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COMMON HYPERCYCLIC VECTORS FOR COMPOSITION OPERATORS

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ABSTRACT. We study the existence of a common hypercyclic vector for different families of composition operators. We also give a continuous version of Salas theorem on weighted shifts.

KEYWORDS: *Composition operators, hypercyclic vector.*

MSC (2000): 47A16, 47B33.

1. INTRODUCTION

A continuous operator acting on a topological vector space X is called *hypercyclic* provided there exists a vector $x \in X$ such that its orbit $\{T^n x : n \geqslant 1\}$ is dense in X. Such a vector is called a *hypercyclic vector* for T. The set of hypercyclic vectors will be denoted by $\mathrm{HC}(T)$. The first example of hypercyclic operator was given by Birkhoff in 1929 ([4]), who has shown that the operator of translation by a non-zero complex number is hypercyclic on the space of all entire functions. For a complete account on hypercyclicity, we refer to [9].

The main focus of our study is the hypercyclic behavior for composition operators. Let us denote by $H^2(\mathbb{D})$ the Hardy space on the unit disk $\mathbb{D}$, and by $\mathrm{Aut}(\mathbb{D})$ the set of automorphisms of $\mathbb{D}$. For φ in $\mathrm{Aut}(\mathbb{D})$, the hypercyclicity of the composition operator C_φ defined on $H^2(\mathbb{D})$ by $C_\varphi(f) = f \circ \varphi$ is well understood since the work of Bourdon and Shapiro ([6]).

THEOREM 1.1. *C_φ is hypercyclic on $H^2(\mathbb{D})$ if and only if φ has no fixed point in $\mathbb{D}$.*

This theorem strengthens a previous result of Seidel and Walsh ([14]), who proved the same theorem for C_φ acting on the space of holomorphic functions on $\mathbb{D}$.

We will concentrate on the common hypercyclicity of a family of operators. Given a family $(T_\lambda)_{\lambda\in\Lambda}$ of hypercyclic operators on X, we ask whether it is possible to find a single vector x which is hypercyclic for each T_λ. Observe that if the family is countable, and if X is an F-space, a Baire category argument implies that this is always possible: indeed, it turns out that $\mathrm{HC}(T)$ is either empty or a dense G_δ set. For uncountable families, the first positive result was obtained by E. Abakumov and J. Gordon ([1]), improving a theorem of Rolewicz, as follows.

THEOREM 1.2. *Let B be the backward shift acting on ℓ^2, defined by the formula $B(x_i)_{i\geqslant 0} = (x_{i+1})_{i\geqslant 0}$. There exists a common hypercyclic vector for the operators λB, $\lambda > 1$.*

In Section 2, we will recall the construction made in the paper of Abakumov and Gordon. We will deduce a criterion for common hypercyclicity of multiples of a single operator, and we will apply this criterion to adjoints of multipliers. Section 3 is devoted to some positive and negative results for the problem of simultaneous hypercyclicity of composition operators. In particular, Theorem 3.3 below is a simultaneous version of a theorem of Seidel and Walsh. Let us mention that the situation here is more complicated than in Birkhoff's theorem, since you have to handle not only translations, but also homotheties. Finally, in Section 4, we provide some remarks and problems. In particular, we give a continuous analog to some well-known theorems on weighted shifts.

2. ADJOINTS OF MULTIPLIERS

2.1. THE SIZE OF THE SET OF COMMON HYPERCYCLIC VECTORS.

We begin by the following result, suggested by J. Saint-Raymond (a particular case of this result was used in Section 3.4 of [1]).

PROPOSITION 2.1. *Let X be an F-space, $A \subset \mathcal{L}(X)$ such that A is a countable union of compact sets. Then $\bigcap_{T\in A} \mathrm{HC}(T)$ is a G_δ set.*

Proof. Define $M = \big\{(T,x) \in A \times X : x \notin \mathrm{HC}(T)\big\}$, and let $(\mathcal{B}_m)$ be a countable basis of open sets in X. Then

$$M^{\mathrm{c}} = \big\{(T,x) \in A \times X : x \in \mathrm{HC}(T)\big\} = \bigcap_{m\geqslant 1}\bigcup_{n\geqslant 0}\big\{(T,x) : T^n x \in \mathcal{B}_m\big\}.$$

In particular, M^{c} is a G_δ set in $A \times X$. Let us write $M = \bigcup_{k\geqslant 1} F_k$ (respectively $A = \bigcup_{p\geqslant 1} A_p$) where each F_k is closed in $A\times X$ (respectively each A_p is compact). If $\pi : \mathcal{L}(X)\times X \to X$ denotes the projection of $\mathcal{L}(X)\times X$ onto the second coordinate, we deduce that

$$\pi(M) = \pi\Big(\bigcup_{k\geqslant 1} F_k\Big) = \bigcup_{k\geqslant 1}\bigcup_{p\geqslant 1}\pi\big(F_k \cap (A_p \times X)\big).$$

Each set $\pi\big(F_k \cap (A_p \times X)\big)$ is closed in X since A_p is compact and F_k is closed. Therefore, $\pi(M)$ is F_σ. Now, $\pi(M) = \big[\bigcap_{T\in A} \mathrm{HC}(T)\big]^{\mathrm{c}}$, and this gives the result. ∎

The previous proposition does not ensure that $\bigcap_{T\in A} \mathrm{HC}(T)$ is not empty. But as soon as this is the case, we should control the size of this set.

Corollary 2.2. *Let X be an F-space, $A\subset\mathcal{L}(X)$. Assume that:*
(i) *A is a countable union of compact sets;*
(ii) *$\bigcap_{T\in A} \mathrm{HC}(T)\neq\emptyset$;*
(iii) *there exists $S\in A$ which commutes with all $T\in A$.*
Then $\bigcap_{T\in A} \mathrm{HC}(T)$ is residual.

Proof. Pick $x\in\bigcap_{T\in A} \mathrm{HC}(T)$, and S as in (iii). It is straightforward to check that the dense set $\{S^k x : k\geqslant 1\}$ is contained in $\bigcap_{T\in A} \mathrm{HC}(T)$. ∎

2.2. Abakumov-Gordon's construction. Our proofs will be constructive ones. We need the following approximation tool, which is the main construction done in the paper of Abakumov and Gordon.

Lemma 2.3. *There exist an integer $k_0\geqslant 1$ and a function $j:\{n\in\mathbb{N}: n\geqslant k_0\}\to\mathbb{N}$ such that, for any sequence $(\alpha_l)_{l\geqslant 1}$ of positive real numbers, there exists a sequence $(M_k)_{k\geqslant k_0}$ of positive integers and a sequence $(r_k)_{k\geqslant k_0}$ of positive real numbers satisfying:*
(i) *(M_k) is increasing, $M_{k+1}-M_k\to+\infty$;*
(ii) *(r_k) is decreasing, $\frac{r_{k+1}}{r_k}\to 0$;*
(iii) *for any $l\in\mathbb{N}$, $\varepsilon>0$, $\lambda>1$, $K>0$, there exists $k>K$ such that*

$$j(k)=l \quad \textit{and} \quad |\lambda^{M_k}r_k-\alpha_l|<\varepsilon.$$

j is a *choice function.* This lemma can be seen as an uncountable Baire type theorem. It is trivial that

$$\forall\lambda>1,\ \exists (M_k)_{k\in\mathbb{N}}\in\mathbb{N},\ (r_k)_{k\in\mathbb{N}}\in\mathbb{R} \text{ such that } \{\lambda^{M_k}r_k\} \text{ is dense in } \mathbb{R}_+.$$

Lemma 2.3 says

$$\exists (M_k)_{k\in\mathbb{N}}\in\mathbb{N},\ (r_k)_{k\in\mathbb{N}}\in\mathbb{R} \text{ such that } \forall\lambda>1,\ \{\lambda^{M_k}r_k\} \text{ is dense in } \mathbb{R}_+.$$

We will also need an additive version of this result, obtained by setting $X_k=-\ln r_k$.

