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A WEAKLY HYPERCYCLIC OPERATOR THAT IS NOT NORM HYPERCYCLIC

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Abstract. We give a sufficient condition for a bilateral weighted shift to be hypercyclic in the weak topology. Using this condition, we provide one such shift that fails to be hypercyclic in the norm topology. Even more interesting, the shift is bounded below by 1 and consequently every vector has a norm increasing orbit. This result provides a negative answer to a natural question raised by Feldman who asked whether every weakly hypercyclic operator is necessarily norm hypercyclic. On the other hand, if the operator is a unilateral weighted backward shift, we prove the answer is positive. Furthermore, with a simple condition on the weights, there exists a weakly hypercyclic vector that is not a norm hypercyclic vector.

Keywords: *Hypercyclic operator, hypercyclic vector, bilateral weighted shift, unilateral weighted backward shift, weak topology.*

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1. INTRODUCTION

Let X be a separable, infinite dimensional Banach space. We say a bounded linear operator $T : X \to X$ is *hypercyclic*, or specifically *norm hypercyclic*, if there is a vector x in X such that $\mathrm{Orb}(T, x) = \{x, Tx, T^2x, \ldots\}$ is dense in X with respect to the norm topology. Such a vector x is said to be a *hypercyclic vector*, or a *norm hypercyclic vector*, for T. If $\mathrm{Orb}(T, x)$ is dense in X with respect to the weak topology, then the operator T is *weakly hypercyclic* and such a vector x is a *weakly hypercyclic vector* for T. The operator is *cyclic* if the linear span of $\mathrm{Orb}(T, x)$, denoted by $\operatorname{span}\mathrm{Orb}(T, x)$, is norm dense in X and such a vector x is a *cyclic vector* for T. Since $\operatorname{span}\mathrm{Orb}(T, x)$ is a convex set, it is norm dense if and only if it is weakly dense. In other words, the operator T is cyclic if and only if it is weakly cyclic. On the other hand, the norm topology is strictly stronger than the weak topology, and so every hypercyclic operator is a weakly hypercyclic operator. Hence, it is natural for Feldman ([6], Question 2.1) to pose the converse: Is every weakly hypercyclic operator, in fact, a hypercyclic operator and is every weakly hypercyclic vector a hypercyclic vector?

In Section 2, we show some similarities between hypercyclicity and weak hypercyclicity. In Section 3, we give a sufficient condition for a bilateral weighted shift on $\ell^p(\mathbb{Z})$ with $2 \leqslant p < \infty$ to be weakly hypercyclic; see Theorem 3.2 below. Since the sufficient condition is quite involved, we provide a less general condition in Corollary 3.5 that can easily be applied. In this corollary, the finite products of the negative indexed weights of the bilateral weighted shift are bounded. In contrast, Theorem 3.2 shows an interesting phenomenon that these finite products can in fact be unbounded. Using Theorem 3.2, we create a weakly hypercyclic operator that fails to be hypercyclic, and hence it gives a negative answer for both parts of the question posed by Feldman; see Corollary 3.3 below. Even more interesting, this operator is bounded below by 1, and consequently all the orbits of this weakly hypercyclic operator are norm increasing. For a unilateral weighted backward shift, the situation is different. In Section 4, we show hypercyclicity and weak hypercyclicity are equivalent for a unilateral weighted backward shift on ℓ^p with $1 \leqslant p < \infty$; see Theorem 4.1 below. Nevertheless, it does not necessarily imply that every weakly hypercyclic vector is a hypercyclic vector. For $1 < p < \infty$ and a simple condition on the weights of the shift, we show the corresponding unilateral weighted backward shift on ℓ^p has a weakly hypercyclic vector that is not a hypercyclic vector; see Theorem 4.2 below.

Recently, a lot of work on hypercyclicity has been based on the Hypercyclicity Criterion which was originally established by Kitai ([11], Theorem 1.4), and rediscovered by Gethner and Shapiro ([7], Theorem 2.2) in a more general form. One version of the Criterion states:

A bounded linear operator $T : X \to X$ is hypercyclic if there exist a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers, norm dense sets X_0, Y_0, and a map (not necessarily linear or continuous) $S : Y_0 \to Y_0$ such that:

(i) *for each $x \in X_0$, we have $T^{n_k}x \to 0$ in norm as $k \to \infty$;*
(ii) *for each $y \in Y_0$, we have $S^{n_k}y \to 0$ in norm as $k \to \infty$;*
(iii) *$TS = I$ on Y_0.*

Note that if (i) and (ii) hold for norm dense sets X_0, Y_0, then they hold for their linear span. Hence, by a convex set argument, we may assume that X_0 and Y_0 are weakly dense instead of norm dense. Note also that a Banach space is separable if and only if it is separable in the weak topology, by the same convex set argument. Furthermore, a linear map from a Banach space to another Banach space is norm-to-norm continuous if and only if it is weak-to-weak continuous; see, for example, page 166 in [4]. It makes us wonder what happens if we replace the norm topology in the Hypercyclicity Criterion with the weak topology. In fact, the resulting statement fails to hold. For a simple counterexample, let $X = \ell^2$ and define $T : X \to X$ by $T(\beta_0, \beta_1, \beta_2, \ldots) = (\beta_1, \beta_2, \beta_3, \ldots)$. That is, T is the unilateral backward shift. Let $X_0 = Y_0 = \operatorname{span}\{e_\alpha : \alpha \geqslant 0\}$ where e_α is the sequence with 0 in all the entries except a 1 in the α-th entry. Define $S : Y_0 \to Y_0$ by $S(\beta_0, \beta_1, \beta_2, \ldots) = (0, \beta_0, \beta_1, \ldots)$. Hence, $TS = I$ on Y_0, and furthermore $T^n x \to 0$ weakly for all $x \in X_0$ and $S^n y \to 0$ weakly for all $y \in Y_0$. Since $\|T\| = 1$, every orbit is bounded, and hence T cannot be weakly hypercyclic. This phenomenon is not surprising since the proof of the Hypercyclicity Criterion by

Gethner and Shapiro ([7], Theorem 2.2) uses the Baire Category Theorem, which is not available in the weak topology.

2. WEAK HYPERCYCLICITY

In a separable, infinite dimensional Banach space X, the weak topology is strictly weaker than the norm topology. Despite this fact, a weakly hypercyclic operator shares many of the same properties as a hypercyclic operator. For example, it clearly follows from the definitions that every hypercyclic vector for a bounded linear operator $T : X \to X$ is automatically a cyclic vector for T. The same applies to a weakly hypercyclic vector.

PROPOSITION 2.1. *If $T : X \to X$ is a bounded linear operator, then every weakly hypercyclic vector is a cyclic vector for T.*

Proof. If x is a weakly hypercyclic vector, then $\operatorname{Orb}(T, x)$ is weakly dense, and so is $\operatorname{span}\operatorname{Orb}(T, x)$. Since the span is a convex set, the weak closure coincides with the norm closure. This gives us that x is a cyclic vector for T. ∎

Another property of a hypercyclic operator $T : X \to X$ is that it always has an invariant, norm dense, linear subspace in which every nonzero vector is a hypercyclic vector for T. The complex scalar case of this result was established by Herrero ([9], Proposition 4.1) and independently by Bourdon ([3]). The real scalar case was established by Bès ([1]). Again, we have a similar result for weakly hypercyclic operators.

PROPOSITION 2.2. *Let $T : X \to X$ be a bounded linear operator. Then T is a weakly hypercyclic operator if and only if there is an invariant, norm dense, linear subspace in which every nonzero vector is a weakly hypercyclic vector for T.*

Proof. Since $\operatorname{span}\operatorname{Orb}(T, x)$ is convex, it is weakly dense if and only if it is norm dense. Hence, it suffices to show that if T has a weakly hypercyclic vector x, then every nonzero vector in $\operatorname{span}\operatorname{Orb}(T, x)$ is a weakly hypercyclic vector for T.

Recall that X is a locally convex space under the weak topology. Thus, by a result of Bès ([1]), $p(T)$ has a weakly dense range for all nonzero polynomials p. Hence, our result follows from the observation that

$$p(T)X \subseteq \overline{p(T)\operatorname{Orb}(T, x)}^{\mathrm{wk}} = \overline{\operatorname{Orb}(T, p(T)x)}^{\mathrm{wk}}. \quad ∎$$

Several necessary conditions exist for an operator to be weakly hypercyclic. For example, if $\|T\| \leqslant 1$ or $\sup\{\|T^n\| : n \geqslant 1\} < \infty$, then every orbit is norm bounded, and hence can never be norm dense or weakly dense. Hence, if T is weakly hypercyclic, then $\|T\| > 1$ and $\sup\{\|T^n\| : n \geqslant 1\} = \infty$. Another necessary condition for an operator $T : X \to X$ to be weakly hypercyclic is that its adjoint T^* has no eigenvalues. If $T^* x^* = \lambda x^*$ for some nonzero x^* in X^*, then

$$\langle T^n x, x^* \rangle = \langle x, T^{*n} x^* \rangle = \lambda^n \langle x, x^* \rangle,$$

for any x in X. Hence, $\operatorname{Orb}(T, x)$ is not norm dense or weakly dense.

