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THE PLANAR ALGEBRA OF A COACTION

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ABSTRACT. We study actions of *compact quantum groups* on *finite quantum spaces*. According to Woronowicz and to general C^*-algebra philosophy, these correspond to certain coactions $v : A \to A \otimes H$. Here A is a finite dimensional C^*-algebra, and H is a certain special type of Hopf $*$-algebra. If v preserves a positive linear form $\varphi : A \to \mathbb{C}$, a version of Jones' *basic construction* applies. This produces a certain C^*-algebra structure on $A^{\otimes n}$, plus a coaction $v_n : A^{\otimes n} \to A^{\otimes n} \otimes H$, for every n. The elements x satisfying $v_n(x) = x \otimes 1$ are called fixed points of v_n. They form a C^*-algebra $Q_n(v)$. We prove that under suitable assumptions on v the graded union of the algebras $Q_n(v)$ is a spherical C^*-planar algebra.

KEYWORDS: *Subfactors, Hopf algebras, planar algebras.*

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INTRODUCTION

A major discovery of the eighties, due to Jones ([13]), is that an inclusion of von Neumann algebras with trivial centers, also called subfactor, produces a representation of the algebra of Temperley and Lieb ([27]). This can be used for getting several unexpected results about von Neumann algebras. For instance that subfactors of index < 4 are classified by ADE diagrams, and that their indices must be of the form $4\cos^2(\pi/n)$.

The classification program for subfactors, with many people involved over the last 20 years, already reached a few final conclusions. Among them is an axiomatization of a large class of *quantum algebras*, having positivity properties. A first set of axioms, of algebraic nature, was found by Popa in [24]. A set of topological axioms, leading to the notion of planar algebra, was found by Jones in [16].

The colored planar operad $\mathcal{P}$ consists of certain planar diagrams called tangles. Each tangle has several input discs and an output disc, connected by noncrossing strings. The operad law is given by gluing of tangles. A planar algebra is by definition an algebra over $\mathcal{P}$. That is, we have a graded vector space

$Q = Q_0^{\pm}, Q_1, Q_2, Q_3, \ldots$, and for every tangle T we can put elements of Q in the input discs of T and we get an element of Q on the output disc of T.

When the scalars are complex numbers and certain positivity and spherical invariance properties hold, Q is said to be spherical C^*-planar algebra. Results of Jones ([16]) and Popa ([24]) show that every subfactor produces such a planar algebra, and vice versa. In the *amenable* case the correspondence is one-to-one, by a result of Popa ([23]).

It is natural to ask about how these fundamental techniques from subfactors work for compact quantum groups. According to Woronowicz ([31]) such a quantum group is described by a certain special type of Hopf C^*-algebra. So, let H be such a Hopf C^*-algebra, let A be a finite-dimensional C^*-algebra and let $v : A \to A \otimes H$ be a coaction. It is convenient to assume that v is co-faithful, in the sense that its coefficients generate H as a C^*-algebra.

If v leaves invariant a linear form the basic construction produces coactions $v_n : A^{\otimes n} \to A^{\otimes n} \otimes H$ for every n. Here the tensor powers $A^{\otimes n}$ are given the C^*-algebra structure coming from the basic construction. Consider the algebras $Q_n(v)$ of fixed points under the coactions v_n. That is, of elements satisfying $v_n(x) = x \otimes 1$. These form an increasing sequence of finite dimensional C^*-algebras, and their union is a graded $*$-algebra, denoted $Q(v)$.

In most cases of interest $Q(v)$ is known to be a spherical C^*-planar algebra. Moreover, $Q(v)$ encodes important information about (H, v), and several algebraic or analytic properties of (H, v) can be translated in terms of $Q(v)$. In fact, it is expected that a reconstruction map of type $Q(v) \to (H, v)$ exists, as a modification of Woronowicz's Tannakian duality ([30]).

When H is finite dimensional and $v : H \to H \otimes H$ is its comultiplication, this follows from Ocneanu's depth 2 duality; see David ([10]), Longo ([21]) and Szymanski ([26]). A direct proof is obtained by Kodiyalam, Landau and Sunder in [18]. For more results on the depth 2 case see Das ([8]) and Das and Kodiyalam ([9]).

More generally, one can consider the case when H is a Kac type, meaning that the square of its antipode S^2 is the identity. Several explicit results, due to Landau ([19]), Landau and Sunder ([20]), and Bhattacharyya and Landau ([5]) are available here. In the general $S^2 = \mathrm{id}$ case a subfactor is constructed in [2], and its standard invariant is computed by using a method of Wassermann from [28]. By combining this with a result of Jones in [16], it follows that $Q(v)$ is a planar algebra.

In the $S^2 \neq \mathrm{id}$ case things are less explicit. When $H = C(G)_q$ corresponds to a q-deformation with $q > 0$ of a compact Lie group and v comes from a projective representation of G this follows from work of Sawin ([25]). More generally, when $A = M_n(\mathbb{C})$ and v is adjoint to a corepresentation of H, this follows from a many-to-one Tannakian correspondence, established in [1].

The problem with most of the above results is that the planar algebra structure of $Q(v)$ is not quite explicit, because it comes from a subfactor or a standard

λ-lattice, via the fundamental results of Jones ([16]) and Popa ([24]). The other obvious problem is that all these results certainly cover the most interesting cases, but some cases are still left. And finally, a third problem is with the reconstruction map, not available in most cases.

One may wonder about a very general correspondence of the form $(H, A, v) \leftrightarrow Q(v)$, between triples (H, A, v) satisfying certain assumptions and certain planar algebras. Something like *triples satisfying a Perron-Frobenius type condition are in one-to-one correspondence with twisted C^*-subalgebras of depth 1 planar algebras*. Moreover, for this result to be ready to use, one would like to have a direct construction of the correspondence, somehow in the spirit of Ocneanu's depth 2 duality and of Woronowicz's Tannakian duality.

So far, the only fully satisfactory result in this sense seems to be the one in the depth 2 case, where the enlightening paper of Kodiyalam, Landau and Sunder ([18]) is available.

The aim of the present work is to construct a general map of type $(H, A, v) \to Q(v)$.

In Section 1 and Section 2 we apply the basic construction, and we study the equivariance properties of various annular tangles. In the planar algebra setting it is convenient to use bases and indices and to do it right from the beginning. This requires a normalisation of the coefficients of v. We choose the one which makes the spin factor behave uniformly at even and odd levels.

In Section 3 and Section 4 we prove that under suitable assumptions $Q(v)$ is a spherical C^*-planar algebra. When the square of the antipode S^2 is the identity this is a subalgebra of the depth 1 planar algebra $P(A)$ constructed by Jones in [14], by using a certain explicit statistical mechanical sum. In the general case the inclusion $Q(v) \subset P(A)$ appears to be *twisted*, and the partition function of $Q(v)$ comes here from a standard λ-lattice in the sense of Popa ([24]), by using the *bubbling*construction of Jones ([16]).

As a conclusion, in the $S^2 = \mathrm{id}$ case the map $(H, A, v) \to Q(v)$ is constructed quite explicitely, and what is left is to do the converse construction. In the $S^2 \neq \mathrm{id}$ case what we do is rather to compute the domain of $(H, A, v) \to Q(v)$, by a method which is to be improved.

The first version of this paper was written in 2002. This version is the third one, written in 2004, with new notations and many comments added, but basically containing the same material. So far, we have found no improvement in the $S^2 \neq \mathrm{id}$ case.

In the recent paper [4] we obtain the duality for coactions on $A = \mathbb{C}^n$. Here the condition $S^2 = \mathrm{id}$ is automatic. This duality restricts to a correspondence between Hopf C^*-algebras associated to colored graphs with n vertices and planar subalgebras of the spin planar algebra $P(\mathbb{C}^n)$, generated by a self-adjoint 2-box. This latter correspondence makes a link between Hopf C^*-algebras and the classification program initiated by Bisch and Jones in [6] and [7], and can be used for explicit (numeric) computations of Poincaré series of such Hopf C^*-algebras.

Some other possible applications of such dualities are discussed in Section 5 in [4].

1. FORMALISM

The formalism we need is that of a Hopf $*$-algebra with a positive integral.

However, it is more convenient to start with the more enlightening axioms in Woronowicz's paper ([31]). A good reference here is the paper [22] by Maes and Van Daele, containing a short exposition of the subject, with several simplifications, and available at arxiv.org.

The terminology in the definition below is probably quite reasonable, but not standard.

DEFINITION 1.1. *A Hopf C^*-algebra with unit* is a pair $\mathbb{H} = (\mathbb{H}, \Delta)$ consisting of a C^*-algebra with unit $\mathbb{H}$ and a C^*-morphism $\Delta : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$, subject to the following conditions:

(i) coassociativity condition $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$;

(ii) cocancellation law: the sets $\Delta(\mathbb{H})(1 \otimes \mathbb{H})$ and $\Delta(\mathbb{H})(\mathbb{H} \otimes 1)$ are dense in $\mathbb{H} \otimes \mathbb{H}$.

The basic example is $\mathbb{H} = \mathbb{C}(G)$, the algebra of continuous functions on a compact group G, with $\Delta(\varphi) : (g, h) \to \varphi(gh)$. Here coassociativity of Δ follows from associativity of the multiplication $\cdot$ of G, and cocancellation in $(\mathbb{H}, \Delta)$ follows from cancellation in $(G, \cdot)$.

Conversely, assume that $(\mathbb{H}, \Delta)$ is as in Definition 1.1, and that $\mathbb{H}$ is commutative. The Gelfand transform gives an isomorphism $\mathbb{H} \simeq \mathbb{C}(G)$, where G is the spectrum of H. Now the coassociative map Δ gives rise to an associative map $\cdot : G \times G \to G$. In other words, we have here a compact semigroup $(G, \cdot)$, which by (ii) follows to have cancellation. It is then well-known that G must be a compact group.

As a conclusion, the construction $(G, \cdot) \to (\mathbb{C}(G), \Delta)$ is a contravariant equivalence of categories between compact groups and commutative Hopf C^*-algebras with unit. So, a pair $(\mathbb{H}, \Delta)$ as in Definition 1.1 can be thought of as corresponding to a *compact quantum group*.

Among main results of Woronowicz in [31] is the construction of a dense subalgebra $H \subset \mathbb{H}$, consisting of *representative functions* on the compact quantum group. This has a counit $\varepsilon : H \to \mathbb{C}$ and an antipode $S : H \to H$, which satisfy the usual Hopf algebra identities. It is convenient to denote by $m : H \otimes H \to H$ and $u : \mathbb{C} \to H$ its multiplication and unit maps.

DEFINITION 1.2. In this paper $H = (H, m, u, \Delta, \varepsilon, S, *)$ will denote the Hopf $*$-algebra of *representative functions on a compact quantum group*, in the sense that H is the canonical dense subalgebra associated by Woronowicz to a Hopf C^*-algebra with unit $\mathbb{H}$.

As already explained, H will come in fact together with a fundamental corepresentation. So, we will feel free to refer to results of Woronowicz from the fundamental paper [29].

The other piece of data we need is a pair (A, φ) consisting of a finite dimensional C^*-algebra A and a positive linear form $\varphi : A \to \mathbb{C}$. It is well-known that A must be isomorphic to a direct sum of matrix algebras, and φ must be of the form $a \to \mathrm{tr}(qa)$, with $q \in A$ positive.

A basic example here is the algebra $A = \mathbb{C}(X)$ of functions on a finite set X, with linear form $\varphi(f) = \sum f(x)\mu(x)$, where μ is a positive measure on X. It is probably tempting to think of a general pair (A, φ) as corresponding to a *measured finite quantum space*. But the other main example is $A = M_n(\mathbb{C})$ with $\varphi(a) = \mathrm{tr}(qa)$, where q is some positive matrix, and here this interpretation doesn't quite help.

