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GLOBAL GLIMM HALVING FOR C^*-BUNDLES

ETIENNE BLANCHARD and EBERHARD KIRCHBERG

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Abstract. A global notion of Glimm halving for C^*-algebras is considered which implies that every nonzero quotient of an algebra with this property is antiliminal. We prove subtriviality and selection results for Banach spaces of sections vanishing at infinity of a continuous field of Banach spaces. We use them to prove the global Glimm halving property for strictly antiliminal C^*-algebras with Hausdorff primitive ideal space of finite dimension. This implies that a C^*-algebra A with Hausdorff primitive ideal space of finite dimension must be purely infinite if its simple quotients are purely infinite.

Keywords: *C^*-algebra, global Glimm halving property.*

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1. INTRODUCTION

A natural generalization of the notion of *pure infiniteness* to non-simple C^*-algebras is given in the following definition. It is equivalent in the case of simple algebras to the one given by J. Cuntz in [8].

Definition 1.1. ([19]) A C^*-algebra A is said to be *purely infinite* (for short p.i.) if and only if

(i) for every pair of positive elements $a, b \in A_+$ such that b lies in the closed two-sided ideal $\overline{\mathrm{span}(AaA)}$ generated by a and for every $\varepsilon > 0$, there exists an element $d \in A$ such that $\|b - d^*ad\| < \varepsilon$;

(ii) there is no character on A.

By Proposition 3.3(ii) and Theorem 4.16 of [19] this is equivalent to the property that for every $\varepsilon > 0$ and $a \in A$ there are $u, v \in \overline{aAa}$ with $u^*v = 0$ and $u^*u = v^*v = (a - \varepsilon)_+$. (Here $(a - \varepsilon)_+ \in A$ means the positive part of $a - \varepsilon \cdot 1$ in the multiplier algebra $\mathcal{M}(A)$ of A.) Thus $b = uv^*$ satisfies $b \in \overline{aAa}$, $b^2 = 0$ and $(a - \varepsilon)_+^2 = u^*bv \in AbA$.

The Glimm halving lemma (Lemma 6.7.1 of [24]) can be equivalently restated as follows: Given any nonzero positive element a in a C^*-algebra A such that $\overline{aAa}$ is not a commutative algebra, there exists a nonzero element $b \in \overline{aAa}$ with $b^2 = 0$. We put this local observation in a global setting by the following definition.

Definition 1.2. A C^*-algebra A is said to have the *global Glimm halving property* if for every positive $a \in A_+$ and every $\varepsilon > 0$, there exists $b \in \overline{aAa}$ such that $b^2 = 0$ and $(a - \varepsilon)_+$ belongs to the closed (two-sided) ideal of A generated by b.

(In this paper "ideal" will always mean "two-sided ideal".)

Above we have seen that purely infinite C^*-algebras satisfy the global Glimm halving property. If a C^*-algebra A has the global Glimm halving property then A is *strictly antiliminal*, i.e. every quotient of A is antiliminal (or equivalently: every hereditary C^*-subalgebra of A has only zero characters). In Section 4 we show that a strictly antiliminal C^*-algebra A with Hausdorff primitive ideal space of finite dimension satisfies the global Glimm halving property (Theorem 4.3).

In Section 5 we use the global Glimm halving property to derive that C^*-algebras with Hausdorff primitive ideal space of finite dimension are purely infinite if and only if all their simple quotients are purely infinite (Theorem 5.1).

To make the proofs concerning the global Glimm having property as transparent as possible we study in the preliminary Section 2 the Banach $C_0(X)$-modules (respectively $C_0(X)$-algebras) of continuous sections vanishing at infinity of a continuous field of Banach spaces (respectively C^*-algebras). We call their axiomatic characterizations as Banach $C_0(X)$-modules "Banach bundles" and "C^*-bundles" respectively. Among others we obtain simple proofs of the following result:

A Banach $C_0(X)$-module B is a Banach bundle over a locally compact space X if and only if B is a $C_0(X)$-submodule of a commutative C^*-bundle over X with nonzero commutative C^*-algebras as fibers (Theorem 2.7).

We get from Theorem 2.7 the following corollaries:

(i) Separable Banach bundles over compact metric spaces X are subtrivial, i.e. are closed $C(X)$-submodules of $C(X \times [0, 1])$ (Corollary 2.8).

(ii) Suppose that B is a separable Banach bundle over a second countable locally compact space X. Then, for every linear functional φ of norm $\leqslant 1$ on the fiber B_y, there exists a contractive linear and $C_0(X)$-module map ψ from B into $C_0(X)$ such that $\psi(a)(y) = \varphi(a_y)$ for $a \in B$ (Corollary 2.9).

We introduce a noncommutative (operator space) version of Banach bundles over noncomutative C^*-algebras by these properties.

Another ingredient of our proofs is the notion of decomposition dimension. In the case of Hausdorff spaces it coincides with the usual covering dimension, cf. Lemma 3.2. (The covering dimension can be strictly smaller than the decomposition dimension for general T_0 spaces, like primitive ideal spaces.) It allows to construct in some cases in a controlled way classical subbundles of finite dimension of some Banach subbundles of C^*-bundles in the proofs of Theorems 4.3 and 5.1.

Suppose that A is a unital C^*-algebra with primitive ideal space $\mathrm{Prim}(A)$ isomorphic to the Hilbert cube $[0, 1]^\infty$ and simple quotients isomorphic to $\mathcal{O}_2$. Is A purely infinite ? We have the feeling that this question is related to the observation that there are nonstable separable C^*-algebras with the Hilbert cube as primitive ideal space and with simple quotients isomorphic to the compact operators on $\ell_2(\mathbb{N})$, cf. Corollary 3.7. We study some related questions in Sections 3 and 6.

2. PRELIMINARIES

We recall in this section a few basic results of the theory of (not necessarily locally trivial) continuous fields of C^*-algebras, of "Banach bundles" and on representations of $C_0((0,1], M_n)$. Since we need later to work also with Banach bundles (i.e. with those Banach $C_0(X)_{\mathrm{sa}}$-modules which are Banach spaces of continuous sections vanishing at infinity of a continuous field over X of Banach spaces), we also list some basic results on them and outline how the reader can modify the corresponding arguments of [9], [10], [2], [1], [3] and [20] to get elementary proofs of these results. It requires that some of our formulas in the case of C^*-bundles look more complicate than in the cited papers, but this is necessary to obtain sufficient generality also working in the case of Banach bundles. (In the subsection on Banach bundles we give detailed proofs of these general formulas.)

2.1. $C(X)$-ALGEBRAS AND C^*-BUNDLES. Let Y be a not necessarily separated topological space. By $C_{\mathrm{b}}(Y)$ we denote the C^*-algebra of bounded continuous functions on Y with values in the complex numbers $\mathbb{C}$. Given a Hausdorff locally compact space X, let $C_0(X)$ denote the C^*-algebra of continuous functions on X with values in $\mathbb{C}$ and which vanish at infinity. Then we naturally identify $C_{\mathrm{b}}(X)$ with the multiplier C^*-algebra of $C_0(X)$.

DEFINITION 2.1. ([15]) A *$C(X)$-algebra* is a C^*-algebra A endowed with a nondegenerate $*$-morphism from $C_0(X)$ in the center $\mathcal{Z}(\mathcal{M}(A))$ of the multiplier C^*-algebra $\mathcal{M}(A)$ of A.

Here "nondegenerate" means that $C_0(X)A$ is dense in A. Thus A is nothing else but a quotient of $C_0(X,A)$ by a closed ideal and the $C_0(X)$-module structure is defined by this epimorphism. The homomorphism from $C_0(X)$ to $\mathcal{Z}(\mathcal{M}(A))$ need not to be faithful.

The Cohen factorization theorem ([6], [12]), or the description of a $C(X)$-algebra A as a quotient of $C_0(X,A)$ shows that the set of products $C_0(\Omega)A = \{fa : f \in C_0(\Omega), a \in A\}$ is a closed ideal of A if Ω is an open subset of X. In particular $A = C_0(X)_+A$.

If $F \subset X$ is a closed subset we denote by $A_{|F}$ the quotient of A by the *closed* ideal $C_0(X \setminus F)A$. Note that $C_0(\Omega)A$ is also a $C_{\mathrm{b}}(\Omega)$-algebra if Ω is an open subset of X and that $A_{|F}$ is also a $C_{\mathrm{b}}(F)$-algebra, because $C_0(X \setminus F) \subset C_0(X)$ is the kernel of the restriction map $C_0(X) \to C_0(F)$ and $C_0(X \setminus F)$ is contained in the kernel of the quotient-action of $C_0(X)$ on $A_{|F}$.

If y is a point of the Hausdorff space X, we write A_y for $A_{|\{y\}}$, and we call A_y the *fiber* of A at $y \in X$.

Given an element $a \in A$, let a_y be the image of a in the fiber A_y, $y \in X$.

Since $C_0(X)_+A = A$ (and since $C_0(X) \to \mathcal{M}(A)$ is contractive), we can naturally and uniquely extend the action $C_0(X) \to \mathcal{M}(A)$ to a unital $*$-homomorphism from $C_{\mathrm{b}}(X) \cong C(\beta X)$ into the center of $\mathcal{M}(A)$. This yields $(fa)_y = f(y)a_y$ for $f \in C_{\mathrm{b}}(X)$, $a \in A$ and $y \in X$, because $(f - f(y))C_0(X) \subset C_0(X \setminus \{y\})$.

Thus, the function $N(a)\colon y \mapsto \|a_y\| := \|a + C_0(X \setminus \{y\})A\|$ satisfies

$$N(fa) = |f|N(a) \tag{2.1}$$

for $a \in A$ and $f \in C_{\mathrm{b}}(X)$.

M. Rieffel ([26]) has remarked that $N(a)$ is also given by

$$N(a)\colon y \in X \mapsto \|a_y\| = \inf\{\|[1 - f(y)]a + fa\| \,:\, f \in C_0(X)_{\mathrm{sa}}\}, \tag{2.2}$$

and is always upper semicontinuous, because the function $y \in X \mapsto \|[1 - f(y)]a + fa\|$ is continuous for fixed $f \in C_0(X)_{\mathrm{sa}}$.

Let $\mathrm{Prim}(A)$ denote the primitive ideal space of A, cf. Example 2.2.2. Since, by the Dauns–Hofmann theorem (Corollary 4.4.8 in [24]), there is a natural isomorphism from $C_{\mathrm{b}}(\mathrm{Prim}(A))$ onto the center of $\mathcal{M}(A)$, we can equivalently define a $C(X)$-algebra A by a continuous map η from $\mathrm{Prim}(A)$ into the Stone-Čech compactification βX of X such that $\|fa+J\| = |f(\eta(J))|\cdot\|a+J\|$ for $f \in C(\beta X)$, $a \in A$ and every primitive ideal J of A. The nondegeneracy condition $A = C_0(X)A$ in our definition implies that $\eta(\mathrm{Prim}(A)) \subset X$. Thus, $\|a\| = \sup\{N(a)(y) \,:\, y \in X\}$, and for $x \in \mathrm{Im}(\eta)$ (and with the convention $\sup \emptyset := 0$), we have

$$N(a)(x) = \|a_x\| = \sup\{\|a + J\| \,:\, J \in \mathrm{Prim}(A),\, \eta(J) = x\}. \tag{2.3}$$

Obviously, this implies for $a \in A$ and $t \in [0, \infty)$ that η maps the open set $U(a,t) := \{J \in \mathrm{Prim}(A) \,:\, \|a + J\| > t\}$ onto the set $N(a)^{-1}(t, \infty) \subset X$. Since the open sets $U(a,t)$ build a base of the (Jacobson) topology of $\mathrm{Prim}(A)$ (cf. 2.2.2.) and since, by (2.2), the functions $N(a)$ are upper semicontinuous, we get: The above introduced continuous map $\eta\colon \mathrm{Prim}(A) \to X$ *is open if and only if the functions* $N(a)$ *on* X *are continuous for all* $a \in A$. (If η is open then $\eta(\mathrm{Prim}(A))$ is an open subset of X, but e.g. the map $\eta\colon (0,1] \to [0,1]$, induced by $C([0,1]) \to \mathcal{M}(C_0((0,1]))$, has not a closed image.)

DEFINITION 2.2. We say that the $C(X)$-algebra A is a C^**-bundle* over X if the function $N(a)$ is moreover continuous for every $a \in A$ ([20], [2]).

Sometimes we write *continuous* C^*-bundle if we want to underline that the functions $N(a)$ are continuous.

Since we have assumed that $C_0(X)A$ is dense in A, we get $A = C_0(X)_+A$ and then, from (2.1) and $(C_0(X)_+)^2 = C_0(X)_+$, that $N(a)$ is even in $C_0(X)_+$ for $a \in A$.

It is well-known that A is a C^*-bundle over X if and only if A is the C^*-algebra of continuous sections vanishing at infinity of a continuous field of C^*-algebras over X in the sense of Definition 10.3.1 of [9], such that the fibers are the A_x and the $*$-morphism from $C_0(X)$ into $\mathcal{M}(A)$ coincides with the multiplication of continuous sections with functions ([2]; see also Subsections 2.6.3 and 2.6.4).

2.2. EXAMPLES OF C^*-BUNDLES.

2.2.1. If C is a $C(X)$-algebra and D is a C^*-algebra, the spatial tensor product $B = C \otimes D$ is endowed with a structure of $C(X)$-algebra through the map $f \in C_0(X) \mapsto f \otimes 1_{\mathcal{M}(D)} \in \mathcal{M}(C \otimes D)$. This $C(X)$-algebra is not in general a C^*-bundle over X.

If $C = C_0(X)$, the tensor product $B = C_0(X) \otimes D \simeq C_0(X, D)$ is a "trivial" C^*-bundle over X with constant fiber $B_x \cong D$. Thus, if $A \subset B$ is a closed $C_0(X)$-submodule and A is a C^*-subalgebra of B then A is a C^*-bundle over X. If A is only a closed $C_0(X)$- or $C_0(X)_{\mathrm{sa}}$-submodule of B, then A is a complex or real Banach bundle over X in the sense of the below given Definition 2.6; see 2.4.3 and Remark 2.5.

Let A be a separable C^*-bundle over X with exact fibers A_x. If $\mathcal{O}_2$ is the unital Cuntz algebra generated by two isometries s_1, s_2 satisfying the relation $1_{\mathcal{O}_2} = s_1 s_1^* + s_2 s_2^*$ ([7]), then there exists a $C(X)$-linear $*$-monomorphism $A \hookrightarrow C(X) \otimes \mathcal{O}_2$ if and only if A is itself exact as a C^*-algebra, and this happens if and only if for every C^*-algebra D, the $C(X)$-algebra $A \otimes D$ is again a C^*-bundle over X (Theorem A.1 in [3] and [20]). There exists a separable continuous C^*-bundle A over $\{0\} \cup \{1/n : n \in \mathbb{N}\} \subset [0, 1]$ with exact fibers, such that A is not exact ([20]).

2.2.2. Given a C^*-algebra A, one calls *primitive ideal space* of A the space $\mathrm{Prim}(A)$ of kernels of irreducible representations of A. It is a T_0-space for the Jacobson topology (kernel-hull topology). A base of this topology is given by the open sets of the form $\{K \in \mathrm{Prim}(A) : \|a + K\| > 0\}$ for some $a \in A_+$. Since $\|(a - t)_+ + K\| = (\|a + K\| - t)_+$ for $t > 0$ and $a \in A_+$, this means that the Jacobson topology is the coarsest topology on $\mathrm{Prim}(A)$ such that for every $a \in A$, the function $K \in \mathrm{Prim}(A) \mapsto \|a + K\|$ is lower semicontinuous.

On the other hand, for $a \in A$ and $t > 0$, the G_δ-subset $\{K \in \mathrm{Prim}(A) : \|a + K\| \geqslant t\}$ of $\mathrm{Prim}(A)$ is quasicompact, (Proposition 3.3.7 in [9]).

If the space $\mathrm{Prim}(A)$ is in addition Hausdorff, then this yields that $\mathrm{Prim}(A)$ is locally compact and that the functions $N(a) \colon K \in \mathrm{Prim}(A) \mapsto \|a + K\|$ are continuous functions on $\mathrm{Prim}(A)$ which vanish at infinity (Corollary 3.3.9 of [9]). Then the Dauns-Hofmann theorem (Corollary 4.4.8 of [24]) implies that A is naturally a C^*-bundle over $\mathrm{Prim}(A)$ with simple fiber $A_K = A/K$ at $K \in \mathrm{Prim}(A)$; see also 2.6.3.

Note that the non-Hausdorff space $\mathrm{Prim}(A)$ is not always locally compact in the sense that points have closed (!) quasicompact neighborhoods: There exists a separable (non unital) C^*-algebra A with $\mathrm{Prim}(A)$ isomorphic to $(0, 1]$ as a set, but with the T_0-topology defined by the system of open subsets $\{(x, 1] : x \in (0, 1]\}$, which have all the closure $(0, 1]$ (cf. [29]). $(0, 1]$ with this topology is locally quasicompact but is not locally compact.

2.2.3. Suppose that A is a $C(X)$-algebra and let $\pi \colon C_0(X, A) \to A$ denote the C^*-epimorphism which corresponds to the nondegenerate $*$-homomorphism L from $C_0(X)$ into the center of $\mathcal{M}(A)$ (and satisfies $\pi(f \otimes a) = L(f)a$). Then π defines a homeomorphism κ from $\mathrm{Prim}(A)$ onto a closed subset Z of $\mathrm{Prim}(C_0(X, A)) \cong X \times \mathrm{Prim}(A)$. Obviously, the above described continuous map $\eta \colon \mathrm{Prim}(A) \to X$ is the same as $p_1 \circ \kappa$, where p_1 is defined by $p_1(x, y) := x$.