Even though the two types of hypercyclic operators share many of the same properties, not all of the results for hypercyclic operators can be converted to results about weakly hypercyclic operators. The Hypercyclicity Criterion is one such example as we have discussed in Section 1. Another example is the following result:

A bounded linear operator $T : X \to X$ is hypercyclic if and only if for every pair of nonempty, norm open sets U and V, there is a positive integer n such that $T^n(U) \cap V \neq \emptyset$.

A proof of this result can be found in Kitai's dissertation ([11], Theorem 2.1). In fact, the result holds in much greater generality which is known as the Birkhoff Transitivity Theorem; see page 245 in [12]. By replacing the norm topology with the weak topology in the result, one can directly see that the new forward implication still holds true by Proposition 2.2. That is, if $T : X \to X$ is a weakly hypercyclic operator, then for every pair of nonempty, weakly open sets U and V, there is a positive integer n such that $T^n(U) \cap V \neq \emptyset$. Even further, it is easy to see the result still holds if the set U is assumed to be norm open. However, the new backward implication fails to hold. If it did hold, then this would immediately imply that an invertible operator T is weakly hypercyclic if and only if T^{-1} is weakly hypercyclic. This statement is not true. In Corollary 3.6 below, we give an example of an invertible weakly hypercyclic operator whose inverse fails to be weakly hypercyclic.

3. BILATERAL SHIFTS

Let $\{e_\alpha : \alpha \in \mathbb{Z}\}$ be the canonical basis for $\ell^p(\mathbb{Z})$. That is, e_α is the sequence $(\ldots, 0, 0, 1, 0, 0, \ldots)$ where the 1 is in the α-th position. Then, the operator $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ defined by $Te_\alpha = w_\alpha e_{\alpha-1}$ is a *bilateral weighted (backward) shift* where the weight sequence $\{w_\alpha : \alpha \in \mathbb{Z}\}$ is a bounded subset of positive real numbers. Of course, a bilateral weighted backward shift is unitarily equivalent to a bilateral weighted forward shift. We choose the backward direction in our discussion because it is the only possible direction for a weighted shift to hypercyclic on $\ell^p = \ell^p(\mathbb{Z}^+)$, as we study in the next section. Salas ([14], Theorem 2.1) established a necessary and sufficient condition for a bilateral weighted shift on $\ell^p(\mathbb{Z})$ with $1 \leqslant p < \infty$ to be hypercyclic in the norm topology. His condition is stated for a bilateral weighted forward shift. To be consistent, we rephrase his result for a bilateral weighted backward shift:

A bilateral weighted shift T with positive weights $\{w_\alpha : \alpha \in \mathbb{Z}\}$ is hypercyclic if and only if for any given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists an arbitrarily large n such that for all $|\alpha| \leqslant q$,

$$\prod_{t=1}^{n} w_{\alpha+t} > \frac{1}{\varepsilon} \quad \textit{and} \quad \prod_{t=0}^{n-1} w_{\alpha-t} < \varepsilon.$$

The main theorem of this section gives a sufficient condition for a bilateral weighted shift on $\ell^p(\mathbb{Z})$ with $2 \leqslant p < \infty$ to be weakly hypercyclic. Like Salas' condition, this sufficient condition is in terms of the weights of the bilateral shift. Since hypercyclicity implies weak hypercyclicity, our sufficient condition is weaker than Salas' condition. Nevertheless, it is more difficult to state, and so we first need to prove a lemma.

LEMMA 3.1. *For any given real numbers $\lambda > 1$ and $p \geqslant 1$, there exists a bijective map $\nu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following Properties:*

(i) *for each $r \geqslant 1$, the sequence $(\nu(r,s))_{s=1}^{\infty}$ is strictly increasing;*

(ii) *for each $r \geqslant 1$, we have $r \leqslant \nu(r,1)$;*

(iii) *there exists a sequence $(a_r)_{r=1}^{\infty}$ of positive integers such that if $(c_r)_{r=1}^{\infty}$ is a sequence of nonnegative real numbers with $c_r^p \leqslant r\lambda^{pr}$ for each $r \geqslant 1$, then the new sequence $(d_n)_{n=1}^{\infty}$ given by $d_n = d_{\nu(i,j)} = c_i$ satisfies the inequality*

$$\sum_{n=1}^{\nu(r,s)} d_n^p \leqslant (a_r + s)\log(a_r + s), \quad \textit{for each } r, s \geqslant 1.$$

Proof. Select a strictly increasing sequence $(m_i)_{i=1}^{\infty}$ of positive integers that satisfies

$$(1 + 2 + \cdots + i)\lambda^{pi} \leqslant \log m_i. \tag{3.1}$$

Then let $(\alpha_j)_{j=1}^{\infty}$ be the sequence of integers given by

$$(\alpha_1, \alpha_2, \ldots) = (\underbrace{G_1, \ldots, G_1}_{m_2 \text{ copies}}, \underbrace{G_2, \ldots, G_2}_{m_3 \text{ copies}}, \underbrace{G_3, \ldots, G_3}_{m_4 \text{ copies}}, \ldots),$$

where $G_i = (1, 2, \ldots, i)$. Let $\nu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $\nu(r,s) = n$ if $\alpha_n = r$ and $\alpha_j = r$ for precisely s positive integers j which are less than or equal to n. That is, $\nu(r,s)$ is the position of the s-th appearance of the positive integer r in the sequence $(\alpha_j)_{j=1}^{\infty}$. Clearly, ν is a bijective map and for each $r \geqslant 1$, the sequence $(\nu(r,s))_{s=1}^{\infty}$ is strictly increasing in s. Next, observe that

$$\nu(1,1) = 1 \quad \text{and} \quad \nu(r,1) = m_2 + 2m_3 + \cdots + (r-1)m_r + r,$$

and thus Properties (i) and (ii) are satisfied.

To establish Property (iii), we let $a_1 = m_1$ and $a_r = \nu(r,1)$ for each $r \geqslant 2$ and proceed by induction on the positive integer s. Suppose $(c_r)_{r=1}^{\infty}$ and $(d_n)_{n=1}^{\infty}$ are the sequences stated in Property (iii) of our lemma. Actually, $(d_n)_{n=1}^{\infty}$ is given by

$$(d_1, d_2, \ldots) = (\underbrace{F_1, \ldots, F_1}_{m_2 \text{ copies}}, \underbrace{F_2, \ldots, F_2}_{m_3 \text{ copies}}, \underbrace{F_3, \ldots, F_3}_{m_4 \text{ copies}}, \ldots), \tag{3.2}$$

where $F_i = (c_1, c_2, \ldots, c_i)$. Moreover, $\nu(r,s)$ gives the position of the s-th appearance of c_r in the sequence $(d_n)_{n=1}^{\infty}$. For $r = 1$, we have that $\nu(r,1) = \nu(1,1) = 1$, and thus by (3.1)

$$\sum_{n=1}^{\nu(r,1)} d_n^p = c_1^p \leqslant \lambda^p \leqslant \log m_1 \leqslant (a_1 + 1)\log(a_1 + 1).$$

For $r \geqslant 2$, we have that $\nu(r,1) = m_2 + 2m_3 + \cdots + (r-1)m_r + r$, and thus

$$\begin{aligned}
\sum_{n=1}^{\nu(r,1)} d_n^p &= m_2 c_1^p + \cdots + m_r(c_1^p + \cdots + c_{r-1}^p) + (c_1^p + \cdots + c_r^p) \\
&\leqslant (m_2 + m_3 + \cdots + m_r + 1)(c_1^p + \cdots + c_r^p) \\
&\leqslant (\nu(r,1) + 1)((1 + 2 + \cdots + r)\lambda^{pr}) \leqslant (a_r + 1)\log m_r, \qquad \text{by (3.1)} \\
&\leqslant (a_r + 1)\log(a_r + 1).
\end{aligned}$$

Our induction assumption is that for some $s \geqslant 1$, the inequality $\sum_{n=1}^{\nu(r,s)} d_n^p \leqslant (a_r + s)\log(a_r + s)$ holds for each $r \geqslant 1$. We must show that the inequality holds for $s+1$. Note that c_r makes its first appearance in the sequence $(d_n)_{n=1}^{\infty}$ as the last member in the first F_r appearing in the representation (3.2). As a result, we need to separate the induction step into two cases.

Case 1. $s \leqslant m_{r+1}$.

In this case, the s-th appearance of c_r lies in an F_r. Hence,

$$\begin{aligned}
\sum_{n=1}^{\nu(r,s+1)} d_n^p &= \sum_{n=1}^{\nu(r,s)} d_n^p + (c_1^p + c_2^p + \cdots + c_r^p) \\
&\leqslant (a_r + s)\log(a_r + s) + (1 + 2 + \cdots + r)\lambda^{pr} \\
&\leqslant (a_r + s)\log(a_r + s) + \log m_r, \qquad \text{by (3.1)} \\
&\leqslant (a_r + s)\log(a_r + s + 1) + \log a_r \\
&\leqslant (a_r + s + 1)\log(a_r + s + 1).
\end{aligned}$$

Case 2. $s > m_{r+1}$.