Lemma 2.4. *There exist an integer $k_0\geqslant 1$, a function $j:\{n\in\mathbb{N}: n\geqslant k_0\}\to\mathbb{N}$, a sequence $(M_k)_{k\geqslant k_0}$ of positive integers and a sequence $(X_k)_{k\geqslant k_0}$ of real numbers such that:*
(i) *(M_k) is increasing, $M_{k+1}-M_k\to+\infty$;*
(ii) *(X_k) is increasing, $X_{k+1}-X_k\to+\infty$;*
(iii) *for any $l\in\mathbb{N}$, $\varepsilon>0$, $a>0$, $K>0$, there exists $k>K$ such that*

$$j(k)=l \quad \textit{and} \quad \big|M_k a-X_k\big|<\varepsilon.$$

2.3. A criterion for common hypercyclicity. If one wants to show that an operator T is hypercyclic, the most useful tool is the hypercyclicity criterion formulated first by C. Kitai (see [10] for the original statement, or Corollary 1.5 from [8] for a more general one). We give here a sufficient condition for the existence of a common hypercyclic vector for all multiples of an operator with a dense generalized kernel.

THEOREM 2.5. *Let X be a separable Banach space, and $T \in L(X)$. Assume that:*

(i) $V = \bigcup_n \mathrm{Ker}(T^n)$ *is dense in X;*

(ii) *there exists $S : V \to X$ with $TS = \mathrm{Id}_V$ and $\|Sx\| \leqslant \|x\|$ for all x in V.*

Then $\bigcap_{\lambda>1} \mathrm{HC}(\lambda T)$ is a dense G_δ set.

Proof. By Corollary 2.2, it is enough to prove that $\bigcap_{\lambda>1} \mathrm{HC}(\lambda T)$ is non-empty. Fix (v_l) a dense sequence in V, and set $\alpha_l = \|v_l\|$. Lemma 2.3 gives a function j and sequences (M_k) and (r_k). For $k \geqslant k_0$, let us set:

$$d_k = r_k - r_{k+1} \geqslant 0;$$

$$w_k = v_{j(k)} \text{ if } T^{M_{k+1}-M_k} v_{j(k)} = 0, \quad w_k = 0 \text{ otherwise};$$

$$y_k = \frac{d_k}{\|w_k\|} S^{M_k} w_k \text{ if } w_k \neq 0, \quad y_k = 0 \text{ otherwise}.$$

We claim that $f = \sum_{m \geqslant k_0} y_m$ is hypercyclic for each λT, with $\lambda > 1$. First, observe that if $m < k$, $T^{M_k} S^{M_m} w_m = T^{M_k - M_m} w_m = 0$, which implies

$$\|T^{M_k} f\| = \Big\| \sum_{m \geqslant k} \frac{d_m}{\|w_m\|} T^{M_k} S^{M_m} w_m \Big\| \leqslant \sum_{m \geqslant k} d_k = r_k.$$

Take now $\varepsilon > 0$ and $l \in \mathbb{N}$. By Lemma 2.3, there exists $k \in \mathbb{N}$ such that $j(k) = l$ and $w_k = v_l$, $|\lambda^{M_k} r_k - \|v_l\|| \leqslant \varepsilon$, and $\frac{r_{k+1}}{r_k}(\varepsilon + \|v_l\|) \leqslant \varepsilon$. Then

$$\begin{aligned}
\|(\lambda T)^{M_k} f - v_l\| &\leqslant \Big\| \lambda^{M_k} \frac{d_k}{\|v_l\|} v_l - v_l \Big\| + \Big\| \sum_{m>k} \lambda^{M_k} \frac{d_m}{\|w_m\|} T^{M_k} S^{M_m} w_m \Big\| \\
&\leqslant \|v_l\| \Big(\Big| \frac{\lambda^{M_k} r_k}{\|v_l\|} - 1 \Big| + \frac{\lambda^{M_k} r_{k+1}}{\|v_l\|} \Big) + \lambda^{M_k} r_{k+1} \\
&\leqslant \varepsilon + 2\lambda^{M_k} r_{k+1} \leqslant \varepsilon + 2(\varepsilon + \|v_l\|) \frac{r_{k+1}}{r_k} \leqslant 3\varepsilon.
\end{aligned}$$

This achieves to prove that f is hypercyclic for λT. ∎

Hypercyclic operators are strongly connected with the existence of invariant subspaces. The following corollary illustrates this link.

COROLLARY 2.6. *Under the assumptions of Theorem 2.5, there exists a dense subspace of X, invariant by T, whose elements, except 0, are hypercyclic vectors for λT, with $\lambda > 1$.*

Proof. Take x in $\bigcap_{\lambda>1} \mathrm{HC}(\lambda T)$. Then

$$M = \{p(T)x : p \text{ is a polynomial}\}$$

answers the question: the proof given by P.S. Bourdon in [5] also works in this setting. ∎

2.4. Application to adjoints of multipliers.

COROLLARY 2.7. *Let φ be a nonconstant inner function, and M_φ the associated multiplier on $H^2(\mathbb{D})$ (defined by $M_\varphi(f) = \varphi f$). Then $\bigcap_{\lambda>1} \mathrm{HC}(\lambda M_\varphi^*)$ is a residual set.*

By choosing $\varphi(z) = z$, we retrieve Theorem 1.2.

Proof. It is plain that $\ker(M_\varphi^*)^n = (\varphi^n H^2)^\perp$. Let us recall the following result from pp. 34–35 of [11]: let E be a normed space, and (E_n) a sequence of subspaces of E. We define:

$$\underline{\lim}\, E_n = \bigl\{x \in E : \lim_n \operatorname{dist}(x, E_n) = 0\bigr\}.$$

If $E = H^2$, and if $E_n = (\theta_n H^2)^\perp$, where (θ_n) is a sequence of inner functions, then the following equivalence holds

$$\underline{\lim}\bigl(\theta_n H^2(\mathbb{D})\bigr)^\perp = H^2(\mathbb{D}) \Leftrightarrow \forall z \in \mathbb{D},\ \lim_n \theta_n(z) = 0.$$

In our context, $\theta_n = \varphi^n$, and $(\varphi^n H^2)^\perp \subset (\varphi^{n+1} H^2)^\perp$. So, $\underline{\lim}(\varphi^n H^2)^\perp \subset \bigcup_n (\varphi^n H^2)^\perp$. Now, since φ is not constant, for each z in $\mathbb{D}$, $\varphi^n(z) \to 0$, and then

$$H^2(\mathbb{D}) \subset \overline{\bigcup_n \operatorname{Ker}\bigl((M_\varphi^*)^n\bigr)}.$$

Thus (i) of Theorem 2.5 is satisfied.