Thus A is a C^*-bundle over X if and only if the restriction of p_1 to $Z \subset X \times \mathrm{Prim}(A)$ is an *open* map from Z to X.

In the case of a commutative C^*-bundle A over a locally compact space X with *nonzero* fibers A_x this yields the following observation:

Let $Y := \mathrm{Prim}(A)$ be the locally compact space of its maximal ideals. The natural epimorphism from $C_0(X, A) \cong C_0(X \times Y)$ onto A is (as well-known) in one-to-one correspondence with a homeomorphism κ from Y onto a closed subset Z of $X \times Y$.

The subset $Z \subset X \times Y$ satisfies:

(i) Z is closed,

(ii) for every $x \in X$, there exists $y \in Y$ such that $(x, y) \in Z$,

(iii) the map $p_1 : (x, y) \in Z \mapsto x \in X$ is open (with respect to Z).

Conversely, a subset Z of a Tychonoff product $X \times Y$ of (arbitrary) locally compact spaces X and Y defines in a natural way a commutative C^*-bundle $A := C_0(Z)$ over X with nonzero fibres if Z satisfies conditions (i)–(iii) above. The action is given by the epimorphism $C_0(X) \otimes C_0(Y) \to C_0(Z)$ with $f \mapsto f_{|Z}$ coming from (i). (Indeed, the natural embedding $\kappa : z \mapsto (p_1(z), z)$ from Z into $X \times Z \subset X \times X \times Y$ shows that here $\eta = (p_1)_{|Z}$ for $\mathrm{Prim}(C_0(Z)) \cong Z$.)

We use this example in Subsection 2.4.4 to derive subtriviality of all separable Banach bundles.

2.3. Banach $C_0(X)$-modules. Let us now collect some basic properties of general Banach $C_0(X)$-modules, which we need for our ("commutative") generalizations of C^*-bundles to "Banach bundles". The Banach bundles (Definition 2.6) are defined by four requirements on Banach $C_0(X)$-modules which allow to work completely in the framework of Banach modules, if we need later some properties of the Banach $C_0(X)$-modules of continuous sections vanishing at infinity of continuous fields of Banach spaces over X.

In the following, denote X and Y locally compact spaces and A, B, E Banach spaces.

Recall that a *Banach $C_0(X)$-module* is a Banach space A with a bounded algebra homomorphism $L : C_0(X) \to \mathfrak{L}(A)$ (cf. e.g. Definition I Section 9.12 of [5]). Sometimes we call L the *action of $C_0(X)$ on A* and we write fa for $L(f)(a)$ if $a \in A$ and $f \in C_0(X)$. We say that the Banach $C_0(X)$-module A is *nondegenerate* if the action L of $C_0(X)$ on A is nondegenerate, i.e. if A is the closed linear span of $C_0(X)_+A$.

The $C_0(X)$-module A is *countably generated*, if there is a countable subset C of A, such that the $C_0(X)$-submodule generated by C is dense in A.

In the sequel, we call the action L of $C_0(X)$ on A *contractive* if L is a contraction, i.e. if $\|fa\| \leqslant \|f\|\|a\|$ for $a \in A$ and $f \in C_0(X)$. Clearly the contractivity passes to the restrictions $f \mapsto L(f)|B$ of the action of $C_0(X)$ on $L(C_0(X)_+)$-invariant closed linear subspaces B of A. The restrictions provide B with the structure of a Banach $C_0(X)$-module. Such a closed *vector* subspace B with $f \mapsto L(f)|B$ is called a *Banach $C_0(X)$-submodule* of A.

The action of $C_0(X)$ on A/B given by $[L](f) : \pi_B(a) \mapsto \pi_B(fa)$ is also contractive and has the closed ideal $M(L, A, B) := \{f \in C_0(X) : L(f)(A) \subset B\}$ as its kernel. Thus there is an open subset $\Omega_{\max}$ of X with $C_0(\Omega_{\max}) = M(L, A, B)$ such that A/B is naturally a Banach $C_0(F)$-module for every closed subset $F \supset X \setminus \Omega_{\max}$ of X.

If Ω is an open subset of X, then the set $C_0(\Omega)_+A := \{fa : f \in C_0(\Omega)_+, a \in A\}$ is a closed linear subspace of A. In fact, if B is a Banach algebra with a bounded approximate unit and if E is a Banach B-module, then Cohen factorization theorem yields that the product $B \cdot XSE = \{be : b \in B \text{ and } e \in E\}$ is a closed linear subspace of E ([6], [12], Theorem I Section 11.10 in [5], Proposition 1.8 in [2]). Now note that $C_0(\Omega)_+ \cdot C_0(\Omega) = C_0(\Omega)$.

Thus we can define "restrictions" $A_{|F} := A/C_0(X\setminus F)_+\cdot A$ of A to closed subsets F of X, i.e. we can use the same definition as in the case of $C(X)$-algebras A.

Note that $C_0(\Omega)_+A$ is a $C_0(X)$-submodule of A. It is the biggest nondegenerate $C_0(\Omega)$-submodule of A, and $M(L, C_0(\Omega)_+A, A) = C_0(\Omega_{\max})$ contains $C_0(\Omega)$. In particular $A_{|F}$ is in a natural way a Banach $C_0(F)$-module.

As in the case of $C(X)$-algebras, we use the notations $a_{|F}$ for the element $a + C_0(X\setminus F)_+A \in A_{|F}$, A_y for $A_{|\{y\}}$ and a_y for $a_{|\{y\}}$, if $y \in A$. Further we define again the function $N(a)$ by $y \in X \mapsto \|a_y\|$, which can be everywhere zero, but plays an important role in the sequel.

The Cohen factorization, the identities $C_0(U)+C_0(V) = C_0(U\cup V)$, $C_0(U)\cdot C_0(V) = C_0(U\cap V)$ and the inclusion $C_0(U) \subset C_0(W)$ imply the formulas $C_0(U)A+C_0(V)A = C_0(U\cup V)A$, $C_0(U)A\cap C_0(V)A = C_0(U\cap V)A$ and $C_0(U)A \subset C_0(W)A$ for open subsets $U, V, W \subset X$ with $U \subset W$. Consider now closed subsets $G \subset F \subset X$ and an open subset $\Omega \subset X$. Let $U := X\setminus F$, $W := X\setminus G$ and $V := \Omega$. Then, this formulas and the definitions give the following observations (a)–(f) and equations (2.4), (2.5) and (2.6):

(a) The natural map $\pi_{F,G}\colon A_{|F} \to A_{|G}$, given by $\pi_{F,G}(a_{|F}) := a_{|G}$, maps the open unit ball of $A_{|F}$ onto the open unit ball of $A_{|G}$ for closed subsets $G \subset F$ (i.e. is the quotient map) and thus defines an isometric isomorphism $(A_{|F})_{|G} \cong A_{|G}$.

(b) If we take here $G = \{y\}$ for a point $y \in F$, we get $A_y \cong (A_{|F})_y$ and thus

$$N(a_{|F}) = N(a)_{|F}. \tag{2.4}$$

(c) The epimorphism $\pi_F\colon A \to A_{|F}$ maps $C_0(\Omega)A$ onto $C_0(F\cap\Omega)A_{|F}$ if Ω is open and F is closed, because fb, with $b \in A_{|F}$ and $f \in C_0(F\cap\Omega)$, is the image of ga where $a \in A$, $g \in C_0(\Omega)$, $b = a_{|F}$ and $g_{|F\cap\Omega} = f$ (it exists by Tietze extension). Its kernel is $C_0(\Omega)A \cap C_0(X\setminus F)A = C_0(\Omega\setminus F)A$. Thus it defines a natural isomorphism

$$(C_0(\Omega)A)_{|\Omega\cap F} \cong C_0(\Omega\cap F)\cdot(A_{|F})\,. \tag{2.5}$$

(d) If the action of $C_0(X)$ is contractive, then the isomorphism (2.5) is *isometric* and $N_A(a)_{|\Omega} = N_{C_0(\Omega)A}(a)$ for $a \in C_0(\Omega)A$, because then $\|a + C_0(\Omega\setminus F)A\| = \|a + C_0(X\setminus F)A\|$ for $a \in C_0(\Omega)A$ (where $C_0(\emptyset) := 0$, $A_{|\emptyset} := 0$ etc.).

(e) Consider the natural extension L^{e} of $L\colon C_0(X) \to \mathfrak{L}(A)$ to $C(\widehat{X}) = \mathbb{C}1 + C_0(X)$ with $L^{\mathrm{e}}(1) = \mathrm{id}_A$, and let $\kappa(L) := \sup\{\|L^{\mathrm{e}}(1-f)\| \,:\, f \in C_0(X)_+,\ \|f\| \leqslant 1\}$. Then for $a \in A$ and every closed $F \subset X$ one has directly from the definition of $a_{|F}$ that

$$\|a_{|F}\| \leqslant \inf\{\|a - fa\| \,:\, f \in C_0(X\setminus F)_+,\ \|f\| \leqslant 1\} \leqslant \kappa(L)\cdot\|a_{|F}\|, \tag{2.6}$$

(cf. Lemma 1.10 of [2]). It implies that every closed $C_0(X)$-submodule B of A is nondegenerate Banach $C_0(X)$-module, if A is nondegenerate. In particular, then B is a vector subspace of A.

The inequalities (2.6) become equalities if the natural extension L^{e} of L to $C(\widehat{X}) = \mathbb{C}1 + C_0(X)$ is contractive. Then the norms $\|a_{|F}\|$ in $B_{|F}$ are equal to the norms $\|a_{|F}\|$ in $A_{|F}$ for $a \in B$ if B is a Banach $C_0(X)$-submodule of A. Thus, for $a \in B$ the function $N(a)$ is then the same with respect to B as the function $N(a)$ build with respect to A, and the natural algebraic monomorphism $B_{|F} \to A_{|F}$ is an isometric isomorphism onto $\pi_F(B) \subset A_F$. We get:

(f) The $C(\beta X)$-linear map $A \to A_{|F}$ with kernel $C_0(X \setminus F)A \supset C_0(X \setminus F)B$ induces a linear isometry from $B_{|F}$ into $A_{|F}$, which is a $C_0(F)$-module morphism, if the action on A is nondegenerate and contractive. In particular, then $\pi_x(B) \subset A_x$ is isometrically isomorphic to B_x.

LEMMA 2.3. *Suppose that A is a Banach $C_0(X)$-module. The following properties of A are equivalent:*

(i) *The action L of $C_0(X)$ on A is contractive and nondegenerate, i.e. satisfies $\|L\| \leqslant 1$ and $C_0(X)_+ A = A$.*

(ii) *A is the quotient of the Banach $C_0(X)$-module $D := C_0(X) \widehat{\otimes} \ell_1(Z)$ for a suitable set Z by some Banach submodule B of D.*

(iii) *There is a unital, contractive and strongly continuous algebra-morphism L^β from $C(\beta X) \cong C_{\mathrm{b}}(X) = \mathcal{M}(C_0(X)) \subset \mathfrak{L}(C_0(X))$ into $\mathfrak{L}(A)$ which extends the given algebra morphism from $L \colon C_0(X) \to \mathfrak{L}(A)$.*

Proof. Here $\widehat{\otimes}$ means the projective tensor product (=maximal uniform tensor product) of Banach spaces, Z is a dense subset of the open unit ball of A, and the nondegenerate and contractive action $L \colon C_0(X) \to \mathfrak{L}(D)$ is given by $L(f)(g \otimes h) := (fg) \otimes h$. The submodule B is the kernel of the natural Banach $C_0(X)$-module epimorphism onto A which maps the open unit ball onto the open unit ball of A. Note that $D = C_0(X)_+ D_\beta$, where $D_\beta := C(\beta X) \widehat{\otimes} \ell_1(Z)$. This implies that each $C_0(X)$-invariant closed subspace B of D is also a $C(\beta X)$-invariant subspace of the $C(\beta X)$-invariant subspace $C_0(X) D_\beta$: by (2.6) every element a of B must be in the closure of $C_{\mathrm{c}}(X)_+ a \subset B$. ∎

It follows from part (iii) of Lemma 2.3 that the formulas $(fa)_y = f(y)a_y$, (2.1) and (2.2) still apply in this context, and that all the above formulas of this subsection hold, because the identity $C_0(\beta X \setminus \{y\})_+ C_0(X)_+ = C_0(X \setminus \{y\})$ and the inequalities $\|(1 - f)b\| \leqslant \|b\|$ for $b \in A$, $0 \leqslant f \leqslant 1$ in $C(\beta X)$ can be used for the proofs as in the case of $C(X)$-algebras. So the function $N(a)$ is upper semicontinuous for all $a \in A$ by formula (2.2).

Let us summarize some of the later needed facts:

LEMMA 2.4. *Suppose that A is a Banach $C_0(X)$-module with nondegenerate and contractive action of $C_0(X)$, and that B, C are closed $C_0(X)$-submodules of A with $C \subset B$. Then:*

(i) *B and C are vector subspaces of A, i.e. B and C are a Banach $C_0(X)$-submodules of A;*

(ii) *the actions of $C_0(X)$ on B, C, A/C and B/C are nondegenerate and contractive;*

(iii) *B_x is isometrically isomorphic to the image of B in A_x for $x \in X$, i.e. $N_B(b) = N_A(b)$ for $b \in B$;*

(iv) *$(A/C)_x$ is isometrically isomorphic to A_x/C_x and thus, by* (ii), *$(B/C)_x \subset A_x/C_x$ in a natural way;*

(v) *$\operatorname{dist}(a_x, \pi_x(C)) = N_{B/C}(a + C)(x) = N_{A/C}(a + C)(x)$ for $a \in B$.*

Proof. We have already seen everything but (iv) and (v). Clearly B/C is a Banach $C_0(X)$-submodule of A/C, and $(B/C)_x$ is a Banach subspace of $(A/C)_x$ by (iii).

Let M denote the closure of $C + C_0(X \setminus \{x\})A$ (in fact it is closed), and let N denote $C_0(X \setminus \{x\})A$. Then $\pi_C(N) = C_0(X \setminus \{x\})(A/C)$, e.g. by Cohen factorization. Since $\pi_C(N)$ is closed, we get $\pi_C(N) = \pi_C(M)$ and, thus, $\operatorname{dist}(a, M) = \|(\pi_C(a))_x\|$. On the other hand, $\pi_x(C)$ is closed by (iii) and thus, $\pi_x(C) = \pi_x(M)$, $\operatorname{dist}(\pi_x(a), \pi_x(C)) = \operatorname{dist}(a, M)$.

If we now identify $\pi_x(C)$ and C_x naturally by (iii), then we get natural isometric isomorphisms from $(A/C)_x$ onto A_x/C_x, and, in the same way, from $(B/C)_x$ onto B_x/C_x.

This implies (v) by (iii). ∎

REMARK 2.5. To simplify the notations, we have only considered complex Banach $C_0(X)$-modules. Later (in Sections 3 and [4]) we use also the corresponding results on real Banach $C_0(X, \mathbb{R})$-modules.

Indeed, one can also work with *real* Banach $C_0(X, \mathbb{R})$-modules i.e. real Banach spaces A with a bounded algebra homomorphism $L\colon C_0(X, \mathbb{R}) = C(X)_{\mathrm{sa}} \to \mathfrak{L}(A)$. The results in Section 2 remain valid if we replace in the statements and proofs of Subsections 2.3–2.6 $\mathbb{C}$ by $\mathbb{R}$, $C_0(X)$ $(= C_0(X, \mathbb{C}))$ by $C_0(X, \mathbb{R})$ $(= C_0(X)_{\mathrm{sa}})$, and read "Banach spaces" as "real Banach spaces".

In particular, every complex Banach $C_0(X)$-module A defines a real Banach $C_0(X, \mathbb{R})$-module if we restrict L to $C_0(X)_{\mathrm{sa}}$ and if we consider A only as a real Banach space. As the sets $C_0(\Omega)A$, $C_0(\Omega)_{\mathrm{sa}}A$ and $C_0(\Omega)_+A$ coincide for all Ω, we get the same real vector subspace $C_0(\Omega)A$, the same norms $\|a_{|F}\|$ and the same function $N(a)$, if we consider a complex Banach $C_0(X)$-module A with action L as a real Banach $C_0(X)_{\mathrm{sa}}$-module A with action L restricted to $C_0(X)_{\mathrm{sa}}$.

If a (complex) Banach $C_0(X)$-module A is a continuous *real* Banach bundle (cf. Definition 2.6 read with our "real-case" convention), then A is also a continuous *complex* Banach bundle, as assertion 2.4.5 shows.

2.4. BANACH BUNDLES. We generalize the (commutative) C^*-bundles as follows:

DEFINITION 2.6. A *Banach bundle* over a locally compact space X is a Banach space A together with an algebra homomorphism L from $C_0(X)$ into the bounded linear operators $\mathfrak{L}(A)$ on A, the *action of* $C_0(X)$ *on* A, such that, for every $a \in A$:

(i) $\|f \cdot a\| \leqslant \|f\|\,\|a\|$ for $f \in C_0(X)$;

(ii) $C_0(X)_+A$ is dense in A;

(iii) the function

$$N(a)\colon y \in X \mapsto \|a_y\| := \|a + C_0(X \setminus \{y\})_+A\|$$

is a *continuous* function on X;

(iv)

$$\|a\| = \sup\{N(a)(y) \,:\, y \in X\}. \tag{2.7}$$

Note that the Banach bundles correspond to continuous fields of Banach spaces introduced by Dixmier in [9], as will be made clear in 2.6.4. But we prefer to use in this paper the Banach-module picture.

