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MULTIPLIERS OF MINIMAL NORM ON DIRICHLET TYPE SPACES

ANTÓNIO SERRA

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Abstract. We give a necessary and sufficient condition for the existence of non-constant multipliers of minimal norm on complete NP spaces and use it, along with a result of S. Shimorin, to answer a question posed by S. Axler.

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1. INTRODUCTION

The multipliers of a reproducing kernel Hilbert space ([4], [11]) are necessarily bounded functions, whose sup norms are dominated by the operator norms of the associated multiplication operators. This inequality is not necessarily strict. For the best studied case, that of the Hardy space H^2 on the open unit disk $\mathbb{D}$, equality occurs for all multipliers (that are all the bounded holomorphic functions on $\mathbb{D}$). The same happens for the Bergman space A^2. However, for the Dirichlet space D, formed by the holomorphic functions whose derivatives belong to A^2, it is easy to see that the inequality is strict for the identity function (the "Dirichlet shift"). This led S. Axler to ask in [5] whether there are non-constant multipliers on D with operator norm equal to the sup norm (or multipliers of minimal norm, as we call them here, since the sup norm of a multiplier is a lower bound for its operator norm).

Axler's question was answered affirmatively by J. Lech, reportedly using techniques similar to those in [7], but his work is not published.

Here we show that Axler's question has an affirmative answer for a large family of reproducing kernel Hilbert spaces, including all harmonically weighted Dirichlet spaces, of which D is an example. This paper is organized as follows. Sections 2 and 3 summarize some facts about multipliers and NP kernels which will be used in our results. In Section 4 we give a necessary and sufficient condition for the existence of multipliers of minimal norm on NP spaces. Finally in Section 5 we apply our result and a theorem of Shimorin ([14]) to establish the existence of multipliers of minimal norm on harmonically weighted Dirichlet spaces.

2. MULTIPLIERS

All functions are considered to be complex valued unless stated otherwise.

Let $\mathcal{H}$ be a Hilbert space of functions on a set X such that point evaluations are bounded functionals. Then, for each $x \in X$ there exists $k_x \in \mathcal{H}$ such that for $f \in \mathcal{H}$,

$$\langle f, k_x \rangle_{\mathcal{H}} = f(x).$$

A function k defined on $X \times X$ is generally called a *kernel* (on X) and it is said to be positive definite, semi-definite, etc., if for every $x_1, \ldots, x_n \in X$ the matrix $(k(x_i, x_j))_{i,j=1}^n$ is respectively positive definite, semi-definite, etc.

The kernel $k(x, y) = k_y(x)$ is called the *reproducing kernel* of $\mathcal{H}$. It is positive semi-definite because the matrix $(k(x_j, x_i))_{i,j=1}^n$ is the Grammian of the vectors k_{x_j}. The space $\mathcal{H}$ is called a *reproducing kernel Hilbert space.*

Any positive semi-definite kernel on X is the reproducing kernel of a uniquely determined Hilbert space of functions on X, which we will denote by $\mathcal{H}(k)$. This seems to have been first noted by E.H. Moore (*Mem. Amer. Philos. Soc.*, **1**:2(1939), Chapter V). The theory of reproducing kernels was systematically developed in Aronszajn's paper ([4]). A recent and very elegant account is [11].

Here are the reproducing kernels of some familiar spaces:

(i) the Hardy space H^2 on the unit disk $\mathbb{D}$ whose kernel is the "Szegö kernel",

$$\mathrm{Sz}(z, w) = \frac{1}{1 - \overline{w}z};$$

(ii) the Bergman space A^2 on the unit disk:

$$k_A(z, w) = \frac{1}{(1 - \overline{w}z)^2},$$

the square of the Szegö kernel;

(iii) and finally, for the Dirichlet space D:

$$d(z, w) = \frac{1}{\overline{w}z} \log \frac{1}{1 - \overline{w}z}.$$

If $\mathcal{H}$ is a reproducing kernel Hilbert space on X, a *multiplier* of $\mathcal{H}$ is a function φ on X with the property that φf belongs to $\mathcal{H}$ whenever f belongs to $\mathcal{H}$. The operator that maps $f \in \mathcal{H}$ to φf will be denoted by M_φ. It follows from the Closed Graph Theorem that M_φ is bounded. We summarize the essential facts about multipliers to be used in the sequel in the following proposition. Proofs can be found in [4] and [11].