If $f \in V$, and $g \in H^2(\mathbb{D})$, then one has

$$\langle g, M_\varphi^* M_\varphi f\rangle = \langle M_\varphi g, M_\varphi f\rangle = \langle g, f\rangle \quad \text{since } \varphi \text{ is inner.}$$

So we can take $S = M_\varphi$ in part (ii) of Theorem 2.5. ∎

3. COMPOSITION OPERATORS

3.1. Geometry of the disk.

For details on the background material of this section, we refer to [15]. The automorphisms of $\mathbb{D}$ can be classified in function of their fixed points. $\varphi \in \mathrm{Aut}(\mathbb{D})$ is called:

- *parabolic* if φ has a single (attractive) fixed point on $\mathbb{T} = \partial\mathbb{D}$;
- *hyperbolic* if φ has an attractive fixed point on $\mathbb{T}$, and a second fixed point on $\mathbb{T}$;
- *elliptic* if φ has an attractive fixed point in $\mathbb{D}$.

We are concerned by parabolic and hyperbolic automorphisms. It is easier to describe their action on the right half-plane $\mathbb{C}_+$. Denote by $\sigma : \mathbb{D} \to \mathbb{C}_+$, $\sigma(z) = \frac{1+z}{1-z}$ the Cayley map from $\mathbb{D}$ onto $\mathbb{C}_+$. For $\varphi \in \mathrm{Aut}(\mathbb{D})$ with $+1$ as attractive fixed point, set $\psi = \sigma \circ \varphi \circ \sigma^{-1}$. Then one has:

- $\psi(z) = z + ia$ where $a \in \mathbb{R}$, $a \neq 0$, if φ is parabolic (a parabolic automorphism of $\mathbb{D}$ is conjugated to a translation);

• $\psi(z) = \lambda(z - \mathrm{i}b) + \mathrm{i}b$, where $\lambda > 1$ and $b \in \mathbb{R}$, if φ is hyperbolic (a hyperbolic automorphism of $\mathbb{D}$ is conjugated to a positive dilation).

3.2. MAIN STATEMENTS. In view of Theorem 1.1, a natural question appears:

Does there exist a common hypercyclic vector for all composition operators C_φ on $H^2(\mathbb{D})$, where $\varphi \in \mathrm{Aut}(\mathbb{D})$ has no fixed point in $\mathbb{D}$?

Here, you can play with two parameters: you can choose the attractive fixed point, and its attractivity (the scalars λ, a and b of the previous paragraph). The following result shows that it is impossible to have a wide set of attractive fixed points.

THEOREM 3.1. *Let A be a subset of $\mathrm{Aut}(\mathbb{D})$ such that, for any φ in A, φ has no fixed point in $\mathbb{D}$. Let B be the set of attractive fixed points of elements of A*

$$B = \{\omega \in \mathbb{T} : \exists \varphi \in A \text{ such that } \omega \text{ is the attractive fixed point of } \varphi\}.$$

If B has positive Lebesgue measure, then $\bigcap_{\varphi \in A} \mathrm{HC}(C_\varphi) = \emptyset$.

Here, C_φ is considered as a composition operator on $H^2(\mathbb{D})$.

Proof. The theorem is a direct consequence of the following lemma, since a function of $H^2(\mathbb{D})$ admits angular limits almost everywhere on the boundary. ∎

LEMMA 3.2. *Suppose that $\varphi \in \mathrm{Aut}(\mathbb{D})$, and that $\omega \in \mathbb{T}$ is the attractive fixed point of φ. If $f \in H^2(\mathbb{D})$ is a hypercyclic vector for C_φ, then f has no angular limit at ω.*

Proof. We denote $\varphi_n = \varphi \circ \cdots \circ \varphi$ (n times). By Denjoy-Wolff's Theorem, $(\varphi_n(0))$ converges non-tangentially to ω. Now, evaluation at 0 is continuous on $H^2(\mathbb{D})$, and by hypercyclicity of f, there exist integers m and n, as large as necessary, such that:

$$|f \circ \varphi_m(0) - 0| < \frac{1}{4} \quad \text{and} \quad |f \circ \varphi_n(0) - 1| < \frac{1}{4}.$$

In particular, f does not admit any non-tangential limit at ω. ∎

So, essentially we have to fix the attractive fixed point, say $+1$, and the question becomes:

Does there exist a common hypercyclic vector for all composition operators C_φ on $H^2(\mathbb{D})$, where $\varphi \in \mathrm{Aut}(\mathbb{D})$ has $+1$ as attractive fixed point?

We are not able to give a positive or a negative answer to this question. But if we relax the conditions on the space, this will be the case. On the one hand, we can forget the growth condition: if $\varphi \in \mathrm{Aut}(\mathbb{D})$, C_φ is a composition operator on $H(\mathbb{D})$, the F-space of holomorphic functions on $\mathbb{D}$. By the Seidel and Walsh Theorem, we know that such a composition operator is hypercyclic. Under these assumptions, there exists a common hypercyclic vector.

THEOREM 3.3. *Let $\omega \in \mathbb{T}$. There exists a common hypercyclic vector for all composition operators C_φ acting on $H(\mathbb{D})$, where $\varphi \in \mathrm{Aut}(\mathbb{D})$ admits ω as attractive fixed point. Moreover, the set of common hypercyclic vectors is a residual set.*

On the other hand, we can ignore the regularity condition: by results of Nordgren ([12]), C_φ is also a composition operator on $L^2(\mathbb{T})$. An application of Kitai's criterion should prove its hypercyclicity. We directly prove a simultaneous hypercyclicity theorem.

THEOREM 3.4. *Let $\omega \in \mathbb{T}$. There exists a common hypercyclic vector for all composition operators C_φ acting on $L^2(\mathbb{T})$, where $\varphi \in \mathrm{Aut}(\mathbb{D})$ admits ω as attractive fixed point. Moreover, the set of common hypercyclic vectors is a residual set.*

The remaining part of this section is devoted to the proof of the previous theorems. We will assume that $\omega = +1$.

3.3. PROOF OF THE HOLOMORPHIC CASE.

We take the model of the half-plane. Define T_a and $S_{\lambda,b}$ by

$$T_a(f)(z) = f(z + \mathrm{i}a),$$
$$S_{\lambda,b}(f)(z) = f(\lambda(z - \mathrm{i}b) + \mathrm{i}b).$$

It suffices to show that $\bigcap_{a\neq 0} \mathrm{HC}(T_a)$ and $\bigcap_{\substack{\lambda>1\\ b\in\mathbb{R}}} \mathrm{HC}(S_{\lambda,b})$ are dense G_δ sets. Until the end of this section, we fix (δ_k), $0 < \delta_k < 1$, a sequence which converges to 0, and (P_l) a sequence in $H(\mathbb{C}_+)$ such that, for any $\mu \geqslant 1$ and any $\tau \in \mathbb{R}$, $\big(P_l(\mu z - \mu \mathrm{i}\tau)\big)$ is dense in $H(\mathbb{C}_+)$ (for example, (P_l) could be the sequence of all polynomials with coefficients in $\mathbb{Q} + \mathrm{i}\mathbb{Q}$). We handle separately the parabolic and the hyperbolic case.