Examples of Banach bundles are e.g. the C^*-bundles introduced in Definition 2.2: the formula $\|a\| = \sup N(a)$ was derived in Subsection 2.1 from formula (2.3). Clearly, for every Banach space B, the $C_0(X)$-module $C_0(X, B)$ is a Banach bundle over X. It easily follows from (ii) and equation 2.7, that a Banach bundle A is a quotient Banach $C_0(X)$-module of $C_0(X, A)$.

We prove (at the end of Subsection 2.4.4) the following Theorem 2.7, which explains a lot of the general structure, and which might be implicitly contained in other papers:

Theorem 2.7. *A Banach $C_0(X)$-module B is a Banach bundle over a locally compact space X if and only if B is $C_0(X)$-submodule of a commutative C^*-bundle over X (with nonzero commutative fibers).*

Since separable commutative C^*-bundles A over a compact metric space X are $C(X)$-C^*-subalgebras of $C(X, \mathcal{O}_2)$, cf. [3], and since every separable Banach space is isometrically isomorphic to a closed subspace of $C([0,1])$, we immediately get from Theorem 2.7 the following corollary.

Corollary 2.8. *Separable Banach bundles over compact metric spaces X are subtrivial, i.e. are closed $C(X)$-submodules of $C(X \times [0,1])$.*

This and (classical) Hahn-Banach extension yield directly the following:

Corollary 2.9. *Suppose that B is a separable Banach bundle over a second countable locally compact space X. Then for every linear functional φ of norm $\leqslant 1$ on the fiber B_y, there exist a contractive linear and a $C_0(X)$-module map ψ from B into $C_0(X)$ such that $\psi(a)(y) = \varphi(a_y)$ for $a \in B$.*

Equation (2.1) and the nondegeneracy of the action of $C_0(X)$ on A imply that even $N(a) \in C_0(X)_+$ if $N(a)$ is continuous (by requirements (ii) and (iii) in Definition 2.6).

The convex span of $\{N(a) : a \in A\}$ is dense in the positive part of an ideal $J \cong C_0(\Omega)$ of $C_0(X)$, where Ω is the open subset of points $y \in X$ with nonzero quotient spaces A_y. We do *not* require in Definition 2.6 that $A_y \neq 0$ for every $y \in X$, or that the algebra morphism from $C_0(X)$ into $L(A)$ is faithful.

The algebra morphism from $C_0(X)$ into $L(A)$ is an isometry if and only if Ω is dense in X.

Next we list some later used properties of Banach bundles A over a locally compact space X.

2.4.1. *The Banach module $C_0(\Omega)A$ (respectively $A_{|F}$) is a Banach bundle over Ω (respectively over F), and its fiber-norm function N is given by the restriction of the fiber-norm function on A, if $\Omega \subset X$ is open (respectively $F \subset X$ is closed).*

Moreover, the quotient map from $A_{|F}$ onto $A_{|G}$ maps the closed *unit ball of $A_{|F}$ onto the* closed *unit ball of $A_{|G}$, if $G \subset F \subset X$ are closed.*

Proof. Suppose, more generally, that $C_0(X)$ acts contractively on A, that the span of $C_0(X) \cdot A$ is dense in A, and that the A satisfies $\|a\| = \sup N(a)$ for every $a \in A$. Then, for every closed subset F of X and $a \in A$,

$$\|a_{|F}\| = \sup\{N(a)(y) \,:\, y \in F\}. \tag{2.8}$$

Indeed, we have $C_0(X) \cdot A = A$. So there are $h \in C_0(X)_+$ and $b \in A$ with $a = hb$. Then $g := N(b)$ is a bounded upper semicontinuous function by equation (2.2) and $N(a) = hg$ by (2.1). Formulas (2.1) and (2.6) imply that

$$\|a_{|F}\| = \inf\{\sup((1-f)hg)(X) \,:\, f \in C_0(X \setminus F)_+, f \leqslant 1\}.$$

The right side is just $\sup(hg)(F)$ by an (obvious) application of Tietze extension theorem, which shows (2.8).

The nondegeneracy and the contractivity of the action of $C_0(F)$ on $A_{|F}$ follow from these properties of the action of $C_0(X)$ on A, as we have explained above. The formula (2.8) for the norms $\|a_{|F}\|$ and the identity (2.4) imply that, for every closed subset $F \subset X$, the restriction $A_{|F}$ is a Banach bundle over F, if we now suppose in addition that A is a Banach bundle over X.

The action of $C_0(\Omega)$ on the Banach space $C_0(\Omega)A$ is always a nondegenerate action, i.e. $C_0(\Omega)_+C_0(\Omega)A = C_0(\Omega)A$. Moreover $C_0(\Omega)A$ satisfies the contractivity condition (i) for $C_0(\Omega)$, because A satisfies (i) for $C_0(X)$. Thus the norm function of $a \in C_0(\Omega)A$ with respect to $C_0(\Omega)A$ is just the restriction to Ω of the norm function $N(a)$ with respect to A, as we have explained below formula (2.5). Since formula (2.1) holds for the nondegenerate and contractive action of $C_0(X)$ on A, we get that $N(a) = hN(b)$ for suitable $h \in C_0(\Omega)_+$ and $b \in A$. Thus $\|a\| = \sup_X N(a) = \sup_\Omega N(a)$ and the properties (iii) and (iv) for the action of $C_0(\Omega)$ on $C_0(\Omega)A$ follow from (i), (iii) and (iv) of Definition 2.6.

The natural epimorphism from $A_{|F}$ onto $A_{|G}$ maps the closed unit ball of $A_{|F}$ onto the closed unit ball of $A_{|G}$, because, by the nondegeneracy and the contractivity of the action and by (iii) of Lemma 2.3, we can multiply $a \in A$ with elements of $C_{\mathrm{b}}(X)$: Let $b \in A_{|G}$, $\|b\| \leqslant 1$, and $a \in A$ with $a_{|G} = b$. Consider the bounded continuous function $f(x) = g(N(a)(x))$, where $g(t) = 1$ on $[0,1]$ and $g(t) = t^{-1}$ for $t > 1$. Then $(fa)_{|G} = b$ and $\|fa\| = \sup fN(a) \leqslant 1$ by (2.1) and (2.7). ∎

2.4.2. *Let F and G be closed subsets of X. The natural epimorphism $A_{|(F\cup G)} \to A_{|F}$ and $A_{|(F\cup G)} \to A_{|G}$ defines $A_{|(F\cup G)}$ as the pullback of the epimorphism $A_{|F} \to A_{|(F\cap G)}$ and $A_{|G} \to A_{|(F\cap G)}$.*

Proof. Let $\Omega := X \setminus G$. In Subsection 2.3 we have seen that the epimorphism $A \to A_{|F}$ maps $C_0(\Omega)A$ onto $C_0(F \cap \Omega)A_{|F}$, that there is a natural isometric isomorphism $(A_{|F})_{|(F\cap G)} \cong A_{|(F\cap G)}$, and that $a_{|F}$ and $a_{|G}$ have the same image $a_{|(F\cap G)}$ in $A_{|(F\cap G)}$ by definition of the "restrictions".

By formula (2.8), the map $a_{|(F\cup G)} \mapsto (a_{|F}, a_{|G})$ defines a linear isometry from $A_{|(F\cup G)}$ to the Banach space sum $A_{|F} \oplus_\infty A_{|G}$ with supremum norm.

Now let $a \in A_{|F}$ and $b \in A_{|G}$ be such that $a_{|(F\cap G)} = b_{|(F\cap G)}$. We take $c \in A$ with $c_{|G} = b$. Then $d := c_{|F} - a$ is in the kernel $C_0(F \cap \Omega)A_{|F}$ of the epimorphism $A_{|F} \to A_{|(F\cap G)}$. Therefore, we can find $e \in C_0(\Omega)A$ such that $e_{|F} = d$. Let $f := c - e$, then $f_{|F} = a$ and $f_{|G} = b$. Thus (a,b) is in the image of the natural isometry from $A_{|(F\cup G)}$ into $A_{|F} \oplus_\infty A_{|G}$. ∎

2.4.3. *If A is a Banach bundle over X and B is a closed $C_0(X)$-submodule of A, then B is again a Banach bundle over X, $N_B(b) = N_A(b)$ for $b \in B$, and $B_{|F}$ is a closed $C_0(F)$-submodule of $A_{|F}$ for every closed subset F of X.*

Moreover, $B = A$ if and only if $B_x = A_x$ for every $x \in X$.

We call B a *Banach subbundle* of A.

Proof. Since A is a nondegenerate and contractive Banach $C_0(X)$-module, we get from Lemma 2.4 that B is a Banach $C_0(X)$-submodule of A, that the action of $C_0(X)$ on B is nondegenerate and that $N_B(b) = N_A(b)$ for $b \in B$. Thus $\|a\| = \sup N_B(a)$ for $a \in B$ by (2.7), and the Banach $C_0(X)$-module B satisfies the conditions of Definition 2.6. We have seen below the inequalities (2.6) that the natural linear $C_0(F)$-module map from $B_{|F}$ into $A_{|F}$ is an isometry, if the action of $C_0(X)$ on A is nondegenerate and contractive.

Suppose that B is a Banach subbundle of A such that $B_x = A_x$ for every $x \in X$, and let $a \in A$, $\varepsilon > 0$. Then for every $y \in X$, there exists $b^y \in B$ with $N(a - b^y)(y) = 0$. By continuity of $x \mapsto N(a - b^y)(x)$ there is a neighborhood $U(y)$ of y such that $N(a - b^y)(x) < \varepsilon$ for $x \in U(y)$.

Since $N(a) \in C_0(X)$, the set $F := \{x \in X : \varepsilon \leqslant N(a)(x)\}$ is compact, and we can find open subsets $U_1, \ldots, U_n$ of X and $b_1, \ldots, b_n \in B$ such that $F \subset U_1 \cup \cdots \cup U_n$ and $N(a - b_j)(x) < \varepsilon$ for $x \in U_j$. There exist $e_1, \ldots, e_n \in C_0(X)_+$, such that $e_j \in C_0(U_j)_+$ and $e := \sum e_j$ satisfies $0 \leqslant e \leqslant 1$, $e_{|F} = 1$. Then $N(a - ea)(x) = (1 - e(x))N(a)((x) < \varepsilon$ and $N(ea - b)(x) \leqslant \varepsilon$ for $x \in X$, with $b := \sum e_j b_j$, by (2.1). Thus $\|a - b\| \leqslant 2\varepsilon$. ∎

2.4.4. *Suppose that B is a Banach $C_0(X)$-module with a nondegenerate and contractive action of $C_0(X)$ on B. Then there is a commutative $C_0(X)$-algebra A with nonzero fibers and a $C_0(X)$-module homomorphism $\psi \colon B \to A$ such that $\|\psi(b)_x\| = \|b_x\|$ for $x \in X$ and $b \in B$. (Thus $\|\psi(b)\| = \sup N(b)$.)*

If B is a Banach bundle, then A can be chosen as a C^*-bundle with commutative fibers, and such that ψ is isometric.

Proof. Consider the map $\gamma \colon x \in X \mapsto E_x := \{y \in E : y(C_0(X \setminus \{x\})B) = 0\}$ from X into the set of weakly closed convex sets of the unit ball E of the dual space of B, where E_x is (naturally isomorphic to) the unit ball of the dual spaces of B_x. Then E with the $\sigma(B^*, B)$-topology is a compact space. The subset $Z := \{(x, y) : y \in E_x\}$ of $X \times E$ contains $X \times \{0\}$ and is closed in $X \times E$: indeed, Z is the intersection of all the sets $Z_b := \{(x, y) \in X \times E : |y(b)| \leqslant N(b)(x)\}$, where $b \in B$. Since $N(b)$ is lower semicontinuous, Z_b is closed in $X \times E$.

Let $b \in B \mapsto \widehat{b} \in C(E)$ be given by the evaluation maps $\widehat{b}(y) := y(b)$. This defines a natural isometry from $C_0(X, B) \cong C_0(X) \check{\otimes} B$ into $C_0(X) \check{\otimes} C(E) \cong C_0(X \times E)$. The restriction map $f \mapsto f_{|Z}$ defines an epimorphism $\varphi \colon C_0(X \times E) \to C_0(Z)$, which induces a $C_0(X)$-algebra structure on $C_0(Z)$, because the kernel is a closed ideal, which is automatically $C_0(X) \otimes 1$-invariant.

For $f, g \in C_0(X)$, $x \in X$, $y \in E_x$ and $b \in B$ we have $(f \otimes \widehat{g \cdot b})(x, y) = ((fg) \otimes \widehat{b})(x, y)$, because $f(x)y(g \cdot b) = f(x)g(x)y(b)$. Thus

$$(f \otimes \widehat{g \cdot b})_{|Z} = ((fg) \otimes \widehat{b})_Z.$$

Since Cohen factorization applies to the nondegenerate $C_0(X)$-module B, we get a well-defined $C_0(X)$-module homomorphism ψ from B into $C_0(Z)$ by $\psi(b) = \varphi(f \otimes \widehat{a})$ for a factorization $b = fa$, with $a \in B$, $f \in C_0(X)$.

The fiber maps $f \mapsto f_x$ on $C_0(Z)$ are given by the restrictions to $Z \cap (\{x\} \times E) = \{x\} \times E_x$. Thus $\|\psi(b)_x\| = |f(x)| \sup\{|y(a)| : y \in E_x\}$, which means $N(\psi(b)) = N(b)$.

Now, suppose that $N(a)$ is continuous for every $a \in B$. We show that the map $p_1 : (x, y) \in Z \mapsto x \in X$ is open. Then $Z \subset X \times E$ satisfies the requirements of (i)–(iii) of Example 2.2.3 and $C_0(Z)$ is a C^*-bundle over X.

Let $(s, t) \in Z$ and $U \times V$ a neighborhood of (s, t). Suppose that $p_1(Z \cap (U \times V))$ does not contain a neighborhood of s. Then there exists a net (x_β) in U such that x_β converges to s, but $E_{x_\beta} \cap V = \emptyset$.

Upon replacing V by a smaller neighborhood of t, we may assume that V is the set $\left\{y \in E : \sum_{j=1}^{n} |y(b_j) - t(b_j)|^2 < 1\right\}$ for suitable $b_1, \ldots, b_n \in B$.

Since E_{x_β} is compact and convex, for each β there is $w_\beta = (w_{1,\beta}, \ldots, w_{n,\beta}) \in \mathbb{C}^n$ such that $\|w_\beta\|_2 = 1$ and $\mathrm{Re}(y(c_\beta)) \leqslant 1 + \mathrm{Re}(t(c_\beta)) \leqslant 1 + N(c_\beta)(s)$ for all $y \in E_{x_\beta}$, where $c_\beta := \sum w_{j,\beta} b_j$. On the other hand, $N(c_\beta)(x_\beta) = \sup\{\mathrm{Re}(y(c_\beta)) : y \in E_{x_\beta}\}$.

We can pass to suitable subnets (x_α), (w_α) such that (w_α) converges to a vector $w_0 = (w_{1,0}, \ldots, w_{n,0})$, because the unit sphere in $\mathbb{C}^n$ is compact. Then c_α converges in B (in norm) to $a := \sum w_{j,0} b_j$. Thus $N(c_\alpha)$ converges uniformly to $N(a)$. It follows that $\limsup N(a)(x_\alpha) < 1/2 + N(a)(s)$, which contradicts the continuity of $N(a)$. ∎

Proof of Theorem 2.7. If B is a Banach $C_0(X)$-submodule of a C^*-bundle A over X, then B is a Banach bundle over X, by 2.4.3. Conversely, B is isomorphic to a Banach $C_0(X)$-submodule of the commutative C^*-bundle $C_0(Z)$ over X constructed in the proof of assertion 2.4.4. ∎

2.4.5. *Suppose that Y is a locally compact Hausdorff space, that B is a complex vector space which is an algebraic $C_0(Y)$-module, and that $P : B \to C_0(Y)_+$ is a subadditive map from B into $C_0(Y)_+$, which satisfies $P(fa) = |f|P(a)$ and $P(za) = |z|P(a)$ for $a \in B$, $f \in C_0(Y)_{\mathrm{sa}}$ and $z \in \mathbb{C}$.*

Then $\|a\|_P = \sup P(a)$ is a seminorm on B.

If $\|\cdot\|_P$ is a norm and if B is complete with respect to this norm, then P is equal to the fiberwise norm function N, i.e. *$P(a)(y) = \|a + C_0(Y \setminus \{y\})_+ B\|_P$ for $a \in B$ and $y \in Y$, and B is a Banach bundle over Y.*

Proof. The map $a \in B \mapsto \|a\|_P := \sup_{x \in Y} P(a)(x)$ is subadditive and satisfies $\|za\| = |z|\|a\|_P$ for $z \in \mathbb{C}$, because P is subadditive and $\mathbb{C}$-homogeneous. It also satisfies $\|fa\|_P \leqslant \|f\|\,\|a\|_P$ for $f \in C_0(Y)_{\mathrm{sa}}$ and so $\|ga\|_P \leqslant \sqrt{2}\|g\|\,\|a\|_P$ for $g \in C_0(Y)$.

Let us now assume that $\|\cdot\|_P$ is a norm on B. First we show that the action of $C_0(Y)$ on B is nondegenerate (with respect to $\|\cdot\|_P$). Let $\varepsilon > 0$. Since $P(a) \in C_0(Y)_+$, we can find a compact subset $F \subset Y$ such that $P(a)(x) \leqslant \varepsilon$ for $x \in Y \setminus F$. Take a function $g \in C_0(Y)_+$ with $g \leqslant 1$ and $g(x) = 1$ for $x \in F$. Given any $y \in Y$, there is $f \in C_0(Y)_+$ with $f(y) = 1$, $f \leqslant 1$. Then the identities

$P(a-ga)(y) = f(y)P(a-ga)(y) = P((f(1-g)a)(y)$ and $(1-g)(y)P(a)(y) = (|f(1-g)|P(a))(y)$ imply that $P(a-ga)(y) \leqslant \varepsilon$, because $f(1-g) \in C_0(Y)_+$. Thus $\|a-ga\|_P \leqslant \varepsilon$, which means that the action of $C_0(Y)$ on B is nondegenerate.