In this case, the s-th appearance of c_r lies in an F_j for some $j \geqslant r+1$. Hence, $m_{r+1} + \cdots + m_j < s \leqslant m_{r+1} + \cdots + m_j + m_{j+1}$. This yields

$$\begin{aligned}
\sum_{n=1}^{\nu(r,s+1)} d_n^p &= \sum_{n=1}^{\nu(r,s)} d_n^p + (c_{r+1}^p + \cdots + c_j^p + c_1^p + \cdots + c_r^p) \\
&\leqslant (a_r + s)\log(a_r + s) + (1 + 2 + \cdots + j)\lambda^{pj} \\
&\leqslant (a_r + s)\log(a_r + s) + \log m_j, \qquad \text{by (3.1)} \\
&\leqslant (a_r + s)\log(a_r + s + 1) + \log s \\
&\leqslant (a_r + s + 1)\log(a_r + s + 1). \quad \blacksquare
\end{aligned}$$

We are now ready to state and prove the main theorem of this section which is a sufficient condition for a bilateral weighted shift to be weakly hypercyclic. The bijective function ν in Lemma 3.1 plays an important role in the statement of the theorem and the construction of the weakly hypercyclic vector. As a result of the theorem, we point out a simpler sufficient condition in Corollary 3.5 which does not involve the function ν above and the function σ in the statement of the theorem.

Theorem 3.2. *Let $2 \leqslant p < \infty$ and $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ be a bilateral weighted shift defined by $Te_\alpha = w_\alpha e_{\alpha-1}$. Suppose $\lambda = \|T\| > 1$ and $\nu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijective map satisfying all three Properties in Lemma* 3.1 *with respect to p and λ. Then T is weakly hypercyclic if*

(A) *there exists a map $\sigma : \mathbb{N} \to (0, \infty)$ satisfying the following Conditions:*

(A-1) *$\sigma(\nu(r,s)) \leqslant \sigma(\nu(r,s+1))$ for all $r, s \geqslant 1$,*

(A-2) *$\sum_{s=2}^{\infty} [(s\log s)\sigma^q(\nu(r,s))]^{-1} = \infty$ for each $r \geqslant 1$ where $p^{-1} + q^{-1} = 1$, and*

(B) *there exists a sequence* $(k_n)_{n=0}^{\infty}$ *of positive integers satisfying the following Conditions:*

(B-1) $k_0 = 0$ *and* $k_n > \max\{4k_{n-1}, 4n\}$ *for each* $n \geqslant 1$,

(B-2) *for each* $n \geqslant 1$, *we have* $\prod_{t=1}^{k_n} w_{\alpha+t} > n^{1/p}\lambda^{2n+k_{n-1}}$ *for all* $\alpha = -2n, \ldots, -1, 0$,

(B-3) *for* $m = \nu(r,s)$, *we have* $\prod_{t=0}^{k_m-k_n-r-1} w_{\alpha-t} \leqslant \sigma(m)$ *for all* $\alpha = -2m, \ldots, -1, 0$ *and* $n = 0, 1, \ldots, m-1$.

Proof. We introduce the notation $\widehat{g}(\alpha) = \langle g, e_\alpha \rangle$ for any g in $\ell^p(\mathbb{Z})$. To begin our construction of the weakly hypercyclic vector, select a norm dense set $\{h_r : r \geqslant 1\}$ in $\ell^p(\mathbb{Z})$ such that for each $r \geqslant 1$,

$$\|h_r\|_p^p \leqslant r, \quad \text{and} \quad \widehat{h_r}(\alpha) = 0 \quad \text{whenever } |\alpha| > r. \tag{3.3}$$

Let $f_r = T^r h_r$ for each $r \geqslant 1$. Observe that

$$\|f_r\|_p^p \leqslant r\|T\|^{pr} = r\lambda^{pr}.$$

If we set $n = \nu(i,j)$, set $g_n = g_{\nu(i,j)} = f_i$ for all $i, j \geqslant 1$, and set $c_r = \|f_r\|_p$ for all $r \geqslant 1$, then by Lemma 3.1, we have

$$\sum_{n=1}^{\nu(r,s)} \|g_n\|_p^p \leqslant (a_r + s)\log(a_r + s), \tag{3.4}$$

where $(a_r)_{r=1}^{\infty}$ is the sequence given in Property (iii) of Lemma 3.1. Moreover, since $r \leqslant \nu(r,s)$ and T is a bilateral shift, we have

$$\|g_{\nu(r,s)}\|_p^p = \|f_r\|_p^p \leqslant \nu(r,s)\lambda^{p\nu(r,s)}, \quad \text{and} \tag{3.5}$$

$$\widehat{g_{\nu(r,s)}}(\alpha) \neq 0 \quad \text{only if } -2\nu(r,s) \leqslant \alpha \leqslant 0. \tag{3.6}$$

Next, define $S : \operatorname{span}\{e_\alpha : \alpha \in \mathbb{Z}\} \to \operatorname{span}\{e_\alpha : \alpha \in \mathbb{Z}\}$ by letting $Se_\alpha = w_{\alpha+1}^{-1} e_{\alpha+1}$ and extending linearly. Note that $TS = ST = I$ on $\operatorname{span}\{e_\alpha : \alpha \in \mathbb{Z}\}$. Also, observe that if $(k_n)_{n=0}^{\infty}$ is the sequence given in the statement of the theorem, then by (3.5), (3.6), and Condition (B-2), we get

$$\begin{aligned} \|S^{k_n} g_n\|_p^p &= \Big\| \sum_{\alpha=-2n}^{0} \widehat{g_n}(\alpha) S^{k_n} e_\alpha \Big\|_p^p = \sum_{\alpha=-2n}^{0} \Big(|\widehat{g_n}(\alpha)|^p \cdot \prod_{t=1}^{k_n} \frac{1}{w_{\alpha+t}^p} \Big) \\ &\leqslant \frac{1}{n} \left(\frac{1}{\lambda^{k_{n-1}}\lambda^{2n}} \right)^p \|g_n\|_p^p \leqslant \left(\frac{1}{\lambda^{k_{n-1}}\lambda^{n}} \right)^p. \end{aligned} \tag{3.7}$$

Hence, $\sum_{n=1}^{\infty} \|S^{k_n} g_n\|_p < \infty$, and so $g = \sum_{n=1}^{\infty} S^{k_n} g_n$ is a vector in $\ell^p(\mathbb{Z})$. We claim that g is a weakly hypercyclic vector for T. Before we show this, we need to prove three claims involving the vectors

$$\varphi_m = \sum_{n=1}^{m-1} T^{k_m-r} S^{k_n} g_n \quad \text{and} \quad \psi_m = \sum_{n=m+1}^{\infty} T^{k_m-r} S^{k_n} g_n, \tag{3.8}$$

for all $m = \nu(r,s) > 1$.

CLAIM 1. *For any given $r \geqslant 1$, we have $\|\psi_{\nu(r,s)}\|_p \to 0$ as $s \to \infty$.*

Proof of Claim 1. By (3.7) and the property that $(\nu(r,s))_{s=1}^\infty$ is strictly increasing, we get

$$\begin{aligned}\|\psi_{\nu(r,s)}\|_p &\leqslant \sum_{n=\nu(r,s)+1}^{\infty} \|T\|^{k_{\nu(r,s)}-r}\|S^{k_n}g_n\|_p \leqslant \sum_{n=\nu(r,s)+1}^{\infty} \lambda^{k_{\nu(r,s)}}\frac{1}{\lambda^{k_{n-1}}\lambda^n}\\ &\leqslant \sum_{n=\nu(r,s)+1}^{\infty}\frac{1}{\lambda^n} = \frac{1}{\lambda-1}\cdot\frac{1}{\lambda^{\nu(r,s)}} \to 0, \quad \text{as } s\to\infty.\end{aligned}$$

This completes the proof for Claim 1. ∎

We now proceed to estimate the norm of $\varphi_{\nu(r,s)}$. Unlike Claim 1, the estimate involves a_r and the function σ given in the statement of the theorem.

CLAIM 2. *For any $r, s \geqslant 1$, we have*

$$\|\varphi_{\nu(r,s)}\|_p^p \leqslant (a_r+s)\log(a_r+s)\sigma^p(\nu(r,s)).$$

Proof of Claim 2. First, observe that by (3.6) and the definition of S, we have

$$\widehat{S^{k_n}g_n}(\alpha) \neq 0 \quad \text{only if } k_n - 2n \leqslant \alpha \leqslant k_n. \tag{3.9}$$

Now, if $1 \leqslant n < n'$, then by Condition (B-1), we have

$$k_{n'} - 2n' > 4k_{n'-1} - 2n' = k_{n'-1} + (3k_{n'-1} - 2n') \geqslant k_{n'-1} \geqslant k_n.$$

It follows that if we fix $r, s \geqslant 1$ with $m = \nu(r,s) > 1$, then for any integer α, there exists at most one n such that $(T^{k_m-r}S^{k_n}g_n)\widehat{\ }(\alpha) \neq 0$. As a result,

$$\|\varphi_m\|_p^p = \sum_{n=1}^{m-1}\|T^{k_m-r}S^{k_n}g_n\|_p^p. \tag{3.10}$$

By Condition (B-1) and the fact $r \leqslant \nu(r,s) = m$, we have

$$k_m - r - k_n > 4k_{m-1} - r - k_n > (k_{m-1} - r) + (k_{m-1} - k_n) \geqslant 0,$$

for all $n = 1, \ldots, m-1$. Therefore, $T^{k_m-r}S^{k_n}g_n = T^{k_m-r-k_n}g_n$. Hence, by (3.6) and Condition (B-3), we get

$$\begin{aligned}\|T^{k_m-r}S^{k_n}g_n\|_p^p &= \Big\|\sum_{\alpha=-2n}^{0}\widehat{g_n}(\alpha)T^{k_m-r-k_n}e_\alpha\Big\|_p^p\\ &\leqslant \sum_{\alpha=-2n}^{0}\Big(|\widehat{g_n}(\alpha)|^p\cdot\prod_{t=0}^{k_m-r-k_n-1}w_{\alpha-t}^p\Big) \leqslant \sigma^p(m)\|g_n\|_p^p.\end{aligned} \tag{3.11}$$

To conclude the proof of Claim 2, we observe that by (3.4), (3.10) and (3.11),

$$\|\varphi_m\|_p^p \leqslant \sigma^p(m)\sum_{n=1}^{m-1}\|g_n\|_p^p \leqslant \sigma^p(m)\sum_{n=1}^{m}\|g_n\|_p^p \leqslant (a_r+s)\log(a_r+s)\sigma^p(m). \qquad ∎$$

We now proceed to prove one more claim before we return to the main argument of the proof.