3.3.1. PARABOLIC AUTOMORPHISMS.

By Corollary 2.2, it is enough to prove for instance that $\bigcap_{a>0} \mathrm{HC}(T_a)$ is not empty. We fix sequences (M_k) and (X_k) as in Lemma 2.4. For $k \geqslant k_0 + 1$, let us set

$$R_k = \min\Big(\frac{X_{k+1} - X_k}{2}, \frac{X_k - X_{k-1}}{2}\Big).$$

We build by induction rectangles C_k, D_k and Γ_k, for $k \geqslant k_0 + 1$, beginning by the initialization $\Gamma_{k_0} = \{(1, 0)\}$. For $k \geqslant k_0 + 1$, fix C_k the square whose center is $(R_k/2, 0)$ and whose side has length $R_k - \delta_k$. Observe that, for any compact subset K of $\mathbb{C}_+$, for k large enough, K is contained in C_k. Set $D_k = C_k + \mathrm{i}X_k$. The squares (D_k) are disjoint. Moreover, there exists a rectangle Γ_k which contains Γ_{k-1}, D_k, but which has empty intersection with D_{k+1}.

We then define a sequence $(\pi_k)_{k\geqslant k_0}$ of polynomials. First, we set $\pi_{k_0}(z) = 1$. Next, for $k > k_0$, Runge's Theorem gives a polynomial π_k satisfying:

$$|\pi_k(z) - P_l(z - \mathrm{i}X_k)| \leqslant \frac{1}{2^k} \quad \text{if } z \in D_k \quad \text{and} \quad j(k) = l;$$
$$|\pi_k(z) - \pi_{k-1}(z)| \leqslant \frac{1}{2^k} \quad \text{if } z \in \Gamma_{k-1}.$$

The sequence (π_k) converges uniformly on each compact subset of $\mathbb{C}_+$. Let us denote by f its limit. Observe that, for each $z \in \Gamma_k$, we have

$$|f(z) - \pi_k(z)| \leqslant |\pi_k(z) - \pi_{k+1}(z)| + |\pi_{k+1}(z) - \pi_{k+2}(z)| + \cdots \leqslant \frac{1}{2^k}.$$

We claim that f is hypercyclic for each T_a, with $a > 0$. Indeed, fix $l \in \mathbb{N}$, K a compact subset of $\mathbb{C}_+$, and $\eta > 0$ such that $K_1 = K + \overline{B}(0, \eta) \subset \mathbb{C}_+$. Let $0 < \delta < \eta$ with

$$z_1, z_2 \in K_1 \wedge |z_1 - z_2| \leqslant \delta \Rightarrow |P_l(z_1) - P_l(z_2)| \leqslant \varepsilon.$$

There exists an integer k such that $j(k) = l$, $\frac{1}{2^k} \leqslant \varepsilon$, $K_1 \subset C_k$, and $|aM_k - X_k| \leqslant \delta$. Then, for $z \in K$, $z + \mathrm{i}M_k a - \mathrm{i}X_k \in K_1 \subset C_k$, and therefore $z + \mathrm{i}M_k a \in D_k$. This implies that

$$\begin{aligned}
|[T_a(f)]^{M_k}(z) - P_l(z)| &\leqslant \varepsilon + |\pi_k(z + \mathrm{i}M_k a) - P_l(z)| \\
&\leqslant 2\varepsilon + |P_l(z + \mathrm{i}M_k a - \mathrm{i}X_k) - P_l(z)| \leqslant 3\varepsilon.
\end{aligned}$$

3.3.2. Hyperbolic automorphisms. Here, dilations do not commute, and we need to prove that $\bigcap_{\lambda,b} \mathrm{HC}(S_{\lambda,b})$ is dense. First, by applying Lemma 2.3 to the sequence (α_l) identically one, one gets sequences (M_k) and (r_k). For $k \geqslant k_0 + 1$, we set

$$R_k = \delta_k \min\left(\frac{\sqrt{\frac{r_{k-1}}{r_k}} - 1}{\sqrt{\frac{r_{k-1}}{r_k}} + 1}, \frac{1 - \sqrt{\frac{r_{k+1}}{r_k}}}{\sqrt{\frac{r_{k+1}}{r_k}} + 1}\right).$$

We always fix $\Gamma_{k_0} = \{(1,0)\}$, and for $k \geqslant k_0$, let C_k be the hyperbolic disk whose center is $(1,0)$ and whose radius is R_k

$$C_k = \left\{z \in \mathbb{C}_+ : \frac{|z-1|}{|z+1|} \leqslant R_k\right\}.$$

Let D_k be the image of C_k by the homothety of center 0 and of ratio $\frac{1}{r_k}$. Then (D_k) are disjoint sets, and by construction there exists a rectangle Γ_k which contains Γ_{k-1} and D_k, but whose intersection with D_{k+1} is empty (see Figure 1).

Finally, we set $\pi_{k_0}(z) = 1$, and if $k > k_0$, $l = j(k)$, Runge's Theorem gives us a polynomial π_k which satisfies:

$$\begin{aligned}
|\pi_k(z) - P_l(r_k z)| &\leqslant \frac{1}{2^k} \quad \text{if } z \in D_k, \\
|\pi_k(z) - \pi_{k-1}(z)| &\leqslant \frac{1}{2^k} \quad \text{if } z \in \Gamma_{k-1}.
\end{aligned}$$

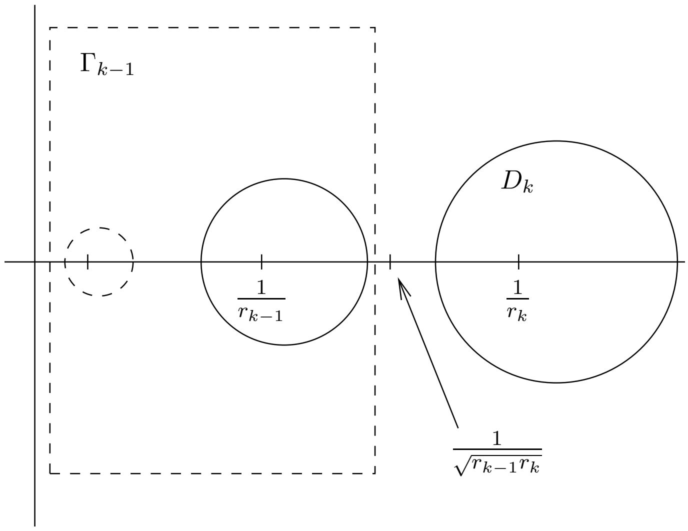

Figure 1. The hyperbolic construction

As previously, (π_k) converges uniformly on each compact of $\mathbb{C}_+$ to a function f, with

$$|f(z) - \pi_k(z)| \leqslant \frac{1}{2^k} \quad \text{if } z \in \Gamma_k.$$

For $\mu > 1$, we claim that $g(z) = f(\mu z)$ is hypercyclic for each $S_{\lambda,b}$, $\lambda > 1$, $b \in \mathbb{R}$. Indeed, fix $l \in \mathbb{N}$, $\varepsilon > 0$, K a compact subset of $\mathbb{C}_+$ and $\eta > 0$ such that $K_1 = K + \overline{B}(0,\eta) \subset \mathbb{C}_+$. Let $0 < \delta < \eta$ with

$$z_1, z_2 \in K_1 \wedge |z_1 - z_2| \leqslant \delta \Rightarrow |P_l(\mu z_1 - \mu \mathrm{i}b) - P_l(\mu z_2 - \mu \mathrm{i}b)| \leqslant \varepsilon.$$