Now suppose such that B is complete with respect to the norm $\|\cdot\|_P$. Since $C_0(Y)_+B$ is dense in B, the contractive algebra morphism from $C_0(Y)_{\mathrm{sa}}$ into $\mathfrak{L}(B, \|\cdot\|_P)$ extends to a contractive algebra morphism from $C_{\mathrm{b}}(Y)_{\mathrm{sa}}$ into $\mathcal{L}(B)$ by the *real* variant of Lemma 2.3 (cf. Remark 2.5 for this).

Thus formula (2.6) applies and $\|a+C_0(Y\setminus\{x\})B\|_P = \|a+C_0(Y\setminus\{x\})_+B\|_P$ is the same as $\inf\{\sup(1-f)P(a) : f \in C_0(Y\setminus\{x\})_+, f \leqslant 1\}$. Since $P(a) \in C_0(Y)_+$, the latter is equal to $P(a)(x)$, and so $P = N$, $\|a\|_P = \sup N(a)$, and $(fa)_y = f(y)a_y$ for every $f \in C_0(Y)$ and $a \in B$, because the map $a \mapsto a_y$ is complex linear and the equality $(ha)_y = h(y)a_y$ holds for $h \in C_0(Y)_{\mathrm{sa}}$. Thus $N(fa) = |f|N(a)$ and $\|fa\|_P \leqslant \|f\|\|a\|$ for $a \in B$ and every $f \in C_0(X)$, which implies that the (complex) Banach $C_0(Y)$-module $(B, \|\cdot\|_P)$ satisfies all conditions of Definition 2.6. ∎

2.4.6. *Suppose that A and B are Banach bundles over a compact space X, that ψ is a $C(X)$-module map from A into B, and that ψ is bounded (as linear map). Then for every $x \in X$, there is a unique linear map ψ_x from A_x into B_x, such that $\|\psi_x\| \leqslant \|\psi\|$ and $\psi_x(a_x) = (\psi(a))_x$ for $a \in A$.*

Suppose moreover, that the kernel of ψ is trivial and $\psi(A)$ is dense in B. Then $\psi(A) = B$ holds if and only if there is $\gamma < \infty$ such that $\|a\| \leqslant \gamma\|\psi_x(a)\|$ for all $x \in X$ and $a \in A_x$ (Then ψ_x is an isomorphism from A_x onto B_x for $x \in X$.)

Proof. The image $\psi(a + C_0(X\setminus\{x\})A)$ is contained in $\psi(a) + C_0(X\setminus\{x\})B$ because ψ is a $C(X)$-module map. Thus $\psi_x : A_x \to B_x$ is well-defined and satisfies $\|\psi_x\| \leqslant \|\psi\|$.

If, moreover, ψ is faithful and surjective, then ψ has a bounded linear inverse $\varphi : B \to A$ by the inverse mapping theorem. It is necessarily a $C(X)$-module map. Since $\psi_x(A_x)$ is dense in B_x (because $\psi(A)$ is dense in B), φ_x is the inverse of ψ_x, and $\gamma := \|\varphi\|$ is the desired bound.

Conversely, by (2.7), $\gamma\|\psi(a)\| \geqslant \|a\|$ for all $a \in A$, if the uniform bound is given on the fibers of A and B. Thus $\psi(A) = B$. ∎

2.4.7. We generalize Theorem 3.3 of [2] in Lemma 2.10 below to countably generated $C(X)$-bundles. In order to do so, we need the following statements:

Suppose that G is a separable subset of A, and let B denote the Banach subbundle of A which is (topologically) generated by G. Then there are

(i) *a σ-compact open subset Ω of X,*

(ii) *a locally compact separable and metrizable space Y,*

(iii) *a continuous (not necessarily open) map ψ from Ω onto Y, such that the inverse images of compact subsets of Y are compact subsets of X (i.e. ψ is proper: $\psi^*(C_0(Y)) \subset C_0(\Omega)$),*

(iv) *a separable closed linear subspace D of B, with $G \subset D$,*

such that D is the closure of $\psi^*(C_0(Y)_+)D$ and, for every $d \in D$ and $x \in \Omega$, $N(d) \in \psi^*(C_0(Y)_+)$ and $\|d_x\|_B = \|d_{\psi(x)}\|_D$.

Then D is in a natural way a Banach bundle over Y, B is the closed linear span of $C_0(X)_+D$ in A, and there are natural isomorphisms $B_{|\psi^{-1}(z)} \cong$

$C(\psi^{-1}(z), D_z)$ for $z \in Y$. The natural map $\pi_x : D \to B_x$ defines an isomorphism $B_x \cong D_{\psi(x)}$ if $x \in \Omega$, and $B_x = 0$ for $x \in X \setminus \Omega$. If $B_x \neq 0$ for every $x \in X$, then $\psi^*(C_0(Y))C_0(X) = C_0(X)$. (But in general $C_0(\Omega)$ is *not* a C^*-bundle over Y, because ψ is not open.)

Moreover, D is the closed linear span of $\psi^*(C_0(Y))G$, D is a separable C^*-bundle over Y with nonzero fibers, and B is the closed linear span of $C_0(X)D$, if (in addition) A is a complex C^*-bundle and if G is a separable C^*-subalgebra of A.

Proof. We can find separable C^*-subalgebras $C_1 \subset C_2 \subset \cdots \subset C_0(X)$, and separable closed linear subspaces $G \subset D_1 \subset D_2 \subset \cdots \subset B$ such that $N(D_n) \subset C_n$, $C_n D_n \subset D_{n+1}$, for $n = 1, 2, \ldots$. Then the closure D of the union of the D_n, the character space Y of the closure C of the union of the C_n, the union Ω of the supports of the functions in C and the natural epimorphism ψ from Ω onto Y are as desired, as one can easily check.

Suppose that A is a C^*-bundle and G is a separable C^*-subalgebra of A. The closed linear subspace E (respectively F) generated by $\psi^*(C_0(Y))G$ (respectively $C_0(X)G$) is a C^*-subalgebra of A, which is a $C_0(Y)$-submodule of D (respectively a $C_0(X)$-submodule of B) and which contains the generating set G of B. Thus $F_x = B_x$ for $x \in X$ and $E_y = D_y$ for $y \in Y$. This implies $F = B$ and $E = D$ by 2.4.3. ∎

Lemma 2.10. *Let A be a countably generated C^*-bundle over a compact space X, such that every fiber A_x is nonzero.*

Then there is a positive $C(X)$-module map ψ from A into $C(X)$ such that the positive linear functionals ψ_x on A_x given by $\psi_x(a_x) = \psi(a)(x)$ for $a \in A$, are faithful for every $x \in X$.

Proof. Let $a_1, a_2, \ldots$ be a generating sequence of A as a Banach $C(X)$-module, and let G be the C^*-algebra which is generated by this sequence. We consider $D \subset B := A$, Y, $\psi \colon \Omega \to Y$ as in 2.4.7. Since $A_x \neq 0$ for every $x \in X$, here $\Omega = X$ and $Y = \psi(X)$ is compact, as observed in 2.4.7. Thus D is a separable C^*-bundle over a metrizable compact space Y, and A is the closed linear span of $C(X)D$.

There exists a unital $C(Y)$-linear positive map μ from D to $C(Y)$ such that for every $y \in Y$, the induced state $\mu_y \colon D_y \to \mathbb{C}$ is faithful on the fiber D_y ([2], Theorem 3.3).

Let ν be the $C(X)$-linear positive map from $C(X) \otimes D$ into $C(X)$, which is the composition of the min-tensor product $\mathrm{id}_{C(X)} \otimes^{\min} \mu$ with the adjoint of the map $x \mapsto (x, \psi(x))$. Then ν maps the elementary tensor $g \otimes d \in C(X) \otimes D$ to $g \cdot (\mu(d)\psi) \in C(X)$. Hence ν is zero on the *linear span* $\mathcal{I}$ of the elements $(g \otimes fd) - (fg \otimes d)$, with $f \in C(Y)$, $g \in C(X)$, $d \in D$.

The reader can easily check that $\mathcal{I}$ is a $*$-ideal of the algebraic tensor product $C(X) \odot D$. Thus the closure K of $\mathcal{I}$ is a closed ideal of $C(X) \otimes D$ with $\nu(K) = 0$.

It is shown in [1] that K is the intersection of the kernels of the natural epimorphisms $C(X) \otimes D \to C(X)_y \otimes D_y$. Since $C(X)_y \otimes D_y$ is isomorphic to $C(\psi^{-1}(y), A_x)$ for $x \in X$ with $\psi(x) = x$ (cf. 2.4.7), it follows that the natural C^*-algebra epimorphism λ from $C(X) \otimes D$ onto A with $\lambda(g \otimes d) = gd$ has K as its kernel. Thus ν is zero on the kernel of λ, and therefore $\nu = \varphi\lambda$ for a (unique) positive $C(X)$-linear map $\varphi \colon A \to C(X)$.

For every $x \in X$ the induced state $\varphi_x : A_x \to \mathbb{C}$ is faithful, because for $a_x \in (A_x)_+ \setminus \{0\}$ there is $d \in D$ with $d_{\psi(x)} = a_x$, if we identify $A_x = \pi_x(A) = \pi_x(D)$ and $\pi_{\psi(x)}(D) = D_{\psi(x)}$ naturally, and note that $\varphi_x(a_x) = \mu_{\psi(x)}(d_{\psi(x)}) > 0$, cf. 2.4.7. ∎

2.5. QUOTIENTS OF BANACH BUNDLES BY EVERYWHERE NONZERO SECTIONS. If A is a Banach $C_0(X)$-module and $B \subset A$ is a closed $C_0(X)$-submodule of A such that A/B is a (continuous) Banach bundle over X, then, for every $d \in A/B$ and $\varepsilon > 0$, there is $a \in A$ with $\|a_x\| \leqslant \|d_x\| + \varepsilon$ for all $x \in X$. (Indeed, $a := (\varepsilon + N(d))e$ is as desired for some e in the open unit ball of A with $e + B = (\varepsilon + N(d))^{-1}d$.)

Unfortunately, quotients of Banach bundles over X by Banach subbundles are not always Banach bundles (see also Remark 6.6). But one of the later needed good cases is the following result.

LEMMA 2.11. *Suppose that X is compact, A is a Banach bundle over X and that $a \in A$ satisfies that $N(a)$ is everywhere (strictly) positive on X.*

Then the submodule $C(X)a$ is closed in A, is module-isomorphic to $C(X)$, and the quotient Banach module $A/C(X)a$ is a (continuous) Banach bundle over X.

Proof. Step 1: *The submodule generated by a is closed in A and is isometrically module-isomorphic to $C(X)$.*

Indeed, $C(X)a$ is a Banach subbundle of A and $\|a_y\|_{C(X)a} = \|a_y\|_A$ for $y \in X$ by 2.4.3. Since X is compact, we have $\inf N(a) > 0$. Thus an isometric module isomorphism from $C(X)$ onto $C(X)a$ is given by $h \mapsto hc$, where $c := N(a)^{-1}a$.

Step 2: *The action of $C(X)$ on the quotient Banach $C(X)$-module $A/C(X)a$ is nondegenerate and contractive, the fibers of $A/C(X)a$ are naturally isomorphic to $A_y/\mathbb{C}a_y$, and*

$$\|d_y\| = \inf\{\|ta_y + b_y\| \,:\, t \in \mathbb{C}\,,\ |t| \leqslant 2\|b_y\|/\|a_y\|\,\}$$

for $y \in X$ and $d = b + C(X)a$, with $b \in A$.

Indeed, by Lemma 2.4, the quotient $E := A/C(X)a$ is a Banach $C(X)$-module with a contractive and nondegenerate action of $C(X)$, with fibers E_x isomorphic to $A_x/(\mathbb{C}\cdot a_x)$ by an isometry which maps $d_x \in E_x$ to $b_x + \mathbb{C}\cdot a_x$ for $d = b + C(X)a$. The norm of $d_x = b_x + \mathbb{C}\cdot a_x$ in $A_x/(\mathbb{C}\cdot a_x)$ is given by $\inf\{\|b_x + ta_x\| \,:\, t \in \mathbb{C}\}$.

The convex continuous function $t \in \mathbb{C} \mapsto \|ta_y + b_y\|$ takes its minimum $\mathrm{dist}(b_y, \mathbb{C}a_y)$ at a point $t_y \in \mathbb{C}$ with $|t_y| \leqslant 2\|b_y\|/\|a_y\|$, as the triangle inequality shows.

Step 3: *The map $y \in X \mapsto \|d_y\|$ is continuous for $d \in A/C(X)a$, and for $b \in A$, $\varepsilon > 0$, there exists $f \in C(X)$ such that*

$$N(fa + b)(y) \leqslant \varepsilon + \inf\{\|ta_y + b_y\| \,:\, t \in \mathbb{C}\}$$

and $|f(y)| \leqslant (2\|b_y\| + \varepsilon)/\|a_y\|$ for every $y \in X$.

Indeed, let $b \in A$, define the compact set $F := \{t \in \mathbb{C} \,:\, |t| \leqslant 2\|b\|/\inf N(a)\}$, and let $E := A/C(X)a$. Then, by Step 2, the function $N_E(b + C(X)a)$ satisfies, for $y \in X$, that

$$N_E(b + C(X)a)(y) = \inf\{\|b_y + t\,a_y\| \,:\, t \in F\}.$$

Moreover, this function is continuous with respect to y: Indeed the function $(t, y) \in (F \times X) \mapsto N(a + tb)(y)$ is continuous, because the map $(c, y) \mapsto N(c)(y)$ is continuous. Since F is compact, it also follows that $y \mapsto \inf\{N(a+tb)(y) : t \in F\}$ is continuous on X.

Now we use Step 2 and the continuity of $N_E(d)$ and of $y \mapsto N(ta + b)(y)$ to construct the function $f \in C(X)$ with the desired properties.

Given $b \in A$ and $\varepsilon > 0$ (fixed), we define $g := N(a)^{-1}N(b)$ and $r := N(a)^{-1}(N(b) + \varepsilon)$. For $x \in X$, there exists $t_x \in \mathbb{C}$ and a neighborhood $U(x)$ of x such that $|t_x| \leqslant 2g(x)$, $N(t_x a+b)(x) = N_E(d)(x)$, $g(x) < r(y)$ and $N(t_x a+b)(y) < \varepsilon + N_E(d)(y)$ for $y \in U(x)$.

Since X is compact, we find $x_1, \dots, x_n \in X$, $t_1, \dots, t_n \in \mathbb{C}$ and neighborhoods $U_1, \dots, U_n$ of the x_i, such that $\mathcal{U} = \{U_1, \dots, U_n\}$ is a covering of X with $|t_i| < 2r(x)$ and $N(t_i a + b)(x) < N_E(d)(x) + \varepsilon$ for $x \in U_i$. Further let $e_1, \dots, e_n \in C(X)_+$ be a decomposition of 1 which is subordinate to $\mathcal{U}$, i.e. $\sum e_i = 1$ and $e_i(x) > 0$ implies $x \in U_i$. Then $f := \sum t_i e_i \in C(X)$ satisfies $|f(x)| < 2r(x)$ and $N(fa + b) \leqslant \varepsilon + N_E(d)$.

Step 4: Let $\varepsilon > 0$. With f as in Step 3 we get

$$\|d\| = \|b + C(X)a\| \leqslant \|fa + b\| = \sup N(fa + b) \leqslant \varepsilon + \sup N_E(d).$$

Since $N_E(d)(y) = \|d_y\| \leqslant \|d\|$, it follows $\|d\| = \sup N_E(d)$. We have seen in Step 3, that $N_E(d)\colon y \in X \mapsto \|d_y\|$ is continuous for $d \in A/C(X)a$. Thus $A/C(X)a$ satisfies (i)-(iv) of Definition 2.6. ∎

LEMMA 2.12. *Suppose that X is compact and that the elements $a_1, \dots, a_n \in A$ are such that $(a_1)_x, \dots, (a_n)_x$ are linearly independent for every $x \in X$.*

Then the $C(X)$-submodule D of A which is algebraically generated by $\{a_1, \dots, a_n\}$ is closed, and the quotient A/D is a Banach bundle.

The Banach bundle D is naturally (but not necessarily isometrically) isomorphic to $C(X)^n$ as a Banach $C(X)$-module.

Moreover, for $b \in A$ and $\varepsilon > 0$, there exists $d \in D$ such that $N_A(d + b) \leqslant \varepsilon + N_{A/D}(b + D)$.

*For every closed subset F of X, $(A/D)_{|F}$ is naturally isometrically isomorphic to $(A_{|F})/(D_{|F})$, and the natural epimorphism from $A_{|F}$ onto $(A/D)_{|F}$ maps the open unit ball onto the open unit ball (*i.e. *is the quotient map of the underlying normed spaces).*

Proof. (Cf. also the proof of Proposition 12 of [10] for a different argument.)

For $n = 1$ this follows from Lemma 2.11 and from Step 3 in its proof.

The general case follows by induction, because $A/(D + C(X)a_{n+1})$ is naturally isomorphic to $(A/D)/(C(X)(a_{n+1} + D))$ if $(a_{n+1})_x$ is not in D_x for every $x \in X$.