CLAIM 3. *Let $r \geqslant 1$, $\varepsilon > 0$ and $x_1, \dots, x_t \in \ell^q(\mathbb{Z})$. Then for any $S \geqslant 1$, there exists $s > S$ such that*

$$|\langle \varphi_{\nu(r,s)}, x_j \rangle| < \varepsilon \quad \textit{whenever } 1 \leqslant j \leqslant t.$$

That is, the zero vector is a weak limit point of the set $\{\varphi_{\nu(r,s)} : s \geqslant 1\}$.

Proof of Claim 3. By way of contradiction, suppose there is an $S \geqslant 1$ such that for each $s > S$,

$$|\langle \varphi_{\nu(r,s)}, x_{j_s} \rangle| \geqslant \varepsilon \quad \text{for some } j_s \text{ with } 1 \leqslant j_s \leqslant t.$$

First, observe that by the definition of $\varphi_{\nu(r,s)}$ in (3.8) and (3.9),

$$\widehat{\varphi_{\nu(r,s)}}(\alpha) \neq 0 \quad \text{only if } \; -k_{\nu(r,s)} + r + k_1 - 2 \leqslant \alpha \leqslant -k_{\nu(r,s)} + r + k_{\nu(r,s)-1}.$$

If s and s' are any two positive integers with $1 \leqslant s < s'$, then by Condition (B-1), we have

$$k_{\nu(r,s')} > 4k_{\nu(r,s')-1} \geqslant k_{\nu(r,s')-1} + 2 + k_{\nu(r,s)}.$$

Hence,

$$-k_{\nu(r,s')} + r + k_{\nu(r,s')-1} < -k_{\nu(r,s)} + r - 2 < -k_{\nu(r,s)} + r + k_1 - 2.$$

This implies that for any given integer α, there is at most one positive integer s with $\widehat{\varphi_{\nu(r,s)}}(\alpha) \neq 0$.

Next, let $A_s = \operatorname{span}\{e_\alpha : \widehat{\varphi_{\nu(r,s)}}(\alpha) \neq 0\}$ for each $s \geqslant 1$. Note that $A_s \cap A_{s'} = \{0\}$ if $s \neq s'$. Also, observe that each A_s is a finite dimensional space which can be viewed as a subspace of $\ell^p(\mathbb{Z})$ as well as its dual $\ell^q(\mathbb{Z})$ where $p^{-1} + q^{-1} = 1$. Thus, if we let $P_{\nu(r,s)} : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ be a linear coordinate projection onto A_s given by

$$P_{\nu(r,s)} e_\alpha = \begin{cases} e_\alpha & \text{if } \alpha \in A_s, \\ 0 & \text{if } \alpha \notin A_s, \end{cases}$$

then its adjoint $P^*_{\nu(r,s)} : \ell^q(\mathbb{Z}) \to \ell^q(\mathbb{Z})$ is a linear coordinate projection onto A_s. Hence, for each $s > S$, we get

$$\begin{aligned} \varepsilon &\leqslant |\langle \varphi_{\nu(r,s)}, x_{j_s} \rangle| = |\langle P_{\nu(r,s)} \varphi_{\nu(r,s)}, x_{j_s} \rangle| \\ &= |\langle \varphi_{\nu(r,s)}, P^*_{\nu(r,s)} x_{j_s} \rangle| \leqslant \|\varphi_{\nu(r,s)}\|_p \|P^*_{\nu(r,s)} x_{j_s}\|_q. \end{aligned}$$

Therefore, by Claim 2, Condition (A-1), and the assumption that $2 \leqslant p < \infty$, we get

$$\begin{aligned} \|P^*_{\nu(r,s)} x_{j_s}\|_q^q &\geqslant \varepsilon^q (\|\varphi_{\nu(r,s)}\|_p^p)^{-q/p} \geqslant \varepsilon^q [(a_r + s)\log(a_r + s)\sigma^p(\nu(r,s))]^{-q/p} \\ &\geqslant \varepsilon^q \sigma^{-q}(\nu(r,s))[(a_r + s)\log(a_r + s)]^{-1} \\ &\geqslant \varepsilon^q \sigma^{-q}(\nu(r, a_r + s))[(a_r + s)\log(a_r + s)]^{-1}. \end{aligned}$$

However, since $A_s \cap A_{s'} = \{0\}$ if $s \neq s'$, the last inequality implies that

$$\begin{aligned} \sum_{j=1}^t \|x_j\|_q^q &\geqslant \sum_{j=1}^t \sum_{s=S+1}^\infty \|P^*_{\nu(r,s)} x_j\|_q^q = \sum_{s=S+1}^\infty \sum_{j=1}^t \|P^*_{\nu(r,s)} x_j\|_q^q \geqslant \sum_{s=S+1}^\infty \|P^*_{\nu(r,s)} x_{j_s}\|_q^q \\ &\geqslant \varepsilon^q \sum_{s=S+1}^\infty [(a_r + s)\log(a_r + s)\sigma^q(\nu(r, a_r + s))]^{-1} = \infty, \quad \text{by Condition (A-2).} \end{aligned}$$

Hence, we have a contradiction and this ends the proof for Claim 3. ∎

Now, we are ready to show g is a weakly hypercyclic vector for T. Recall the set $\{h_r : r \geqslant 1\}$ is a norm dense set, and so it suffices to show $\{h_r : r \geqslant 1\} \subseteq \overline{\mathrm{Orb}(T,g)}^{\mathrm{wk}}$. For that we let $r \geqslant 1$, $\varepsilon > 0$ and $x_1, \ldots, x_t$ be t nonzero vectors in $\ell^q(\mathbb{Z})$. Let $\gamma = \max\{\|x_j\|_q : 1 \leqslant j \leqslant t\}$. By Claim 1, there exists $S \geqslant 1$ such that

$$\|\psi_{\nu(r,s)}\|_p < \frac{\varepsilon}{2\gamma} \quad \text{for all } s \geqslant S. \tag{3.12}$$

Then, by Claim 3, there exists $s_0 > S$ such that

$$|\langle \varphi_{\nu(r,s_0)}, x_j\rangle| < \frac{\varepsilon}{2} \quad \text{whenever } 1 \leqslant j \leqslant t. \tag{3.13}$$

If we let $N = k_{\nu(r,s_0)} - r$, then by (3.8)

$$\begin{aligned} T^N g &= \varphi_{\nu(r,s_0)} + T^N S^{k_{\nu(r,s_0)}} g_{\nu(r,s_0)} + \psi_{\nu(r,s_0)} \\ &= \varphi_{\nu(r,s_0)} + T^N S^{k_{\nu(r,s_0)}} T^r h_r + \psi_{\nu(r,s_0)} = \varphi_{\nu(r,s_0)} + h_r + \psi_{\nu(r,s_0)}. \end{aligned}$$

Hence, for all $j = 1, \ldots, t$, we have

$$\begin{aligned} |\langle T^N g - h_r, x_j\rangle| &\leqslant |\langle \varphi_{\nu(r,s_0)}, x_j\rangle| + \|\psi_{\nu(r,s_0)}\|_p \|x_j\|_q \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\gamma}\gamma, \qquad \text{by (3.12) and (3.13)} \\ &= \varepsilon, \end{aligned}$$

which completes the whole proof. ∎

A bilateral weighted shift that satisfies the conditions of Theorem 3.2 may in fact be a hypercyclic operator. For example, consider the bilateral weighted shift on $\ell^p(\mathbb{Z})$ with $2 \leqslant p < \infty$ given by $Te_\alpha = 3e_{\alpha-1}$ if $\alpha \geqslant 1$ and $Te_\alpha = 3^{-1}e_{\alpha-1}$ if $\alpha \leqslant 0$. With $\sigma \equiv 1$, the operator satisfies the conditions of Theorem 3.2, but it is also a hypercyclic operator as one may easily check using Salas' condition stated before Lemma 3.1. Nevertheless, this is not always the situation.