By Lemma 2.3, there exists an integer k such that $j(k) = l$, $\frac{1}{2^k} \leqslant \varepsilon$, $\mu\lambda^{M_k} r_k(K - \mathrm{i}b) + \mu r_k \mathrm{i}b \subset C_k$, and, moreover, if M is such that $z \in K \Rightarrow |z| \leqslant M$, then

$$\mu|\lambda^{M_k} r_k - 1|(M + |b|) + \mu r_k |b| < \delta.$$

Then, if $z \in K$, one has $\mu\lambda^{M_k}(z - \mathrm{i}b) + \mu \mathrm{i}b \in D_k \subset \Gamma_k$, and so

$$\begin{aligned}
|[S_{\lambda,b}(g)]^{M_k}(z) - P_l(\mu z - \mu \mathrm{i}b)| &= |f\big(\mu\lambda^{M_k}(z - \mathrm{i}b) + \mu \mathrm{i}b\big) - P_l(\mu z - \mu \mathrm{i}b)| \\
&\leqslant \varepsilon + |\pi_k(\mu\lambda^{M_k}(z - \mathrm{i}b) + \mu \mathrm{i}b) - P_l(\mu z - \mu \mathrm{i}b)| \\
&\leqslant 2\varepsilon + |P_l(\mu\lambda^{M_k} r_k(z - \mathrm{i}b) + \mu r_k \mathrm{i}b) - P_l(\mu z - \mu \mathrm{i}b)| \leqslant 3\varepsilon,
\end{aligned}$$

where the last inequality comes from

$$|\mu\lambda^{M_k} r_k(z - \mathrm{i}b) + \mu r_k \mathrm{i}b - \mu z - \mu \mathrm{i}b| \leqslant \mu|\lambda^{M_k} r_k - 1|\big(|z| + |b|\big) + \mu r_k |b| < \delta.$$

Therefore, $\{f(\mu z) : \mu \geqslant 1\} \subset \bigcap_{\substack{\lambda > 1 \\ b \in \mathbb{R}}} \mathrm{HC}(S_{\lambda,b})$, and $\{f(\mu z) : \mu \geqslant 1\}$ is dense in $H(\mathbb{C}_+)$ since f is hypercyclic for $S_{2,0}$.

3.4. PROOF OF THE L^2-CASE. Let λ_i be the probability measure on $\mathbb{R}$ defined by $\mathrm{d}\lambda_i(t) = \pi^{-1}(1+t^2)^{-1}\,\mathrm{d}t$ (λ_i is the image of the Lebesgue measure on $\mathbb{T}$ by σ). Notice that $f \in L^2(\mathbb{T}) \Leftrightarrow f\circ\sigma^{-1} \in L^2(\mathbb{R}, \mathrm{d}\lambda_i)$, and that

$$\int_{-\infty}^{+\infty} |f\circ\sigma^{-1}(\mathrm{i}t)|^2\,\mathrm{d}\lambda_i(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\mathrm{e}^{\mathrm{i}\theta})|^2\,\mathrm{d}\theta.$$

Let us change the notation to avoid the integration on $\mathrm{i}\mathbb{R}$. For $\lambda > 1$, $a \in \mathbb{R}$, $a \neq 0$, and $b \in \mathbb{R}$, we now set

$$T_a(f)(x) = f(x+a),$$
$$S_{\lambda,b}(f)(x) = f(\lambda(x-b)+b).$$

We prove a slightly more precise result.

THEOREM 3.5. *Let $p \geqslant 1$, $\alpha > \frac{1}{2}$, and consider T_a and $S_{\lambda,b}$ as operators on $L^p\Big(\mathbb{R}, \frac{\mathrm{d}t}{(1+t^2)^\alpha}\Big)$. Then $\bigcap_{a\neq 0} \mathrm{HC}(T_a)$ and $\bigcap_{\substack{\lambda>1\\ b\in\mathbb{R}}} \mathrm{HC}(S_{\lambda,b})$ are dense G_δ sets in $L^p\Big(\mathbb{R}, \frac{\mathrm{d}t}{(1+t^2)^\alpha}\Big)$.*

Taking $p = 2$ and $\alpha = 1$ gives exactly Theorem 3.4. The following lemma will be useful for our purpose.

LEMMA 3.6. *Let $(v_k)_{k\geqslant 1}$ be a non-decreasing sequence of positive numbers, which tends to $+\infty$. Then there exists a non-decreasing sequence $(u_k)_{k\geqslant 1}$ of positive numbers which tends to $+\infty$, and such that:*

(i) $\sum_{k\geqslant 1} \frac{u_k}{k^3} < +\infty$;

(ii) $\frac{u_k}{v_k^3} \to 0$ *for* $k \to +\infty$;

(iii) $\sum_{m>k} \frac{u_m}{((m-k)+v_m)^3} \to 0$ *for* $k \to +\infty$.

Proof. For $k \geqslant 1$, we set $u'_k = \inf(k, v_{[k/2]}, v_{[k/2]+1}, \dots, v_k)$, and $u_k = \inf_{l\geqslant k} u'_l$. Assertions (i) and (ii) are trivial. For (iii)

$$\sum_{m>2k} \frac{u_m}{((m-k)+v_m)^3} = \sum_{m>k} \frac{u_{m+k}}{(m+v_{m+k})^3} \leqslant \sum_{m>k} \frac{1}{m^2} \to 0;$$
$$\sum_{k<m\leqslant 2k} \frac{u_m}{((m-k)+v_m)^3} \leqslant v_k \sum_{m\leqslant k} \frac{1}{(m+v_k)^3} \leqslant \frac{C}{v_k} \to 0. \qquad \blacksquare$$

3.4.1. PARABOLIC AUTOMORPHISMS. First, we prove that $\bigcap_{a>0} \mathrm{HC}(T_a)$ is not empty (and therefore is a dense G_δ set) for $\alpha = 2$. We set $\mathrm{d}\mu = \frac{\mathrm{d}t}{(1+t^2)^2}$, and let $C > 0$ be a constant such that, for $x > 0$, $\int_x^{+\infty} \mathrm{d}\mu \leqslant \frac{C}{x^3}$. We consider sequences (M_k), (X_k) as in Lemma 2.4. In particular, we will assume that $X_k \geqslant k$.