DEFINITION 1.3. Let (A, φ) be a finite dimensional C^*-algebra together with a positive faithful linear form. A *coaction* of H on (A, φ) is a morphism of $*$-algebras $v : A \to A \otimes H$, subject to the following conditions:

(i) coassociativity condition $(v \otimes \mathrm{id})v = (\mathrm{id} \otimes \Delta)v$;
(ii) counitality condition $(\mathrm{id} \otimes \varepsilon)v = \mathrm{id}$;
(iii) copreservation of φ condition $(\varphi \otimes \mathrm{id})v = \varphi(\cdot)1$.

If (iii) is not satisfied we just say that v is a coaction of H on A.

The purpose of this section is to reformulate these axioms, for further use in establishing results about propagation of v in the Jones tower for $\mathbb{C} \subset A$. The precise structure of the Jones tower for $\mathbb{C} \subset A$ is that of a spherical C^*-planar algebra. The following are known.

(1) Bases and indices are needed so far in understanding this planar algebra structure, meaning that an approach with *global formulae* is not available yet. In fact, a planar algebra is quite an abstract notion, and the action of tangles on tensors is best understood by keeping in mind rules like *indices are allowed to travel on strings* or *two different indices make the whole thing vanish when they meet* etc. This is why indices are necessary.

(2) Some quite unobvious choices of bases, normalisations, notations etc. are needed as well. See e.g. the comments of Jones in [16] and [14]. The idea here is that the planar meaning of various *deformation* parameters is very unclear. The *spin vector* used by Jones in [14], which already requires a tricky normalisation, turns to have a quite clear planar interpretation, in terms of *horizontal* structure. In this paper the set of *parameters* will be even bigger. This will require several careful normalisations, and the problem of finding a reasonable planar interpretation of these parameters will be eventually left open in the general case.

Now (1) tells us to look for a reformulation of Definition 1.3, in terms of coefficients of v, with respect to some basis of A. This is an a priori quite standard task: coassociativity corresponds to the well-known condition $\Delta(v_{ij}) = \sum v_{ik} \otimes v_{kj}$

and so on. However, because of (2), we have be extremely careful in the choice of the basis and coefficients.

We will use a normalisation which may seem a bit strange, but which does work, in the sense that formulae in the Jones tower will look quite similar at even and odd levels. Of course, this choice of simplifying things in higher formulae to come might cause the very first formulae — in statements and proofs — to look more complicated than needed. This will be indeed the case.

DEFINITION 1.4. Let (A,φ) be as above. Choose a system of matrix units $X \subset A$ making φ diagonal, with the following multiplication convention:

$$\begin{pmatrix} j \\ i \end{pmatrix}\begin{pmatrix} l \\ k \end{pmatrix} = \delta_{jk}\begin{pmatrix} l \\ i \end{pmatrix}.$$

We denote by q_i the fourth roots of the weights of φ, chosen positive:

$$\varphi\begin{pmatrix} j \\ i \end{pmatrix} = \delta_{ij}q_i^4 \quad q_i > 0.$$

Any linear map $v : A \to A \otimes H$ will be written in the following form:

$$v\begin{pmatrix} j \\ i \end{pmatrix} = \sum \begin{pmatrix} l \\ k \end{pmatrix} \otimes q_k^{-1}q_iq_jq_l^{-1}V\begin{pmatrix} l & j \\ k & i \end{pmatrix}.$$

This is, to any linear map v we associate in this way a matrix V, and vice versa.

It is convenient to define the coefficients $V(\begin{smallmatrix} l & j \\ k & i \end{smallmatrix})$ for all indices i,j,k,l, by saying that they are equal to zero if $(\begin{smallmatrix} j \\ i \end{smallmatrix})$ or $(\begin{smallmatrix} l \\ k \end{smallmatrix})$ do not exist. In fact, best here would be to use the groupoid structure of X, but since we do not have results for more general groupoids, we do not do it.

As for the sum sign in Definition 1.4, this is by definition over all elements $(\begin{smallmatrix} l \\ k \end{smallmatrix}) \in X$. More generally, in any formula of type $A = \sum B$ or $\sum B = A$ with $A, B \in H$ the sum will be over all indices which appear in B and don't appear in A.

The normalisation in Definition 1.4 is the one which will appear to work well in the Jones tower. For, we must first do the above-mentioned reformulation of Definition 1.3.

PROPOSITION 1.5. *A linear map $v : A \to A \otimes H$ is a coaction of H on A if and only if V satisfies the following Conditions:*

$$\varepsilon V\begin{pmatrix} l & j \\ k & i \end{pmatrix} = \delta_{ki}\delta_{lj}1, \qquad \Delta V\begin{pmatrix} l & j \\ k & i \end{pmatrix} = \sum V\begin{pmatrix} l & h \\ k & g \end{pmatrix} \otimes V\begin{pmatrix} h & j \\ g & i \end{pmatrix},$$

$$V\begin{pmatrix} l & j \\ k & i \end{pmatrix}^* = V\begin{pmatrix} k & i \\ l & j \end{pmatrix}, \quad \sum q_i^2 V\begin{pmatrix} l & i \\ k & i \end{pmatrix} = \delta_{kl}q_k^2,$$

$$\sum q_s^{-2}V\begin{pmatrix} s & h \\ k & g \end{pmatrix}V\begin{pmatrix} l & j \\ s & i \end{pmatrix} = \delta_{hi}q_i^{-2}V\begin{pmatrix} l & j \\ k & g \end{pmatrix}.$$

This sequence of five Conditions will be denoted (ε), (Δ), $(*)$, (u°), $({}^\circ\mathrm{m})$.

Proof. This is well-known, modulo our normalisations for V, so the only thing to check is that all q values in the statement are the good ones. It is possible to prove this either by using global formulae, or with a direct matrix computation. We prefer to present this latter approach, as a warm-up for more involved computations to come, where bases and matrix computations seem to be unavoidable, cf. Considerations (1) and (2) at page 123.

By using the defining formula of v we get

$$(\mathrm{id}\otimes\varepsilon)v\begin{pmatrix} j\\ i\end{pmatrix}=\sum\begin{pmatrix} l\\ k\end{pmatrix}\otimes q_k^{-1}q_iq_jq_l^{-1}\varepsilon V\begin{pmatrix} l & j\\ k & i\end{pmatrix},$$

so the condition $(\mathrm{id}\otimes\varepsilon)v=\mathrm{id}$ holds if and only if V satisfies

$$\varepsilon V\begin{pmatrix} l & j\\ k & i\end{pmatrix}=q_kq_i^{-1}q_j^{-1}q_l\delta_{ki}\delta_{lj},$$

for any i,j,k,l, i.e. if and only if V satisfies (ε). We have

$$\begin{aligned}(v\otimes\mathrm{id})v\begin{pmatrix} j\\ i\end{pmatrix}&=\sum v\begin{pmatrix} l\\ k\end{pmatrix}\otimes q_k^{-1}q_iq_jq_l^{-1}V\begin{pmatrix} l & j\\ k & i\end{pmatrix}\\
&=\sum\begin{pmatrix} h\\ g\end{pmatrix}\otimes q_g^{-1}q_kq_lq_h^{-1}V\begin{pmatrix} h & l\\ g & k\end{pmatrix}\otimes q_k^{-1}q_iq_jq_l^{-1}V\begin{pmatrix} l & j\\ k & i\end{pmatrix}\\
&=\sum\begin{pmatrix} h\\ g\end{pmatrix}\otimes q_g^{-1}q_h^{-1}q_iq_jV\begin{pmatrix} h & l\\ g & k\end{pmatrix}\otimes V\begin{pmatrix} l & j\\ k & i\end{pmatrix}\\
&=\sum\begin{pmatrix} l\\ k\end{pmatrix}\otimes q_k^{-1}q_l^{-1}q_iq_jV\begin{pmatrix} l & h\\ k & g\end{pmatrix}\otimes V\begin{pmatrix} h & j\\ g & i\end{pmatrix},\end{aligned}$$

so $(v\otimes\mathrm{id})v=(\mathrm{id}\otimes\Delta)v$ is equivalent to (Δ). We have

$$v(1)=\sum v\begin{pmatrix} i\\ i\end{pmatrix}=\sum\begin{pmatrix} l\\ k\end{pmatrix}\otimes q_k^{-1}q_i^2q_l^{-1}V\begin{pmatrix} l & i\\ k & i\end{pmatrix}$$

so $v(1)=1\otimes 1$ is equivalent to (u°). Also, from the formulae

$$\begin{aligned}v\begin{pmatrix} h\\ g\end{pmatrix}v\begin{pmatrix} j\\ i\end{pmatrix}&=\sum\begin{pmatrix} s\\ k\end{pmatrix}\begin{pmatrix} l\\ S\end{pmatrix}\otimes q_k^{-1}q_gq_hq_s^{-1}q_S^{-1}q_iq_jq_l^{-1}V\begin{pmatrix} s & h\\ k & g\end{pmatrix}V\begin{pmatrix} l & j\\ S & i\end{pmatrix}\\
&=\sum\begin{pmatrix} l\\ k\end{pmatrix}\otimes(q_i^2q_s^{-2})(q_hq_i^{-1})q_k^{-1}q_gq_jq_l^{-1}V\begin{pmatrix} s & h\\ k & g\end{pmatrix}V\begin{pmatrix} l & j\\ s & i\end{pmatrix},\\
v\left(\begin{pmatrix} h\\ g\end{pmatrix}\begin{pmatrix} j\\ i\end{pmatrix}\right)&=\sum\begin{pmatrix} l\\ k\end{pmatrix}\otimes\delta_{hi}q_k^{-1}q_gq_jq_l^{-1}V\begin{pmatrix} l & j\\ k & g\end{pmatrix},\end{aligned}$$

we get that v is multiplicative if and only if $({}^\circ\mathrm{m})$ holds. We have

$$v\begin{pmatrix} i\\ j\end{pmatrix}=\sum\begin{pmatrix} k\\ l\end{pmatrix}\otimes q_l^{-1}q_jq_iq_k^{-1}V\begin{pmatrix} k & i\\ l & j\end{pmatrix}$$

so v is involutive if and only if $(*)$ holds. ∎

PROPOSITION 1.6. *Assume that v is a coaction of H on A. Then the following three Conditions* (S), $(\,^\circ\mathrm{u})$ *and* (m°)

$$SV\begin{pmatrix} l & j \\ k & i \end{pmatrix} = q_k^2 q_i^{-2} q_j^2 q_l^{-2} V\begin{pmatrix} i & k \\ j & l \end{pmatrix}, \quad \sum q_i^2 V\begin{pmatrix} i & l \\ i & k \end{pmatrix} = \delta_{kl} q_k^2 ,$$

$$\sum q_s^{-2} V\begin{pmatrix} h & s \\ k & g \end{pmatrix} V\begin{pmatrix} l & j \\ i & s \end{pmatrix} = \delta_{hi} q_i^{-2} V\begin{pmatrix} l & j \\ k & g \end{pmatrix},$$

are equivalent, and are satisfied if and only if v preserves φ.