PROPOSITION 2.1.

(i) *An operator T on $\mathcal{H}(k)$ is a multiplier M_φ if and only if T is bounded and $T^* k_x = \overline{\varphi(x)} k_x$ for all $x \in X$.*

(ii) *A function φ on X is a multiplier of $\mathcal{H}(k)$ with operator norm smaller or equal to $r > 0$ if and only if*

$$l(x, y) = (r^2 - \varphi(x)\overline{\varphi(y)})k(x, y) \tag{2.1}$$

is a positive semi-definite kernel on X.

It follows from part 1 that $|\varphi(x)| \leqslant \|M_\varphi\|$ for all multipliers φ.

Finally, we state here a result from [3] which will be useful in Section 4. Let ρ denote the pseudo-hyperbolic metric on $\mathbb{D}$, that is,

$$\rho(w, z) = \left|\frac{z - w}{1 - \overline{w}z}\right|$$

for every z and w in $\mathbb{D}$.

PROPOSITION 2.2. ([3]) *Let k be a positive definite kernel on a set X. Then the function*

$$d_k(x, y) = \sqrt{1 - \frac{|k(x, y)|^2}{k(x, x)k(y, y)}} \tag{2.2}$$

is a distance on X and if φ is a contractive multiplier of $\mathcal{H}(k)$ then

$$\rho(\varphi(x), \varphi(y)) \leqslant d_k(x, y). \tag{2.3}$$

3. NP KERNELS

Let k be a positive definite kernel on a set X. We say that k is a *Nevanlinna-Pick kernel*, abbreviated to *NP kernel* if:

(i) there exists a point $c \in X$ such that $k(x, c) = 1$ for all $x \in X$;
(ii) k has no zeros on $X \times X$;
(iii) the kernel $1 - \frac{1}{k}$ is positive semi-definite on X.

Condition (i) implies that the constant functions form an isometric embedding of the complex plane $\mathbb{C}$ in $\mathcal{H}(k)$. A kernel which satisfies (ii) is called *irreducible* (see [3]). Condition (iii) was introduced by Agler in [1] and studied by Quiggin in [9].

The following proposition states the key property of NP kernels which will be used here. A stronger form of this property characterizes NP kernels.

Let k be a positive semi-definite kernel on a set X and let $Y \subset X$ be any subset of X. The restriction of k to $Y \times Y$ is a positive semi-definite kernel on Y. Abusively, we will denote that kernel on Y by $k|Y$.

Proposition 3.1. ([9]) *Let k be an NP kernel on a set X and let $Y \subset X$ be any subset of X. If φ is a function on Y and a contractive multiplier of $\mathcal{H}(k|Y)$ then there exists an extension $\tilde{\varphi}$ of φ which is a contractive multiplier of $\mathcal{H}(k)$.*

The Szegö kernel is easily seen to be an NP kernel on $\mathbb{D}$: $\mathrm{Sz}(z,0) = 1$, it is irreducible and

$$1 - \frac{1}{\mathrm{Sz}(z,w)} = \overline{w}z,$$

which is a (rank one) positive semi-definite kernel on $\mathbb{D}$. Proposition 3.1 is, in this case, a generalization of a famous theorem of Pick ([8]):

Theorem 3.2. *Let $z_1, \dots, z_n$ be points in $\mathbb{D}$ and $\lambda_1, \dots, \lambda_n$ be complex values (in $\mathbb{D}$ too). Then, there exists a function f in the unit ball of H^∞ such that $f(z_j) = \lambda_j$ for $1 \leqslant j \leqslant n$ if and only if the matrix*

$$\left(\frac{1 - \lambda_i \overline{\lambda}_j}{1 - z_i \overline{z}_j}\right)_{i,j=1}^{n} \tag{3.1}$$

is positive semi-definite.

Note that the (i,j)-entry of the matrix in (3.1) equals

$$(1 - \lambda_i \overline{\lambda}_j)\mathrm{Sz}(z_i, z_j);$$

by Proposition 2.1, part (ii), this matrix is positive semi-definite if and only if the function $z_j \mapsto \lambda_j$ is a contractive multiplier for the restriction of the Szegö kernel to $\{z_1, \dots, z_n\} \subset \mathbb{D}$.