For $k \geqslant k_0$, let us define $R_k = \inf\left(\frac{X_{k+1}-X_k}{2}, \frac{X_k-X_{k-1}}{2}, \frac{X_k}{2}\right)$. Without lost of generality, we can always assume that (R_k) is increasing. Next, (u_k) is defined by applying Lemma 3.6 to the sequence (v_k) with $v_k = R_k - 2$. We fix (f_l) a dense sequence in $L^p(\mathbb{R}, \mathrm{d}\mu)$ of compactly supported bounded functions, with $\|f_l\|_\infty^p \leqslant u_l$. For $k \geqslant k_0$ and $l = j(k)$, let us set:

$$\begin{aligned} w_k &= f_l \quad \text{if } \operatorname{supp} f_l \subset [-R_k; R_k], \quad w_k = 0 \text{ otherwise;} \\ h_k(x) &= w_k(x - X_k). \end{aligned}$$

Then (h_k) have mutually disjoint supports, and we define finally $f = \sum_{k \geqslant k_0} h_k$. First of all, $f \in L^p(\mathbb{R}, \mathrm{d}\mu)$. Indeed,

$$\|f\|_p^p \leqslant \sum_{k \geqslant k_0} \int_{X_k/2}^{+\infty} |w_k(x - X_k)|^p \,\mathrm{d}\mu \leqslant C \sum_{k \geqslant k_0} \frac{2^3 u_k}{X_k^3} < +\infty.$$

We claim that f is hypercyclic for T_a, with $a > 0$. Indeed, let $l \in \mathbb{N}$, $\varepsilon > 0$ and $0 < \delta < 1$ whose value will be precised later. There exists $k \geqslant k_0$, as large as necessary, such that $j(k) = l$, $\operatorname{supp} f_l \subset [-R_k, R_k]$, $|M_k a - X_k| \leqslant \delta$, and $X_{m+1} - X_m \geqslant 1$ for $m \geqslant k$. Then

$$\left\|T_a^{M_k} f - f_l\right\|_p \leqslant \left\|T_a^{M_k} h_k - f_l\right\|_p + \Big\|\sum_{m>k} T_a^{M_k} h_m\Big\|_p + \Big\|\sum_{m<k} T_a^{M_k} h_m\Big\|_p.$$

We estimate now the three terms in the right hand side. First,

$$\|T_a^{M_k} h_k - f_l\|_p = \|T_{M_k a - X_k} f_l - f_l\|_p \leqslant \varepsilon$$

as soon as δ is small enough.

Secondly,

$$\begin{aligned} \Big\|\sum_{m>k} T_a^{M_k} h_m\Big\|_p^p &\leqslant \sum_{m>k} \int_{X_m - R_m - M_k a}^{+\infty} |h_m(x + M_k a)|^p \,\mathrm{d}\mu \\ &\leqslant C \sum_{m>k} \frac{u_m}{(X_m - X_k - R_m - 1)^3}. \end{aligned}$$

Observe that

$$X_m - X_k \geqslant X_m - X_{m-1} + \cdots + X_{k+1} - X_k \geqslant 2R_m + m - k - 1.$$

We deduce that

$$\Big\|\sum_{m>k} T_a^{M_k} h_m\Big\|_p^p \leqslant C \sum_{m>k} \frac{u_m}{((m-k) + R_m - 2)^3},$$

and this last quantity is smaller than ε if k is large enough.

Thirdly,

$$\Big\|\sum_{m<k} T_{M_k a} h_m\Big\|_p^p \leqslant \sum_{m<k} u_k \int\limits_{-M_k a+X_m-R_m}^{-M_k a+X_m+R_m} \mathrm{d}\mu$$

$$\leqslant C\frac{u_k}{(M_k a - X_{k-1}-R_{k-1})^3} \qquad \text{(disjoint supports)}$$

$$\leqslant C\frac{u_k}{(X_k - X_{k-1}-R_{k-1}-1)^3}$$

$$\leqslant C\frac{u_k}{(R_k-1)^3} \to 0 \text{ for } k\to+\infty.$$

It remains to prove the case $\alpha \neq 2$. We use a slightly modified classical lemma (see p. 111, "The hypercyclic comparison principle" from [15]), whose proof is straightforward.

LEMMA 3.7. *Let $X \subset Y$ be topological vector spaces, $(\varphi_\lambda)_{\lambda\in\Lambda}$ a family of continuous operators on X and Y. Assume that:*

(i) *the inclusion is continuous;*

(ii) *X is dense in Y,*

(iii) *$f \in X$ is a common hypercyclic vector for the family $(\varphi_\lambda)_{\lambda\in\Lambda}$, considered as operators on X.*

Then f is a common hypercyclic vector for the family $(\varphi_\lambda)_{\lambda\in\Lambda}$, considered as operators on Y.

So, if $\alpha > 2$, we apply the lemma with

$$X = L^p\Big(\mathbb{R}, \frac{\mathrm{d}t}{(1+t^2)^2}\Big), \quad Y = L^p\Big(\mathbb{R}, \frac{\mathrm{d}t}{(1+t^2)^\alpha}\Big).$$

If $\frac{1}{2}<\alpha<2$, set $\varepsilon = \alpha - \frac{1}{2}$. If $f \in L^p\Big(\mathbb{R}, \frac{\mathrm{d}t}{(1+t^2)^\alpha}\Big)$, Hölder's inequality gives

$$\bigg(\int\limits_{\mathbb{R}} \frac{|f|^p}{(1+t^2)^{1/4+3\varepsilon/4}} \frac{\mathrm{d}t}{(1+t^2)^{1/4+\varepsilon/4}}\bigg)^{1/p} \leqslant C\bigg(\int\limits_{\mathbb{R}} \frac{|f|^{2p}}{(1+t^2)^{1/2+3\varepsilon/2}}\mathrm{d}t\bigg)^{1/2p}.$$

Repeated applications of this inequality show that

$$\exists q \geqslant 1,\ \exists \beta \geqslant 2 \text{ such that } L^q\Big(\mathbb{R}, \frac{\mathrm{d}t}{(1+t^2)^\beta}\Big) \subset L^p\Big(\mathbb{R}, \frac{\mathrm{d}t}{(1+t^2)^\alpha}\Big),$$

and the lemma works.

3.4.2. HYPERBOLIC AUTOMORPHISMS. We just prove the case $\alpha = 2$. First, apply Lemma 2.3 with the sequence (α_l) identically one, to obtain sequences (M_k) and (r_k). A variant of Lemma 3.6 gives a nondecreasing sequence (u_k), tending to $+\infty$, and such that:

(i) $\sum\limits_{k\geqslant k_0} \frac{u_k}{k^3} < +\infty$;

(ii) $u_k\sqrt{\frac{r_k}{r_{k-1}}} \to 0$ for $k\to+\infty$;

(iii) for all $b\in\mathbb{R}$, $\sum_{m>k}\frac{u_m}{\left(2^{m-k}\sqrt{\frac{r_k}{r_{k-1}}}-r_kb+b\right)^3}\to 0$ for $k\to+\infty$.

We fix $(f_l)_{l\geqslant 1}$ a sequence of continuous functions, with $\operatorname{supp} f_l\subset\left[-l;-\frac{1}{l}\right]\cup\left[\frac{1}{l};l\right]$, $\|f_l\|_\infty^p\leqslant u_l$, and such that, for any y in $\mathbb{R}$ and any $\mu\geqslant 1$, $(f_l(\mu x+\mu y))_{l\geqslant 1}$ is dense in $L^p(\mathbb{R},\mathrm{d}\mu)$.

For $k>k_0$, let us set $I_k=\left]\frac{1}{\sqrt{r_kr_{k-1}}};\frac{1}{\sqrt{r_kr_{k+1}}}\right[$, and $J_k=-I_k$; (I_k) and (J_k) are two families of mutually disjoint intervals. We define the function f by $f(x)=f_{j(k)}(r_kx)$ if x belongs to $I_k\cup J_k$, and $f(x)=0$ if x is outside $\bigcup_k I_k\cup J_k$.