Proof. We keep proving things by performing matrix computations. We have

$$(\varphi\otimes\mathrm{id})v\begin{pmatrix} l \\ k \end{pmatrix} = \sum \varphi\begin{pmatrix} j \\ i \end{pmatrix} q_i^{-1} q_k q_l q_j^{-1} V\begin{pmatrix} j & l \\ i & k \end{pmatrix} = q_k q_l \sum q_i^2 V\begin{pmatrix} i & l \\ i & k \end{pmatrix}$$

so the condition $(\varphi\otimes\mathrm{id})v = \varphi(\cdot)1$ holds if and only if V satisfies $(\,^\circ\mathrm{u})$. It remains to prove that if (ε), (Δ), (u°), $(\,^\circ\mathrm{m})$ are satisfied, then (S), (m°) and $(\,^\circ\mathrm{u})$ are equivalent. First, by applying S to (u°) we get that (S) implies $(\,^\circ\mathrm{u})$:

$$\delta_{kl} q_k^2 = \sum q_i^2 SV\begin{pmatrix} l & i \\ k & i \end{pmatrix} = \sum q_i^2 q_k^2 q_l^{-2} V\begin{pmatrix} i & l \\ i & k \end{pmatrix}.$$

Assume that $(\,^\circ\mathrm{u})$ holds. By combining it with $(\,^\circ\mathrm{m})$ we get

$$\sum q_i^2 q_j^{-2} V\begin{pmatrix} j & l \\ i & k \end{pmatrix} V\begin{pmatrix} i & h \\ j & g \end{pmatrix} = \sum q_i^2 \delta_{lg} q_g^{-2} V\begin{pmatrix} i & h \\ i & k \end{pmatrix} = \delta_{lg} q_g^{-2} \delta_{kh} q_k^2$$

and this can be rewritten in the following form:

$$\sum q_i^2 q_k^{-2} q_l^2 q_j^{-2} V\begin{pmatrix} j & l \\ i & k \end{pmatrix} V\begin{pmatrix} i & h \\ j & g \end{pmatrix} = \delta_{lg}\delta_{kh} 1 .$$

If e_{xy} with $x, y \in X$ is the system of matrix units in $\mathcal{L}(A)$ we get

$$\left(\sum e_{\binom{k}{l}\binom{i}{j}} \otimes q_i^2 q_k^{-2} q_l^2 q_j^{-2} V\begin{pmatrix} j & l \\ i & k \end{pmatrix}\right)\left(\sum e_{\binom{i}{j}\binom{h}{g}} \otimes V\begin{pmatrix} i & h \\ j & g \end{pmatrix}\right) = 1\otimes 1 .$$

On the other hand, Conditions (ε) and (Δ) say that $V \in \mathcal{L}(A)\otimes H$ is a corepresentation, i.e. that it satisfies

$$(\mathrm{id}\otimes\Delta)V = V_{12}V_{13}, \quad (\mathrm{id}\otimes\varepsilon)V = 1 ,$$

so by considering $(\mathrm{id}\otimes E)V$ with E given by the Hopf algebra axiom

$$E = m(S\otimes\mathrm{id})\Delta = m(\mathrm{id}\otimes S)\Delta = \varepsilon(\cdot)1$$

we get that $(\mathrm{id}\otimes S)V$ is an inverse for V. Thus the above formula gives $(\mathrm{id}\otimes S)V$, and by identifying coefficients we get (S). Thus $(\,^\circ\mathrm{u})$ implies (S). It remains to prove that (S) is equivalent to (m°). Assume that (S) is satisfied. Then

$$\begin{aligned} &S\Big(\sum q_s^{-2} V \begin{pmatrix} s & h \\ k & g \end{pmatrix} V \begin{pmatrix} l & j \\ s & i \end{pmatrix}\Big) \\ &\qquad = \sum q_s^{-2} SV \begin{pmatrix} l & j \\ s & i \end{pmatrix} SV \begin{pmatrix} s & h \\ k & g \end{pmatrix} \\ &\qquad = \sum q_s^{-2} q_s^2 q_i^{-2} q_j^2 q_l^{-2} q_k^2 q_g^{-2} q_h^2 q_s^{-2} V \begin{pmatrix} i & s \\ j & l \end{pmatrix} V \begin{pmatrix} g & k \\ h & s \end{pmatrix}. \end{aligned}$$

On the other hand by applying S to the right term of (°m) we get

$$S\left(\delta_{hi} q_i^{-2} V \begin{pmatrix} l & j \\ k & g \end{pmatrix}\right) = \delta_{hi} q_i^{-2} q_k^2 q_g^{-2} q_j^2 q_l^{-2} V \begin{pmatrix} g & k \\ j & l \end{pmatrix}.$$

By using (°m) we get after cancelling q's that

$$\delta_{hi} V \begin{pmatrix} g & k \\ j & l \end{pmatrix} = \sum q_h^2 q_s^{-2} V \begin{pmatrix} i & s \\ j & l \end{pmatrix} V \begin{pmatrix} g & k \\ h & s \end{pmatrix}$$

and this is (m°). Finally, the proof of (m°) implies (S) is similar to the proof of (°u) implies (S). Indeed, by combining (m°) and (u°) we get

$$\sum q_i^2 q_k^{-2} q_l^2 q_j^{-2} V \begin{pmatrix} j & l \\ i & k \end{pmatrix} V \begin{pmatrix} i & h \\ j & g \end{pmatrix} = \delta_{lg} \delta_{kh} 1$$

and this gives a right inverse for V, hence the Formula (S) for the antipode. ∎

2. BASIC CONSTRUCTION, EQUIVARIANCE RESULTS, AND THE UNTWISTED CASE

For any n the set X^n is a basis of the linear space $A^{\otimes n}$. With *loop* notations

$$\begin{pmatrix} j_1 \\ i_1 \end{pmatrix} \otimes \begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} j_{2s} \\ i_{2s} \end{pmatrix} = \begin{pmatrix} j_{2s} & i_{2s} & \ldots & i_{s+1} \\ i_1 & j_1 & \ldots & j_s \end{pmatrix}$$

$$\begin{pmatrix} j_1 \\ i_1 \end{pmatrix} \otimes \begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} j_{2s-1} \\ i_{2s-1} \end{pmatrix} = \begin{pmatrix} j_{2s-1} & i_{2s-1} & \ldots & j_s \\ i_1 & j_1 & \ldots & i_s \end{pmatrix}$$

for this basis, depending on the parity of n, the linear extension of

$$\begin{pmatrix} j_1 & \ldots & j_n \\ i_1 & \ldots & i_n \end{pmatrix} \begin{pmatrix} l_1 & \ldots & l_n \\ k_1 & \ldots & k_n \end{pmatrix} = \delta_{j_1 k_1} \ldots \delta_{j_n k_n} \begin{pmatrix} l_1 & \ldots & l_n \\ i_1 & \ldots & i_n \end{pmatrix}$$

is an associative multiplication on $A^{\otimes n}$. Together with the antilinear extension of

$$\begin{pmatrix} j_1 & \ldots & j_n \\ i_1 & \ldots & i_n \end{pmatrix}^* = \begin{pmatrix} i_1 & \ldots & i_n \\ j_1 & \ldots & j_n \end{pmatrix}$$

this gives a finite dimensional C^*-algebra structure on $A^{\otimes n}$. Note that $A^{\otimes 1} = A$ and that $A^{\otimes 2}$ is a matrix algebra. In fact the algebras $A^{\otimes n}$ are obtained from A by performing the basic construction to the inclusion $\mathbb{C} \subset A$. See the book of Goodman, de la Harpe and Jones ([11]).

We use for $A^{\otimes n}$ the same conventions for sums etc. as those for A.

Conditions (ε) and (Δ) show that the matrix

$$u = \sum e_{\binom{l}{k}\binom{j}{i}} \otimes V\begin{pmatrix} l & j \\ k & i \end{pmatrix} \in \mathcal{L}(A) \otimes H$$

is a corepresentation, so we can consider its tensor powers:

$$u^{\otimes n} = u_{1,n+1}u_{2,n+1}\dots u_{n,n+1} \in \mathcal{L}(A^{\otimes n}) \otimes H.$$

Let V_n be the matrix of coefficients of $u^{\otimes n}$, defined by

$$u^{\otimes n} = \sum e_{\binom{l_1\dots l_n}{k_1\dots k_n}\binom{j_1\dots j_n}{i_1\dots i_n}} \otimes V_n \begin{pmatrix} l_1 & \dots & l_n & j_1 & \dots & j_n \\ k_1 & \dots & k_n & i_1 & \dots & i_n \end{pmatrix}.$$

Define a linear form $\widetilde{\varphi}_n$ by

$$\widetilde{\varphi}_n \begin{pmatrix} j_1 & \dots & j_n \\ i_1 & \dots & i_n \end{pmatrix} = \delta_{(i_1\dots i_n)(j_1\dots j_n)} q^4_{(i_1\dots i_n)}$$

where the weights are given by the function

$$q_{(i_1\dots i_n)} = q_{i_1} q_{i_2}^{-1} q_{i_3} \dots q_{i_n}^{\mp 1},$$

where $\pm 1 = (-1)^n$. Note that $q_{(i)} = q_i$, so $\widetilde{\varphi}_1 = \varphi$ on $A^{\otimes 1} = A$.

PROPOSITION 2.1. *If $v : A \to A \otimes H$ is a coaction of H on A which preserves φ then the linear map $v_n : A^{\otimes n} \to A^{\otimes n} \otimes H$ given by*

$$v_n \begin{pmatrix} j_1 & \dots & j_n \\ i_1 & \dots & i_n \end{pmatrix} = \sum \begin{pmatrix} l_1 & \dots & l_n \\ k_1 & \dots & k_n \end{pmatrix} \otimes q^{-1}_{(k_1\dots k_n)} q_{(i_1\dots i_n)} q_{(j_1\dots j_n)} q^{-1}_{(l_1\dots l_n)} \cdot V_n \begin{pmatrix} l_1 & \dots & l_n & j_1 & \dots & j_n \\ k_1 & \dots & k_n & i_1 & \dots & i_n \end{pmatrix}$$

is a coaction of H on $A^{\otimes n}$ which preserves $\widetilde{\varphi}_n$.

Proof. The tensor powers of u are given by

$$\begin{aligned} u^{\otimes n} &= \sum e_{\binom{l_1}{k_1}\binom{j_1}{i_1}} \otimes \dots \otimes e_{\binom{l_n}{k_n}\binom{j_n}{i_n}} \otimes V\begin{pmatrix} l_1 & j_1 \\ k_1 & i_1 \end{pmatrix} \dots V\begin{pmatrix} l_n & j_n \\ k_n & i_n \end{pmatrix} \\ &= \sum e_{\binom{l_1}{k_1}\otimes\dots\otimes\binom{l_n}{k_n},\binom{j_1}{i_1}\otimes\dots\otimes\binom{j_n}{i_n}} \otimes V\begin{pmatrix} l_1 & j_1 \\ k_1 & i_1 \end{pmatrix} \dots V\begin{pmatrix} l_n & j_n \\ k_n & i_n \end{pmatrix} \\ &= \sum e_{\binom{k_2}{k_1}\otimes\dots\otimes\binom{l_1}{l_2},\binom{i_2}{i_1}\otimes\dots\otimes\binom{j_1}{j_2}} \otimes V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} \dots V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix} \\ &= \sum e_{\binom{l_1\dots l_n}{k_1\dots k_n}\binom{j_1\dots j_n}{i_1\dots i_n}} \otimes V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} \dots V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix} \end{aligned}$$

and this gives the formula of V_n. More precisely, we have $V_1 = V$ and

$$V_2\begin{pmatrix} l_1 & l_2 & j_1 & j_2 \\ k_1 & k_2 & i_1 & i_2 \end{pmatrix} = V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}$$

$$V_3\begin{pmatrix} l_1 & l_2 & l_3 & j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 & i_1 & i_2 & i_3 \end{pmatrix} = V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V\begin{pmatrix} l_3 & j_3 \\ k_3 & i_3 \end{pmatrix} V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}$$

$$V_4\begin{pmatrix} l_1\, l_2\, l_3\, l_4\, j_1\, j_2\, j_3\, j_4 \\ k_1\, k_2\, k_3\, k_4\, i_1\, i_2\, i_3\, i_4 \end{pmatrix} = V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V\begin{pmatrix} k_4 & i_4 \\ k_3 & i_3 \end{pmatrix} V\begin{pmatrix} l_3 & j_3 \\ l_4 & j_4 \end{pmatrix} V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}$$

$$\cdots$$

Since ε and Δ are multiplicative, (ε) and (Δ) for V imply (ε) and (Δ) for V_2:

$$\begin{aligned}
\varepsilon V_2\begin{pmatrix} l_1 & l_2 & j_1 & j_2 \\ k_1 & k_2 & i_1 & i_2 \end{pmatrix} &= \varepsilon V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} \varepsilon V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix} \\
&= \delta_{k_1 i_1}\delta_{k_2 i_2}\delta_{l_2 j_2}\delta_{l_1 j_1} 1 = \delta_{(k_1 k_2)(i_1 i_2)}\delta_{(l_1 l_2)(j_1 j_2)} 1\,, \\
\Delta V_2\begin{pmatrix} l_1 & l_2 & j_1 & j_2 \\ k_1 & k_2 & i_1 & i_2 \end{pmatrix} &= \Delta V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} \Delta V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix} \\
&= \sum V\begin{pmatrix} k_2 & h_1 \\ k_1 & g_1 \end{pmatrix} V\begin{pmatrix} l_1 & h_2 \\ l_2 & g_2 \end{pmatrix} \otimes V\begin{pmatrix} h_1 & i_2 \\ g_1 & i_1 \end{pmatrix} V\begin{pmatrix} h_2 & j_1 \\ g_2 & j_2 \end{pmatrix} \\
&= \sum V_2\begin{pmatrix} l_1 & l_2 & h_2 & g_2 \\ k_1 & k_2 & g_1 & h_1 \end{pmatrix} \otimes V_2\begin{pmatrix} h_2 & g_2 & j_1 & j_2 \\ g_1 & h_1 & i_1 & i_2 \end{pmatrix}.
\end{aligned}$$

