

## SET THEORY AND CYCLIC VECTORS

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*Communicated by Kenneth R. Davidson*

ABSTRACT. Let  $H$  be a separable, infinite dimensional Hilbert space and let  $S$  be a countable subset of  $H$ . Then most positive operators on  $H$  have the property that every nonzero vector in the span of  $S$  is cyclic, in the sense that the set of operators in the positive part of the unit ball of  $B(H)$  with this property is comeager for the strong operator topology.

Suppose  $\kappa$  is a regular cardinal such that  $\kappa \geq \omega_1$  and  $2^{<\kappa} = \kappa$ . Then it is relatively consistent with ZFC that  $2^\omega = \kappa$  and for any subset  $S \subset H$  of cardinality less than  $\kappa$  the set of positive operators in the unit ball of  $B(H)$  for which every nonzero vector in the span of  $S$  is cyclic is comeager for the strong operator topology.

KEYWORDS: *Cyclic vector, invariant subspace, Martin's axiom.*

MSC (2000): 03E35, 03E50, 47A15, 47A16.

### 1. INTRODUCTION

Let  $H$  be a separable, infinite dimensional Hilbert space and let  $B(H)$  be the set of bounded linear operators  $A : H \rightarrow H$ . A closed subspace  $E$  of  $H$  is *invariant* for such an operator  $A$  if  $A(E) \subset E$ . The invariant subspace problem (ISP) for Hilbert spaces asks whether there exists an operator  $A \in B(H)$  whose only closed invariant subspaces are  $\{0\}$  and  $H$ .

It was shown by Enflo ([2]) that there exists a bounded operator on a Banach space that has no proper closed invariant subspaces. Read ([7]) showed that the Banach space could be taken to be  $l^1 = l^1(\mathbb{N})$ . A simplified version of Read's example is given in [1]. The ISP remains open for Hilbert spaces; it is also unknown whether there exists a separable, infinite dimensional Banach space on which every bounded operator has a proper closed invariant subspace.

This note was motivated by the observation that the ISP for Hilbert spaces can be rephrased as a question about the existence of a generic filter on a certain poset. (This material is not needed for Section 2.) The construction is this. Let

$P$  be the poset consisting of all partially defined operators  $A$  on the Hilbert space  $l^2 = l^2(\mathbb{N})$  with the properties

- (a)  $\text{dom}(A)$  is a finite dimensional subspace of  $l^2$ ;
- (b) if  $E$  is a subspace of  $\text{dom}(A)$  and  $A(E) \subset E$  then  $E = \{0\}$ ; and
- (c)  $\|A\| < 1$ .

Order  $P$  by reverse inclusion. For any unit vectors  $v, w \in l^2$  define

$$D_{v,w} = \{A \in P : \text{there exists } n \text{ such that } A^n(v) \text{ is defined and } \langle A^n v, w \rangle \neq 0\}.$$

It is not too hard to see that every  $D_{v,w}$  is dense in  $P$ , and a filter of  $P$  which intersects every  $D_{v,w}$  defines a bounded operator with no proper invariant subspaces. Conversely, if there is such an operator it can be scaled to have norm  $< 1$ , and then its finite dimensional restrictions define a filter of  $P$  which intersects every  $D_{v,w}$ . Thus, the ISP for Hilbert space can be cast in set-theoretic terms: it is equivalent to the existence of a  $D$ -generic filter on  $P$ , where  $D = \{D_{v,w} : v, w \in l^2 \text{ and } \|v\| = \|w\| = 1\}$ .

The poset  $P$  is not ccc, but this is not essential; for example, it can be replaced by the countable poset of all finite matrices with rational entries, ordered by a reasonable notion of approximate inclusion. Thus, one can apply Martin's axiom (see, e.g., [6]) to obtain the consistency of an operator which meets "many" of the sets  $D_{v,w}$ . This raises the possibility that the ISP for Hilbert space may be independent of ZFC. However, the assertion that  $A \in B(l^2)$  has no invariant subspaces is  $\Sigma_1$  (it can be reformulated as "for all unit vectors  $v, w \in l^2$  there exists  $n$  such that  $\langle A^n v, w \rangle \neq 0$ ") and hence absolute, so if the ISP is independent, this cannot be shown using a straightforward forcing argument.

## 2. A RELATIVE CONSISTENCY RESULT

As we indicated above, although Martin's axiom alone will not suffice to prove the consistency of an operator with no proper closed invariant subspaces (unless this can already be proven in ZFC), it does allow one to prove partial results in this direction. In this section we present perhaps the strongest natural result along these lines. It was in fact originally proven directly from Martin's axiom, but here we give a better proof based on a suggestion of Kenneth Kunen.