Corollary 3.3. *For $2 \leqslant p < \infty$, there exists a bilateral weighted shift T on $\ell^p(\mathbb{Z})$ that is weakly hypercyclic and satisfies the inequality $\|Tf\|_p \geqslant \|f\|_p$ for all $f \in \ell^p(\mathbb{Z})$. Hence, T is not hypercyclic.*

Proof. Let $\nu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijective function given by Lemma 3.1 with respect to p and $\lambda = 2$. Define $\sigma : \mathbb{N} \to (0,\infty)$ by $\sigma(n) = 1$. By the integral test, $\sum_{s=2}^{\infty} [(s \log s)\sigma^q(\nu(r,s))]^{-1} = \infty$ for each $r \geqslant 1$. Hence, σ satisfies Conditions (A-1) and (A-2) in Theorem 3.2. Now, consider the bilateral weighted shift $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ given by

$$Te_\alpha = \begin{cases} e_{\alpha-1} & \text{if } \alpha \leqslant 0, \\ 2e_{\alpha-1} & \text{if } \alpha \geqslant 1. \end{cases}$$

That is, $w_\alpha = 1$ for all $\alpha \leqslant 0$ and $w_\alpha = 2$ for all $\alpha \geqslant 1$. Note that $\lambda = 2 = \|T\|$. Next, observe that if we fix $n \geqslant 1$, then for any $m \geqslant 2n$, we get

$$\prod_{t=1}^{m} w_{\alpha+t} = 2^{m+\alpha} \quad \text{for all } \alpha = -2n, \ldots, -1, 0, \tag{3.14}$$

and

$$\prod_{t=0}^{m-1} w_{\alpha-t} = 1 \qquad \text{for all } \alpha = -2n, \dots, -1, 0. \tag{3.15}$$

To create a sequence of positive integers $(k_n)_{n=0}^{\infty}$ which satisfies Conditions (B-1)-(B-3) in Theorem 3.2, first set $k_0 = 0$. Then, by (3.14), find a strictly increasing sequence of positive integers $(k'_n)_{n=1}^{\infty}$ that satisfies Condition (B-2). Lastly, by (3.15), inductively define $(k_n)_{n=1}^{\infty}$ by choosing $k_n > \max\{4k_{n-1}, 4n, k'_n\}$ that satisfies Condition (B-3). In fact, one can simply take $k_n = 5^n$. Thus, T is a weakly hypercyclic operator. Also, by the definition of T, we have

$$\|f\|_p \leqslant \|Tf\|_p \quad \text{for every } f \text{ in } \ell^p(\mathbb{Z}). \tag{3.16}$$

Therefore, T cannot be hypercyclic. ∎

It is interesting to note that the weakly hypercyclic bilateral weighted shift T in Corollary 3.3 satisfies (3.16), and hence every nonzero vector has a norm increasing orbit. Since weakly convergent sequences are necessarily bounded by the Principle of Uniform Boundedness, this tells us that no sequence in any weakly dense orbit $\mathrm{Orb}(T, x)$ converges weakly to any vector outside the orbit. As a result, one must use nets to describe the convergence property. Moreover, a review of the construction of the weakly hypercyclic vector g as presented in Theorem 3.2 yields $\|T^{n+1}g\|_p > \|T^n g\|_p$, and so $(T^n g)_{n=1}^{\infty}$ is a strictly norm increasing, weakly dense sequence. This leads to the following result.

Corollary 3.4. *For $2 \leqslant p < \infty$, there exists a strictly norm increasing, weakly dense sequence in $\ell^p(\mathbb{Z})$.*

A norm increasing, weakly dense sequence can be constructed without using the weakly hypercyclic operator provided in Corollary 3.3. For instance, when $p = 2$, the space $\ell^p(\mathbb{Z}) = H$ is a Hilbert space, and we may construct the sequence in H in the following manner.

Let $\{e_\alpha : \alpha \geqslant 1\}$ be an orthonormal basis of H. Choose a set $\{h_s : s \geqslant 1\}$ that is norm dense in $\{h \in H : \|h\| \geqslant 1\}$ with the properties

$$1 \leqslant \|h_s\|^2 \leqslant s, \quad \text{and} \quad \widehat{h_s}(\alpha) = 0 \quad \text{whenever } \alpha > s. \tag{3.17}$$

Consider the set $\{f_{r,s} : r \geqslant 1 \text{ and } 1 \leqslant s \leqslant r\}$ where the vector $f_{r,s} = h_s + \sqrt{r} e_{r+s}$. Note that by (3.17), we have

$$\|f_{r,s}\|^2 = \|h_s\|^2 + \|\sqrt{r} e_{r+s}\|^2 = \|h_s\|^2 + r,$$

and so $r + 1 \leqslant \|f_{r,s}\|^2 \leqslant 2r$ for any $r \geqslant 1$ and $1 \leqslant s \leqslant r$. This implies that for any positive integer n, there is only a finite number of vectors in the set $\{f_{r,s} : r \geqslant 1 \text{ and } 1 \leqslant s \leqslant r\}$ whose norm is less than n. Thus, this set can be ordered into a sequence $(g_n)_{n=1}^{\infty}$ with $\|g_n\| \leqslant \|g_{n+1}\|$.

To show the sequence $(g_n)_{n=1}^{\infty}$ is weakly dense, it suffices to show $\{h_s : s \geqslant 1\} \subseteq \overline{\{f_{r,s} : r \geqslant 1 \text{ and } 1 \leqslant s \leqslant r\}}^{\mathrm{wk}}$. Observe that for any fixed $s \geqslant 1$, we have $0 \in \overline{\{\sqrt{r} e_{r+s} : r \geqslant s\}}^{\mathrm{wk}}$, and thus

$$h_s \in \overline{\{h_s + \sqrt{r} e_{r+s} : r \geqslant s\}}^{\mathrm{wk}} \subseteq \overline{\{f_{r,s} : r \geqslant 1 \quad \text{and} \quad 1 \leqslant s \leqslant r\}}^{\mathrm{wk}}.$$

If we further require the weakly dense sequence $(g_n)_{n=1}^{\infty}$ in H to be strictly norm increasing, choose a strictly increasing sequence of positive integers $(k_n)_{n=1}^{\infty}$ such that

$$\widehat{g_i}(k_n) = 0 \quad \text{for all } i = 1, \ldots, n.$$

Consider the new sequence $(g_n + \beta_n e_{k_n})_{n=1}^{\infty}$ where $\beta_n = n/(n+1)$. This new sequence is still weakly dense and satisfies $\|g_n\| < \|g_{n+1}\|$.

Now turning our attention to the proof of Corollary 3.3, we can easily see a sufficient condition for a bilateral weighted shift to satisfy Theorem 3.2.

COROLLARY 3.5. *For $2 \leqslant p < \infty$, a bilateral weighted shift $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is weakly hypercyclic if its weight sequence $\{w_\alpha : \alpha \in \mathbb{Z}\}$ satisfies the following conditions:*

(i) $\inf\{w_\alpha : \alpha \geqslant 1\} > 1$;

(ii) $\sup\left\{\prod_{\alpha=m}^{n} w_\alpha : m \leqslant n \leqslant 0\right\} < \infty$.

The bilateral weighted shift in the corollary above is weakly hypercyclic but not norm hypercyclic if we further impose the condition that $\inf\left\{\prod_{\alpha=m}^{n} w_\alpha : m \leqslant n \leqslant 0\right\} > 0$. The bilateral weighted shift in Corollary 3.3 satisfies the conditions in Corollary 3.5. However, this condition is not necessary. For example, we may let $\sigma(\nu(r,s)) = (\log\log(r+s+1))^{1/q}$ and carefully choose the weights w_α so that they satisfy the conditions in Theorem 3.2. Then we can create a weakly hypercyclic bilateral weighted shift which fails to be norm hypercyclic and the finite products of consecutive negative indexed weights are unbounded.

Though we know that a purely weakly hypercyclic operator exists, it does not necessarily carry all the properties of a hypercyclic operator. For instance, a result of Kitai ([11], Corollary 2.2) showed that an invertible operator $T : X \to X$ is hypercyclic if and only if T^{-1} is hypercyclic. Theorem 3.2 allows us to provide a counterexample to this result in the weak topology.

COROLLARY 3.6. *For $2 \leqslant p < \infty$, there exists an invertible bilateral weighted shift on $\ell^p(\mathbb{Z})$ that is weakly hypercyclic but its inverse is not weakly hypercyclic.*

Proof. Consider the same bilateral weighted shift T given in Corollary 3.3. Then, T^{-1} is given by

$$T^{-1}e_\alpha = \begin{cases} e_{\alpha+1} & \text{if } \alpha \leqslant -1, \\ 2^{-1}e_{\alpha+1} & \text{if } \alpha \geqslant 0. \end{cases}$$

Note that $\|T^{-1}\| = 1$, and so T^{-1} is not weakly hypercyclic. ∎

All of the results in this section are operator theoretic in nature, but the ideas can easily be translated to function theory on an annulus. For the details, one may refer to the survey article of Shields ([16]). For instance, the result in Corollary 3.3 can be translated into a result on the Hardy space $H^2(A)$ of the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$ consisting of all the functions $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ analytic on A with

$$\|f\|_A^2 = \frac{1}{2\pi} \lim_{r \searrow 1} \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^2 \,\mathrm{d}\theta + \frac{1}{2\pi} \lim_{t \nearrow 2} \int_0^{2\pi} |f(t\mathrm{e}^{\mathrm{i}\theta})|^2 \,\mathrm{d}\theta$$
$$= \sum_{-\infty}^{\infty} (1 + 2^{2n})|a_n|^2 < \infty.$$