Since S and $*$ are antimultiplicative, (S) and ($*$) for V imply (S) and ($*$) for V_2:

$$\begin{aligned}
SV_2\begin{pmatrix} l_1 & l_2 & j_1 & j_2 \\ k_1 & k_2 & i_1 & i_2 \end{pmatrix} &= SV\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix} SV\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} \\
&= q_{l_2}^2 q_{j_2}^{-2} q_{j_1}^2 q_{l_1}^{-2} q_{k_1}^2 q_{i_1}^{-2} q_{i_2}^2 q_{k_2}^{-2} V\begin{pmatrix} j_2 & l_2 \\ j_1 & l_1 \end{pmatrix} V\begin{pmatrix} i_1 & k_1 \\ i_2 & k_2 \end{pmatrix} \\
&= q_{(k_1 k_2)}^2 q_{(i_1 i_2)}^{-2} q_{(j_1 j_2)}^2 q_{(l_1 l_2)}^{-2} V_2\begin{pmatrix} i_1 & i_2 & k_1 & k_2 \\ j_1 & j_2 & l_1 & l_2 \end{pmatrix}, \\
V_2\begin{pmatrix} l_1 & l_2 & j_1 & j_2 \\ k_1 & k_2 & i_1 & i_2 \end{pmatrix}^* &= V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}^* V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix}^* \\
&= V\begin{pmatrix} l_2 & j_2 \\ l_1 & j_1 \end{pmatrix} V\begin{pmatrix} k_1 & i_1 \\ k_2 & i_2 \end{pmatrix} = V_2\begin{pmatrix} k_1 & k_2 & i_1 & i_2 \\ l_1 & l_2 & j_1 & j_2 \end{pmatrix}.
\end{aligned}$$

By using (m°) and (u°) for V we get (u°) for V_2:

$$\begin{aligned}
\sum q_{(i_1 i_2)}^2 V_2\begin{pmatrix} l_1 & l_2 & i_1 & i_2 \\ k_1 & k_2 & i_1 & i_2 \end{pmatrix} &= \sum q_{i_1}^2 q_{i_2}^{-2} V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V\begin{pmatrix} l_1 & i_1 \\ l_2 & i_2 \end{pmatrix} \\
&= \sum q_{i_1}^2 \delta_{k_2 l_2} q_{l_2}^{-2} V\begin{pmatrix} l_1 & i_1 \\ k_1 & i_1 \end{pmatrix} \\
&= \delta_{k_2 l_2} q_{l_2}^{-2} \delta_{k_1 l_1} q_{k_1}^2 = \delta_{(k_1 k_2)(l_1 l_2)} q_{(k_1 k_2)}^2.
\end{aligned}$$

By using (°m) and (°u) for V we get (°m) for V_2:

$$\sum q_{(s_1s_2)}^{-2} V_2\begin{pmatrix}s_1&s_2&h_1&h_2\\k_1&k_2&g_1&g_2\end{pmatrix} V_2\begin{pmatrix}l_1&l_2&j_1&j_2\\s_1&s_2&i_1&i_2\end{pmatrix}$$
$$= \sum q_{s_1}^{-2}q_{s_2}^2 V\begin{pmatrix}k_2&g_2\\k_1&g_1\end{pmatrix} V\begin{pmatrix}s_1&h_1\\s_2&h_2\end{pmatrix} V\begin{pmatrix}s_2&i_2\\s_1&i_1\end{pmatrix} V\begin{pmatrix}l_1&j_1\\l_2&j_2\end{pmatrix}$$
$$= \sum \delta_{h_1i_1} q_{i_1}^{-2}q_{s_2}^2 V\begin{pmatrix}k_2&g_2\\k_1&g_1\end{pmatrix} V\begin{pmatrix}s_2&i_2\\s_2&h_2\end{pmatrix} V\begin{pmatrix}l_1&j_1\\l_2&j_2\end{pmatrix}$$
$$= \delta_{h_2i_2} q_{i_2}^2 \delta_{h_1i_1} q_{i_1}^{-2} V\begin{pmatrix}k_2&g_2\\k_1&g_1\end{pmatrix} V\begin{pmatrix}l_1&j_1\\l_2&j_2\end{pmatrix}$$
$$= \delta_{(h_1h_2)(i_1i_2)} q_{(i_1i_2)}^{-2} V_2\begin{pmatrix}l_1&l_2&j_1&j_2\\k_1&k_2&g_1&g_2\end{pmatrix}.$$

Since ε and Δ are multiplicative, (ε) and (Δ) for V imply (ε) and (Δ) for V_3:

$$\varepsilon V_3\begin{pmatrix}l_1&l_2&l_3&j_1&j_2&j_3\\k_1&k_2&k_3&i_1&i_2&i_3\end{pmatrix} = \varepsilon V\begin{pmatrix}k_2&i_2\\k_1&i_1\end{pmatrix} \varepsilon V\begin{pmatrix}l_3&j_3\\k_3&i_3\end{pmatrix} \varepsilon V\begin{pmatrix}l_1&j_1\\l_2&j_2\end{pmatrix}$$
$$= \delta_{k_1i_1}\delta_{k_2i_2}\delta_{k_3i_3}\delta_{l_3j_3}\delta_{l_2j_2}\delta_{l_1j_1}1$$
$$= \delta_{(k_1k_2k_3)(i_1i_2i_3)}\delta_{(l_1l_2l_3)(j_1j_2j_3)}1,$$

$$\Delta V_3\begin{pmatrix}l_1&l_2&l_3&j_1&j_2&j_3\\k_1&k_2&k_3&i_1&i_2&i_3\end{pmatrix}$$
$$= \Delta V\begin{pmatrix}k_2&i_2\\k_1&i_1\end{pmatrix} \Delta V\begin{pmatrix}l_3&j_3\\k_3&i_3\end{pmatrix} \Delta V\begin{pmatrix}l_1&j_1\\l_2&j_2\end{pmatrix}$$
$$= \sum V\begin{pmatrix}k_2&h_1\\k_1&g_1\end{pmatrix} V\begin{pmatrix}l_3&h_2\\k_3&g_2\end{pmatrix} V\begin{pmatrix}l_1&h_3\\l_2&g_3\end{pmatrix} \otimes V\begin{pmatrix}h_1&i_2\\g_1&i_1\end{pmatrix} V\begin{pmatrix}h_2&j_3\\g_2&i_3\end{pmatrix} V\begin{pmatrix}h_3&j_1\\g_3&j_2\end{pmatrix}$$
$$= \sum V_3\begin{pmatrix}l_1&l_2&l_3&h_3&g_3&h_2\\k_1&k_2&k_3&g_1&h_1&g_2\end{pmatrix} \otimes V_3\begin{pmatrix}h_3&g_3&h_2&j_1&j_2&j_3\\g_1&h_1&g_2&i_1&i_2&i_3\end{pmatrix}.$$

Since S and $*$ are antimultiplicative, (S) and ($*$) for V imply (S) and ($*$) for V_3:

$$SV_3\begin{pmatrix}l_1&l_2&l_3&j_1&j_2&j_3\\k_1&k_2&k_3&i_1&i_2&i_3\end{pmatrix}$$
$$= SV\begin{pmatrix}l_1&j_1\\l_2&j_2\end{pmatrix} SV\begin{pmatrix}l_3&j_3\\k_3&i_3\end{pmatrix} SV\begin{pmatrix}k_2&i_2\\k_1&i_1\end{pmatrix}$$
$$= q_{l_2}^2 q_{j_2}^{-2} q_{j_1}^2 q_{l_1}^{-2} q_{k_3}^2 q_{i_3}^{-2} q_{j_3}^2 q_{l_3}^{-2} q_{k_1}^2 q_{i_1}^{-2} q_{i_2}^2 q_{k_2}^{-2} V\begin{pmatrix}j_2&l_2\\j_1&l_1\end{pmatrix} V\begin{pmatrix}i_3&k_3\\j_3&l_3\end{pmatrix} V\begin{pmatrix}i_1&k_1\\i_2&k_2\end{pmatrix}$$
$$= q_{(k_1k_2k_3)}^2 q_{(i_1i_2i_3)}^{-2} q_{(j_1j_2j_3)}^2 q_{(l_1l_2l_3)}^{-2} V_3\begin{pmatrix}i_1&i_2&i_3&k_1&k_2&k_3\\j_1&j_2&j_3&l_1&l_2&l_3\end{pmatrix},$$

$$\begin{aligned} V_3\begin{pmatrix} l_1 & l_2 & l_3 & j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 & i_1 & i_2 & i_3 \end{pmatrix}^* &= V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}^* V\begin{pmatrix} l_3 & j_3 \\ k_3 & i_3 \end{pmatrix}^* V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix}^* \\ &= V\begin{pmatrix} l_2 & j_2 \\ l_1 & j_1 \end{pmatrix} V\begin{pmatrix} k_3 & i_3 \\ l_3 & j_3 \end{pmatrix} V\begin{pmatrix} k_1 & i_1 \\ k_2 & i_2 \end{pmatrix} \\ &= V_3\begin{pmatrix} k_1 & k_2 & k_3 & i_1 & i_2 & i_3 \\ l_1 & l_2 & l_3 & j_1 & j_2 & j_3 \end{pmatrix}. \end{aligned}$$

By using (u$^\circ$), (m$^\circ$) and (u$^\circ$) again for V we get (u$^\circ$) for V_3:

$$\begin{aligned} &\sum q^2_{(i_1 i_2 i_3)} V_3\begin{pmatrix} l_1 & l_2 & l_3 & i_1 & i_2 & i_3 \\ k_1 & k_2 & k_3 & i_1 & i_2 & i_3 \end{pmatrix} \\ &\qquad = \sum q^2_{i_1} q^{-2}_{i_2} q^2_{i_3} V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V\begin{pmatrix} l_3 & i_3 \\ k_3 & i_3 \end{pmatrix} V\begin{pmatrix} l_1 & i_1 \\ l_2 & i_2 \end{pmatrix} \\ &\qquad = \sum q^2_{i_1} q^{-2}_{i_2} \delta_{k_3 l_3} q^2_{k_3} V\begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V\begin{pmatrix} l_1 & i_1 \\ l_2 & i_2 \end{pmatrix} \\ &\qquad = \sum q^2_{i_1} \delta_{k_3 l_3} q^2_{k_3} \delta_{k_2 l_2} q^{-2}_{l_2} V\begin{pmatrix} l_1 & i_1 \\ k_1 & i_1 \end{pmatrix} \\ &\qquad = \delta_{k_3 l_3} q^2_{k_3} \delta_{k_2 l_2} q^{-2}_{l_2} \delta_{k_1 l_1} q^2_{k_1} = \delta_{(k_1 k_2 k_3)(l_1 l_2 l_3)} q^2_{(k_1 k_2 k_3)}. \end{aligned}$$