It is easy to see that the operator  $A \in B(H)$  has no proper closed invariant subspaces if and only if every nonzero vector is cyclic, i.e., for every nonzero  $v \in H$  the span of the sequence  $(A^n v)$  is dense in  $H$ . Thus, the more cyclic vectors  $A$  has, the "closer" it gets to being a counterexample to the ISP.

If  $X$  is a Banach space then we let  $[X]_1$  denote its closed unit ball. The *strong operator topology* on  $B(H)$  is generated by the basic open sets

$$\mathcal{O}_{B,E,\varepsilon} = \{A \in B(H) : \|A|_E - B\| < \varepsilon\}$$

taken over  $\varepsilon > 0$ ,  $E$  a finite dimensional subspace of  $H$ , and  $B : E \rightarrow H$  a linear map. If  $H$  is separable then this topology is second countable and its restriction to  $[B(H)]_1$  is metrizable by  $d(A, A') = \sum 2^{-n} \|(A - A')(v_n)\|$ , where  $(v_n)$  is a countable dense subset of  $[H]_1$ . Moreover, this metric is complete, so  $[B(H)]_1$  with the relative strong operator topology is a Polish space.

Let  $[B(H)]_1^+$  denote the set of positive operators  $A \in [B(H)]_1$ , i.e., those self-adjoint operators which satisfy  $0 \leq \langle Av, v \rangle \leq 1$  for all  $v \in [H]_1$ . (In the case of complex scalars, self-adjointness follows from the second condition.) If  $H$  is separable then  $[B(H)]_1^+$  is a Polish space via the same metric which shows that  $[B(H)]_1$  is Polish.

LEMMA. *Let  $H$  be a separable, infinite dimensional Hilbert space and let  $E$  be a finite dimensional subspace of  $H$ . Then the set of operators in  $[B(H)]_1^+$  for which every nonzero vector in  $E$  is cyclic is comeager for the relative strong operator topology.*

*Proof.* Let  $(x_n)$  be an orthonormal basis of  $H$ . For  $m \in \mathbb{N}$  and  $\delta > 0$  let  $U_{m,\delta}$  be the set of operators  $A \in [B(H)]_1^+$  such that

$$d(x_m, \text{span}\{A^k v : k \in \mathbb{N}\}) < \delta$$

for every nonzero  $v \in E$ . We will show that  $U_{m,\delta}$  is open and dense in  $[B(H)]_1^+$  for every  $m$  and  $\delta$ . Intersecting the sets  $U_{m,\delta}$  over all  $m$  and all  $\delta$  of the form  $\delta = 1/n$  yields the set of  $A \in [B(H)]_1^+$  for which  $\text{span}\{A^k v : k \in \mathbb{N}\} = H$  for every nonzero  $v \in E$ ; so we will have shown that this set is a countable intersection of open, dense sets, as desired.

Fix  $m$  and  $\delta$  for the remainder of the proof. We first show that  $U_{m,\delta}$  is open. Let  $A \in U_{m,\delta}$  and for each unit vector  $v \in E$  let  $f(v)$  be the smallest integer such that

$$d(x_m, \text{span}\{A^k v : 0 \leq k \leq f(v)\}) < \delta.$$

Then the function  $f$  is upper semicontinuous on the unit sphere of  $E$  (which is compact), so  $f$  is bounded. Let  $N$  be an upper bound for  $f$  and let  $F = \text{span}\{A^k v : v \in E, 0 \leq k \leq N\}$  and

$$\delta' = \sup\{d(x_m, \text{span}\{A^k v : 0 \leq k \leq N\}) : v \in E, \|v\| = 1\}.$$

Since  $E$  is finite dimensional, so is  $F$ . Also, by compactness of the unit sphere  $\delta' < \delta$ . Now for  $\varepsilon > 0$  let  $U_\varepsilon$  be the set of operators  $B \in [B(H)]_1^+$  which satisfy  $\|(B - A)w\| < \varepsilon$  for all  $w \in [F]_1$ . This set is strong operator open for each positive  $\varepsilon$ . For each unit vector  $v \in E$  let  $g(v)$  be the supremum of the set of  $\varepsilon > 0$  such that  $B \in U_\varepsilon$  implies

$$d(x_m, \text{span}\{B^k v : 0 \leq k \leq N\}) \leq \frac{1}{2}(\delta' + \delta) < \delta.$$

Then every vector in the unit sphere of  $F$  has a neighborhood in which  $g$  is bounded away from 0, so  $g$  must be bounded below by some positive  $\varepsilon$ , and we have  $A \in U_\varepsilon \subset U_{m,\delta}$  for this  $\varepsilon$ . Thus  $U_{m,\delta}$  is strong operator open.