We claim that f belongs to $L^p(\mathbb{R},\mathrm{d}\mu)$. Indeed, the following inequalities hold:

$$\begin{aligned}\int_0^{+\infty}|f(x)|^p\,\mathrm{d}\mu&\leqslant\sum_{k>k_0}\int_{\frac{1}{\sqrt{r_kr_{k-1}}}}^{+\infty}|f_{j(k)}|^p\,\mathrm{d}\mu\\&\leqslant\sum_{k>k_0}u_k\mu\Big(\Big[\frac{1}{\sqrt{r_kr_{k-1}}};+\infty\Big[\Big)<+\infty.\end{aligned}$$

Fix $\lambda>1$, $b\in\mathbb{R}$. We now prove that f is hypercyclic for $S_{\lambda,b}$. Let $l\in\mathbb{N}$, $\varepsilon>0$, and $0<\delta<\frac{1}{2}$ whose precise value will be determined later. There exists $k>k_0$ such that $j(k)=l$, $|\lambda^{M_k}r_k-1|<\delta$, and, for $m\geqslant k$, $\sqrt{\frac{r_{m-1}}{r_m}}\geqslant 2$. Then

$$\begin{aligned}&\int_{-\infty}^{+\infty}|S_{\lambda,b}^{M_k}(f)(x)-f_l(x-b)|^p\,\mathrm{d}\mu\\
&\leqslant\sum_{m<k}\int_{\lambda^{M_k}(x-b)+b\in I_m}|f_{j(m)}(\lambda^{M_k}r_m(x-b)+r_mb)-f_l(x-b)|^p\,\mathrm{d}\mu\\
&\quad+\int_{\lambda^{M_k}(x-b)+b\in I_k}|f_l(\lambda^{M_k}r_k(x-b)+r_kb)-f_l(x-b)|^p\,\mathrm{d}\mu\\
&\quad+\sum_{m>k}\int_{\lambda^{M_k}(x-b)+b\in I_m}|f_{j(m)}(\lambda^{M_k}r_m(x-b)+r_mb)-f_l(x-b)|^p\,\mathrm{d}\mu+S_1'+S_2'+S_3'\\
&\leqslant S_1+S_2+S_3+S_1'+S_2'+S_3',\end{aligned}$$

where S_i' is the same as S_i, replacing I_m by J_m. Now:

(1) $S_1\leqslant 2^pu_k\mu\Big(\bigcup_{m<k}\frac{I_m-b}{\lambda^{M_k}}+b\Big)$. Since

$$\bigcup_{m<k}\frac{I_m-b}{\lambda^{M_k}}+b\subset\Big[b-\frac{b}{\lambda^{M_k}};b+\frac{1}{\lambda^{M_k}\sqrt{r_kr_{k-1}}}-\frac{b}{\lambda^{M_k}}\Big],$$

we obtain that

$$S_1\leqslant 2^pu_k\frac{1}{\lambda^{M_k}\sqrt{r_kr_{k-1}}}\leqslant 2^{p+1}u_k\sqrt{\frac{r_k}{r_{k-1}}}.$$

For k large enough, $|S_1| \leqslant \varepsilon$.

(2) We have $|\lambda^{M_k} r_k(x-b) + r_k b - (x-b)| \leqslant \delta|x-b| + r_k|b|$. By uniform continuity of f_l, if δ is small enough, and k is large enough, $|S_2| \leqslant \varepsilon$.

(3) We have

$$\begin{aligned} S_3 &\leqslant 2^p \sum_{m>k} u_m \mu\Big(\frac{I_m - b}{\lambda^{M_k}} + b\Big) \leqslant A_1 \sum_{m>k} u_m \frac{1}{\Big(\frac{r_k}{\sqrt{r_m r_{m-1}}} - r_k b + b\Big)^3} \\ &\leqslant A_2 \sum_{m>k} \frac{u_m}{\Big(2^{m-k}\sqrt{\frac{r_k}{r_{k-1}}} - r_k b + b\Big)^3}, \end{aligned}$$

where this last inequality comes from

$$\frac{r_k}{\sqrt{r_m r_{m-1}}} = \sqrt{\frac{r_k}{r_{k+1}}} \times \cdots \times \sqrt{\frac{r_{m-1}}{r_m}} \times \sqrt{\frac{r_k}{r_{m-1}}} \geqslant 2^{m-k}\sqrt{\frac{r_k}{r_{k-1}}}.$$

For k large enough, S_3 is smaller than ε.

S_i' can be treated by the same method as S_i. Therefore, f is hypercyclic for $S_{\lambda,b}$. Now, as in the holomorphic case, it is not difficult to prove that in fact, for each $\mu \geqslant 1$, $g(x) = f(\mu x)$ is hypercyclic for all $S_{\lambda,b}$. This achieves to prove that the set of common hypercyclic vectors is dense.

4. FINAL REMARKS

4.1. Our interest on hypercyclicity originates from the following question: in [7], J. Gordon and H. Hedenmalm characterized the composition operators on the Hilbert space of square summable Dirichlet series $\mathcal{H} = \Big\{f(s) = \sum_{n\geqslant 1} a_n n^{-s} : \|f\|^2 := \sum |a_n|^2 < +\infty\Big\}$. In [2] and [3], we began a comparison between the properties of the operator C_ϕ and of its symbol ϕ. Pursuing this project, we wanted to characterize the hypercyclic composition operators on $\mathcal{H}$. The answer is very simple.

Proposition 4.1. *No composition operator on $\mathcal{H}$ is hypercyclic.*

Proof. Let C_ϕ be such a composition operator, induced by $\phi(s) = c_0 s + \varphi(s)$, c_0 being an integer, and φ a Dirichlet series. If $c_0 = 1$, C_ϕ is a contraction, and therefore is never hypercyclic. If $c_0 = 0$, by Lemma 11 of [2], $\phi_2(\mathbb{C}_+) \subset \mathbb{C}_{1/2+\varepsilon}$. Now, take f in $\mathcal{H}^2$. Then

$$|f\circ\phi_n(+\infty)|^2 \leqslant \|f\|^2\zeta(2\mathrm{Re}\,\phi_n(+\infty)) \leqslant \|f\|^2 \max\big(\zeta(1+2\varepsilon), \zeta(2\mathrm{Re}\,\phi(+\infty))\big).$$

In particular, $(C_\phi^n(f))$ cannot be dense in $\mathcal{H}$. ∎

We do not know if there exists a supercyclic composition operator on $\mathcal{H}$.

4.2. In [8], G. Godefroy and J. Shapiro proved that if φ is a holomorphic bounded function on $\mathbb{D}$, then M_φ^* is hypercyclic on $H^2(\mathbb{D})$ if and only if $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$. In view of Corollary 2.7, we ask whether there exists a common hypercyclic vector for all λM_{φ^*}, where $\lambda\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

4.3. We have proved that if the set B of fixed points of symbols has positive measure, then there is no common hypercyclic vectors for composition operators on $H^2(\mathbb{D})$, and that if the set is a single point, or even if it is countable, common hypercylic vectors exist for composition operators on $L^2(\mathbb{T})$ or $H(\mathbb{D})$. It could be interesting to consider an intermediate case, like $B =$ a zero measure Cantor set.

4.4. In view of Theorem 3.5, one may study the weights ω on $\mathbb{R}$ for which the translation operator $Tf(x) = f(x+1)$ and the homothety operator $Sf(x) = f(2x)$ are hypercyclic on $L^1(\mathbb{R}, \omega)$.