The natural map from $(A/D)_{|F}$ into $(A_{|F})/(D_{|F})$ is isometric, because this is true for $F = \{y\}$. ∎

2.5.1. *If $y \in X$, $e_1, \dots, e_k \in A$ and the dimension of the span of $(e_1)_y, \dots, (e_k)_y$ has dimension $\geqslant n$, then there exists a neighborhood U of y such that for $x \in U$, the dimension of the span of $(e_1)_x, \dots, (e_k)_x$ is $\geqslant n$.*

Proof. By the (vector) basis extension theorem, we may assume that $k = n$. Then $(e_1)_y, \ldots, (e_n)_y$ are linearly independent. For $n = 1$, the result follows from the continuity of $N(e_1)$. By induction the result follows from Lemma 2.12 and the case $n = 1$. ∎

2.5.2. We say that a compact subset F of X has the *sphere-section extension property* (for short s.e.p.) with respect to A, if for every closed subset $G \subset F$ and every $b \in A$ with $N(b)(x) = 1$ for $x \in G$, there exists $c \in A$ such that $N(c)(x) = 1$ for $x \in F$ and $c_{|G} = b_{|G}$. Note that the condition $N(c)(x) = 1$ on F can be softened to $N(c)(x) > 0$ on F. Take indeed $f \in C(X)_+$ with $f_{|F} = (N(c)_{|F})^{-1}$. Then $d := fc$ satisfies $d_{|G} = b_{|G}$ and $N(d)(x) = 1$ for $x \in F$. Note also that the empty set has obviously the s.e.p.

Let $F_0 \subset F$ be compact subsets of X, such that F_0 has the sphere-section extension property and for every $x \in F \setminus F_0$, there exists a compact neighborhood K of x in the locally compact space $F \setminus F_0$, such that K has the sphere-section extension property.

Then F has the sphere-section extension property.

Proof. Obviously, the s.e.p. passes to closed subsets of a compact set with s.e.p. The pull-back Property 2.4.2 of Banach bundles allows the reader to check that $F_1 \cup F_2$ has the s.e.p. if F_1 and F_2 have the s.e.p. It follows that every compact subset H of $F \setminus F_0$ has the s.e.p.

Let now G be a closed subset of F and let $b \in A$ satisfy $N(b)_{|G} = 1$. As F_0 has the s.e.p., there exists $c \in A$ with $N(c)_{|F_0} = 1$ and $c_{|(F_0 \cap G)} = b_{|(F_0 \cap G)}$. By Property 2.4.2, there exists $d \in A$ such that $d_{|F_0} = c_{|F_0}$ and $d_{|G} = b_{|G}$. It follows $N(d)(x) = 1$ for $x \in F_0 \cup G$.

Consider the compact subset $H := \{x \in F : N(d)(x) \leqslant 1/2\}$ in $F \setminus (F_0 \cup G) \subset F \setminus F_0$. On the boundary ∂H of H the function $N(d)$ takes the value $1/2$. Since H has the s.e.p. , there exists $e \in A$ such that $N(e)_{|H} = 1/2$ and $e_{|\partial H} = d_{|\partial H}$. Let M denote the closure of $F \setminus H$. Then $M \cap H = \partial H$. There exists $f \in A$, such that $f_{|H} = e_{|H}$ and $f_{|M} = d_{|M}$ (by Property 2.4.2). Thus $N(f)_{|F} \geqslant 1/2$ and $f_{|G} = b_{|G}$. ∎

2.6. Examples of Banach bundles.

2.6.1. Hilbert $C_0(X)$-modules (or real Hilbert $C_0(X)_{\mathrm{sa}}$-modules) A in the sense of Kasparov ([14]) are exactly the complex (or real) Banach bundles A over X where the fiber-norm function $N \colon A \to C_0(X)_+$ satisfies the parallelogram law

$$N(a+b)^2 + N(a-b)^2 = 2(N(a)^2 + N(b)^2),$$

i.e. where the quotient spaces $A_x = A/C_0(X \setminus \{x\})$ are Hilbert spaces with the quotient norms.

To see this, apply for the Banach bundle A the standard (real or complex) polar formula to the quadratic form $\beta_N : a \in A \mapsto N(a)^2 \in C_0(X)$ to get the desired $C_{\mathrm{b}}(X)$-hermitian form $(a, b) \mapsto \langle a, b\rangle$ on $A \times A$ with values in $C_0(X)$ and with $\|a\|^2 = \|\langle a, a\rangle\|_\infty$.

Conversely, if A is a Hilbert module, let $P(a) := (\langle a, a\rangle)^{1/2}$ and use 2.4.5 to see that A is a Banach bundle with $N(a) = P(a)$.

2.6.2. Suppose that B is a (real or complex) Banach space (respectively a C^*-algebra) and that Y is a set. We consider Y as discrete locally compact space and let $Z := \beta Y$ denote the Stone-Čech compactification of Y. Note that $C(Z) \cong \ell_\infty(Y)$.

Then the Banach $\ell_\infty(Y)$-module $A := \ell_\infty(Y, B)$ is a Banach bundle (respectively C^*-bundle) over Z. (This follows from 2.4.5 if one considers the map N from $\ell_\infty(Y, B)$ into $\ell_\infty(Y)$ which is given by $(b_y) \mapsto (\|b_y\|)$.)

If $Y = \mathbb{N}$, then we write $\ell_\infty(B)$. For the Banach bundle $A := \ell_\infty(B)$ over $\beta(\mathbb{N})$ we have the natural fibers $A_n \cong B$ if $n \in \mathbb{N}$ and A_ω is the "ultrapowers" $B_\omega := \ell_\infty(B)/J_\omega$ if $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$. Here J_ω is the closed space (respectively closed ideal) of bounded sequences (b_n) in B with $\lim_\omega (\|b_n\|) = 0$.

One can also consider closed subspaces (respectively C^*-subalgebras) B_y of B for every $y \in Y$ and the closed subspace $\prod_{y \in Y} B_y$ is a $\ell_\infty(Y)$-submodule of A and hence is also a Banach bundle (respectively C^*-bundle) over $Z := \beta Y$. (One can also directly apply 2.4.5 to see that for any family $(B_y)_{y \in Y}$ of Banach spaces the direct product of Banach space $A := \prod_{y \in Y} B_y$ is a Banach bundle over Z.)

2.6.3. Suppose that X is locally compact. Let $Y := X_{\mathrm{d}}$ be the set X with discrete topology and assume that to each $y \in X$ is attached a Banach space (respectively a C^*-algebra) B_y. Let $A = \prod_{y \in X} B_y$ denote the Banach space of bounded maps $b \colon y \in X \to b_y \in B_y$. As pointed out in Example 2.6.2 it is a Banach bundle (respectively a C^*-bundle) over $Z = \beta Y$.

Now consider a linear subspace D of A which is a $C_0(X)$-submodule of A and satisfies that $N(d) \in C_0(X)_+$ for $d \in D$, where we consider $C_0(X)$ naturally as a subalgebra of $\ell_\infty(X) \cong C(Z)$.

Then the closure E of D (in A) is a Banach bundle over X, because the $C_0(X)$-module E, and $P := N$ satisfy the requirements of 2.4.5.

If, moreover, the B_y are Hilbert spaces, then the closure E of D is a Hilbert bundle, i.e. a Hilbert $C_0(X)$-module in the sense of Kasparov, ([14]), by example 2.6.1, because the parallelogram law holds for the norm of B_x an thus for N on A.

2.6.4. *A Banach $C_0(X)$-module A is a Banach bundle over X if and only if A is the Banach space of continuous sections vanishing at infinity of a continuous field of Banach spaces over X, cf. Definition 10.1.2 in* [9], *such that the fibers are the A_x and the algebra morphism from $C_0(X)$ into $\mathfrak{L}(A)$ coincides with the multiplication of continuous sections by functions.*

Proof. It follows from 2.6.3 that the algebras of continuous sections vanishing at infinity of a continuous field of Banach spaces over X is a Banach bundle in the sense of our Definition 2.6.

Conversely, if A is a Banach bundle over X, then we can consider A as a vector subspace of the (unbounded) Cartesian product $\prod_{y \in X} A_x$ and can multiply there the elements of A with unbounded continuous functions f on X (where it is possible). We get a subspace Γ of $\prod_{y \in X} A_x$.

It is easy to check with the help of Section 10.2 of [9] that $\Gamma \subset \prod_{y \in X} A_x$ is a continuous field of Banach spaces over X with fibers A_x in the sense of Definition 10.1.2 of [9] (more precisely: it is the set of *all* continuous sections of this field), and that A is just the Banach space of continuous sections vanishing at infinity. ∎

2.7. PROJECTIVITY OF $M_n(C_0(0,1])$. Let $\{e_{i,j}\}_{i,j\in\mathbb{N}}$ denote the canonical system of matrix units of the C^*-algebra $\mathcal{K} := \mathcal{K}(\ell_2(\mathbb{N}))$ of compact operators acting on the separable infinite dimensional Hilbert space $\ell_2(\mathbb{N})$. These operators satisfy the relations $e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$ and $e^*_{i,j} = e_{j,i}$.

As the function $h_0 \colon t \in (0,1] \mapsto t \in \mathbb{C}$ generates $C_0((0,1])$, one gets that for $n > 1$, $C_0((0,1]) \otimes M_n(\mathbb{C})$ is the universal C^*-algebra generated by $n-1$ contractions $f_2, \ldots, f_n$ satisfying the relations

$$f_i f_j = 0 \quad \text{and} \quad f_i^* f_j = \delta_{i,j}\, f_2^* f_2 \quad \text{for} \quad 2 \leqslant i,j \leqslant n\,. \tag{2.9}$$

The natural C^*-algebra epimorphism Φ from $C_0((0,1])\otimes M_n(\mathbb{C})$ onto $C^*(f_2,\ldots,f_n)$ is uniquely determined by

$$\Psi : h_0 \otimes e_{j,1} \mapsto f_j \quad \text{for } 1 < j \leqslant n.$$

Note that $f_j := g_j(g_1)^*$, $1 < j \leqslant n$ satisfy (2.9) if $g_1, \ldots, g_n$ just satisfy $g_i^* g_j = \delta_{i,j}\, g_1^* g_1$.

The C^*-algebra $C_0((0,1], M_n(\mathbb{C}))$ is *projective*, i.e. for every closed ideal $J \subset A$ and every $*$-homomorphism of C^*-algebras $\psi \colon C_0((0,1], M_n(\mathbb{C})) \to A/J$ there is a $*$-homomorphism $\varphi \colon C_0((0,1], M_n(\mathbb{C})) \to A$ with $\pi_J\varphi = \psi$ (cf. Theorem 10.2.1 of [21] and [22]).

3. FINITE DIMENSIONAL HAUSDORFF SPACES

We recall here some properties of finite dimensional compact spaces, prove that the ordinary covering-dimension is the same as the later used decomposition-dimension, and we study a counter-example, originally due to Dixmier and Douady, for global stability of a Banach bundle in the case when the locally compact Hausdorff base space is not finite dimensional (Corollary 3.7).

Recall that a compact Hausdorff space X has *(covering-) dimension* $\mathrm{Dim}(X) \leqslant n \in \mathbb{N}$ if for every finite open covering of X there is another covering of X by open subsets which refines the given covering and is such that the intersection of every $n+2$ distinct sets of this covering is always empty, i.e. a given finite open covering admits a refinement whose nerve is a simplicial complex of dimension $\leqslant n$.

Here an open covering $\mathcal{V}$ of X is a *refinement* of an open covering $\mathcal{U}$ of X if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$.

The typical example of a *metrizable* finite dimensional space is a closed subset of the Euclidean space $\mathbb{R}^n = \mathbb{R}\times\cdots\times\mathbb{R}$, $n < \infty$, which is (at most) n-dimensional as topological product of n copies of the 1-dimensional space $\mathbb{R}$ ([13], Theorem III.4). This also happens for (affinely) n-dimensional polyhedra.

Let us define open cubes as follows:

(i) $O_r =]r-(2/3), r+(2/3)[\subset O'_r =]r-1, r+1[\subset \mathbb{R}$ for $r \in \mathbb{Z}$,

(ii) $O_s = O_{s_1} \times \cdots \times O_{s_n} \subset O'_s = O'_{s_1} \times \cdots \times O'_{s_n} \subset \mathbb{R}^n$ for $s = (s_1, \ldots, s_n) \in \mathbb{Z}^n$.

If $r \mapsto \dot{r}$ is the quotient map $\mathbb{Z}^n \to (\mathbb{Z}/2\mathbb{Z})^n$, then $\mathbb{R}^n = \bigcup\{Z_t : t \in (\mathbb{Z}/2\mathbb{Z})^n\}$, where each open set $Z_t = \bigcup\{O_s : \dot{s} = t\} \subset \mathbb{R}^n$ is the disjoint union of the cubes O_s, $\dot{s} = t$, and these cubes have moreover disjoint closures $\overline{O}_s \subset O'_s$.

Since we can scale this construction by $\varepsilon > 0$, we get that, for any finite open covering $X = \bigcup_i \Omega_i$ of a compact subset $X \subset \mathbb{R}^n$, there is a covering of X which refines this covering and consists of intersections of X with open cubes holding the above disjointness properties.

Hurewicz and Wallman have shown that every *metrizable* compact space X of *finite* dimension n is homeomorphic to a closed subset of $[0,1]^{2n+1}$ ([13], Theorem V.3). As a consequence, there is a monotonous function $n \mapsto \Psi(n)$ on $\mathbb{N}$, e.g. $\Psi(n) = 2^{2n+1}$, such that every n-dimensional compact space has *decomposition-dimension* $\leqslant m = \Psi(n) - 1$ in the sense of the following definition:

Definition 3.1. We say that a topological space X has the *decomposition-dimension* $\leqslant m$ if for every finite covering $\mathcal{O}$ of the topological space X, there is a finite open covering $\mathcal{U} = \{U_1, \ldots, U_q\}$ which refines $\mathcal{O}$ and for which there exists a map $\iota\colon \{1, \ldots, q\} \to \{1, \ldots, m+1\}$, such that for each $1 \leqslant k \leqslant m+1$, the open set $Z_k = \bigcup_{j \in \iota^{-1}(k)} U_j$ is the *disjoint* union of the open sets U_j, $j \in \iota^{-1}(k)$.

3.1 We shall later use that one can actually take $\Psi(n) = n+1$ in the previous statement (and this is the best possible choice, because it is obvious that X has covering-dimension $\leqslant n$ if X has decomposition-dimension $\leqslant n$). We could however not find explicit references for this. But it is likely that a proof of the following lemma is implicitly contained in the proof of Theorem V.1 of [13]. For completeness we add a proof which is our elementary reformulation of an idea of W. Winter.

Lemma 3.2. *Let X be a compact Hausdorff space of topological dimension $\leqslant n$, let $\mathcal{O} = \{O_1, \ldots, O_p\}$ be an open covering of X and let $\mathcal{U} = \{U_1, \ldots, U_q\}$ be an open covering of X which is an refinement of $\mathcal{O}$ such that every intersection of $n+2$ different elements of $\mathcal{U}$ is empty.*

Then there is a finite open covering $\mathcal{V}$ of X which is a refinement of $\mathcal{U}$ (and thus of $\mathcal{O}$) and is such that the set $\mathcal{V}$ can be partitioned into $n+1$ subsets, consisting of elements with pairwise disjoint closures.

The lemma says that a compact Hausdorff space X has covering-dimension $\leqslant n$ if and only if it has decomposition-dimension $\leqslant n$.

In general the covering $\mathcal{U}$ itself cannot be partitioned in the desired way, i.e. there is no pure combinatorial proof of Lemma 3.2, as a study of the covering $\mathcal{U}_1$ of P in the below given proof can show. (In fact our proof only needs X to be normal.) There are T_0-spaces, e.g. primitive ideal spaces $\mathrm{Prim}(A)$ of separable C^*-algebras A, having covering-dimension which is strictly smaller than its decomposition dimension, cf. remark before Proposition 3.4 in [18].

Proof. Note that the covering $\mathcal{U}$ exists by definition of the topological dimension n. Note also that it is enough to construct a finite refinement $\mathcal{W} := \{W_1, \ldots, W_r\}$ of $\mathcal{U}$ and a partition of the set $\mathcal{W}$ into $n+1$ subsets such that the elements of the partition consist of pairwise disjoint elements: Let $f_1, \ldots, f_r \in C(X)$ be a partition of unity subordinate to $\mathcal{W}$, i.e. $f_i \in C_0(W_i)_+$ and $\sum f_i = 1$ in $C(X)$. Then $\mathcal{V} = \{V_1, \ldots, V_r\}$, with $V_i := f_i^{-1}((1/(n+2), 1]) \subset W_i$, is a covering of X and has a partition into $n+1$ subsets, consisting of elements with pairwise disjoint closures.

By Tietze extension, we find a partition of unity $\eta_1, \ldots, \eta_q$ subordinate to $\mathcal{U}$. Let $\psi(x) := (\eta_1(x), \ldots, \eta_q(x))$, which is in the $q-1$-dimensional standard simplex $\sigma_{q-1} \subset \mathbb{R}^q$, i.e. the the convex span of standard basis elements of $\mathbb{R}^q$. Each value $\psi(x)$ of ψ has at most $n+1$ coordinates different from zero. It follows that the image is in the union P of the $q!/((n+1)!(q-n-1)!)$ simplices σ_S of affine dimension n, which are the intersections of σ_{q-1} with the $n+1$-dimensional subspaces of $\mathbb{R}^q$ spanned by a subset S of $n+1$ different elements of the natural basis of $\mathbb{R}^q$. Thus the image of ψ is contained in the polyhedron P (of affine dimension n).

Let $H_i := \{(y_1, \ldots, y_q) \in P \,:\, y_i > 0\}$. Then $\psi^{-1}(H_i) = \{x \in X \,:\, \eta_i(x) > 0\}$, is contained in U_i.