Now we must show that  $U_{m,\delta}$  is strong operator dense in  $[B(H)]_1^+$ . Fix  $A \in [B(H)]_1^+$ , a finite dimensional subspace  $F$  of  $H$  which contains  $E$  and  $x_m$ , and  $\varepsilon > 0$ . We will find an operator  $B \in U_{m,\delta}$  such that  $\|(B - A)w\| < \varepsilon$  for all  $w \in [F]_1$ . Let  $F' = \text{span}(F + A(F))$ , let  $n = \dim(F')$ , let  $P_{F'}$  be the orthogonal projection of  $H$  onto  $F'$ , and let  $A' = P_{F'}AP_{F'}$ . Note that  $A'v = Av$  for all  $v \in F$ . Next, choose an integer  $r > 4n/\delta^2$  and let  $X$  be a subspace of  $H$  of dimension  $nr$  which contains  $F'$ . We can identify  $X$  with the tensor product space  $\mathbb{K}^n \otimes \mathbb{K}^r$

(where  $\mathbb{K}$  is the scalar field,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ ) in such a way that  $F'$  is identified with  $\mathbb{K}^n \otimes \{(r^{-1/2}, \dots, r^{-1/2})\}$ .

Since  $A$  is positive and  $\|A\| \leq 1$ , the same is true of  $A'$ . Thus  $F'$  is spanned by eigenvectors of  $A'$ , each of which belongs to an eigenvalue between 0 and 1, inclusive. Let  $v'_1, \dots, v'_n$  be an orthonormal set of eigenvectors belonging to the eigenvalues  $\lambda_1, \dots, \lambda_n$ . (The  $\lambda_i$  need not be distinct.) According to the above identification we have  $v'_i = v_i \otimes (r^{-1/2}, \dots, r^{-1/2})$  for some orthonormal basis  $\{v_i\}$  of  $\mathbb{K}^n$ .

Let  $\{w_j : 1 \leq j \leq r\}$  be the standard orthonormal basis of  $\mathbb{K}^r$ . Then the vectors  $v_i \otimes w_j$  constitute an orthonormal basis of  $X \cong \mathbb{K}^n \otimes \mathbb{K}^r$ . Define  $B' \in B(X)$  by setting  $B'(v_i \otimes w_j) = \lambda_i v_i \otimes w_j$ . Then  $B'$  is positive,  $\|B'\| \leq 1$ , and  $B'|F = A'|F = A|F$ .

To complete the proof, we will show that there exists  $B \in [B(X)]_1^+$  such that  $\|B - B'\| < \varepsilon$  and  $BP_X \in U_{m,\delta}$ . We will define  $B$  by choosing an orthonormal basis  $\{e_{ij}\}$  of  $X$  and corresponding values  $0 \leq \sigma_{ij} \leq 1$  and setting  $Be_{ij} = \sigma_{ij}e_{ij}$ . If each  $e_{ij}$  is sufficiently close to  $v_i \otimes w_j$  and each  $\sigma_{ij}$  is sufficiently close to  $\lambda_i$  then we will have  $\|B - B'\| < \varepsilon$ . Thus, we must show that there exist  $\{e_{ij}\}$  and  $\{\sigma_{ij}\}$  arbitrarily close to  $\{v_i \otimes w_j\}$  and  $\{\lambda_i\}$  which achieve  $BP_X \in U_{m,\delta}$ .

First, we claim that there exist orthonormal bases  $\{e_{ij}\}$  arbitrarily close to the basis  $\{v_i \otimes w_j\}$  with the property that every  $n$ -element subset of the set  $\{P_{F'}(e_{ij})\}$  is linearly independent. That is, any  $n$  vectors in the basis orthogonally project to independent vectors in  $F'$ . This is true because, for any  $n$  indices  $i_1 j_1, \dots, i_n j_n$  the family of bases  $\{e_{ij}\}$  for which the vectors  $P_{F'}(e_{i_1 j_1}), \dots, P_{F'}(e_{i_n j_n})$  are dependent is a variety of codimension 1 in the manifold of all orthonormal bases. Thus, the family of bases for which some  $n$  elements project onto a dependent set is a union of  $\binom{rn}{n}$  meager sets, and hence meager. So, we can perturb the basis  $\{v_i \otimes w_j\}$  by an arbitrarily small amount and achieve this condition.