If we define a new norm on $H^2(A)$ by

$$\|f\|^2 = \sum_{-\infty}^{0} |a_n|^2 + \sum_{1}^{\infty} 2^{2n}|a_n|^2,$$

then one can easily check that $\|f\| \leqslant \|f\|_A \leqslant \sqrt{2}\,\|f\|$. Hence, $\|\cdot\|$ and $\|\cdot\|_A$ are equivalent norms. If we define

$$e_\alpha = \frac{z^\alpha}{\|z^\alpha\|} = \begin{cases} z^\alpha & \text{if } \alpha \leqslant 0, \\ 2^{-\alpha} z^\alpha & \text{if } \alpha \geqslant 1, \end{cases}$$

then $\{e_\alpha : \alpha \in \mathbb{Z}\}$ is an orthonormal basis of $H^2(A)$ under $\|\cdot\|$. Consider the operator $M_z : H^2(A) \to H^2(A)$ defined by $(M_z f)(z) = z f(z)$ for all f in $H^2(A)$. One can easily check that

$$M_z e_\alpha = \frac{z^{\alpha+1}}{\|z^\alpha\|} = \frac{\|z^{\alpha+1}\|}{\|z^\alpha\|} \frac{z^{\alpha+1}}{\|z^{\alpha+1}\|} = \begin{cases} e_{\alpha+1} & \text{if } \alpha \leqslant -1, \\ 2e_{\alpha+1} & \text{if } \alpha \geqslant 0, \end{cases}$$

and the adjoint M_z^* of M_z satisfies

$$M_z^* e_\alpha = \begin{cases} e_{\alpha-1} & \text{if } \alpha \leqslant 0, \\ 2e_{\alpha-1} & \text{if } \alpha \geqslant 1. \end{cases}$$

It follows that M_z^* is unitarily equivalent to the operator T in the proof of Corollary 3.3. Hence, the operator M_z^* is weakly hypercyclic but not norm hypercyclic.

4. UNILATERAL BACKWARD SHIFTS

Let $\{e_\alpha : \alpha \geqslant 0\}$ be the canonical basis for $\ell^p = \ell^p(\mathbb{Z}^+)$. That is, e_α is the sequence $(0, \ldots, 0, 1, 0, \ldots)$ where the 1 is in the α-th position. Then the operator $T : \ell^p \to \ell^p$ defined by $Te_\alpha = w_\alpha e_{\alpha-1}$ for $\alpha \geqslant 1$ and $Te_0 = 0$ is a *unilateral weighted backward shift* where the weight sequence $\{w_\alpha : \alpha \geqslant 1\}$ is a bounded subset of positive real numbers. These shifts provided the first examples of hypercyclic operators on a Banach space, as shown by Rolewicz ([13], Theorem 1) in 1969. In fact, he showed that if the all the weights are a constant value strictly larger than 1, then T is hypercyclic. Then, in 1995, Salas ([14], Theorem 2.8) gave a necessary and sufficient condition on the weights of a finite direct sum of unilateral weighted backward shifts to be hypercyclic. In particular, for a single shift T, he showed that *T is hypercyclic if and only if* $\sup\{w_1 w_2 \cdots w_n : n \geqslant 1\} = \infty$. This condition, as we see in the next theorem, is equivalent to weak hypercyclicity as well.

THEOREM 4.1. *Let $1 \leqslant p < \infty$, and let $T : \ell^p \to \ell^p$ be a weighted unilateral backward shift. Then T is hypercyclic if and only if T is weakly hypercyclic.*

Proof. Clearly hypercyclicity implies weak hypercyclicity. For the converse, we suppose g is a weakly hypercyclic vector for T and use the coefficients $\widehat{g}(\alpha)$ of g to construct a hypercyclic vector for T. Let $\{h_i : i \geqslant 1\}$ be a norm dense set in ℓ^p for which there is a strictly increasing sequence of positive integers $(m_i)_{i=1}^{\infty}$ such that

$$\widehat{h}_i(\alpha) = 0 \quad \text{whenever } \alpha > m_i.$$

For each $i, k \geqslant 1$, let $U(i,k)$ denote the basic weakly open set

$$U(i,k) = \Big\{h \in \ell^p : |\langle h - h_i, e_\alpha\rangle| < \frac{1}{k} \text{ for all } \alpha = 0, 1, \ldots, m_i\Big\}.$$

CLAIM. *For any given $\varepsilon > 0$ and integers $i, k, N \geqslant 1$, there exists $n \geqslant N$ such that if $\varphi = \sum_{\alpha=n}^{n+m_i} \widehat{g}(\alpha)e_\alpha$, then $T^n\varphi \in U(i,k)$ and $\|\varphi\|_p < \varepsilon$.*

Proof of Claim. Since $\|g\|_p^p = \sum_{\alpha=0}^{\infty} |\widehat{g}(\alpha)|^p < \infty$, there exists $M \geqslant N$ such that $\sum_{\alpha=M}^{\infty} |\widehat{g}(\alpha)|^p < \varepsilon^p$. Next, choose $n \geqslant M$ such that $T^n g \in U(i,k)$. By letting $\varphi = \sum_{\alpha=n}^{n+m_i} \widehat{g}(\alpha)e_\alpha$, we get $\widehat{T^n g}(\alpha) = \widehat{T^n\varphi}(\alpha)$ for $\alpha = 0, 1, \ldots, m_i$, and hence $T^n\varphi \in U(i,k)$. Moreover, since $n \geqslant M$, we get $\|\varphi\|_p < \varepsilon$. This finishes the proof of the claim. ∎

Let $\{(i_r, k_r) : r \geqslant 1\}$ be an enumeration of $\mathbb{N} \times \mathbb{N}$. By the Claim, we can select $n_1 \geqslant 1$ and $\varphi_1 \in \operatorname{span}\{e_{n_1}, \ldots, e_{n_1+m_{i_1}}\}$ such that

$$T^{n_1}\varphi_1 \in U(i_1, k_1) \quad \text{and} \quad \|\varphi_1\|_p < \frac{1}{2}.$$

Then, for $r \geqslant 2$, inductively let $n_r > \sum_{j=1}^{r-1}(m_{i_j} + n_j)$ and $\varphi_r \in \operatorname{span}\{e_{n_r}, \ldots, e_{n_r+m_{i_r}}\}$ such that

$$T^{n_r}\varphi_r \in U(i_r, k_r) \quad \text{and} \quad \|\varphi_r\|_p < \frac{1}{2^r\|T\|^{n_{r-1}}}. \tag{4.1}$$

Since T is weakly hypercyclic, $\|T\| > 1$, and so the sum $\psi = \sum_{j=1}^{\infty} \varphi_j$ is absolutely convergent. We want to show that ψ is a hypercyclic vector for T. It suffices to show $\{h_i : i \geqslant 1\} \subseteq \overline{\mathrm{Orb}(T,\psi)}^{\mathrm{norm}}$. Note that for any $r \geqslant 1$, we have

$$\begin{aligned}
\|T^{n_r}\psi - h_{i_r}\|_p &= \Big\| T^{n_r}\varphi_r - h_{i_r} + \sum_{j=r+1}^{\infty} T^{n_r}\varphi_j \Big\|_p \\
&\leqslant \|T^{n_r}\varphi_r - h_{i_r}\|_p + \sum_{j=r+1}^{\infty} \|T\|^{n_r}\|\varphi_j\|_p \\
&\leqslant \Big(\sum_{\alpha=0}^{m_{i_r}} |\widehat{T^{n_r}\varphi_r}(\alpha) - \widehat{h_{i_r}}(\alpha)|^p \Big)^{\frac{1}{p}} + \sum_{j=r+1}^{\infty} \frac{\|T\|^{n_r}}{2^j \|T\|^{n_{j-1}}}, \quad \text{by (4.1)} \\
&< \frac{1}{k_r}(m_{i_r}+1)^{1/p} + \frac{1}{2^r}, \qquad \text{by (4.1).}
\end{aligned}$$

For any $i \geqslant 1$ and $\varepsilon > 0$, choose a sufficiently large r so that

$$i_r = i, \quad \frac{1}{2^r} < \frac{\varepsilon}{2}, \quad \text{and} \quad \frac{1}{k_r} < \frac{\varepsilon}{2(m_i+1)^{1/p}}.$$

Thus,

$$\|T^{n_r}\psi - h_i\|_p \leqslant \frac{1}{k_r}(m_i+1)^{1/p} + \frac{1}{2^r} < \varepsilon. \quad \blacksquare$$

At first glance, the proof seems to suggest that a unilateral weighted backward shift is hypercyclic if and only if it has an unbounded orbit, but this is not the case. To construct a counterexample, we let $n_0 = 1$, $w_1 = 1/4$ and $w_2 = w_3 = \cdots = w_{11} = 4^{1/10}$. Inductively, for all integers $k \geqslant 1$, we let $n_k = (5k+1)(k+1)$ and set $w_{n_k} = 4^{-k-1}$ and set $w_{n_k+1} = w_{n_k+2} = \cdots = w_{n_k+10(k+1)} = 4^{1/10}$. Since $n_k + 10(k+1) + 1 = n_{k+1}$ by definition, the weights w_α are defined for all integers $\alpha \geqslant 1$. Note that if $k \geqslant 0$ and $0 \leqslant i \leqslant 10(k+1)$, then

$$w_{n_k} w_{n_k+1} \cdots w_{n_k+i} \leqslant 1.$$

It follows that $w_1 w_2 \cdots w_n \leqslant 1$ for all $n \geqslant 1$. Hence, if $T : \ell^2 \to \ell^2$ is the unilateral weighted backward shift with weight sequence $\{w_\alpha : \alpha \geqslant 1\}$, then T is not norm hypercyclic, by the result of Salas which is stated before Theorem 4.1. To show that T has an unbounded orbit, consider the vector $f = \sum_{k=0}^{\infty} 2^{-k} e_{n_k-1}$ in ℓ^2. For any integer $m \geqslant 1$, by orthogonality, we have

$$\|T^{10m} f\|_2 \geqslant \|T^{10m} 2^{-m} e_{n_m-1}\|_2 = \|2^{-m} w_{n_m-10m} \cdots w_{n_m-1} e_{n_m-10m-1}\|_2.$$

Since $n_{m-1} + 10m + 1 = n_m$, we see that

$$\|T^{10m} f\|_2 \geqslant 2^{-m} w_{n_{m-1}+1} w_{n_{m-1}+2} \cdots w_{n_{m-1}+10m} = 2^{-m} 4^m = 2^m,$$

and hence T has an unbounded orbit.