The (relatively) open subsets $H_i \cap P$ define a covering $\mathcal{U}_1$, which we are going to refine to a covering $\mathcal{W}_1$ of P, which has a map $\iota\colon \mathcal{W}_1 \to \{1, \ldots, n+1\}$ such that $Y \cap Z = \emptyset$ if $Y, Z \in \mathcal{W}_1$, $Y \neq Z$ and $\iota(Y) = \iota(Z)$. Such a map ι is called a $(n+1)$-*coloring* (of the covering $\mathcal{W}_1$).

Then $\mathcal{W} := \{\psi^{-1}(Y) \,:\, Y \in \mathcal{W}_1\}$ is the desired refinement of $\mathcal{U}$ with a $(n+1)$-coloring.

The construction of $\mathcal{W}_1$ is very simple (following an idea of W. Winter): $\mathcal{W}_1$ consists of the open stars (of the new vertices = old barycentres) of the first barycentric subdivision of the (affine) n-dimensional polyhedron P together with its standard $(n+1)$-coloring.

This is the complete proof. But since the reader is possibly not familiar with this terminology, we finish the proof with an elementary description of $\mathcal{W}_1$ and $\iota\colon \mathcal{W}_1 \to \{1, \ldots, n+1\}$.

Let from now on $e_j := (\delta_{i,j})$, $i, j = 1, \ldots, q$ denote the standard basis of $\mathbb{R}^q$. For any subset S of $\{1, \ldots, q\}$ with cardinality $|S| \leqslant n+1$, let $e(S) := |S|^{-1} \sum_{j \in S} e_j \in \mathbb{R}^q$. (This is the *barycentre* corresponding to the first barycentric subdivision of P.) Then $e(S) = e(T)$ if and only if $S = T$. Further, let $\sigma(S_1, \ldots, S_{n+1})$ denote the convex span of $e(S_1), \ldots, e(S_{n+1})$, and if $|S| = n+1$, let σ_S denote the convex span of $\{e_j \,:\, j \in S\}$. The above defined polyhedron P is the union of the simplices σ_S and every simplex σ_S is the union of the (affine) n-dimensional simplices $\sigma(S_1, \ldots, S_{n+1})$, with $S_1 \subset S_2 \subset \cdots \subset S_{n+1}$, $|S_k| = k$ and $S_{n+1} = S$.

If we define Σ as the collection (i.e. finite set) of the simplices $\sigma(S_1, \ldots, S_{n+1})$ with $S_1 \subset S_2 \subset \cdots \subset S_{n+1}$ and $|S_k| = k$ $(k = 1, \ldots, n+1)$, then $P = \bigcup \Sigma$.

The star $\mathrm{st}(y)$ of a point $y \in P$ is defined as the subset

$$\mathrm{st}(y) := P \setminus \bigcup \{\sigma \in \Sigma \,:\, y \notin \sigma\}.$$

Then $\mathrm{st}(y)$ is a (relatively) open subset of P and contains y, because the simplices $\sigma \in \Sigma$ are compact.

Now the reader is ready to check the following properties (i)–(vii) step by step:

(i) $y \in \sigma \in \Sigma$ if and only if $e(T) \in \sigma \in \Sigma$ for the support $T \subset \{1, \ldots, q\}$ of y, considered as a function on $\{1, \ldots, q\}$.

(ii) σ_S is covered by the stars $\mathrm{st}(e(T))$ for $T \subset S$, because $\mathrm{st}(y) = \mathrm{st}(e(T))$ if $y \in \sigma_S$ has T as its support, by (i).

(iii) $\mathrm{st}(y)$ is (in our special situation) the union those half-open rays (intervals) $[y, z)$ beginning in y and ending in $z \in P$ which are contained in P. Here $[y, z) := \{y + t(z - y) \, : \, t \in [0, 1)\}$.

(iv) $\mathrm{st}(e(S)) \subset H_k$ for every $k \in S$, by (iii).

(v) $S = T$ if $\mathrm{st}(e(S)) = \mathrm{st}(e(T))$, by (iv), because if e.g. $k \in S \setminus T$ then $e(S) \subset H_k$ but $e(T) \notin H_k$.

(vi) If $e(S)$ and $e(T)$ are both contained in the same (n-dimensional) simplex σ of Σ, then $S = T$, or $|S| \neq |T|$ and $S \subset T$ or $T \subset S$.

(vii) If $S \neq T$ but $|S| = |T|$, then $\mathrm{st}(e(S))$ and $\mathrm{st}(e(T))$ are disjoint by (vi).

Summing up, we get that $\tau \colon S \mapsto \mathrm{st}(e(S))$ defines a one-to-one map from the subsets $S \subset \{1, \ldots, n+1\}$ of cardinality $|S| \leqslant n+1$ into the open subsets of P, such that $\mathcal{W}_1 := \mathrm{Im}(\tau)$ is an open covering of P which is a refinement of $\mathcal{U}_1$ and the map

$$\iota \colon Y \in \mathcal{U}_1 \mapsto |\tau^{-1}(Y)| \in \{1, \ldots, n+1\}$$

is an $(n+1)$-coloring of the covering $\mathcal{U}_1$. ∎

REMARK 3.3. *Suppose that X is compact and at most n-dimensional, that $k > n$, that $Y \subset X$ is closed, and that $\varphi \colon Y \to \mathbb{R}^k \setminus \{0\}$ is a continuous map.*

Then there exists a continuous map $\overline{\varphi} \colon X \to \mathbb{R}^k \setminus \{0\}$ with $\overline{\varphi}_{|Y} = \varphi$.

(Indeed, since $\mathbb{R}^k \setminus \{0\}$ is homeomorphic to $S^{k-1} \times \mathbb{R}$, this follows from Theorem VI.4 of [13] and the Tietze extension theorem.)

3.2. Corollary 3.5 below is a reformulation of a result of Dixmier and Douady ([10], Theorem 5 and [9], Lemma 10.8.7) in our terminology. A generalization of it will be one of the key ingredients for our constructions in the next section. The original paper [10] considers continuous fields of Hilbert spaces over paracompact spaces and use different methods for the proofs. We deduce it from a more refined result on Banach bundles (in the case of a compact base-space). We note that the Weyl–von Neumann theorem of [25] (in conjunction with the Kasparov stabilization theorem) also induces Corollary 3.5. (We discuss some of this aspects at the end of this subsection.)

Let us give here a similar result with an elementary proof which is more qualitative and is more near to the corresponding results on locally trivial topological vector bundles.

PROPOSITION 3.4. *Suppose that X is a compact Hausdorff space of dimension $\leqslant n$ and that E is a real (respectively complex) Banach bundle over X, such that every fiber of E has real dimension $> n+k-1$ (respectively complex dimension $> [n/2]+k$).*

Then for every integer $1 \leqslant l \leqslant k$, for every sequence $1 > \varepsilon_1 \geqslant \varepsilon_2 \geqslant \cdots \geqslant \varepsilon_l > 0$ and every l sections $\zeta_1, \ldots, \zeta_l$ in E there exist sections $\xi_1, \ldots, \xi_l$ in E such that, for all $x \in X$, and $j = 1, \ldots, l$:

(i) $1 \leqslant \|(\xi_j)_x\| < 1 + \varepsilon_j$;

(ii) *the distance from $(\zeta_j)_x$ to the span of $(\xi_1)_x, \ldots, (\xi_j)_x$ is smaller than ε_j;*

(iii) *$(\xi_j)_x$ has distance $= 1$ from the span of $(\xi_1)_x, \ldots, (\xi_{j-1})_x$ for $1 \leqslant j \leqslant n$ (in particular the ξ_j are linearly independent).*

Proof. We consider the real case. The proof of the complex case is similar.

It suffices to consider the case $j = 1$, because if $\xi_1, \ldots, \xi_j$ have been found with the desired properties, then the quotient E/F_j of E by the $C(X)$-linear span F_j of $\xi_1, \ldots, \xi_j$ defines a Banach bundle over$X$ with fibers $E_x/(F_j)_x$ of dimension $> n+(k-j)-1$, cf. Lemma 2.12. If $j < l \leqslant k$, we can consider $\zeta_{j+1} + F_j$ in E/F_j and solve the problem there for this single section. The solution in E/F_j can be lifted afterwards to an element $\xi_{j+1} \in E$ with $1 \leqslant N(\xi_{j+1}) < 1 + \varepsilon_{j+1}$; see Lemma 2.12.

We consider the case $k = l = j = 1$. For $x \in X$, we find $n+1$ elements $e_1, \ldots, e_{n+1}$ in E such that their image in the fiber E_x spans an $(n+1)$-dimensional subspace. By 2.5.1, the linear independence of the $(e_1)_y, \ldots, (e_{n+1})_y$ must also happen for y in a neighborhood of x. Since X is compact, this shows that we can find $g_1, \ldots, g_m \in E$ such that, for every $x \in X$ the linear span of $(g_1)_x, \ldots, (g_m)_x$ is a subspace of E_x of dimension $\geqslant n+1$. Let F denote the closed $C(X)_{\mathrm{sa}}$-submodule of E which is generated by $\{g_1, \ldots, g_m\}$. Then every fiber F_x has dimension n_x with $n < n_x \leqslant m$. By 2.5.1, the function $x \mapsto n_x$ is lower semicontinuous. Thus we get closed subspaces $X_0 := \emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_l = X$, such that the dimension of F_x is $n_i > n$ for x in the locally compact spaces $Y_i := X_{i+1} \setminus X_i$, $0 \leqslant i < l$. By 2.5.1 and Lemma 2.12, every point x of Y_i has a closed neighborhood U (with relatively open interior in Y_i containing x), such that $F_{|U} \cong C(U, \mathbb{R}^{n_i})$. Therefore, it follows from Remark 3.3 that F and U have the sphere-section extension (cf. 2.5.2), because U has dimension $\leqslant n < n_i$. By induction we get from assertion 2.5.2 that X has the sphere-section extension property with respect to F.

In the following we omit the subscript 1, i.e. we write ζ, ξ, ε for ζ_1, ξ_1, ε_1.

Let $Z := \{x \in X : \|(\zeta)_x\| < \varepsilon/2\}$. Take $G := X \setminus Z$ and let $f \in C(X)_+$ with $f_{|G} = (N(\zeta)_{|G})^{-1}$. Then $N(f\zeta)_{|G} = 1$. Since F and X have the sphere-section extension Property 2.5.2, we find $\xi \in F \subset E$ with $N(\xi) = 1$ and $(f\zeta)_{|G} = \xi_{|G}$.

Then $(N(\zeta)\xi)_{|G} = \zeta_{|G}$ and $N(N(\zeta)\xi - \zeta)(x) < \varepsilon$ for $x \in Z$, thus $\|\zeta + C(X)\xi\| < \varepsilon$. ∎

Corollary 3.5. *Let X be a compact space of finite dimension, and let E be a real Hilbert bundle which is countably generated as Banach $C(X,\mathbb{R})$-module and which has fibers E_x of infinite dimension. Then there exists a $C(X,\mathbb{R})$-linear isometric isomorphism α from E onto $C(X,\, \ell_2(\mathbb{N};\, \mathbb{R}))$.*

The same proof applies in the complex case and gives a $C(X)$-module isomorphism $E \cong C(X,\, \ell_2(\mathbb{N};\, \mathbb{C}))$).

The proof of Dixmier and Douady uses a subtriviality result (which translates in our terminology to a special case of the Kasparov stabilization theorem) and relies on a selection construction of Michael for special continuous fiber bundles over a finite dimensional space, ([23], Theorem 2.1). We use the inductive argument in the proof of Proposition 3.4 for a proof of Corollary 3.5. One can imagine that the proof cannot be essentially simpler than that given below, by the fact mentioned in Remark 6.9.

Proof of Corollary 3.5. Let $(\zeta_j)_{j\in\mathbb{N}}$ be a sequence in the unit ball of E, which generates E topologically as a Banach $C(X,\mathbb{R})$-module, such that every element in the sequence appears infinitely often in the sequence. Now let $\varepsilon_j := 4^{-j}$. Then the induction procedure in the proof of Proposition 3.4 does not stop, because all fibers of E are of infinite dimension. We obtain a sequence of elements $\xi_j \in E$, which satisfy (i)–(iii) of Proposition 3.4. Thus also $(\xi_j)_{j\in\mathbb{N}}$ generates E as a Banach $C(X,\mathbb{R})$-module by (ii) and by our choice of the sequences ζ_j and ε_j.

By (i), (iii) and by our choice of ε_j, the Gram–Schmidt orthonormalization process for (ξ_j) works well, and defines a fiberwise orthonormal sequence (e_j) in E, which generates E as Banach $C(X,\mathbb{R})$-module. The sequence (e_j) defines the desired isometric $C(X,\mathbb{R})$-module isomorphism from $C(X,\, \ell_2(\mathbb{R}))$ onto E. ∎

3.3. The above assertion does not hold in full generality, as was proved by Dixmier and Douady ([10], Paragraph II.17). We remind here the simplest example for this phenomenon: Let $\mathbb{K} := \mathbb{C}$ or $\mathbb{R}$, $H_0 := \ell_2(\mathbb{K})$, $H := \ell_2(\mathbb{K}) \oplus_2 \mathbb{K} \cong H_0$, and let X be the unit ball of H_0 endowed with the weak topology. By a theorem of Keller ([16]), the norm-compact convex sets of infinite dimension in a Hilbert space are all homeomorphic to the Hilbert cube $[0,1]^\infty$. Since the compact linear map $(\alpha_1, \alpha_2, \ldots) \mapsto (2^{-1}\alpha_1, 2^{-2}\alpha_2, \ldots)$ maps X homeomorphically onto a norm-compact subset of H_0, we get the well-known fact that X is homeomorphic to the Hilbert cube.

We consider the map

$$\eta\colon x \in X \mapsto \eta(x) := (x\,,\, \sqrt{(1-\|x\|^2)}\,) \in H$$

as an element η of $\ell_\infty(X,\, H)$. (The interesting idea of this construction is that η is *not* in $C(X,H)$.) Then we can define a $C(X,\mathbb{K})$-module map T from $C(X,H_0) \oplus C(X,\mathbb{K})$ into $\ell_\infty(X,H)$ which assigns to $(\xi,f) \in C(X,H_0) \oplus C(X,\mathbb{K})$ the map

$$T(\xi,f)\colon x \in X \mapsto (\xi(x) + f(x)x,\ f(x)\sqrt{(1-\|x\|^2)}\,) \in H$$

i.e. $T(\xi,f) = \xi + f\eta \in \ell_\infty(X,H)$ under the natural embedding of $C(X,\mathbb{K})$ into $\ell_\infty(X,\mathbb{K})$ and of $C(X,H_0)$ into $\ell_\infty(X, H_0 \oplus 0) \subset \ell_\infty(X,H)$.

The function $x \mapsto N(\zeta)(x) := \|\zeta(x)\|$ is continuous for $\zeta = T(\xi, f)$, because $\|\zeta(x)\| = \|(\xi(x) + f(x)x, f(x)\sqrt{1 - \|x\|^2})\|$ and the function

$$x \in X \mapsto \|\xi(x)\|^2 + 2\mathrm{Re}(\overline{f(x)}\,\langle \xi(x), x\rangle) + |f(x)|^2$$

is continuous for the weak topology on X.

Thus assertions 2.4.5 and 2.6.1 apply, and the norm-closure D of $(C(X, H_0) \oplus 0) + C(X, \mathbb{K})\eta$ in $\ell_\infty(X, H)$ is a Hilbert bundle (Hilbert $C(X, \mathbb{K})$-module), and we have $\|\zeta(x)\|_H = \|\zeta_x\|_{D_x}$ for $\zeta \in D$. Note that $\eta \in D$, but η is not contained in $C(X, H)$.

Proposition 3.6. (Reformulation of Lemma 15, Proposition 19 of [10]) *Each section ζ in the orthocomplement $E \subset D$ of the section $\eta \in D$ satisfies $\zeta(x) = 0$ for at least one point $x \in X$.*

Proof. (Proof of Lemma 14 of [10] transferred to our terminology.) Suppose that the section $\zeta' \in E$ is everywhere nonzero. Let $\delta := \inf N(\zeta')/3$. Since $\zeta' \in D$, $N(\zeta')$ is continuous and $\delta > 0$. By definitions of D and E, we find $\xi \in C(X, H_0)$ and $g \in C(X, \mathbb{K})$, such that $\|\zeta' - T(\xi, g)\| < \delta$. Let $f = -\langle \xi, \eta\rangle \in C(X, \mathbb{K})$. Then $\zeta := T(\xi, f)$ is in E and $\|\zeta' - \zeta\| < 2\delta$. Thus, the continuous function $N(\zeta)(x)$ is everywhere nonzero. The equality $0 = \langle \zeta, \eta\rangle\,(x) = f(x) + \langle \xi(x), x\rangle$ then implies that $\xi(x) \neq 0$ for all $x \in X$ and so the map $x \in X \mapsto \xi(x)/\|\xi(x)\| \in H_0$ is a well defined continuous map from X with weak topology to X with norm topology. The Schauder fixed point theorem applied to the convex compact set X and this map yields the existence of a fixed point $x_0 = \xi(x_0)/\|\xi(x_0)\|$, whence $\eta(x_0) = (x_0, 0)$. But this contradicts the two hypotheses $\zeta(x_0) = \xi(x_0) + f(x_0)\,\eta(x_0) \neq 0$ and $\langle \zeta, \eta\rangle(x_0) = \langle \xi(x_0), x_0\rangle + f(x_0) = 0$. ∎

3.4. Let us focus on the involved problem in the complex case. Let $\mathcal{L}(E)$ be the C^*-algebra of $C(X)$-linear operators acting on E, which admit an adjoint. For $x \in X$, denote by θ_x the quotient map $\mathcal{L}(E) \to \mathcal{L}(E_x) \cong \mathcal{L}(\ell_2(\mathbb{C}))$. The above given example has the following properties:

Corollary 3.7. *With the above notations,*

(i) *the unit $e \in \mathcal{L}(E)$ is not properly infinite but has properly infinite images $\theta_x(e)$ in $\mathcal{L}(E_x)$;*

(ii) *$\mathcal{K}(E)$ is a separable C^*-bundle over the Hilbert cube with stable fibers; but $\mathcal{K}(E)$ is not a stable C^*-algebra.*

Proof. (i) Given $x \in X$ and $\zeta \in E$, the relation $\zeta_x \neq 0$ implies by continuity that $\|\zeta(y)\| > 0$ for y in a neighborhood of x. Thus, there exists by compactness a finite family $\xi_1, \ldots, \xi_m \in E$ such that $f := N(\xi_1)^2 + \cdots + N(\xi_m)^2 \in C(X)_+$ is strictly positive. Suppose now that e is properly infinite. Then one can find isometries $s_1, s_2, \ldots, s_m \in \mathcal{L}(E)$ which commute with $C(X)$ and with mutually orthogonal ranges. Thus $f = N(\xi)^2$ with $\xi = s_1\xi_1 + \cdots + s_m\xi_m \in E$, but this contradicts Proposition 3.6.