By using ($^\circ$m), ($^\circ$u) and ($^\circ$m) again for V we get ($^\circ$m) for V_3:

$$\begin{aligned} &\sum q^{-2}_{(s_1 s_2 s_3)} V_3\begin{pmatrix} s_1 & s_2 & s_3 & h_1 & h_2 & h_3 \\ k_1 & k_2 & k_3 & g_1 & g_2 & g_3 \end{pmatrix} V_3\begin{pmatrix} l_1 & l_2 & l_3 & j_1 & j_2 & j_3 \\ s_1 & s_2 & s_3 & i_1 & i_2 & i_3 \end{pmatrix} \\ &= \sum q^{-2}_{s_1} q^2_{s_2} q^{-2}_{s_3} \\ &\quad \cdot V\begin{pmatrix} k_2 & g_2 \\ k_1 & g_1 \end{pmatrix} V\begin{pmatrix} s_3 & h_3 \\ k_3 & g_3 \end{pmatrix} V\begin{pmatrix} s_1 & h_1 \\ s_2 & h_2 \end{pmatrix} V\begin{pmatrix} s_2 & i_2 \\ s_1 & i_1 \end{pmatrix} V\begin{pmatrix} l_3 & j_3 \\ s_3 & i_3 \end{pmatrix} V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix} \\ &= \sum q^2_{s_2} q^{-2}_{s_3} \delta_{h_1 i_1} q^{-2}_{i_1} V\begin{pmatrix} k_2 & g_2 \\ k_1 & g_1 \end{pmatrix} V\begin{pmatrix} s_3 & h_3 \\ k_3 & g_3 \end{pmatrix} V\begin{pmatrix} s_2 & i_2 \\ s_2 & h_2 \end{pmatrix} V\begin{pmatrix} l_3 & j_3 \\ s_3 & i_3 \end{pmatrix} V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix} \\ &= \sum q^{-2}_{s_3} \delta_{h_1 i_1} q^{-2}_{i_1} \delta_{h_2 i_2} q^2_{i_2} V\begin{pmatrix} k_2 & g_2 \\ k_1 & g_1 \end{pmatrix} V\begin{pmatrix} s_3 & h_3 \\ k_3 & g_3 \end{pmatrix} V\begin{pmatrix} l_3 & j_3 \\ s_3 & i_3 \end{pmatrix} V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix} \\ &= \delta_{h_1 i_1} q^{-2}_{i_1} \delta_{h_2 i_2} q^2_{i_2} \delta_{h_3 i_3} q^{-2}_{i_3} V\begin{pmatrix} k_2 & g_2 \\ k_1 & g_1 \end{pmatrix} V\begin{pmatrix} l_3 & j_3 \\ k_3 & g_3 \end{pmatrix} V\begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix} \\ &= \delta_{(h_1 h_2 h_3)(i_1 i_2 i_3)} q^{-2}_{(i_1 i_2 i_3)} V_3\begin{pmatrix} l_1 & l_2 & l_3 & j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 & g_1 & g_2 & g_3 \end{pmatrix}. \end{aligned}$$

The proof for arbitrary n even is similar to the proof for $n = 2$ and for arbitrary n odd, to the proof for $n = 3$. ∎

A linear map $T : A^{\otimes n} \to A^{\otimes m}$ is v_∞-equivariant if the following diagram commutes:

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{T} & A^{\otimes m} \\ v_n \downarrow & & \downarrow v_m \\ A^{\otimes n}\otimes H & \xrightarrow{T\otimes \mathrm{id}} & A^{\otimes m}\otimes H \end{array} .$$

For $n = 0$ a map T is v_∞-equivariant if and only if $T(1)$ is fixed by v_m.

LEMMA 2.2. *The following linear maps*

$$I_n \begin{pmatrix} j_1 & \dots & j_{n-1} \\ i_1 & \dots & i_{n-1} \end{pmatrix} = \sum \begin{pmatrix} j_1 & \dots & j_{n-1} & l \\ i_1 & \dots & i_{n-1} & l \end{pmatrix},$$

$$\widetilde{e}_n = \sum q_i^{\pm 2} q_j^{\pm 2} \begin{pmatrix} g_1 & \dots & g_{n-2} & j & j \\ g_1 & \dots & g_{n-2} & i & i \end{pmatrix},$$

$$\widetilde{E}_n \begin{pmatrix} j_1 & \dots & j_n \\ i_1 & \dots & i_n \end{pmatrix} = \delta_{i_n j_n} q_{i_n}^{\mp 4} \begin{pmatrix} j_1 & \dots & j_{n-1} \\ i_1 & \dots & i_{n-1} \end{pmatrix},$$

where $\pm 1 = (-1)^n$, *are* v_∞*-equivariant.*

Proof. The coactions v_2 and v_3 are given by the following formulae:

$$v_2 \begin{pmatrix} j_1 & j_2 \\ i_1 & i_2 \end{pmatrix} = \sum \begin{pmatrix} l_1 & l_2 \\ k_1 & k_2 \end{pmatrix} \otimes q_{k_1}^{-1} q_{k_2} q_{i_1} q_{i_2}^{-1} q_{j_1} q_{j_2}^{-1} q_{l_1}^{-1} q_{l_2} V \begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V \begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix},$$

$$v_3 \begin{pmatrix} j_1 & j_2 & j_3 \\ i_1 & i_2 & i_3 \end{pmatrix} = \sum \begin{pmatrix} l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \otimes V \begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V \begin{pmatrix} l_3 & j_3 \\ k_3 & i_3 \end{pmatrix} V \begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}$$
$$\cdot\ q_{k_1}^{-1} q_{k_2} q_{k_3}^{-1} q_{i_1} q_{i_2}^{-1} q_{i_3} q_{j_1} q_{j_2}^{-1} q_{j_3} q_{l_1}^{-1} q_{l_2} q_{l_3}^{-1}.$$

By using (m°) we get that I_2 is v_∞-equivariant:

$$\begin{aligned} v_2 I_2 \begin{pmatrix} j_1 \\ i_1 \end{pmatrix} &= \sum \begin{pmatrix} l_1 & l_2 \\ k_1 & k_2 \end{pmatrix} \otimes q_{k_1}^{-1} q_{k_2} q_{i_1} q_{j_1} q_{l_1}^{-1} q_{l_2} q_l^{-2} V \begin{pmatrix} k_2 & l \\ k_1 & i_1 \end{pmatrix} V \begin{pmatrix} l_1 & j_1 \\ l_2 & l \end{pmatrix} \\ &= \sum \begin{pmatrix} l_1 & l_2 \\ k_1 & k_2 \end{pmatrix} \otimes \delta_{k_2 l_2} q_{k_2}^{-2} q_{k_1}^{-1} q_{k_2} q_{i_1} q_{j_1} q_{l_1}^{-1} q_{l_2} V \begin{pmatrix} l_1 & j_1 \\ k_1 & i_1 \end{pmatrix} \\ &= \sum \begin{pmatrix} l_1 & l \\ k_1 & l \end{pmatrix} \otimes q_{k_1}^{-1} q_{i_1} q_{j_1} q_{l_1}^{-1} V \begin{pmatrix} l_1 & j_1 \\ k_1 & i_1 \end{pmatrix} = (I_2 \otimes \mathrm{id}) v \begin{pmatrix} j_1 \\ i_1 \end{pmatrix}. \end{aligned}$$

By using (u°) we get that I_3 is v_∞-equivariant:

$$v_3 I_3 \begin{pmatrix} j_1 & j_2 \\ i_1 & i_2 \end{pmatrix} = \sum \begin{pmatrix} l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \otimes V \begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V \begin{pmatrix} l_3 & l \\ k_3 & l \end{pmatrix} V \begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}$$
$$\cdot\ q_{k_1}^{-1} q_{k_2} q_{k_3}^{-1} q_{i_1} q_{i_2}^{-1} q_{j_1} q_{j_2}^{-1} q_{l_1}^{-1} q_{l_2} q_{l_3}^{-1} q_l^2$$

$$= \sum \begin{pmatrix} l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \otimes V \begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V \begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}$$
$$\cdot\ \delta_{l_3 k_3} q_{k_3}^2 q_{k_1}^{-1} q_{k_2} q_{k_3}^{-1} q_{i_1} q_{i_2}^{-1} q_{j_1} q_{j_2}^{-1} q_{l_1}^{-1} q_{l_2} q_{l_3}^{-1} q_l^2$$
$$= \sum \begin{pmatrix} l_1 & l_2 & l \\ k_1 & k_2 & l \end{pmatrix} \otimes V \begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V \begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}$$
$$\cdot\ q_{k_1}^{-1} q_{k_2} q_{i_1} q_{i_2}^{-1} q_{j_1} q_{j_2}^{-1} q_{l_1}^{-1} q_{l_2} q_l^2$$
$$= (I_3 \otimes \mathrm{id}) v_2 \begin{pmatrix} j_1 & j_2 \\ i_1 & i_2 \end{pmatrix}.$$

By using (u°) twice we get that $\widetilde{e}_2$ is v_∞-equivariant:

$$v_2(\widetilde{e}_2) = \sum \begin{pmatrix} l_1 & l_2 \\ k_1 & k_2 \end{pmatrix} \otimes q_i^2 q_j^2 q_{k_1}^{-1} q_{k_2} q_{l_1}^{-1} q_{l_2} V \begin{pmatrix} k_2 & i \\ k_1 & i \end{pmatrix} V \begin{pmatrix} l_1 & j \\ l_2 & j \end{pmatrix}$$
$$= \sum \begin{pmatrix} l_1 & l_2 \\ k_1 & k_2 \end{pmatrix} \otimes \delta_{k_1 k_2} \delta_{l_1 l_2} q_{k_1}^2 q_{l_1}^2 = \widetilde{e}_2 \otimes 1.$$

By using (m°) twice and (u°) we get that $\widetilde{e}_3$ is v_∞-equivariant:

$$v_3(\widetilde{e}_3) = \sum \begin{pmatrix} l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \otimes V \begin{pmatrix} k_2 & i \\ k_1 & g_1 \end{pmatrix} V \begin{pmatrix} l_3 & j \\ k_3 & i \end{pmatrix} V \begin{pmatrix} l_1 & g_1 \\ l_2 & j \end{pmatrix}$$
$$\cdot\ q_i^{-2} q_j^{-2} q_{k_1}^{-1} q_{k_2} q_{k_3}^{-1} q_{g_1}^2 q_{l_1}^{-1} q_{l_2} q_{l_3}^{-1}$$
$$= \sum \begin{pmatrix} l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \otimes \delta_{l_2 l_3} q_i^{-2} q_{l_2}^{-2} q_{k_1}^{-1} q_{k_2} q_{k_3}^{-1} q_{g_1}^2 q_{l_1}^{-1} V \begin{pmatrix} k_2 & i \\ k_1 & g_1 \end{pmatrix} V \begin{pmatrix} l_1 & g_1 \\ k_3 & i \end{pmatrix}$$
$$= \sum \begin{pmatrix} l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \otimes \delta_{l_2 l_3} \delta_{k_2 k_3} q_{k_2}^{-2} q_{l_2}^{-2} q_{k_1}^{-1} q_{g_1}^2 q_{l_1}^{-1} V \begin{pmatrix} l_1 & g_1 \\ k_1 & g_1 \end{pmatrix}$$
$$= \sum \begin{pmatrix} l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \otimes \delta_{k_1 l_1} \delta_{l_2 l_3} \delta_{k_2 k_3} q_{k_2}^{-2} q_{l_2}^{-2} = \widetilde{e}_3 \otimes 1.$$