Though Theorem 4.1 tells us that for a unilateral weighted backward shift, hypercyclicity and weak hypercyclicity are equivalent, it does not imply every weakly hypercyclic vector is a hypercyclic vector. Under a simple condition on the weights of a unilateral weighted backward shift on ℓ^p with $1 < p < \infty$, there exists a weakly hypercyclic vector that fails to be a hypercyclic vector.

THEOREM 4.2. *Let $1 < p < \infty$. If $T : \ell^p \to \ell^p$ is a unilateral weighted backward shift whose positive weights $\{w_\alpha : \alpha \geqslant 1\}$ satisfy the condition* $\inf\{w_\alpha : \alpha \geqslant 1\} > 1$*, then there exists a weakly hypercyclic vector for T that is not a hypercyclic vector for T.*

Proof. Select a norm dense set $\{h_i : i \geqslant 1\}$ in ℓ^p for which there is a strictly increasing sequence of positive integers $(m_i)_{i=1}^{\infty}$ such that for each $i \geqslant 1$,

$$\widehat{h_i}(\alpha) = 0 \quad \text{whenever } \alpha > m_i.$$

Let $\beta = \inf\{w_\alpha : \alpha \geqslant 1\} > 1$ and $\lambda = \sup\{w_\alpha : \alpha \geqslant 1\} = \|T\|$. Set $n_0 = m_0 = 0$ and inductively choose a sequence of positive integers $(n_i)_{i=1}^{\infty}$ that satisfies

$$n_i > m_i + \sum_{j=0}^{i-1}(m_j + n_j) \quad \text{and} \quad \beta^{n_i} > 2^i \lambda^{n_{i-1}} (\|h_i\|_p^p + 1)^{1/p}$$

for each $i \geqslant 1$. Let $k_i = n_{i+1} - n_i - 1$. Define a linear map $S : \operatorname{span}\{e_\alpha : \alpha \geqslant 0\} \to \operatorname{span}\{e_\alpha : \alpha \geqslant 0\}$ by taking $Se_\alpha = w_{\alpha+1}^{-1} e_{\alpha+1}$.

CLAIM. *For each $i \geqslant 1$, we have $\|S^{n_i}(h_i + e_{k_i})\|_p < (2^i \lambda^{n_{i-1}})^{-1}$.*

Proof of Claim. Observe that for any h in $\operatorname{span}\{e_\alpha : \alpha \geqslant 0\}$, we have $\|Sh\|_p \leqslant \beta^{-1}\|h\|_p$. Thus,

$$\|S^{n_i}(h_i + e_{k_i})\|_p^p \leqslant \Big(\frac{1}{\beta^{n_i}}\Big)^p \|h_i + e_{k_i}\|_p^p.$$

Since $k_i > m_i$, we have $\widehat{h_i}(k_i) = 0$, and so

$$\|h_i + e_{k_i}\|_p^p = \|h_i\|_p^p + 1.$$

Our claim then follows from the choice of n_i. ∎

By the claim, the sum $h = \sum_{i=1}^{\infty} S^{n_i}(h_i + e_{k_i})$ is absolutely convergent. We want to show that h is a weakly hypercyclic vector for T. In fact, we prove $\operatorname{Orb}(T, h)$ is weakly sequentially dense. First observe that by the definition of T and $(k_i)_{i=1}^{\infty}$, we have that for any $j \geqslant 2$,

$$T^{n_j} S^{n_i}(h_i + e_{k_i}) = T^{n_j - n_i}(h_i + e_{k_i}) = 0, \quad \text{for all } i = 1, \ldots, j-1.$$

Thus, for any fixed g in ℓ^p, x in ℓ^q with $p^{-1} + q^{-1} = 1$ and $j \geqslant 2$, we get

$$\begin{aligned}
|\langle T^{n_j} h - g, x\rangle| &= \Big|\Big\langle h_j + e_{k_j} - g + \sum_{i=j+1}^{\infty} T^{n_j} S^{n_i}(h_i + e_{k_i}), x\Big\rangle\Big| \\
&\leqslant |\langle h_j - g, x\rangle| + |\langle e_{k_j}, x\rangle| + \sum_{i=j+1}^{\infty} |\langle T^{n_j} S^{n_i}(h_i + e_{k_i}), x\rangle| \\
&< \|h_j - g\|_p \|x\|_q + |\langle e_{k_j}, x\rangle| + \sum_{i=j+1}^{\infty} \lambda^{n_j} \|S^{n_i}(h_i + e_{k_i})\|_p \|x\|_q \\
&< \|h_j - g\|_p \|x\|_q + |\langle e_{k_j}, x\rangle| + \|x\|_q \sum_{i=j+1}^{\infty} \frac{1}{2^i}, \qquad \text{by the Claim} \\
&= \|h_j - g\|_p \|x\|_q + |\langle e_{k_j}, x\rangle| + \frac{\|x\|_q}{2^j}.
\end{aligned}$$

Since the above estimation holds for the entire sequence $(n_j)_{j=1}^{\infty}$, we can now choose a strictly increasing sequence $(j_m)_{m=1}^{\infty}$ such that

$$\|h_{j_m} - g\|_p \to 0 \quad \text{as } m \to \infty.$$

Hence, $T^{n_{j_m}} h \to g$ weakly as $m \to \infty$. Therefore, $\mathrm{Orb}(T, h)$ is sequentially dense.

To finish the proof, we now prove that h is not a hypercyclic vector for T by showing that $\|T^n h\|_p \geqslant 1$ for each $n \geqslant n_1$. Fix $n \geqslant n_1$. Then there exists $j \geqslant 1$ such that $n_j \leqslant n < n_{j+1}$ and so, $n = n_j + m$ for some m with $0 \leqslant m \leqslant n_{j+1} - n_j - 1 = k_j$. This gives us that

$$\|T^m e_{k_j}\|_p^p \geqslant 1, \tag{4.2}$$

because $\beta > 1$.

Now, observe that if $f_i = h_i + e_{k_i}$, then $\widehat{S^{n_i} f_i}(\alpha) \neq 0$ only if $n_i \leqslant \alpha \leqslant n_{i+1} - 1$. This implies that for any given integer $\alpha \geqslant 0$, there exists at most one i with $(T^n S^{n_i} f_i)\widehat{\ }(\alpha) \neq 0$. Hence,

$$\|T^n h\|_p^p = \sum_{i=1}^{\infty} \|T^n S^{n_i} f_i\|_p^p \geqslant \|T^n S^{n_j} f_j\|_p^p = \|T^m (h_j + e_{k_j})\|_p^p.$$

Also, since $\widehat{h_j}(k_j) = 0$, we get from (4.2)

$$\|T^m (h_j + e_{k_j})\|_p^p = \|T^m h_j\|_p^p + \|T^m e_{k_j}\|_p^p \geqslant 1,$$

which concludes the theorem. ∎

Since the operator T in Theorem 4.2 is weakly hypercyclic, there exists a norm dense set of weakly hypercyclic vectors for T by Proposition 2.2. However, some of the vectors in this set may be norm hypercyclic vectors as well. On the other hand, with the weakly hypercyclic vector h given in the proof of Theorem 4.2, we can easily see that any vector of the form $h + \sum_{\alpha=0}^{m} a_\alpha e_\alpha$ is a weakly hypercyclic vector that fails to be a norm hypercyclic vector. Since the set of these types of vectors is norm dense, we have the following refinement of Theorem 4.2:

The set of weakly hypercyclic vectors for T that fail to be norm hypercyclic vectors is norm dense.

The weakly hypercyclic vector created in the proof of Theorem 4.2 has the property that its orbit is weakly sequentially dense. Nevertheless, one may modify the proof to give a weakly hypercyclic vector whose orbit is not sequentially dense. To do that, we may use the set $\{k^{1/q} e_{n_k} : k \geqslant 1\}$ which has 0 as a weak limit point and follow the ideas in the proof of Theorem 3.2.