Now, having chosen  $\{e_{ij}\}$  so as to satisfy the previous claim, we conclude by showing that any choice of *distinct* values  $\sigma_{ij}$  such that each difference  $|\sigma_{ij} - \lambda_i|$  is sufficiently small will ensure  $BP_X \in U_{m,\delta}$ . To see this, observe first that for any nonzero  $v \in F'$  at most  $n-1$  of the inner products  $\langle v, e_{ij} \rangle$  are zero. Otherwise,  $n$  of the vectors  $e_{ij}$  would be orthogonal to  $v$ , and hence  $n$  of the vectors  $P_{F'}(e_{ij})$  would be orthogonal to  $v$ , which would imply linear dependence since  $\dim(F') = n$ . This contradicts the previous claim. Now for each nonzero  $v \in F'$  let

$$F_v = \text{span}\{e_{ij} : \langle v, e_{ij} \rangle \neq 0\}.$$

Distinctness of the  $\sigma_{ij}$  implies that the vectors  $B^k v$  are linearly independent for  $0 \leq k < \dim(F_v)$ ; since  $F_v$  clearly contains  $\text{span}\{B^k v : k \in \mathbb{N}\}$  (it contains  $v$  and is invariant for  $B$ ) this shows that the two are equal. Thus, we must show that  $d(x_m, F_v) < \delta$ . But  $x_m \in F'$ , so  $|\langle x_m, v_i \otimes w_j \rangle| \leq r^{-1/2}$  for every  $i$  and  $j$ . We may therefore assume that  $|\langle x_m, e_{ij} \rangle| \leq 2r^{-1/2} < \delta/\sqrt{n}$  for every  $i$  and  $j$ . Thus, if  $x'_m = x_m - P_{F_v}(x_m)$  then

$$\|x'_m\|^2 = \sum_{e_{ij} \notin F_v} |\langle x_m, e_{ij} \rangle|^2 < n \cdot (\delta^2/n) = \delta^2,$$

i.e.,  $\|x'_m\| < \delta$ , since  $F_v$  contains all but at most  $n-1$  of the vectors  $e_{ij}$ . This shows that  $d(x_m, F_v) < \delta$ , as desired. ■

Since a countable intersection of comeager sets is comeager, the lemma immediately implies the following result.

**THEOREM 1.** *Let  $H$  be a separable, infinite dimensional Hilbert space and let  $S$  be a countable subset of  $H$ . Then the set of operators in  $[B(H)]_1^+$  for which every nonzero vector in the span of  $S$  is cyclic is comeager in  $[B(H)]_1^+$ .*

**THEOREM 2.** *Let  $H$  be a separable, infinite dimensional Hilbert space and let  $\kappa$  be a regular cardinal such that  $\kappa \geq \omega_1$  and  $2^{<\kappa} = \kappa$ . Then it is relatively consistent with ZFC that  $2^\omega = \kappa$  and for any subset  $S \subset H$  of cardinality  $< \kappa$  the set of operators in  $[B(H)]_1^+$  for which every nonzero vector in the span of  $S$  is cyclic is comeager in  $[B(H)]_1^+$ .*

*Proof.* It is relatively consistent with ZFC + MA that  $2^\omega = \kappa$  ([6], Theorem 6.3). Let  $S \subset H$  be a subset of cardinality  $< \kappa$ . Then the span of  $S$  is a union of fewer than  $\kappa$  finite dimensional subspaces of  $H$ , and the lemma implies that for each such subspace  $E$  the set of operators in  $[B(H)]_1^+$  for which every nonzero vector in  $E$  is cyclic is comeager. The proof is completed by observing that MA implies that the intersection of fewer than  $2^\omega$  comeager sets in a Polish space is comeager ([3], Corollary 22C). ■

Theorem 1 is related to the main theorem of [5]. That result implies, for instance, that for any countable linearly independent subset  $S$  of a separable, infinite dimensional Hilbert space  $H$  there exists  $A \in B(H)$  for which every  $v \in S$  is cyclic.

Sophie Grivaux has pointed out to me that in Theorem 1 one can explicitly construct a single operator for which every nonzero vector in the span of  $S$  is cyclic. Namely, first find an orthonormal basis of  $H$  whose span contains  $S$  (this can be accomplished by applying the Gramm-Schmidt algorithm to  $S$ ); then one can show directly that the operator  $V + V^*$ , where  $V$  is the unilateral shift for the new basis, has the desired property. Moreover, using this technique one can give a fairly simple proof that the set of operators in  $B(H)$  for which every nonzero vector in the span of  $S$  is cyclic is comeager in  $B(H)$ . (Related results appear in [4].) However, density in  $[B(H)]_1^+$  seems to be a more difficult result, and this is needed in Theorem 2 because  $[B(H)]_1^+$  with the strong operator topology is a Polish space, while  $B(H)$  is not.

*Acknowledgements.* This research is supported by NSF grant DMS-0070634.

I wish to thank Sophie Grivaux, Kenneth Kunen, and Peter Rosenthal for helpful discussions.

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Received August 1, 2002.