The Bergman kernel is not an NP kernel while the Dirichlet kernel is. The proof of these facts can be found in [9] and elsewhere. In [14], Shimorin established that the reproducing kernels of a large family of Hilbert spaces are NP kernels. We will use that result later on.

Conditions (i)-(iii) are not independent and they do not correspond to the most general case of an NP kernel. Here we present them in this manner for the sake of brevity and simplicity. In the general case conditions (i) and (ii) are suppressed and condition (iii) suitably modified; however, in [3] it is shown that if k is an NP kernel on X then X admits a partition $\{X_\iota\}_{\iota \in I}$ such that k is irreducible on each X_ι, and

$$\mathcal{H}(k) = \bigoplus_{\iota \in I} \mathcal{H}(k|X_\iota).$$

Moreover if k is an irreducible kernel, condition (i) can be forced without changing the multiplier algebra: choose a point $c \in X$ and replace k by the kernel

$$k_c(x,y) = \frac{k(x,y)k(c,c)}{k(x,c)k(c,y)}.$$

The origin of the concept of an NP kernel is the preprint [1], where Agler applied Sarason's commutant lifting approach from [12] to general reproducing kernels. An application to a Sobolev space was published in [2] but the first published work studying this subject in its full generality was [9]. A complete account of the theory of NP kernels is [3], which contains many original results and applications as well as a very complete bibliography.

4. MULTIPLIERS AND MINIMAL NORM

We begin by stating some facts about positive semi-definite kernels.

PROPOSITION 4.1. *If k and l are two positive semi-definite kernels on a set X, then so is their pointwise product.*

This is a direct consequence of a celebrated theorem of Schur ([13]) which states that the entrywise product (also called "Schur product") of two positive semi-definite matrices is a positive semi-definite matrix.

LEMMA 4.2. *Let k be a positive semi-definite kernel on X, and let $a \in X$ be such that $k(a,a) > 0$. Then, the kernel*

$$k^{(a)}(x,y) = k(x,y) - \frac{k(x,a)k(a,y)}{k(a,a)}$$

is also a positive semi-definite kernel on X.

The proof is obtained by noting that $k^{(a)}(x,y)$ equals $\langle v_y, v_x \rangle_{\mathcal{H}(k)}$ where

$$v_x = k_x - \frac{k(a,x)}{k(a,a)} k_a,$$

and likewise for v_y.

The boundedness of the diagonal of a reproducing kernel is reflected in the boundedness of the functions of the generated Hilbert space. This is a known fact of which we include a proof here, for the sake of completeness.

PROPOSITION 4.3. *The diagonal $k(x,x)$ of a reproducing kernel k is unbounded if and only if there exists an unbounded function in $\mathcal{H}(k)$.*

Proof. Sufficiency: Let $f \in \mathcal{H}(k)$ be an unbounded function. By the Cauchy–Schwarz inequality,

$$|f(x)|^2 = |\langle f, k_x \rangle|^2 \leqslant \|f\|^2 \|k_x\|^2 = \|f\|^2 k(x,x),$$

whence

$$k(x,x) \geqslant \frac{|f(x)|^2}{\|f\|^2},$$

thus proving the unboundedness of the diagonal.

Necessity: If all functions in $\mathcal{H}(k)$ are bounded then consider the set of point evaluation functionals $\{\langle \cdot, k_x \rangle, x \in X\}$. On each function $f \in \mathcal{H}(k)$ the evaluations of these functionals will form the range of f, which is bounded. Then by the Principle of Uniform Boundedness, this must be a norm bounded set and the norm of the functional $\langle \cdot, k_x \rangle$ is $\sqrt{k(x,x)}$. ∎

Greene, Richter and Sundberg prove in [6] that the column functions k_x of an NP kernel k are multipliers and supply an estimate for their multiplier norms. Proposition 4.4 below is a slight improvement of their result.

For φ a multiplier of k, let $\|\varphi\|$ denote its multiplier norm.

PROPOSITION 4.4. *Let k be an NP kernel. Then, for all $a \in X$ the function $k(\cdot, a)$ is an invertible multiplier whose multiplier norm satisfies*

$$\|k(\cdot, a)\| \leqslant \frac{1}{1 - \sqrt{1 - \frac{1}{k(a,a)}}}. \tag{4.1}$$

In particular, the functions $k(\cdot, a)$ are bounded on X.