DEFINITION 4.2. A positive continuous bounded function ω on $\mathbb{R}$ is called a *weight admissible for translation* provided there exists $C > 0$ such that, for all $a \in \mathbb{R}$,

$$\int_{a-1}^{a} \omega(x)\,\mathrm{d}x \leqslant C \int_{a}^{a+1} \omega(x)\,\mathrm{d}x.$$

It is called *admissible for homothety* if there exists $C > 0$ such that, for each $x, y \in \mathbb{R}$ with $0 \leqslant x \leqslant y$ or $x \leqslant y \leqslant 0$,

$$\int_{x/2}^{y/2} \omega(x)\,\mathrm{d}x \leqslant C \int_{x}^{y} \omega(x)\,\mathrm{d}x.$$

If ω is admissible for translation (respectively admissible for homothety), the translation operator T (respectively the homothety operator S) is continuous on $L^1(\mathbb{R}, \omega)$.

THEOREM 4.3. *Let ω be a continuous bounded positive function on $\mathbb{R}$.*

(i) *If ω is admissible for translation, then T is hypercyclic on $L^1(\mathbb{R}, \omega)$ if and only if there exists a sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that*

$$\int_{n_k-q}^{n_k+q} \omega(x)\,\mathrm{d}x \to 0 \quad \textit{and} \quad \int_{-n_k-q}^{-n_k+q} \omega(x)\,\mathrm{d}x \to 0 \quad \textit{for } k \to +\infty$$

for each $q > 0$.

(ii) *If ω is admissible for homothety, S is hypercyclic on $L^1(\mathbb{R}, \omega)$ if and only if there exists $(n_k)_{k\in\mathbb{N}}$ a sequence of integers such that*

$$\int_{2^{n_k}a}^{2^{n_k}b} \omega(x)\,\mathrm{d}x \to 0 \quad \textit{and} \quad \int_{-2^{n_k}b}^{-2^{n_k}a} \omega(x)\,\mathrm{d}x \to 0 \quad \textit{for } k \to +\infty$$

for each $0 < a \leqslant b$.

This statement is the continuous version of Salas Theorem ([13]) on weighted shifts.

Proof. (i) The condition is sufficient: we apply the hypercyclicity criterion, as it is formulated in [9]. Let (P_j) be a dense sequence in $L^1(\mathbb{R},\omega)$ of compactly supported bounded functions. If $\operatorname{supp} P_j \subset [-q,q]$, then

$$\|T^{n_k}P_j\| = \int_{-n_k-q}^{-n_k+q} |P_j(x+n_k)|\omega(x)\,\mathrm{d}x \leqslant \|P_j\|_\infty \int_{-n_k-q}^{-n_k+q} \omega(x)\,\mathrm{d}x \to 0$$

for $k \to +\infty$. Take $Af(x) = f(x-1)$. A is a (possibly unbounded) right inverse of T. It is straightforward that $\|A^{n_k}P_j\| \to 0$ when $k \to +\infty$.

The condition is necessary: by a diagonal argument, it suffices to prove that, for all $\varepsilon > 0$ and all $q > 0$, there exists N arbitrarily large such that

$$\int_{N-q}^{N+q} \omega(x)\,\mathrm{d}x \leqslant \varepsilon \quad \text{and} \quad \int_{-N-q}^{-N+q} \omega(x)\,\mathrm{d}x \leqslant \varepsilon.$$

We set $A_1 = \inf_{[-q,q]} \omega$, $A_2 = \sup_{\mathbb{R}} \omega$. Since the set of hypercyclic vectors for T is dense, there is a hypercyclic vector $f \in L^1(\mathbb{R},\omega)$ such that

$$\|f - 1_{[-q,q]}\| \leqslant \frac{\varepsilon A_1}{2A_2}. \tag{4.1}$$

We can also find N arbitrarily large, $N > 2q$, such that

$$\|T^N f - 1_{[-q,q]}\| \leqslant \frac{\varepsilon A_1}{2A_2}. \tag{4.2}$$

Since $N \geqslant 2q$, inequality (4.1) implies $\int_{N-q}^{N+q} |f(x)|\omega(x)\,\mathrm{d}x \leqslant \frac{\varepsilon}{2}$, whereas inequality (4.2) gives $\int_{-q}^{q} |f(x+N) - 1|\omega(x)\,\mathrm{d}x \leqslant \frac{\varepsilon A_1}{2A_2}$, which in turn proves $\int_{N-q}^{N+q} |f(x) - 1|\omega(x)\,\mathrm{d}x \leqslant \frac{\varepsilon}{2}$. Thus,

$$\int_{N-q}^{N+q} \omega(x)\,\mathrm{d}x \leqslant \varepsilon.$$

We proceed with the same method for the other inequality.

(ii) We prove that the condition is sufficient by using again Kitai's criterion, now with continuous functions whose supports are contained in intervals like $[-A,-\delta] \cup [\delta, A]$, with $0 < \delta \leqslant A$. For the necessity, we fix $0 < a \leqslant b$, and

$A_1 = \inf\limits_{[a,b]} \omega$, $A_2 = \sup\limits_{\mathbb{R}} \omega$. There exists $f \in L^1(\mathbb{R}, \omega)$ and N arbitrarily large (in particular, $2^N a > b$) with

$$\|f - 1_{[a,b]}\| \leqslant \frac{\varepsilon A_1}{2A_2}, \tag{4.3}$$

$$\|S^N f - 1_{[a,b]}\| \leqslant \frac{\varepsilon A_1}{2A_2}. \tag{4.4}$$

As previously, (4.3) gives $\int\limits_{2^N a}^{2^N b} |f(x)|\omega(x)\,\mathrm{d}x \leqslant \frac{\varepsilon}{2}$, and (4.4) implies $\int\limits_{2^N a}^{2^N b} |f(x) - 1|\omega(x)\,\mathrm{d}x \leqslant \frac{\varepsilon}{2}$. This in turn implies

$$\int\limits_{2^N a}^{2^N b} \omega(x)\,\mathrm{d}x \leqslant \varepsilon. \quad \blacksquare$$

EXAMPLE 4.4. For the weight $\omega(x) = \frac{1}{1+|x|}$, the translation operator T is hypercyclic, whereas the homothety operator S is not.

Now, suppose that the weight ω is symmetric ($\omega(-x) = -\omega(x)$), that it decreases to 0 at infinity, and that $1 \in L^1(\mathbb{R}, \omega)$. If $(T_a)_{a>0}$ denotes the semigroup of translations and (S_λ) the semigroup of homotheties, each T_a or S_λ acts boundedly on $L^1(\mathbb{R}, \omega)$, and by Theorem 4.3 above, it is individually hypercyclic. Minor modifications of the proof of Theorem 3.5 actually prove that there exists a common hypercyclic vector for the whole family $(T_a)_{a>0} \cup (S_\lambda)_{\lambda>1}$. For instance, in the course of the proof of the parabolic case, one should now impose that the sequence (u_l) goes to infinity and that it satisfies

$$\sum_{k>k_0} u_k \int\limits_{X_k - R_k}^{X_k + R_k} \omega(t)\,\mathrm{d}t < +\infty.$$

Details are left to the reader.

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