Examining the weakly hypercyclic vector created in Theorem 4.2, we can see the fact that it is a weakly hypercyclic vector depends heavily on the fact that the sequence $(e_{n_k})_{k=1}^{\infty}$ converges weakly to the zero vector in ℓ^p with $1 < p < \infty$, but this is false when $p = 1$. Thus, the same proof does not work for a unilateral weighted shift in ℓ^1.

5. FINAL REMARKS

Throughout the previous sections, we have seen both similarities and differences between weak hypercyclicity and norm hypercyclicity of a bounded linear operator T on a separable, infinite dimensional Banach space X. The similarities follow from the fact that the weak closure and the norm closure of a convex set are the same. Many of the differences often arise from the loss of the Baire Category Theorem in the weak topology, or from the fact that a weakly convergent net need not be bounded. These differences seem to make a lot of the techniques in norm hypercyclicity unavailable for the weak hypercyclicity. Consequently, it is natural to wonder whether some other well known properties of norm hypercyclic operators are shared by weakly hypercyclic operators. For instance, Kitai ([11], Theorem 2.8) showed that every component of the spectrum of a hypercyclic operator T must intersect the unit circle. The techniques she used in the proof of this result are not available for the weak topology. This leads to the question:

Question 5.1. If T is weakly hypercyclic, must the spectrum of T intersect the unit circle?

If T is the weakly hypercyclic bilateral weighted shift in the proof of Corollary 3.3, then T is invertible and by Theorem 5 in [16], the spectrum of T is the closed annulus $\{z \in \mathbb{C} : 1 \leqslant |z| \leqslant 2\}$. Notice the boundary contains the whole unit circle. In fact, it appears that a shift satisfying the sufficient condition in Theorem 3.2 will contain the unit circle in its spectrum. These observations naturally lead to the question whether one can give a negative answer to Question 5.1 by a careful argument involving the weights of a bilateral weighted shift.

The next question is motivated by an open problem of Herrero ([10], Problem 1) who asked whether $T \oplus T$ is hypercyclic whenever T is hypercyclic. A partial answer was obtained by Bès and Peris ([2], Theorem 2.3) who showed that T satisfies a general version of the Hypercyclicity Criterion if and only if $T \oplus T$ is hypercyclic. It remains open whether every hypercyclic operator satisfies this general version of the Hypercyclicity Criterion. Since the weak topology version of the Criterion does not hold as we have shown in Section 1, it is natural to ask the following question.

Question 5.2. If T is weakly hypercyclic, does it follow that $T \oplus T$ is weakly hypercyclic on $X \oplus X$?

When X is a Hilbert space, Kitai ([11], Corollary 4.5) proved that a hyponormal operator T on X is never hypercyclic by estimating the norm of the vector $T^n f$ for any f in X. On the other hand, a norm increasing orbit can be weakly dense as we have seen in Corollary 3.3. Since an operator satisfying the sufficient condition in Theorem 3.2 appears not to be hyponormal, we have the following question.

QUESTION 5.3. Does there exist a weakly hypercyclic hyponormal operator on a Hilbert space?

Now we turn our attention to the proof of Theorem 3.2. It can easily be shown that the conclusion still holds if we replace Condition (A-2) with

$$\sum_{s=2}^{\infty}(s\log s)^{-q/p}\sigma^{-q}(\nu(r,s))=\infty. \tag{5.1}$$

Moreover, under this new condition, it may appear that Theorem 3.2 would hold for the case with $1<p<2$. However, in this case there does not exist a function $\sigma:\mathbb{N}\to(0,\infty)$ that satisfies both Conditions (A-1) and (5.1), and so the proof of the theorem does not help in the construction of a weakly hypercyclic bilateral weight shift on $\ell^p(\mathbb{Z})$.

When $p=1$, the construction in the proof of Theorem 3.2 fails in a more complicated way. First, the dual of $\ell^1(\mathbb{Z})$ is $\ell^\infty(\mathbb{Z})$ which contains sequences that are not summable in a manner that allows us to have an analogue of Claim 3 in the proof. Furthermore, in the proof of Theorem 3.2, we have used the fact that for a fixed r, the zero vector is a weak limit point of the set $\{\varphi_{\nu(r,s)}:s\geqslant 1\}$. We have also used the fact that for every $\alpha\in\mathbb{Z}$ there is at most one s such that $\widehat{\varphi_{\nu(r,s)}}(\alpha)\neq 0$. In $\ell^1(\mathbb{Z})$, these two facts imply that a subsequence of $(\varphi_{\nu(r,s)})_{s=1}^\infty$ converges in norm to the zero vector, as we now prove.

PROPOSITION 5.4. *Let $(\varphi_n)_{n=1}^\infty$ be a sequence of vectors in $\ell^1(\mathbb{Z})$ with the property that for any $\alpha\in\mathbb{Z}$, there is at most one n such that $\widehat{\varphi_n}(\alpha)\neq 0$. Then the following are equivalent:*

(i) *a subsequence $(\varphi_{n_k})_{k=1}^\infty$ of $(\varphi_n)_{n=1}^\infty$ converges to the zero vector in norm;*
(ii) *a subsequence $(\varphi_{n_k})_{k=1}^\infty$ of $(\varphi_n)_{n=1}^\infty$ converges weakly to the zero vector;*
(iii) *the zero vector is a weak limit point of the set $\{\varphi_n:n\geqslant 1\}$.*

Proof. The equivalence of (i) and (ii) is because a sequence in $\ell^1(\mathbb{Z})$ converges weakly to zero if and only if it converges in norm to zero; see, for example, page 135 in [4]. It only remains to show (iii) implies (i). For each $n\geqslant 1$, let $A_n=\{\alpha\in\mathbb{Z}:\widehat{\varphi_n}(\alpha)\neq 0\}$. From the hypothesis, we have $A_m\cap A_n=\emptyset$ whenever $m\neq n$. Next, for each $n\geqslant 1$ and $\alpha\in A_n$, let $\beta_{n,\alpha}\in\mathbb{C}$ such that

$$\beta_{n,\alpha}\widehat{\varphi_n}(\alpha)=|\widehat{\varphi_n}(\alpha)|.$$

Let φ be the sequence in $\ell^\infty(\mathbb{Z})$ given by $\widehat{\varphi}(\alpha)=\beta_{n,\alpha}$ if $\alpha\in A_n$ for some n and $\widehat{\varphi}(\alpha)=0$ otherwise. Note that $\|\varphi\|_\infty=1$ and for any $n\geqslant 1$,

$$\langle\varphi_n,\varphi\rangle=\sum_{\alpha\in A_n}\beta_{n,\alpha}\widehat{\varphi_n}(\alpha)=\sum_{\alpha\in A_n}|\widehat{\varphi_n}(\alpha)|=\|\varphi_n\|_1.$$

Since the zero vector is a weak limit point, there exists a strictly increasing sequence $(n_k)_{k=1}^\infty$ such that

$$\|\varphi_{n_k}\|_1=|\langle\varphi_{n_k},\varphi\rangle|<k^{-1},$$

for all $k\geqslant 1$, and hence the subsequence $(\varphi_{n_k})_{k=1}^\infty$ converges in norm to the zero vector. ∎

Our discussion above seems to indicate that the construction in the proof of Theorem 3.2 may not work for $\ell^p(\mathbb{Z})$ when $1 \leqslant p < 2$. Hence, we are led to the following question.

Question 5.5. For $1 \leqslant p < 2$, does there exist a weakly hypercyclic bilateral weighted shift on $\ell^p(\mathbb{Z})$ that is not norm hypercyclic?

The sufficient condition for weak hypercyclicity provided by Theorem 3.2 does not seem to be a necessary condition as well. On the other hand, there is a necessary and sufficient condition for a bilateral weighted shift to be norm hypercyclic, given by Salas ([14], Theorem 2.1).

Question 5.6. Does there exist a necessary and sufficient condition for a bilateral weighted shift to be weakly hypercyclic?

For a unilateral weighted backward shift on ℓ^p with $1 \leqslant p < \infty$, we have seen in Section 4 that norm hypercyclicity and weak hypercyclicity are equivalent. Furthermore, for a special class of unilateral weighted backward shifts, there exist weakly hypercyclic vectors that fail to be hypercyclic vectors in the norm topology. These results naturally lead to the next two questions.

Question 5.7. For every norm hypercyclic operator on a Hilbert space, must there exist a weakly hypercyclic vector that fails to be a norm hypercyclic vector?

Of course, a positive answer to Question 5.7 will imply that there is no operator on ℓ^2 for which every vector is a norm hypercyclic vector.

Question 5.8. Are there other classes of operators for which norm hypercyclicity and weak hypercyclicity are equivalent?

The weak topology techniques we have used in this paper are specialized for shift operators, especially when we produce a weakly hypercyclic bilateral shift in Corollary 3.3. This shift, as we pointed out at the end of Section 3, is unitarily equivalent to the adjoint of a multiplication operator on the Hardy space of an annulus. Hence, it is interesting to study and to phrase the weak hypercyclicity phenomenon in terms of function theory. For instance, it is desirable to have a sufficient condition for weak hypercyclicity for certain classes of operators, say the adjoints of multiplication operators or the composition operators on a Banach space of analytic functions. These directions, along with the questions raised above, may call for a broad base of weak topology techniques in functional analysis as well.

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Note added in proof. Question 5.1 was answered in the positive by Dilworth and Troitsky ([5]), and Question 5.3 was answered in the negative by Sanders ([15]).