By using (°m) we get that $\widetilde{E}_2$ is v_∞-equivariant:

$$(\widetilde{E}_2 \otimes \mathrm{id}) v_2 \begin{pmatrix} j_1 & j_2 \\ i_1 & i_2 \end{pmatrix} = \sum \begin{pmatrix} l_1 \\ k_1 \end{pmatrix} \otimes q_g^{-2} q_{k_1}^{-1} q_{i_1} q_{i_2}^{-1} q_{j_1} q_{j_2}^{-1} q_{l_1}^{-1} V \begin{pmatrix} g & i_2 \\ k_1 & i_1 \end{pmatrix} V \begin{pmatrix} l_1 & j_1 \\ g & j_2 \end{pmatrix}$$
$$= \sum \begin{pmatrix} l_1 \\ k_1 \end{pmatrix} \otimes \delta_{i_2 j_2} q_{i_2}^{-4} q_{k_1}^{-1} q_{i_1} q_{j_1} q_{l_1}^{-1} V \begin{pmatrix} l_1 & j_1 \\ k_1 & i_1 \end{pmatrix}$$
$$= v \widetilde{E}_2 \begin{pmatrix} j_1 & j_2 \\ i_1 & i_2 \end{pmatrix}.$$

By using (°u) we get that $\widetilde{E}_3$ is v_∞-equivariant:

$$(\widetilde{E}_3 \otimes \mathrm{id}) v_3 \begin{pmatrix} j_1 & j_2 & j_3 \\ i_1 & i_2 & i_3 \end{pmatrix} = \sum \begin{pmatrix} l_1 & l_2 \\ k_1 & k_2 \end{pmatrix} \otimes V \begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V \begin{pmatrix} g & j_3 \\ g & i_3 \end{pmatrix} V \begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}$$
$$\cdot\ q_g^2 q_{k_1}^{-1} q_{k_2} q_{i_1} q_{i_2}^{-1} q_{i_3} q_{j_1} q_{j_2}^{-1} q_{j_3} q_{l_1}^{-1} q_{l_2}$$

$$= \sum \begin{pmatrix} l_1 & l_2 \\ k_1 & k_2 \end{pmatrix} \otimes V \begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix} V \begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}$$
$$\cdot\ \delta_{i_3 j_3} q_{i_3}^4 q_{k_1}^{-1} q_{k_2} q_{i_1} q_{i_2}^{-1} q_{j_1} q_{j_2}^{-1} q_{l_1}^{-1} q_{l_2}$$
$$= v_2 \widetilde{E}_3 \begin{pmatrix} j_1 & j_2 & j_3 \\ i_1 & i_2 & i_3 \end{pmatrix}.$$

The proof for arbitrary n even is similar to the proof for $n = 2$ and for arbitrary n odd, to the proof for $n = 3$. ∎

Let $h : H \to \mathbb{C}$ be the Haar integral constructed by Woronowicz in [29]. This is a unital linear form having the following *bi-invariance* Property:

$$(h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = h(\cdot)1.$$

If $v : A \to A \otimes H$ is a coaction then $\Gamma_n = (\mathrm{id} \otimes h)v_n$ is an idempotent of $\mathcal{L}(A^{\otimes n})$ and its image are the fixed points of v_n. This follows from the computation

$$v_n \Gamma_n(x) = v_n(\mathrm{id} \otimes h)v_n(x) = (\mathrm{id} \otimes \mathrm{id} \otimes h)(\mathrm{id} \otimes \Delta)v_n(x) = \Gamma_n(x) \otimes 1.$$

A pair of linear maps $(T, T^q) : A^{\otimes n} \to A^{\otimes m}$ is called *weakly v_∞-equivariant* if the following diagram commutes:

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{T^q} & A^{\otimes m} \\ \Gamma_n \downarrow & & \downarrow \Gamma_m \\ A^{\otimes n} & \xrightarrow{T} & A^{\otimes m} \end{array}.$$

The interest in this notion is that it makes the following diagram factor:

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{T} & A^{\otimes m} \\ \cup & & \cup \\ \mathrm{Im}(\Gamma_n) & \longrightarrow & \mathrm{Im}(\Gamma_m) \end{array}.$$

We say that an operator T is *weakly v_∞-equivariant* if the pair (T, T) is weakly v_∞-equivariant. This happens for instance if T is v_∞-equivariant, because we can glue the v_∞-equivariance diagram of T to the following trivial diagram:

$$\begin{array}{ccc} A^{\otimes n} \otimes H & \xrightarrow{T \otimes \mathrm{id}} & A^{\otimes m} \otimes H \\ \mathrm{id} \otimes h \downarrow & & \downarrow \mathrm{id} \otimes h \\ A^{\otimes n} & \xrightarrow{T} & A^{\otimes m} \end{array}.$$

The following linear map, called modular map of φ

$$\theta \begin{pmatrix} j \\ i \end{pmatrix} = q_i^4 q_j^{-4} \begin{pmatrix} j \\ i \end{pmatrix}$$

is the unique linear map $\theta : A \to A$ such that $\varphi(ab) = \varphi(b\theta(a))$ for any $a, b \in A$.

Consider the automorphism $\sigma : H \to H$ constructed by Woronowicz in [29], which satisfies $h(ab) = h(b\sigma(a))$ for any a, b.

LEMMA 2.3. *Assume that the following* modularity *Condition is satisfied:*

$$(\theta \otimes \mathrm{id})v\theta = (\mathrm{id} \otimes \sigma)v.$$

(i) *The following pair of maps is weakly v_∞-equivariant:*

$$J_n \begin{pmatrix} j_3 & \dots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} = \sum \begin{pmatrix} l & k & j_3 & \dots & j_{n+1} \\ l & k & i_3 & \dots & i_{n+1} \end{pmatrix},$$

$$J_n^q \begin{pmatrix} j_3 & \dots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} = \sum q_l^{-8} q_k^8 \begin{pmatrix} l & k & j_3 & \dots & j_{n+1} \\ l & k & i_3 & \dots & i_{n+1} \end{pmatrix}.$$

(ii) *If φ has the trace property $\varphi(ab) = \varphi(ba)$ then J_n is weakly v_∞-equivariant.*
(iii) If H is commutative then J_n is v_∞-equivariant.

Proof. (i) By using the formulae of v and θ we get

$$(\theta \otimes \mathrm{id})v\theta \begin{pmatrix} j \\ i \end{pmatrix} = \sum \begin{pmatrix} l \\ k \end{pmatrix} \otimes q_k^4 q_l^{-4} q_i^4 q_j^{-4} \cdot q_k^{-1} q_i q_j q_l^{-1} V \begin{pmatrix} l & j \\ k & i \end{pmatrix}$$

so the modularity condition is equivalent to the following Condition (σ):

$$\sigma V \begin{pmatrix} l & j \\ k & i \end{pmatrix} = q_k^4 q_l^{-4} q_i^4 q_j^{-4} V \begin{pmatrix} l & j \\ k & i \end{pmatrix}.$$

By using the formula of J_n^q we get

$$\begin{aligned} v_{n+1} J_n^q \begin{pmatrix} j_3 & \dots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} &= v_{n+1} \sum q_l^{-8} q_k^8 \begin{pmatrix} l & k & j_3 & \dots & j_{n+1} \\ l & k & i_3 & \dots & i_{n+1} \end{pmatrix} \\ &= \sum \begin{pmatrix} l_1 & l_2 & l_3 & \dots & l_{n+1} \\ k_1 & k_2 & k_3 & \dots & k_{n+1} \end{pmatrix} \\ &\qquad \otimes q^{-1}_{(k_3 \dots k_{n+1})} q_{(i_3 \dots i_{n+1})} q_{(j_3 \dots j_{n+1})} q^{-1}_{(l_3 \dots l_{n+1})} Z \end{aligned}$$

with Z given by the following formula:

$$Z = \sum q_l^{-6} q_k^6 q_{k_1}^{-1} q_{k_2} q_{l_1}^{-1} q_{l_2} V \begin{pmatrix} k_2 & k \\ k_1 & l \end{pmatrix} V \begin{pmatrix} k_4 & i_4 \\ k_3 & i_3 \end{pmatrix} \dots V \begin{pmatrix} l_3 & j_3 \\ l_4 & j_4 \end{pmatrix} V \begin{pmatrix} l_1 & l \\ l_2 & k \end{pmatrix}.$$

By applying the Haar integral to Z we get

$$h(Z) = h\left(V \begin{pmatrix} k_4 & i_4 \\ k_3 & i_3 \end{pmatrix} \dots V \begin{pmatrix} l_3 & j_3 \\ l_4 & j_4 \end{pmatrix} T \right)$$

with T given by the following formula:

$$T = \sum q_l^{-6} q_k^6 q_{k_1}^{-1} q_{k_2} q_{l_1}^{-1} q_{l_2} V \begin{pmatrix} l_1 & l \\ l_2 & k \end{pmatrix} \sigma V \begin{pmatrix} k_2 & k \\ k_1 & l \end{pmatrix}.$$

By using (σ), (m°) and (u°) we can compute T:

$$
\begin{aligned}
T &= \sum q_l^{-6} q_k^6 q_{k_1}^{-1} q_{k_2} q_{l_1}^{-1} q_{l_2} \cdot q_{k_1}^4 q_l^4 q_k^{-4} q_{k_2}^{-4} V \begin{pmatrix} l_1 & l \\ l_2 & k \end{pmatrix} V \begin{pmatrix} k_2 & k \\ k_1 & l \end{pmatrix} \\
&= \sum q_l^{-2} q_k^2 q_{k_1}^3 q_{k_2}^{-3} q_{l_1}^{-1} q_{l_2} V \begin{pmatrix} l_1 & l \\ l_2 & k \end{pmatrix} V \begin{pmatrix} k_2 & k \\ k_1 & l \end{pmatrix} \\
&= \sum \delta_{k_1 l_1} q_{l_1}^{-2} q_k^2 q_{k_1}^3 q_{k_2}^{-3} q_{l_1}^{-1} q_{l_2} V \begin{pmatrix} l_2 & k \\ k_2 & k \end{pmatrix} \\
&= \delta_{k_1 l_1} q_{l_1}^{-2} \delta_{k_2 l_2} q_{k_2}^2 q_{k_1}^3 q_{k_2}^{-3} q_{l_1}^{-1} q_{l_2} = \delta_{k_1 l_1} \delta_{k_2 l_2} \,.
\end{aligned}
$$

Thus by applying $\mathrm{id} \otimes h$ to the formula of $v_{n+1} J_n^q$ we get

$$
\begin{aligned}
(\mathrm{id} \otimes h) v_{n+1} J_n^q & \begin{pmatrix} j_3 & \dots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} \\
&= \sum \begin{pmatrix} k_1 & l_2 & l_3 & \dots & l_{n+1} \\ k_1 & l_2 & k_3 & \dots & k_{n+1} \end{pmatrix} h \left(V \begin{pmatrix} k_4 & i_4 \\ k_3 & i_3 \end{pmatrix} \dots V \begin{pmatrix} l_3 & j_3 \\ l_4 & j_4 \end{pmatrix} \right) \\
&\qquad \cdot \ q^{-1}_{(k_3 \dots k_{n+1})} q_{(i_3 \dots i_{n+1})} q_{(j_3 \dots j_{n+1})} q^{-1}_{(l_3 \dots l_{n+1})} \\
&= (\mathrm{id} \otimes h)(J_n \otimes \mathrm{id}) v_{n-1} \begin{pmatrix} j_3 & \dots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix},
\end{aligned}
$$

so the weak v_∞-equivariance diagram commutes.

(ii) In terms of weights, the fact that φ is a trace means that q_i^4 depends only on the matrix block containing $\binom{i}{i}$. Thus the spin factor $q_l^{-8} q_k^8$ in the formula of J_n^q cancels, because $\binom{l}{l}$ and $\binom{k}{k}$ are in the same matrix block of A (cf. loop notation).

(iii) The Haar integral and its modular map are used in proof of (i) for rotating a product of elements of H. If H is commutative its product does the same job. ∎

We have all ingredients needed for the case of trace-preserving coactions. Let $\mathcal{P}$ be the colored planar operad constructed by Jones in [16]. A planar algebra is a sequence of vector spaces $P = P_0^{\pm}, P_1, P_2, P_3, \dots$ with a colored operad morphism $\pi : \mathcal{P} \to \mathrm{Hom}(P)$, where $\mathrm{Hom}(P)$ is the colored operad of multilinear maps between P_n's.