(ii) follows from (i): Indeed, if $\mathcal{K}(E)$ is stable, then the unit element e of $\mathcal{L}(E) \cong \mathcal{M}(\mathcal{K}(E))$ is properly infinite. ∎

QUESTIONS 3.8. (i) Is the unit $e \in \mathcal{L}(E)$ a *finite* projection?

(ii) Does there exists a positive integer n such that $\mathcal{L}(E \oplus \cdots \oplus E) = \mathcal{L}(E \otimes \mathbb{C}^n)$ has a properly infinite unit? (The point is here that the base space X is contractible.) Note that this (together with (i)) is the case with $n = 2$ for certain "Bott-type" examples of Hilbert bundles E over the highly non-contractible space $(S^2)^\infty$, which have been constructed by Dixmier and Douady ([10]) or Rørdam ([27], [28]). Rørdam ([28]) defines a sequence of projections Q_n in $\mathcal{M}(C((S^2)^\infty, \mathcal{K}))$ such that $E_n := Q_n(C((S^2)^\infty, \ell_2))$ does not have non-singular sections, that $E_n \oplus E_n \cong C((S^2)^\infty, \ell_2)$ and (eventually) Q_{n_0} must be finite for sufficiently large n_0.

(iii) One can show that each fiber $\mathcal{T}(E_x)$ of the (by Pimsner) generalized Fock-Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ built over E is isomorphic to the C^*-algebra $\mathcal{O}_\infty$.

Does there exist an isomorphism of $C(X)$-algebras $\mathcal{T}(E) \cong C(X) \otimes \mathcal{O}_\infty$?

(It would be also interesting to know the answer of the same question for the space $X := S^2 \times S^2 \times \cdots$ and the generalized Fock-Toeplitz $C(X)$-algebra $\mathcal{T}(E_{n_0})$ built over the $C(X)$-Hilbert module E_{n_0} of Rørdam in [28], cf. (ii).)

4. A GLOBAL GLIMM HALVING FOR C^*-BUNDLES

We study in this section a global version of the Glimm halving (see end of Subsection 2.7, Lemma 4.6.6 of [30], Lemma 6.7.1 of [24]) for non-simple C^*-algebras (Definition 1.2), and prove that this property holds for C^*-algebras with Hausdorff finite dimensional primitive ideal space and which do not admit any nonzero type I quotient (Theorem 4.3).

REMARK 4.1. The global Glimm halving property (Definition 1.2) of a C^*-algebra A implies by induction that for all $a \in A_+$, $\varepsilon > 0$ and $n \geqslant 2$, there exists a $*$-morphism $\pi_n \colon C_0((0,1]) \otimes M_n(\mathbb{C}) \to \overline{aAa}$ such that $(a-\varepsilon)_+$ is in the ideal generated by the image of π_n. (In particular A can not have any irreducible representation which contains the compact operators in its image, hence A is *strictly antiliminal*, i.e. every nonzero quotient of A is antiliminal.)

Indeed, let $\delta := \varepsilon/3$ and let $f_2, \ldots, f_n$ be $n-1$ contractions in $\overline{aAa}$ satisfying (2.9) and such that $(a-\delta)_+$ is in the ideal generated by $f_1 := (f_2^* f_2)^{1/2}$. Then there exists $\nu > 0$ such that $(a-2\delta)_+$ is in the ideal generated by $(f_1 - 2\nu)_+$. Take a contraction b of $\overline{(f_1-\nu)_+ A (f_1-\nu)_+}$ such that $(f_1 - 2\nu)_+$ is in the ideal generated by b and such that $b^2 = 0$. Then $(a-\varepsilon)_+$ is in the ideal generated by $g_2, \ldots, g_{n+1}$ where $g_2 := b$, $g_{1+j} := f_j h(f_1) b$ for $j > 1$ and $h \in C_0((0,1])$ is the function with $h_{|(0,\nu/2]} = 0$, $h(t) = t^{-1}$ for $t \in [\nu, 1]$, linear on $[\nu/2, \nu]$. Note that $g_i g_j = 0$ and $g_i^* g_j = \delta_{i,j}\, b^* b$ for $1 < i, j \leqslant n+1$ and that $(a-3\delta)_+$ is in the ideal generated by $b^* b$.

PROPOSITION 4.2. *A C^*-algebra A satisfies the global Glimm halving property in the following cases:*

(i) *A is simple and not isomorphic to the compact operators on a Hilbert space.*

(ii) *The commutant* $A' \cap \mathcal{M}(A)_\omega$ *contains for every* $a \in A_+$ *elements* d, e *with* $0 \leqslant e \leqslant 1$, $ea = a$, $d^2 = 0$ *and* e *is in the ideal of* $A' \cap \mathcal{M}(A)_\omega$ *generated by* d.

(iii) A *is approximately divisible, i.e. for all* $n \in \mathbb{N}$, *there is a sequence of unital* $*$-*homomorphism* $\varphi_k : M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C}) \to \mathcal{M}(A)$ *such that* $\|\varphi_k(e)a - a\varphi_k(e)\| \to 0$ *for all* e *in* $M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C})$ *and all* $a \in A$ *(*[19], *Definition* 5.5*).*

(iv) A *is an infinite tensor product* $A \cong A_0 \otimes B_1 \otimes B_2 \otimes \cdots$ *with simple unital* C^**-algebras* $B_n \neq \mathbb{C}$. *for* $n = 1, 2, \ldots$.

We do not know whether $C([0,1]^\infty) \otimes C_r^*(F_2) \otimes C_r^*(F_2) \otimes \cdots$ is approximately divisible. It is an open question whether strictly antiliminal algebras of real rank zero have the global Glimm halving property.

Proof. (i) follows from the Glimm halving lemma and the simplicity of A. (ii) is stronger than the global Glimm halving property: let $a \in A_+$, $\varepsilon > 0$ and let $d, e \in \mathcal{M}(A)_\omega$ be as in (ii). Then $b := da$ satisfies $b^2 = 0$ and $a = ea$ is in the ideal of $\mathcal{M}(A)_\omega$ generated by b. Thus $(a - \varepsilon)_+$ belongs to the ideal of A generated by b.

Further (iii) or (iv) implies (ii). ∎

THEOREM 4.3. *Let* A *be a continuous* C^**-bundle over a finite dimensional locally compact Hausdorff space* X *and assume that each fiber* A_x *is simple and not of type* I.

Then the global Glimm halving Property 1.2 *holds for* A.

In conjunction with Remark 4.1 this says that a C^*-algebra A with Hausdorff primitive ideal space of finite dimension satisfies the global Glimm halving property if and only if A has no simple quotient of type I.

We shall use two lemmas in order to prove this theorem.

LEMMA 4.4. *Let* X *be a finite dimensional compact space and let* A *be a* C^**-bundle over* X, *whose fibers* A_x *have (complex) dimension* $> 1 + \mathrm{Dim}(X)$.

Then there exist elements $a, b \in A_+$ *such that* $ab = 0$, *but* $a_y \neq 0$ *and* $b_y \neq 0$ *for every* $y \in X$.

Proof. Let $n := \mathrm{Dim}(X)$. For each fiber, A_y one can find $n + 2$ elements $e_1, \ldots, e_{n+2}$ of A_{sa}, such that the span of $(e_1)_y, \ldots, (e_{n+2})_y$ is of dimension $n + 2$. This linear independence also happens in a neighborhood of $y \in X$ by 2.5.1. By compactness of X, we find a finite subset E of A_{sa}, with the property that the images E_x of E in the fibers of A generate real linear subspaces of $(A_x)_{\mathrm{sa}}$ of dimension $> n + 1$. Let $G \subset A$ be the separable C^*-subalgebra of A which is generated by E. By 2.4.3, 2.4.7 and Lemma 2.10, the closure B of the span of $C(X)G$ is a countably generated C^*-subbundle of A and there is a positive $C(X)$-module morphism ψ from B into $C(X)$ such that the positive functionals ψ_y on B_y are faithful for $y \in X$, where ψ_y is defined by $\psi_y(a_y) := \psi(a)(y)$.

As the C^*-norm is monotonous on $(B_y)_+$ and each fiber B_y has complex dimension $> n + 1$, there is $b \in B_+$ such that $N(b)(y) > 0$ and thus, $\psi(b)(y) > 0$ for every $y \in X$.

Let $e := \psi(b)^{-1}b$. Then the map $a \mapsto P(a) := a - \psi(a)e$ is a bounded $C(X)$-module projection from B onto the kernel of ψ, which satisfies $P(a^*) = P(a)^*$ and $P(a)_y = a_y - \psi_y(a_y)e_y$. Thus the intersection of B_{sa} with the kernel of ψ is a real Banach subbundle C of B_{sa}, and the fibers of C have real dimension

$> n$. Accordingly, by Proposition 3.4, there exists $d \in C$ such that $d_y \neq 0$ and $\psi_y(d_y) = 0$ for every $y \in X$. Let $a := d_+$ and $b := d_-$ be the polar decomposition of d in B. Then $a, b \in B_+ \subset A_+$, $ab = 0$ and $d_y = a_y - b_y$ is the polar decomposition of d_y in B_y.

The elements a_y and b_y must be nonzero for every $y \in X$, because $\psi_y(d_y) = 0$, $d_y \neq 0$ and ψ_y is faithful. ∎

Lemma 4.5. *Let A be a continuous C^*-bundle over a finite dimensional compact Hausdorff space X such that for every $y \in X$, A_y is nonzero and does not contain any hereditary C^*-subalgebra of (linear) dimension* 1.

Then there exists a sequence of mutually orthogonal elements $(a_n)_{n\in\mathbb{N}}$ in A_+ satisfying $(a_n)_x \neq 0$ for every pair $(n, x) \in \mathbb{N} \times X$.

Proof. The assumptions imply that every nonzero hereditary C^*-subalgebra D_y of A_y is of infinite linear dimension (in fact this is even equivalent to the conditions on the fibers of A). Thus we can use Lemma 4.4 for an induction argument as follows.

If we have already found mutually orthogonal elements $a_1, a_2, \ldots, a_n, b_n$ in A_+ which have nonzero values $(a_j)_y, (b_n)_y$ in every fiber A_y of A, then we can consider the hereditary C^*-subalgebra $D_n := \overline{b_n A b_n}$ of A. By Lemma 4.4 we find orthogonal $a_{n+1}, b_{n+1} \in D_n$ which have nonzero values in every fiber A_y of A. Then $a_1, \ldots, a_n, a_{n+1}, b_{n+1}$ are mutually orthogonal and have nonzero values in each fiber. ∎

Proof of Theorem 4.3. Fix a nonzero positive element $a \in A_+$, a constant $0 < \varepsilon < \|a\|$, and let us construct an element $b \in \overline{aAa}$ such that $b^2 = 0$ and $(a - \varepsilon)_+ \in \overline{AbA}$.

Let $\delta := \varepsilon/3$, $m := \mathrm{Dim}(X) + 1$ and let Y be the closure of the support of $(N(a) - \delta)_+$.

Notice that Y is a compact subset of X of dimension $< m$, that for $x \in Y$, $a_x \neq 0$ and there is no projection p in A_x with $pA_xp \cong \mathbb{C}p$, because otherwise the simple C^*-algebra A_x would be of type I. Thus, by Lemma 4.5, there are pairwise orthogonal positive elements $a_0, \ldots, a_m$ in the hereditary C^*-subalgebra $B := \overline{aAa}_{|Y}$ of $A_{|Y}$ which satisfy the relation $(a_j)_x \neq 0$ for $x \in Y$, $0 \leqslant j \leqslant m$.

Since B is a continuous C^*-bundle over Y with nonzero simple fibers, one can find for every $x \in Y$ an open neighborhood U of $x \in Y$ and elements $d(1, U), \ldots, d(m, U)$ in B such that, for all $1 \leqslant k \leqslant m$, one has

$$d(k, U) \in \overline{a_0 B a_k} \quad \text{and} \quad \forall\, y \in U\ , \quad \|d(k, U)_y\| = 1\,. \tag{4.1}$$

The compactness of Y leads to a finite open covering $\mathcal{U} = \{U_1, \ldots, U_n\}$ of Y and elements $d(k, U_j)$ satisfying (4.1) for $1 \leqslant k \leqslant m$ and $1 \leqslant j \leqslant n$ (with U_j in places of U). By Lemma 3.2, one can moreover assume, up to taking a suitable refinement of $\mathcal{U}$, that there exists a map $\iota\colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that for each $1 \leqslant k \leqslant m$, the open set

$$Y_k = \bigcup_{j\in\iota^{-1}(k)} U_j$$

is the *disjoint* union of the open sets U_j, $j \in \iota^{-1}(k)$.

Take a partition of unity $1_{C(Y)} = \sum_{0\leqslant j\leqslant n} \eta_j$ subordinate to the open covering $\mathcal{U}$, and set for $k \in \{1,\ldots,m\}$

$$d_k := \sum_{j\in\iota^{-1}(k)} \eta_j\, d(k,U_j) \in \overline{(a_0 A_{|Y} a_k)}\,.$$

Then $\|(d_k)_x\| = \max\{\eta_j(x) \,:\, j \in \iota^{-1}(k)\}$ for $x \in Y$, and the sum $d = \sum_k d_k \in B$ satisfies the desired equality

$$d^2 = \sum_{i,j} d_i d_j = 0,$$

because $a_i a_0 = 0$ for all $i \geqslant 1$. Moreover, for any point $x \in Y$,

$$\|d_x\|^2 = \Big\|\Big(\sum_k d_k d_k^*\Big)_x\Big\| \geqslant \max_j \eta_j(x)^2 > 0\,.$$

We multiply $d \in A_{|Y}$ by $(N(a)-2\delta)_+$, to get an element b of $\overline{aAa}$ with $b^2 = 0$ such that $(a-\varepsilon)_+$ is in the ideal generated by b. ∎

5. AN APPLICATION TO PURELY INFINITE C^*-ALGEBRAS

Recall that a C^*-algebra A is purely infinite if A has no nonzero character and for $a, b \in A_+$, $\varepsilon > 0$ with b in the closed ideal of A generated by a, there exists $d \in A$ with $\|b - d^*ad\| < \varepsilon$.

Theorem 5.1. *Suppose that* Prim(A) *is a Hausdorff space of finite topological dimension. Then A is purely infinite if and only if every simple quotient of A is purely infinite.*

Proof. The simple quotients of A are purely infinite if A is purely infinite by Proposition 4.3 of [19].

Conversely, suppose that all simple quotients of A are purely infinite. The compact operators on a Hilbert space are not purely infinite ([19], Proposition 4.4). Thus A satisfies the assumptions of Theorem 4.3 and has the global Glimm halving property.

Let $X := \mathrm{Prim}(A)$, $n := \mathrm{Dim}(X) < \infty$, and let $a, b \in A_+$ be positive elements, where b is in the closed ideal of A generated by a. For $\varepsilon > 0$, there are $\eta > 0$ and $g_1, \ldots, g_m \in A$ such that $\|b - c\| < \varepsilon$ for $c := \sum_k g_k^*(a - 3\eta)_+ g_k$, because $(b-\varepsilon/2)_+^{1/2}$ is in the minimal dense ideal of the closure of the linear span of $\{g(a-4\eta)_+ h \,:\, \eta > 0, g, h \in A\}$.

Since A has the global Glimm halving property, we find $a_2, \ldots, a_{n+1}$ in $(a-\eta)_+ A (a-\eta)_+$ which satisfy the relations (2.9) and such that $(a-2\eta)_+$ is in the ideal generated by $a_1 := (a_2^* a_2)^{1/2}$. Let $a_0 := a_1^2$. Then c_x is in the ideal generated by $(a_0)_x$ for every $x \in X$. Let $g_\eta(t) := (1/\eta)[(\max(t-\eta,0) - \max(t-2\eta,0)]$ then $g_\eta(a)a_i = a_i$ for $1 \leqslant i \leqslant n+1$.

The function $p(x) := g_\varepsilon(\|c_x\|)^{1/2}$ is a continuous function on X with compact support and satisfies $\|p^2 c - c\| = \sup(1-p^2)N(c) < \varepsilon$, because the fiber norm function $x \mapsto N(c)(x) := \|c_x\|$ is in $C_0(X)_+$. One can find for each point x in the compact closure F of $\{x \in X : p(x) > 0\}$ an element $d(x) \in A$ such that $\|[c - d(x)^* a_0 d(x)]_x\| < \varepsilon$, whence by upper semi-continuity of the norm-functions, there is an open neighborhood $U_x \ni x$ on which $\|[c - d(x)^* a_0 d(x)]_y\| < \varepsilon$ for all y in U_x.