Proof. Define

$$F(x, y) = 1 - \frac{1}{k(x, y)}$$

and let $a \in X$. The kernel F is positive semi-definite by condition (iii) in Section 3, and so is

$$G(x, y) = F^{(a)}(x, y) = F(x, y) - \frac{F(x, a)F(a, y)}{F(a, a)},$$

by Lemma 4.2. Then,

$$\left(1 - \frac{F(x, a)F(a, y)}{F(a, a)}\right) k(x, y) = 1 + G(x, y)k(x, y)$$

is positive semi-definite which shows that $F(\cdot, a)$ is a multiplier of norm no greater than $\sqrt{F(a, a)}$, and $F(a, a) < 1$. Then,

$$k(\cdot, a) = \frac{1}{1 - F(\cdot, a)} = \sum_{n \geqslant 0} F(\cdot, a)^n$$

so it is a multiplier of norm no greater than

$$\frac{1}{1 - \sqrt{F(a, a)}}$$

whose inverse is a multiplier as well. ∎

The estimate we present for the multiplier norm of $k(\cdot, a)$ can not be improved: in the case of the Szegö kernel, equality holds for all $a \in \mathbb{D}$.

We arrive now to our main result. Let B denote the closed unit ball of the multiplier algebra of $\mathcal{H}(k)$.

THEOREM 4.5. *Let k be an NP kernel such that $\mathcal{H}(k)$ contains an unbounded function. Then given any sequence (w_n) in $\mathbb{D}$ there exists $\varphi \in B$ whose range contains w_n for every n.*

In particular, there exists a non-constant multiplier whose sup norm on X equals its multiplier norm.

First we prove a stronger version of the theorem for finite sequences.

LEMMA 4.6. *Under the conditions of Theorem 4.5, given a sequence (w_n) in $\mathbb{D}$, there exists a sequence (x_n) in X such that for each n the matrix*

$$W_n(x_1, \ldots, x_n) = ((1 - w_i \overline{w}_j) k(x_i, x_j))_{i,j=1}^n \tag{4.2}$$

is positive definite.

The positive semi-definiteness of the matrix (4.2) suffices to imply the existence of a multiplier $\varphi_n \in B$ such that $\varphi_n(x_i) = w_i$ for $1 \leqslant i \leqslant n$ because k is an NP kernel.

Proof. We will use induction to show that for each n, given $x_1, \ldots, x_n$ such that

$$W_n(x_1, \ldots, x_n) > 0,$$

there exists $x \in X$ such that

$$W_{n+1}(x_1, \ldots, x_n, x) > 0.$$

For $n = 1$, any point $x \in X$ can be taken for x_1 (reproducing kernels are assumed non-degenerate). Assume now that for $n \geqslant 1$ we have points $x_1, \ldots, x_n$ for which (4.2) is positive definite. All we need is to find an $x \in X$ such that

$$b_{n+1}(x) = \det W_{n+1}(x_1, \ldots, x_n, x)$$

is positive. In fact, $W_{n+1}(x_1, \ldots, x_n, x)$ will be positive definite if and only if all of its principal minors ($b_j(x_j)$, for $1 \leqslant j \leqslant n$ and $b_{n+1}(x)$) are positive, and those of rank less than n must be so by the induction hypothesis. But computing $b_{n+1}(x)$ by the Laplace rule using row $n + 1$ we see that

$$b_{n+1}(x) = (1 - |w_{n+1}|^2)k(x, x)b_n(x_n) + s(x)$$

where $s(x)$ is a fixed polynomial function whose coefficients are polynomials in the entries of W_{n+1} other than the lower right corner, all of which have absolute value no bigger than

$$\max_{1 \leqslant j \leqslant n} (\sup |2k(\cdot, x_j)|),$$

and these suprema are finite by Proposition 4.4 and the observation that follows it. So, $s(x)$ is bounded. But, by Proposition 4.3 $k(x, x)$ is not bounded, so we can certainly choose x such that this determinant is positive (it must be real because this is a hermitian matrix). ∎

We can now prove Theorem 4.5.