If $Q_n \subset P_n$ is a sequence of subspaces, the restriction of multilinear maps between P_n's to multilinear maps between Q_n's is a partially defined colored operad morphism $\mathrm{Res} : \mathrm{Hom}(P) \to \mathrm{Hom}(Q)$. If the domain of Res contains the image of π the composition of Res and π makes Q a planar algebra, called subalgebra of P.

The annular category A is defined as follows. The objects are the positive integers and the space $A(i, j)$ of arrows from i to j is formed by tangles in $\mathcal{P}$ with *output* disc having $2j$ marked points and one *input* disc, having $2i$ marked points. Composition of arrows is given by gluing of annuli. The restriction of π to A is a morphism from A to the category $\mathcal{L}(P)$ having as arrows from i to j the linear maps from P_i to P_j.

Let (A, φ) be as in Section 1 and assume that φ has the trace property $\varphi(ab) = \varphi(ba)$. Let $\pi : \mathcal{P} \to \mathrm{Hom}(P(A, \varphi))$ be the planar algebra associated to the bipartite graph of A, with spin vector $a \mapsto \varphi(1_a)$. The sequence of vector spaces of $P(A, \varphi)$ will be canonically identified with the sequence of tensor powers of A; see Jones ([14]).

THEOREM 2.4. (i) *The linear maps in the image by π of the annular category are weakly v_∞-equivariant. If H is commutative, they are v_∞-equivariant.*

(ii) *The spaces of fixed points of the coactions v_n form a subalgebra of $P(A, \varphi)$.*

Proof. (i) Gluing of commutative diagrams shows that weak v_∞-equivariance is stable by composition, so the annular tangles whose image by π are weakly v_∞-equivariant from a subcategory $B \subset A$. We want to prove that $B = A$.

Consider the inclusion tangle in $A(n-1, n)$, expectation tangle in $A(n, n-1)$, Jones projection tangle in $A(0, n)$ and shift tangle in $A(n-1, n+1)$ (see [16] for pictures). By [14] their images by π are given by the formulae in Lemma 2.1 and Lemma 2.2, suitably rescaled. Lemma 2.1 shows that the first three tangles are in B. In terms of weights, the fact that φ is a trace means that q_i depends only on the matrix block containing $\binom{i}{i}$, so the spin factor in the formula (S) of the antipode cancels. In particular the square of the antipode is the identity on the coefficients of v, and by replacing H with its $*$-subalgebra generated by these coefficients we may assume that $S^2 = \mathrm{id}$. By [29], the Haar integral is a trace, so the modularity condition in Lemma 2.2 is satisfied. Thus the shift tangle is in B.

The sets $A(0, n)$ of Temperley-Lieb tangles being generated by inclusions and Jones projections, they are in B. For $x, y \in A(0, n)$ let $M(x, y) \in A(n, n)$ be the 3-multiplication n-tangle of $\mathcal{P}$ with the upper circle filled with y and the lower circle filled with x. The corresponding linear map is $\widetilde{M}(x, y) : p \mapsto \widetilde{x}p\widetilde{y}$ and since fixed points of v_n are stable under multiplication, $M(x, y)$ is in B.

Let $T \in A(i, j)$. By using boxes instead of discs, as in [16], isotope T, then cut it horizontally in three parts such that the middle part contains the inner box plus vertical strings only. By adding contractible circles at right, we can arrange such that the number of points on the middle cuts is greater than j. By adding more contractible circles at right, each of them consisting of *up* and *down* semicircles plus two outside *expectation* strings connecting them, we get an equality of the form $T^{\circ\circ\ldots\circ} = EM(x, y)IJ$, where I is a composition of inclusion tangles, J is a composition of shift tangles, E is a composition of expectation tangles, x and y are in $A(k, k)$ for some big k and $T^{\circ\circ\ldots\circ}$ is obtained from T by adding contractible circles. Thus $T^{\circ\circ\ldots\circ}$ is in B, so T is in B. Same proof works for the second part, with *weak v_∞-equivariant* replaced by *v_∞-equivariant*.

(ii) Since weakly v_∞-equivariant maps send fixed points to fixed points, part (i) shows that $\pi(A)$ is in the domain of Res. By Proposition 1.18 in [16] this implies that $\pi(\mathcal{P})$ is in the domain of Res and this proves the assertion. ∎

3. TWISTED STRUCTURE

We assume that φ is a δ-form, in the sense that $\varphi(1) = 1$ and $\mathrm{Tr}(B^{-4}) = \delta^2$ for any matrix block B of the unique Q such that $\varphi = \mathrm{Tr}(Q^4\cdot)$. See [3] for examples and comments. In terms of the basis, the unitality of φ translates into the following formula

$$(\ddagger) \qquad \sum q_j^4 = 1.$$

We use the equivalence relation $i \sim j$ if $\binom{i}{i}$ and $\binom{j}{j}$ are in the same matrix block of A. For any i in the set of indices we have the following formula

$$(\dagger) \qquad \sum_{j\sim i} q_j^{-4} = \delta^2.$$

By using δ we define normalised forms, expectations and Jones projections by

$$\varphi_n \begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \cdots & i_n \end{pmatrix} = \delta^{\frac{1}{2}\mp\frac{1}{2}-n} \delta_{(i_1\ldots i_n)(j_1\ldots j_n)} q^4_{(i_1\ldots i_n)},$$

$$E_n \begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \cdots & i_n \end{pmatrix} = \delta_{i_n j_n} \delta^{-1\mp 1} q_{i_n}^{\mp 4} \begin{pmatrix} j_1 & \cdots & j_{n-1} \\ i_1 & \cdots & i_{n-1} \end{pmatrix},$$

$$e_n = \sum \delta^{-1\pm 1} q_i^{\pm 2} q_j^{\pm 2} \begin{pmatrix} g_1 & \cdots & g_{n-2} & j & j \\ g_1 & \cdots & g_{n-2} & i & i \end{pmatrix},$$

where $\pm 1 = (-1)^n$. Define also a linear form ψ_2 by

$$\psi_2 \begin{pmatrix} j_1 & j_2 \\ i_1 & i_2 \end{pmatrix} = \delta_{(i_1 i_2)(j_1 j_2)} \delta^{-2} q_{i_1}^{-4} q_{i_2}^4.$$

The modular map θ_n of $\widetilde{\varphi}_n$ is given by the following formula (see Section 2):

$$\theta_n \begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \cdots & i_n \end{pmatrix} = q^4_{(i_1\ldots i_n)} q^{-4}_{(j_1\ldots j_n)} \begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \cdots & i_n \end{pmatrix}.$$

In this section we prove the following technical result.

PROPOSITION 3.1. *If $Q_n \subset A^{\otimes n}$ is a sequence of C^*-algebras satisfying:*
(i) $I_n(Q_{n-1}) \subset Q_n$, $E_n(Q_n) \subset Q_{n-1}$, $J_n(Q_{n-1}) \subset Q_{n+1}$ *and* $e_n \in Q_n$ *for any* n*;*
(ii) $\theta_n(x) = x$ *for any* $x \in Q_n$ *and any* n*;*
(iii) $\varphi_2(x) = \psi_2(x)$ *for any* $x \in Q_2$.
then there exists a unique C^*-planar algebra structure on the sequence Q_n such that the inclusions, shifts, traces, expectations and Jones projections are the restrictions of I_n, J_n, φ_n, E_n and e_n. This C^*-planar algebra is spherical and of modulus δ.

This will be proved by using the *bubbling* result of Jones in [16] applied to a certain lattice of C^*-algebras satisfying the axioms of Popa in [24]. For checking the axioms we have to verify all relevant formulae satisfied by I_n, J_n, φ_n, E_n and

e_n. This kind of computation appears in many places in the subfactor literature; see e.g. the books [11] or [17].

In this paper the set of parameters is somehow maximal, so we will give self-contained complete proofs for everything. It is possible to use some Hopf algebra dualities in order to cut from computations, but this rather complicates things and we prefer to use the obvious symmetries only. In fact, the interesting thing would be to have an explicit construction of the partition function, as in [14] and we don't know if this is possible.

Note also that all formulae to be verified are irrelevant once the result is proved, because they can be easily verified on pictures.

We first associate to A and to the numbers q_i a system of C^*-algebras satisfying some of Popa's axioms. Define linear maps

$$J_n^- : A^{\otimes n-1} \to A \otimes A^{\otimes n-1}, \quad J_n^+ : A \otimes A^{\otimes n-1} \to A^{\otimes n+1},$$

by the following formulae in terms of the basis:

$$J_n^- \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix} = \sum \begin{pmatrix} g \\ g \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix},$$

$$J_n^+ \left(\begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix} \right) = \sum \begin{pmatrix} h & j_2 & \cdots & j_{n+1} \\ h & i_2 & \ldots & i_{n+1} \end{pmatrix}.$$

These are inclusions of C^*-algebras.

LEMMA 3.2. *We have* $J_n^+ J_n^- = J_n$ *and the following Diagram* (I) *commutes:*

$$\begin{array}{cccccccccc}
\mathbb{C} & \xrightarrow{I_1} & A & \xrightarrow{I_2} & A^{\otimes 2} & \xrightarrow{I_3} & A^{\otimes 3} & \xrightarrow{I_4} & A^{\otimes 4} & \cdots \\
\uparrow J_0^+ & & \uparrow J_1^+ & & \uparrow J_2^+ & & \uparrow J_3^+ & & & \\
 & & \mathbb{C} & \xrightarrow{\mathrm{id}\otimes I_0} & A \otimes \mathbb{C} & \xrightarrow{\mathrm{id}\otimes I_1} & A \otimes A & \xrightarrow{\mathrm{id}\otimes I_2} & A \otimes A^{\otimes 2} & \cdots \\
 & & & & \uparrow J_1^- & & \uparrow J_2^- & & \uparrow J_3^- & \\
 & & & & \mathbb{C} & \xrightarrow{I_1} & A & \xrightarrow{I_2} & A^{\otimes 2} & \cdots \\
 & & & & & & \uparrow J_0^+ & & \uparrow J_1^+ & \\
 & & & & & & \mathbb{C} & \xrightarrow{\mathrm{id}\otimes I_0} & A \otimes \mathbb{C} & \cdots \\
 & & & & & & & & \cdots &
\end{array}$$

where the symbols $\mathrm{id} \otimes I_0$ *and* J_0^+ *denote the unital embedding of* $\mathbb{C}$ *into* A.

