introduce the condition

$$\mathrm{Ker} \left\{ 1 - \frac{2\mathrm{i}}{\gamma+\mathrm{i}} P_M - \mathrm{e}^{\mathrm{i}\theta} \frac{H_0-\mathrm{i}}{H_0+\mathrm{i}} \right\} = \{0\}, \tag{4.1}$$

and call it *admissibility condition.* We also call a pair $\{M, \gamma\}$ satisfying condition (4.1) an *admissible pair.* This condition coincides with the conditions (2.1) and (3.1) if $\theta = 0$ and $\theta = \pi$, respectively.

The main result in this section is the following theorem, which we give without proof, since the proof is similar to the proofs of Theorems 2.1 and 2.7.

THEOREM 4.1. *Let θ and H_0 be as above. Then, for any $H \in \mathcal{C}_{\rm sa}(\mathcal{H})$ there exists an admissible pair $\{M, \gamma\}$ such that the following equivalent relations* (4.2)–(4.5) *hold:*

$$\frac{H+\mathrm{i}}{H-\mathrm{i}} = \mathrm{e}^{-\mathrm{i}\theta}\frac{H_0+\mathrm{i}}{H_0-\mathrm{i}}\Big(1-\frac{2\mathrm{i}}{\gamma+\mathrm{i}}P_M\Big) \tag{4.2}$$

$$\frac{H+\mathrm{i}}{H-\mathrm{i}} = \mathrm{e}^{-\mathrm{i}\theta}\frac{H_0+\mathrm{i}}{H_0-\mathrm{i}}\Big(P_{M^\perp}+\frac{\gamma-\mathrm{i}}{\gamma+\mathrm{i}}P_M\Big), \tag{4.3}$$

$$\frac{1}{H-\mathrm{i}}-\frac{1}{H_0-\mathrm{i}} = \frac{H_0+\mathrm{i}}{H_0-\mathrm{i}}\Big(\frac{\mathrm{e}^{-\mathrm{i}\theta}-1}{2\mathrm{i}}-\frac{\mathrm{e}^{-\mathrm{i}\theta}}{\gamma+\mathrm{i}}P_M\Big) \tag{4.4}$$

$$\frac{1}{H-\mathrm{i}}-\frac{1}{H_0-\mathrm{i}} = \frac{H_0+\mathrm{i}}{H_0-\mathrm{i}}\Big(\frac{\mathrm{e}^{-\mathrm{i}\theta}-1}{2\mathrm{i}}P_{M^\perp}+\frac{1}{2\mathrm{i}}\Big[\mathrm{e}^{-\mathrm{i}\theta}\frac{\gamma-\mathrm{i}}{\gamma+\mathrm{i}}-1\Big]P_M\Big). \tag{4.5}$$

The correspondence $H \leftrightarrow \{M,\gamma\}$ is a bijection between $\mathcal{C}_{\rm sa}(\mathcal{H})$ and the set of all admissible pairs $\{M,\gamma\}$.

Let us denote by $H(M,\gamma;\theta)$ the unique selfadjoint operator determined by any admissible pair $\{M,\gamma\}$ and parameter θ. With $\theta = 0$ formula (4.4) coincides with (2.2), and with $\theta = \pi$ it is (3.3) ($N = M; \beta = \gamma$). In other words

$$H(M,\gamma;0) = H(M,\gamma), \quad H(M,\gamma;\pi) = H'(M,\gamma).$$

The following theorem is a generalization of Theorem 2.3 and can be proven using the same method.

THEOREM 4.2. *Let H_0 be fixed and let H be the selfadjoint operator determined by $\{M,\gamma;\theta\}$ via* (4.4) *or* (4.5). *Then the resolvent of H is given by*

$$\begin{aligned}\frac{1}{H-z}&-\frac{1}{H_0-z}\\ &= \frac{H_0+\mathrm{i}}{H_0-z}\frac{1}{1+(\mathrm{i}-z)\Big(\frac{\mathrm{e}^{-\mathrm{i}\theta}-1}{2\mathrm{i}}-\frac{\mathrm{e}^{-\mathrm{i}\theta}}{\gamma+\mathrm{i}}P_M\Big)\frac{H_0+\mathrm{i}}{H_0-z}}\Big(\frac{\mathrm{e}^{-\mathrm{i}\theta}-1}{2\mathrm{i}}-\frac{\mathrm{e}^{-\mathrm{i}\theta}}{\gamma+\mathrm{i}}P_M\Big)\frac{H_0-\mathrm{i}}{H_0-z}.\end{aligned}$$

In the above formula the operator in the denominator of the big fraction is boundedly invertible in $\mathcal{H}$ if and only if $z \in \rho(H)\cap\rho(H_0)$.