Proof. Let (w_n) be a sequence in $\mathbb{D}$. By Lemma 4.6 there exists a sequence (x_n) in X such that for each $n \in \mathbb{N}$ there exists $\varphi_n \in B$ such that $\varphi_n(x_j) = w_j$ for $j \leqslant n$. All these functions take values in $\mathbb{D}$, which has compact closure, so by Tychonoff's Theorem the sequence (φ_n) has an accumulation point φ. It is easy to check that φ must belong to B (by the positive semi-definiteness of the NP matrix associated with φ for any finite set of points), and clearly $\varphi(x_n) = w_n$. ∎

The following proposition holds for all reproducing kernels satisfying condition (i) in the definition of an NP kernel given above and it is actually a corollary of Proposition 2.2. However, when applied to NP kernels, it shows that the implication in Theorem 4.5 is actually an equivalence.

PROPOSITION 4.7. *If $\mathcal{H}(k)$ contains only bounded functions then every non-constant multiplier has multiplier norm strictly larger than the supremum norm.*

Proof. By Proposition 4.3, there exists $R > 0$ such that $k(x, x) < R$ for all x. Let φ be a multiplier of norm 1. It follows from Proposition 2.2 that

$$\left| \frac{\varphi(x) - \varphi(c)}{1 - \overline{\varphi(c)}\varphi(x)} \right| \leqslant d_k(x, c) = \sqrt{1 - \frac{1}{k(x, x)}} \leqslant \sqrt{1 - \frac{1}{R}} < 1,$$

so the range of φ is contained in the pseudo-hyperbolic ball of radius $\sqrt{1 - 1/R}$ about $\varphi(c)$. It is thus bounded away from the unit circle unless $\varphi(c)$ is itself unimodular, in which case φ must be constant, by Proposition 2.2. ∎

The Sobolev space W^1 on $[0,1]$ is the space formed by those absolutely continuous functions whose derivatives belong to $L^2([0,1])$. It is a Hilbert space with the inner product

$$\langle f, g\rangle = \langle f, g\rangle_{L^2} + \langle f', g'\rangle_{L^2},$$

for $f, g \in W^1$. All the functions in W^1 are bounded. It is easy to see that point evaluations are bounded functionals and so W^1 is a reproducing kernel Hilbert space. The results proved above imply that W^1 does not have multipliers of minimal norm.

In [9] it is proved that a certain family of weighted Sobolev spaces on the interval have NP kernels. For the case of W^1 the proof can also be found in [3].

This space has the distinction of coinciding with its own multiplier algebra. However, not all reproducing kernel Hilbert spaces consisting only of bounded functions coincide with their multiplier algebras. Easy examples are the restrictions of the Szegö kernel to infinite subsets of $\mathbb{D}$ bounded away from the boundary.

5. APPLICATION TO AXLER'S QUESTION

The Szegö kernel on $\mathbb{D}$ is an immediate example of a complete NP kernel whose diagonal is unbounded. A less trivial example is the Dirichlet kernel ([1], [9])

$$\frac{1}{\overline{w}z} \log \frac{1}{1-\overline{w}z}. \tag{5.1}$$

In [5] Axler asked if there exist in the Dirichlet space multipliers of minimal norm. The results in the previous section and the fact that the Dirichlet kernel is an NP kernel give an affirmative answer to this question. Actually it turns out that these multipliers exist for all so-called harmonically weighted Dirichlet spaces. These spaces were introduced by S. Richter in [10], where he used them to classify two-isometries (Hilbert space operators T that satisfy $T^{*2}T^2 - 2T^*T + I = 0$) up to unitary equivalence. The spaces are defined as follows.

Let μ be a positive finite Borel measure on the unit circle $\partial\mathbb{D}$. Denote by $P\mu(z)$ the Poisson integral of μ evaluated at z,

$$P\mu(z) = \int_0^{2\pi} \frac{1-|z|^2}{|\mathrm{e}^{\mathrm{i}\theta} - z|^2} \mathrm{d}\mu(\theta).$$

The holomorphic functions on $\mathbb{D}$ for which the integral

$$D_\mu(f) = \frac{1}{\pi} \int\int_{\mathbb{D}} |f'(x+\mathrm{i}y)|^2 P\mu(x+\mathrm{i}y)\mathrm{d}x\mathrm{d}y$$

is finite is a linear subspace of the Hardy space H^2. It forms a Hilbert space, denoted by $D(\mu)$ for the norm

$$\|f\|^2_{D(\mu)} = \|f\|^2_{H^2} + D_\mu(f).$$

The Dirichlet space D is $D(\mathrm{d}\theta/2\pi)$, i.e., take for μ the normalized Lebesgue measure on $\partial\mathbb{D}$.