There is a finite open covering $\mathcal{U} = \{U_1, \dots, U_p\}$ of F and elements $d^{(1)}, \dots, d^{(p)} \in A$ satisfying $\|[b - (d^{(j)})^* a_0 d^{(j)}]_y\| < \varepsilon$ for all $1 \leqslant j \leqslant p$ and y in U_j. By Lemma 3.2, one can moreover assume, up to taking a suitable refinement of $\mathcal{U}$, that there exits a map $\iota\colon \{1, \dots, p\} \to \{1, \dots, n+1\}$ such that for each $1 \leqslant i \leqslant n+1$, the open set

$$Y_i = \bigcup_{j \in \iota^{-1}(i)} U_j$$

is the *disjoint* union of the open sets U_j, $j \in \iota^{-1}(i)$, because F has dimension $\leqslant n$. Now take $e_j \in C_0(U_j)_+ \subset C(X)_+$ with $\sum_{1 \leqslant j \leqslant p} e_j \leqslant 1$ and $(\sum e_j)_{|F} = 1$, and define, for $i \in \{1, \dots, n+1\}$, $\eta_i := \sum_{j \in \iota^{-1}(i)} e_j$ and

$$d_i := \sum_{j \in \iota^{-1}(i)} (e_j)^{1/2} d^{(j)} \in A.$$

Then $\| [\eta_i c - d_i^* a_0 d_i]_y\| < \eta_i(y)\varepsilon$ if $\eta_i(y) > 0$ and $1 \leqslant i \leqslant n+1$.

Since $\Big(\sum_{1 \leqslant i \leqslant n+1} \eta_i \Big)_{|F} = 1$, we get $N(c - f^*(a_0 \otimes 1_{n+1}) f)_{|F} \leqslant \varepsilon$ for the column $f = (d_1, \dots, d_{n+1})^T \in M_{n+1,1}(A)$.

Let $h(t) := (g_\eta(t)/t)^{1/2}$ and $d := \sum_{1 \leqslant i \leqslant n+1} p \cdot h(a) a_i d_i$. Then

$$N(p^2 c - d^* a d) = p^2 N(c - f^*(a_0 \otimes 1_{n+1}) f) \leqslant \varepsilon,$$

because $d_i^* a_i^* h(a) a h(a) a_k d_k = \delta_{ik} d_i^* a_0 d_i$. Thus $\|b - d^* a d\| < 3\varepsilon$. ∎

6. More on Banach bundles and its noncommutative version

Since we did not want to interrupt the stream of arguments in Sections 2 and 3, we have transferred a few additional observations and remarks to this section.

Remark 6.1. For the validity of formulas (2.1) and (2.2) it is very important that we assume the action of $C_0(X)$ on A to be nondegenerate, because their proofs (implicitly) need the formula $C_0(\beta X \setminus \{y\}) C_0(X) = C_0(\widehat{X} \setminus \{y\}) C_0(X) = C_0(X \setminus \{y\})$, where $\widehat{X}$ is the one-point compactification of X. Here is an example where (2.1) and (2.2) do not hold, but $\|a\| = \|N(a)\|$ for $a \in A$: If we consider $A := C([0,1])$, $X := (0,1]$ with action given by $C_0((0,1]) \subset C([0,1])$ then for $y = 1$, $f \in C_0((0,1])$ with $f(t) = t$, $a \in A$ with $a(t) = 1 - t$, we get $A_y \cong \mathbb{C} \oplus \mathbb{C}$, $a_y = (1, 0)$, $(fa)_y = (0, 0) \neq f(y) a_y$, $N(a)(y) = \|a_y\| = 1$, $N(fa)(y) = 0 \neq f(y) N(a)(y) = 1$. The right side of equation (2.2) is equal to $|a(1)| = 0 \neq 1 = \|a_y\|$.

Remark 6.2. To understand the difference between continuous fields in the sense of Chapter 10 of [9] and our definition of C^*-bundles (or Banach bundles), let us consider $A := C_0(\mathbb{R})$ as a C^*-bundle over the space $\mathbb{R}$. Then $A_t = \mathbb{C}$ for $t \in \mathbb{R}$ and the corresponding continuous field is $\mathcal{E} = ((A_t)_{t\in\mathbb{R}}, C(\mathbb{R}))$ where the unbounded continuous functions in $C(\mathbb{R})$ are considered as elements of $\prod_{t\in\mathbb{R}} A_t$.

REMARK 6.3. (Concerning assertion (e) in Subsection 2.3.) There are examples of degenerate Banach $C_0((0,1])$-modules A with dimensions of fibers $\leqslant 2$ and contractive action $L\colon C_0((0,1]) \to L(A)$, such that $\|a_{|F}\|$ is strictly bigger than $\inf\{\|a - fa\| \,:\, f \in C_0(X \setminus F)_+,\ \|f\| \leqslant 1\}$ for suitable $a \in A$ and suitable closed $F \subset (0,1]$ (cf. Remark 6.1). Thus $\kappa(L) > 1$ and the contractivity of L does not imply the contractivity of L^{e} in general. This phenomenon can not happen if the action is also nondegenerate; see Lemma 2.3.

REMARK 6.4. A "bounded" modification of Lemma 2.3 shows that the non-degeneracy of the action $L\colon C_0(X) \to \mathfrak{L}(A)$ is the crucial point for the proof of the formulas $(fa)_y = f(y)a_y$, (2.1) and (2.2), see Remark 6.1.

REMARK 6.5. The closed subgroup $B := \mathbb{Z}\{1\} \cup C_0((0,1])$ is a closed $C_0((0,1])$-submodule of (the degenerate Banach $C_0((0,1])$-module) $A := C([0,1])$ in the sense of the algebraic definition of modules of given rings, but B is *not* a vector subspace of A! For nondegenerate Banach $C_0(X)$-modules A, every closed algebraic $C_0(X)$-submodule B of A is automatically also a vector subspace by the inequalities (2.6).

REMARK 6.6. (i) Continuity of all the functions $N(a)$ does *not* imply in general equation (2.7), even if the action of $C_0(X)$ on A is contractive and nondegenerate: For instance, the $C([0,1])$-module $L^2(0,1)$ satisfies $N(a) = 0$ for every $a \in L^2(0,1)$. (Thus (i),(ii) and (iii) do not imply (iv) of Definition 2.6.)

(ii) One can show that (i) of Definition 2.6 is implied by (ii), (iii) and (iv) (cf. the arguments in the proof of assertion 2.4.5).

(iii) If a Banach $C_0(X)$-module A satisfies only the conditions (i), (ii) and (iv) of Definition 2.6 then A is a quotient of a (continuous) Banach bundle over X, because these conditions imply that the linear map $\sum_j f_j \otimes a_j \to \sum f_j a_j$ extends to a contraction $\pi\colon C_0(X, A) \cong C_0(X) \otimes^{\min} A \;\to\; A$ and defines an isometric isomorphism from $C_0(X,A)/D$ onto A for $D := \ker(\pi)$.

Conversely, for every Banach space B and every closed $C_0(X)$-submodule D of $C_0(X,B)$ the quotient Banach $C_0(X)$-module $A := C_0(X,B)/D$ satisfies (i), (ii) and (iv) of Definition 2.6. (Indeed, an obvious modification of the arguments in the last part of the proof of assertion 2.4.3 and Lemma 2.4(v) shows that for $f \in C_0(X,B)$ holds: $N_A(f + D)(x) = \mathrm{dist}(f(x), D_x)$, and $\mathrm{dist}(f(x), D_x) < 1$ for all $x \in X$ implies the existence of $d \in D$ with $\|f - d\| \leqslant 1$.)

REMARK 6.7. It is known that the continuous fields of Banach spaces (respectively of C^*-algebras) are in natural correspondence to the bundles $\pi\colon P \to X$ of Banach spaces (respectively C^*-algebras) over X in the ordinary sense of topology. Those are the topological spaces P together with an open and continuous map π from P onto X (i.e. P is a general topological bundle over X with fibers $P_x := \pi^{-1}(x)$) and continuous maps $\mathbb{C} \times P \mapsto P$, $(a,b) \in P \times_\pi P \mapsto a + b \in P$, $a \in P \mapsto \|a\| \in \mathbb{R}_+$ (and maps $a \in P \mapsto a^* \in P$, $(a,b) \in P \times_\pi P \mapsto ab \in P$ in the case of C^*-bundles), which map the fibers $\mathbb{C} \times P_x$ of $\mathbb{C} \times P$ (respectively the fibers

$P_x \times P_x$ of the fiber product $P \times_\pi P := \{(a,b) : a, b \in P, \pi(a) = \pi(b)\})$ into P_x for every $x \in X$, and which define on P_x the structure of a Banach space (respectively of a C^*-algebra). We require that the continuous map $(a,b) \in P \times_\pi P \mapsto (\pi(a), \|b\|) \in X \times \mathbb{R}_+$ is open. Then our requirements on P are equivalent to Definition 13.4 of [11].

By a result of A. Douady and L. dal Soglio-Hérault for every $a \in P$ there exists a continuous cross section $f \colon X \to P$ with $f(\pi(a)) = a$ such that the function $x \in X \mapsto \|f(x)\|$ is in $C_0(X)_+$ if X is locally compact (cf. [11], Appendix C).

The continuous cross sections $f \colon X \to P$ with $N(f) \colon x \in X \to \|f(x)\|$ in $C_0(X)_+$ build a Banach $C_0(X)$-module B with norm $\|f\| := \sup_{x\in X} \|f(x)\|$. It is a Banach bundle in the sense of Definition 2.6 by Subsection 2.4.5.

Conversely, given a Banach bundle B in the sense of 2.6, one finds on the set $P := \bigcup_{x\in X} B_x$ (disjoint union of fibers) a unique topology such that $\pi \colon P \to X$ becomes a continuous bundle of Banach spaces in the above described sense, such that B is naturally isomorphic to the Banach $C_0(X)$-module of continuous sections $f \colon X \to P$ vanishing at infinity (cf. Section 13.18 of [11]).

It is this correspondence which justifies our terminology "Banach bundle" over X in Definition 2.6. We use the picture of $C_0(X)$-modules, because there are ideas for quantizations of this kind of "Bundles" over noncommutative "spaces" as e.g. defined in the following remark.

Remark 6.8. The considerations in Section 2 (in particular in assertion 2.4.4) suggest that the following definition of separable "quantized" bundles over noncommutative C^*-algebras should be the right one: suppose that A is a separable C^*-algebra and B is a separable operator space which is a Banach A-module with a nondegenerate and completely contractive action of A on B.

We call B an *operator space bundle* over A, if there exist:

(i) a separable C^*-algebra C and a closed ideal J of $A \otimes C$;

(ii) a completely isometric A-module map from B into $(A \otimes C)/J$;

(iii) a completely isometric A-module isomorphism from $(A \otimes C)/J$ into $\mathcal{M}(A \otimes \mathcal{K})$.

Above we have shown, that in the case of a commutative C^*-algebra A this characterizes separable C^*-bundles over $X = \mathrm{Prim}(A)$, if the Banach space B in question is considered as an operator space with its minimal operator space structure, which is e.g. given by the embedding $B \subset C(E)$, as considered in the proof of 2.4.4.

We say that an operator space bundle B is *subtrivial* if there exists a completely isometric linear A-module map from B into $A \otimes \mathfrak{L}(\mathcal{H})$.

Is B subtrivial if A is exact and B has exactness constant one?

The question has a positive answer if A is commutative (by a generalization of [3]).

Remark 6.9. Let X be a metrizable compact space X, which is not finite as a set, and let E be a real Hilbert bundle over X, which is countably generated as Banach $C(X, \mathbb{R})$-module and has fibers of infinite dimension. Then for every non-isolated point y of X, there is a section $\zeta \in E$ such that $\zeta_y = 0$, but, for every section $\eta \in E$ in the orthogonal complement of ζ, $\eta_y = 0$. This can be deduced from the special case where X is the one-point compactification $\mathbb{N} \cup \{+\infty\}$ of the positive integers $\mathbb{N}$. There it is easy to see. Thus the Michael selection does not work for orthogonal complements of sections with a singularity, and the Gram–Schmidt orthonormalization anyway is not defined for them. (This argument also works in the complex case.)

REMARK 6.10. Let $X \cong [0,1]^\infty$ and let $T\colon C(X,H_0) \oplus C(X) \to D \subset \ell_\infty(X,H)$ be the $C(X)$-module map constructed above Proposition 3.6. Then the image $(C(X,H_0) \oplus 0) + C(X)\eta$ of $C(X,H_0) \oplus C(X)$ in D is *not closed*. This results from assertion 2.4.6, because the $C(X)$-module map $T\colon (\xi, f) \mapsto T(\xi, f)$ is injective and has norm $\leqslant \sqrt{2}$ (where we identify the algebraic sum $C(X,H_0) \oplus C(X)$ naturally with $C(X,H)$): The implication $\{\, T(\xi,f) = 0 \Rightarrow f = 0 \,\}$ follows here from the density in X of the $y \in X$ with $\|y\| < 1$. Further $\|T\| \leqslant \sqrt{2}$, because $s + t \leqslant \sqrt{2(s^2+t^2)}$, for $s,t \in \mathbb{R}_+$. If $x \in X$ and $\|x\| = 1$, then the map T_x from $C(X,H)_x \cong H$ to $D_x \cong H_0 \oplus 0 \subset H$ of 2.4.6 is explicitly given by $(\xi, \alpha) \mapsto (\xi - \alpha x, 0)$, and is not injective. Thus the image of T can not be closed by 2.4.6.

REMARK 6.11. Let E be the Hilbert $C(X)$-module, for $X = [0,1]^\infty$, as defined above in Proposition 3.6, i.e. the orthogonal complement of η in D. Since $E \oplus C(X,\ell_2) \cong C(X,\ell_2)$ by the Kasparov stabilization theorem [14] (or by the more special of Theorem 4 in [10]) we get that $\mathcal{K}(E)$ is isomorphic to a full corner of $C(X,\mathcal{K})$ which is given by $eC(X,\mathcal{K})e$, where $e = 1 - SS^*$ for an isometry $S \in \mathcal{M}(C(X,\mathcal{K}))$. In particular, $\mathcal{K}(E)$ *satisfies the condition of Fell* (see Definition 10.5.7 of [9]) and $E \cong eC_0(X,\ell_2)$.

Note that the $C(X)$-module map $\psi\colon \xi \in C(X,H_0) \mapsto T(\xi, f_\xi) \in E \subset C(X,\ell_2)$, where $f_\xi(x) := -\langle \xi(x), x\rangle$, has fiberwise image equal to E_x and defines for $x \in X$ on $H_0 \cong \ell_2$ the quadratic form $\|y\|^2 \geqslant q_x(y) = \|y\|^2 - |\langle y, x\rangle|^2 = \|\psi_x(y)\|^2$ (the latter norm taken in $E_x \subset D_x$). Thus (by Halmos-Nagy dilation) there are a $*$-strongly continuous map $x \in X \mapsto S_x \in \mathcal{L}(\ell_2)$ and a strongly (but not $*$-strongly) continuous map $x \in X \mapsto I_x \in \mathcal{L}(\ell_2)$ into the isometries of ℓ_2, such that $\langle (1 - S_xS_x^*)I_x(y), I_x(y)\rangle = q_x(y)$ and $(1 - S_xS_x^*)I_x(\ell_2)$ is dense in $(1 - S_xS_x^*)\ell_2$. It would be desirable to have explicit formulas for minimal realizations of these maps.

QUESTION 6.12. (“Locally of infinite dimension” versus “locally trivial”)

Let Y denote a compact metric space such that every neighborhood of every point of Y has infinite dimension. Does there exist:

(i) A (separable) Hilbert bundle F_a over Y such that F_a contains a trivial Hilbert bundle with fibers of infinite dimension, but F_a is not itself trivial (and, thus, the trivial bundle is not complemented in F_a by Kasparov stable isomorphism theorem)?

(ii) A Hilbert bundle F_b over Y with fibers of infinite dimension, such that F_b has no non-singular section?

(iii) A *full* projection in $\mathcal{M}(C(Y,\mathcal{K}))$ which is finite (respectively is infinite but is not properly infinite, respectively such that, in addition, the corresponding Hilbert bundles satisfy likewise (i) or (ii))?

(An example F_a for (i) in the case of the Hilbert cube $[0,1]^\infty \cong X$ was introduced in [10]. We have used this F_a to define the example E for (ii) in the case of X (cf. Proposition 3.6). Clearly, this applies for every Y which contains the Hilbert cube as a subspace, because the Hilbert cube is an abstract retract.)

More generally, one can consider full corners D of a stable separable C^*-algebra A, such that D has no unital quotient and no finite 2-quasitrace, but D has one of the following properties:

(iv) D contains a stable full hereditary C^*-subalgebra, but is not stable and the support projection e of D in $\mathcal{M}(A)$ is not full in $\mathcal{M}(A)$.

(v) There is $a \in A_+$, such that D does not contain an element which is Cuntz equivalent to a.

(vi) The support projection e of D in $\mathcal{M}(A)$ is full in $\mathcal{M}(A)$ but is not properly infinite, i.e. D is not stable.

(vii) e is properly infinite, but is not full in $\mathcal{M}(A)$.

Which C^*-algebras A have the property that one of the cases (iv)–(vii) can not appear for every full corner of D?

Is there a C^*-algebra A in which one of the cases (iv)–(vii) appears and not all of them?

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ETIENNE BLANCHARD
Institut de Mathématiques
Projet Algèbres d'opérateurs (Plateau 7E)
175, rue du Chevaleret
F–75013 Paris
FRANCE
E-mail: Etienne.Blanchard@math.jussieu.fr

EBERHARD KIRCHBERG
Institut für Mathematik
Humboldt Universität zu Berlin
Unter den Linden 6
D–10099 Berlin
GERMANY
E-mail: kirchberg@mathematik.hu-berlin.de

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