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REFERENCES

1. N.I. AKHIEZER, I.M. GLAZMAN, *Theory of Linear Operators in Hilbert Space*, Pitman, Boston 1981.
2. S. ALBEVERIO, F. GESZTESY, R. HØEGH-KROHN, H. HOLDEN, *Solvable Models in Quantum Mechanics*, Springer, 1988.

3. S. ALBEVERIO, P. KURASOV, *Singular Perturbations of Differential Operators*, London Math. Soc. Lecture Notes Ser., vol. 271, Cambridge Univ. Press, Cambridge 2000.
4. F.A. BEREZIN, L.D. FADDEEV, Remark on the Schrödinger equation with singular potential, *Dokl. Akad. Nauk SSSR* **137**(1961), 1011–1014.
5. F. GESZTESY, K.A. MAKAROV, E. TSEKANOVSKII, An addendum to Krein's formula, *J. Math. Anal. Appl.* **222**(1998), 594–606.
6. A. KISELEV, B. SIMON, Rank-one perturbations at infinitesimal coupling, *J. Funct. Anal.* **128**(1995), 245–252.
7. M. KRASNOSEL'SKII, On selfadjoint extensions of Hermitian operators, *Ukrain. Mat. Zh.*, **1**(1949), 21–38.
8. M.G. KREIN, On Hermitian operators whose deficiency indices are 1, *Comptes Rendue (Doklady) Acad. Sci. URSS (N.S.)* **43**(1944), 323–326.
9. M.G. KREIN, On Hermitian operators whose deficiency indices are equal to one. II, *Comptes Rendue (Doklady) Acad. Sci. URSS (NS)* **44**(1944), 131–134.
10. M.G. KREIN, Concerning the resolvents of an Hermitian operator with the deficiency-index (m, m), *Comptes Rendue (Doklady) Acad. Sci. URSS (NS)* **52** (1946), 651–654.
11. M.G. KREIN, On a remarkable class of Hermitian operators, *Comptes Rendue (Doklady) Acad. Sci. URSS (NS)* **44**(1944), 175–179.
12. P. KURASOV, S.T. KURODA, *Krein's Formula and Perturbation Theory*, Res. Rep. Math., N 6, Department of Mathematics, Stockholm Univ., Stockholm 2000.
13. S.T. KURODA, H. NAGATANI, $\mathcal{H}_{-2}$-construction and some applications, in *Mathematical Results in Quantum Mechanics*, J. Dittrich, P. Exner, M. Tartar, ed., Oper. Theory Adv. Appl., vol. 108, Birkhäuser Verlag, Basel 1999, pp. 99–105.
14. S.T. KURODA, H. NAGATANI, $\mathcal{H}_{-2}$-construction of general type and its applications to point interactions, *J. Evol. Equ.* **1**(2001), 421–440.
15. M. NAIMARK, On spectral functions of a symmetric operator, *Bull. Acad. Sci. URSS* **7**(1943), 285–296.
16. B. PAVLOV, The theory of extensions and explicitly solvable models, *Uspekhi Mat. Nauk* **42**(1987), 99–131.
17. S.N. SAAKJAN, Theory of resolvents of a symmetric operator with infinite defect numbers, *Akad. Nauk. Armjan. SSR Dokl.* **41**(1965), 193–198.
18. B. SIMON, Spectral analysis of rank one perturbations and applications, in *Mathematical Quantum Theory.* II. *Schrödinger Operators (Vancouver, BC, 1993)*, CRM Proc. Lecture Notes, vol. 8, Amer. Math. Soc., Providence, RI, 1995, pp. 109–149,

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