Shimorin proved the following theorem in [14].

THEOREM 5.1. ([14]) *The reproducing kernels of all the spaces $D(\mu)$ are NP kernels.*

In view of this and the previous results, the following proposition establishes the existence of multipliers of minimal norm on each of these spaces.

PROPOSITION 5.2. *All $D(\mu)$ spaces contain unbounded functions.*

Proof. For $w \in \mathbb{D}$ let

$$f(z,w) = \log\left(\frac{1}{1-\overline{w}z}\right).$$

Note that

$$d(z,w) = \frac{1}{\overline{w}z} f(z,w)$$

is the kernel function of the Dirichlet space D.

Let μ be a positive finite Borel measure on $\partial\mathbb{D}$. We have

$$\begin{aligned}\|f(\cdot,w)\|^2_{D(\mu)} &= |w|^2\|d(\cdot,w)\|^2_{H^2} + D_\mu(f(\cdot,w))\\ &\leqslant |w|^2\|d(\cdot,w)\|^2_{D} + D_\mu(f(\cdot,w))\\ &= f(w,w) + D_\mu(f(\cdot,w)).\end{aligned}$$

Let h denote the Poisson integral of μ and $z = x + \mathrm{i}y$. We have:

$$\begin{aligned}D_\mu(f(\cdot,w)) &= \frac{1}{\pi}\iint_{\mathbb{D}}\left|\frac{\partial f(z,w)}{\partial z}\right|^2 h(z)\mathrm{d}x\mathrm{d}y = \frac{1}{\pi}\int_0^1\int_0^{2\pi}\left|\frac{\overline{w}}{1-\overline{w}r\mathrm{e}^{\mathrm{i}\theta}}\right|^2 h(r\mathrm{e}^{\mathrm{i}\theta})r\mathrm{d}\theta\mathrm{d}r\\ &= \frac{1}{\pi}\int_0^1\int_0^{2\pi}\left|\frac{\overline{w}}{\mathrm{e}^{\mathrm{i}\theta}-wr}\right|^2 h(r\mathrm{e}^{\mathrm{i}\theta})r\mathrm{d}\theta\mathrm{d}r = \int_0^1\frac{2|w|^2 r}{1-|w|^2r^2}h(r^2w)\mathrm{d}r.\end{aligned}$$

Let $w = \rho\mathrm{e}^{\mathrm{i}\alpha}$. We have

$$\begin{aligned}\int_0^{2\pi}\int_0^1\frac{2|w|^2r}{1-|w|^2r^2}h(r^2w)\mathrm{d}r\mathrm{d}\alpha &= \int_0^1\int_0^{2\pi}\frac{2\rho^2r}{1-\rho^2r^2}h(r^2\rho\mathrm{e}^{\mathrm{i}\alpha})\mathrm{d}\alpha\mathrm{d}r\\ &= \int_0^1\frac{2\rho^2r}{1-\rho^2r^2}2\pi h(0)\mathrm{d}r = 2\pi h(0)\log\left(\frac{1}{1-\rho^2}\right).\end{aligned}$$

So, for each $\rho \in]0,1[$ there is a $\alpha \in [0,2\pi[$ such that for $w = \rho\mathrm{e}^{\mathrm{i}\alpha}$ we have

$$D_\mu(f(\cdot,w)) \leqslant h(0)f(w,w).$$

Thus, letting $|w|$ tend to 1 and choosing an appropriate argument for w,

$$\frac{f(w,w)}{\|f(\cdot,w)\|_{D(\mu)}} \geqslant \sqrt{\frac{f(w,w)}{1+h(0)}}$$

tends to infinity, and from the Cauchy-Schwarz inequality we have that

$$|g(x)| \leqslant \|g\|\|k_x\|$$

when g belongs to a Hilbert space with reproducing kernel k. When k is the reproducing kernel of $D(\mu)$, the above calculations show that $k(w,w)$ will tend to infinity for an adequate choice of values of w. The conclusion now follows from Proposition 4.3. ∎

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ANTÓNIO SERRA
Dep. Matemática – I.S.T.
Av. Rovisco Pais
1049-001 Lisboa
PORTUGAL

E-mail: aserra@math.ist.utl.pt

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