Proof. The first assertion follows from the following computation:

$$J_n^+ J_n^- \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix} = \sum \begin{pmatrix} h & g & j_3 & \cdots & j_{n+1} \\ h & g & i_3 & \ldots & i_{n+1} \end{pmatrix} = J_n \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix}.$$

The commutation of the n-th square in the first row follow from

$$\begin{array}{ccc} \sum\begin{pmatrix} h & j_2 & \dots & j_n \\ h & i_2 & \dots & i_n \end{pmatrix} & \stackrel{I_{n+1}}{\longrightarrow} & \sum\begin{pmatrix} h & j_2 & \dots & j_n & l \\ h & i_2 & \dots & i_n & l \end{pmatrix} \\ \uparrow J_{n-1}^+ & & \uparrow J_n^+ \\ \begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \dots & j_n \\ i_3 & \dots & i_n \end{pmatrix} & \stackrel{\mathrm{id}\otimes I_{n-1}}{\longrightarrow} & \sum \begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \dots & j_n & l \\ i_3 & \dots & i_n & l \end{pmatrix} \end{array}.$$

The commutation of the n-th square in the second row follow from

$$\begin{array}{ccc} \sum\begin{pmatrix} g \\ g \end{pmatrix} \otimes \begin{pmatrix} j_3 & \dots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} & \stackrel{\mathrm{id}\otimes I_n}{\longrightarrow} & \sum\begin{pmatrix} g \\ g \end{pmatrix} \otimes \begin{pmatrix} j_3 & \dots & j_{n+1} & l \\ i_3 & \dots & i_{n+1} & l \end{pmatrix} \\ \uparrow J_n^- & & \uparrow J_{n+1}^- \\ \begin{pmatrix} j_3 & \dots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} & \stackrel{I_n}{\longrightarrow} & \sum \begin{pmatrix} j_3 & \dots & j_{n+1} & l \\ i_3 & \dots & i_{n+1} & l \end{pmatrix} \end{array}.$$

From vertical 2-periodicity we get that the whole diagram is commutative. ∎

Define linear maps

$$E_n^- : A \otimes A^{\otimes n-1} \to A^{\otimes n-1}, \quad E_n^+ : A^{\otimes n+1} \to A \otimes A^{\otimes n-1},$$

by the following formulae in terms of the basis:

$$E_n^- \left(\begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \dots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} \right) = \delta_{i_2 j_2} q_{i_2}^4 \begin{pmatrix} j_3 & \dots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix},$$

$$E_n^+ \begin{pmatrix} j_1 & \dots & j_{n+1} \\ i_1 & \dots & i_{n+1} \end{pmatrix} = \delta_{i_1 j_1} \delta^{-2} q_{i_1}^{-4} \begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \dots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix}.$$

LEMMA 3.3. *The linear maps in the following Diagram* (E)

$$\begin{array}{ccccccccc} \mathbb{C} & \stackrel{E_1}{\longleftarrow} & A & \stackrel{E_2}{\longleftarrow} & A^{\otimes 2} & \stackrel{E_3}{\longleftarrow} & A^{\otimes 3} & \stackrel{E_4}{\longleftarrow} & A^{\otimes 4} & \dots \\ & & \downarrow E_0^+ & & \downarrow E_1^+ & & \downarrow E_2^+ & & \downarrow E_3^+ & \\ & & \mathbb{C} & \stackrel{\mathrm{id}\otimes E_0}{\longleftarrow} & A \otimes \mathbb{C} & \stackrel{\mathrm{id}\otimes E_1}{\longleftarrow} & A \otimes A & \stackrel{\mathrm{id}\otimes E_2}{\longleftarrow} & A \otimes A^{\otimes 2} & \dots \\ & & & & \downarrow E_1^- & & \downarrow E_2^- & & \downarrow E_3^- & \\ & & & & \mathbb{C} & \stackrel{I_1}{\longleftarrow} & A & \stackrel{I_2}{\longleftarrow} & A^{\otimes 2} & \dots \\ & & & & & & \downarrow E_0^+ & & \downarrow E_1^+ & \\ & & & & & & \mathbb{C} & \stackrel{\mathrm{id}\otimes E_0}{\longleftarrow} & A \otimes \mathbb{C} & \dots \\ & & & & & & & & \dots & \end{array}$$

are unital bimodule morphisms with respect to the inclusions in (I).

Proof. The unit for the multiplication of $A^{\otimes n}$ is

$$1_n = \sum \begin{pmatrix} l_1 & \dots & l_n \\ l_1 & \dots & l_n \end{pmatrix}.$$

By using (†) we get that E_{2n} is unital:

$$\sum E_{2n}\begin{pmatrix} l_1 & \dots & l_{2n} \\ l_1 & \dots & l_{2n}\end{pmatrix} = \sum_{l_{2n}\sim l_{2n-1}} \delta^{-2} q_{l_{2n}}^{-4}\begin{pmatrix} l_1 & \dots & l_{2n-1} \\ l_1 & \dots & l_{2n-1}\end{pmatrix} = 1_{2n-1}.$$

By using (‡) we get that E_{2n+1} is unital:

$$\sum E_{2n+1}\begin{pmatrix} l_1 & \dots & l_{2n+1} \\ l_1 & \dots & l_{2n+1}\end{pmatrix} = \sum q_{l_{2n+1}}^{4}\begin{pmatrix} l_1 & \dots & l_{2n} \\ l_1 & \dots & l_{2n}\end{pmatrix} = 1_{2n}.$$

By using (†) we get that E_n^+ is unital:

$$\sum E_n^+\begin{pmatrix} l_1 & \dots & l_{n+1} \\ l_1 & \dots & l_{n+1}\end{pmatrix} = \sum_{l_1\sim l_2} \delta^{-2} q_{l_1}^{-4}\begin{pmatrix} l_2 \\ l_2\end{pmatrix}\otimes\begin{pmatrix} l_3 & \dots & l_{n+1} \\ l_3 & \dots & l_{n+1}\end{pmatrix} = 1_A\otimes 1_{n-1}.$$

By using (‡) we get that E_n^- is unital:

$$\sum E_n^-\left(\begin{pmatrix} l_2 \\ l_2\end{pmatrix}\otimes\begin{pmatrix} l_3 & \dots & l_{n+1} \\ l_3 & \dots & l_{n+1}\end{pmatrix}\right) = \sum q_{l_2}^{4}\begin{pmatrix} l_3 & \dots & l_{n+1} \\ l_3 & \dots & l_{n+1}\end{pmatrix} = 1_{n-1}.$$

The right bimodule property for E_n can be checked as follows:

$$\begin{aligned}
&E_n\left(I_n\begin{pmatrix} j_1 & \dots & j_{n-1} \\ i_1 & \dots & i_{n-1}\end{pmatrix}\begin{pmatrix} J_1 & \dots & J_n \\ I_1 & \dots & I_n\end{pmatrix}\right)\\
&\quad = E_n\left(\sum\begin{pmatrix} j_1 & \dots & j_{n-1} & l \\ i_1 & \dots & i_{n-1} & l\end{pmatrix}\begin{pmatrix} J_1 & \dots & J_n \\ I_1 & \dots & I_n\end{pmatrix}\right)\\
&\quad = \delta_{(j_1\dots j_{n-1})(I_1\dots I_{n-1})}E_n\begin{pmatrix} J_1 & \dots & J_{n-1} & J_n \\ i_1 & \dots & i_{n-1} & I_n\end{pmatrix}\\
&\quad = \delta_{(j_1\dots j_{n-1})(I_1\dots I_{n-1})}\delta_{I_nJ_n}\delta^{-1\mp 1}q_{I_n}^{\mp 4}\begin{pmatrix} J_1 & \dots & J_{n-1} \\ i_1 & \dots & i_{n-1}\end{pmatrix}\\
&\quad = \delta_{I_nJ_n}\delta^{-1\mp 1}q_{I_n}^{\mp 4}\begin{pmatrix} j_1 & \dots & j_{n-1} \\ i_1 & \dots & i_{n-1}\end{pmatrix}\begin{pmatrix} J_1 & \dots & J_{n-1} \\ I_1 & \dots & I_{n-1}\end{pmatrix}\\
&\quad = \begin{pmatrix} j_1 & \dots & j_{n-1} \\ i_1 & \dots & i_{n-1}\end{pmatrix}\widetilde{E}_n\begin{pmatrix} J_1 & \dots & J_n \\ I_1 & \dots & I_n\end{pmatrix}.
\end{aligned}$$

The proof of the other formula $E_n(xI_n(y)) = E_n(x)y$ is similar. For E_n^- we have

$$E_n^-\left(J_n^-\begin{pmatrix} j_3 & \dots & j_{n+1} \\ i_3 & \dots & i_{n+1}\end{pmatrix}\left(\begin{pmatrix} J_2 \\ I_2\end{pmatrix}\otimes\begin{pmatrix} J_3 & \dots & J_{n+1} \\ I_3 & \dots & I_{n+1}\end{pmatrix}\right)\right)$$

$$= E_n^- \left(\sum \begin{pmatrix} g \\ g \end{pmatrix} \begin{pmatrix} J_2 \\ I_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix} \begin{pmatrix} J_3 & \cdots & J_{n+1} \\ I_3 & \ldots & I_{n+1} \end{pmatrix} \right)$$
$$= \delta_{(j_3 \ldots j_{n+1})(I_3 \ldots I_{n+1})} E_n^- \left(\begin{pmatrix} J_2 \\ I_2 \end{pmatrix} \otimes \begin{pmatrix} J_3 & \cdots & J_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix} \right)$$
$$= \delta_{(j_3 \ldots j_{n+1})(I_3 \ldots I_{n+1})} \delta_{I_2 J_2} q_{I_2}^4 \begin{pmatrix} J_3 & \cdots & J_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix}$$
$$= \delta_{I_2 J_2} q_{I_2}^4 \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix} \begin{pmatrix} J_3 & \cdots & J_{n+1} \\ I_3 & \ldots & I_{n+1} \end{pmatrix}$$
$$= \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix} E_n^- \left(\begin{pmatrix} J_2 \\ I_2 \end{pmatrix} \otimes \begin{pmatrix} J_3 & \cdots & J_{n+1} \\ I_3 & \ldots & I_{n+1} \end{pmatrix} \right).$$

The proof of the other formula $E_n^-(xJ_n^-(y)) = E_n^-(x)y$ is similar. For E_n^+ we have

$$E_n^+ \left(J_n^+ \left(\begin{pmatrix} J_2 \\ I_2 \end{pmatrix} \otimes \begin{pmatrix} J_3 & \cdots & J_{n+1} \\ I_3 & \ldots & I_{n+1} \end{pmatrix} \right) \begin{pmatrix} j_1 & \cdots & j_{n+1} \\ i_1 & \ldots & i_{n+1} \end{pmatrix} \right)$$
$$= E_n^+ \left(\sum \begin{pmatrix} h & J_2 & \cdots & J_{n+1} \\ h & I_2 & \ldots & I_{n+1} \end{pmatrix} \begin{pmatrix} j_1 & \cdots & j_{n+1} \\ i_1 & \ldots & i_{n+1} \end{pmatrix} \right)$$
$$= \delta_{(J_2 \ldots J_{n+1})(i_2 \ldots i_{n+1})} E_n^+ \begin{pmatrix} j_1 & j_2 & \cdots & j_{n+1} \\ i_1 & I_2 & \ldots & I_{n+1} \end{pmatrix}$$
$$= \delta_{(J_2 \ldots J_{n+1})(i_2 \ldots i_{n+1})} \delta_{i_1 j_1} \delta^{-2} q_{i_1}^{-4} \begin{pmatrix} j_2 \\ I_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ I_3 & \ldots & I_{n+1} \end{pmatrix}$$
$$= \delta_{i_1 j_1} \delta^{-2} q_{i_1}^{-4} \begin{pmatrix} J_2 \\ I_2 \end{pmatrix} \begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} J_3 & \cdots & J_{n+1} \\ I_3 & \ldots & I_{n+1} \end{pmatrix} \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix}$$
$$= \left(\begin{pmatrix} J_2 \\ I_2 \end{pmatrix} \otimes \begin{pmatrix} J_3 & \cdots & J_{n+1} \\ I_3 & \ldots & I_{n+1} \end{pmatrix} \right) E_n^+ \begin{pmatrix} j_1 & \cdots & j_{n+1} \\ i_1 & \ldots & i_{n+1} \end{pmatrix}.$$

The proof of the other formula $E_n^+(xJ_n^+(y)) = E_n^+(x)y$ is similar. ∎

LEMMA 3.4. *Consider the sequence of elements e_n.*
(i) *We have $e_{2s} = J_{2s-1}J_{2s-3}\ldots J_5J_3(e_2)$ for any s.*
(ii) *We have $e_{2s+1} = J_{2s}J_{2s-2}\ldots J_6J_4J_2^+(d_2)$ for any s, with d_2 given by*

$$d_2 = \sum \delta^{-2} q_i^{-2} q_j^{-2} \begin{pmatrix} j \\ i \end{pmatrix} \otimes \begin{pmatrix} j \\ i \end{pmatrix}.$$

(iii) *The sequence e_n defines a representation of the Temperley-Lieb algebra of modulus δ on the inductive limit of the algebras in the first row of* (I).

Proof. By definition of e_n and J_{n-2} we have

$$e_n = \sum \delta^{-1\pm 1} q_i^{\pm 2} q_j^{\pm 2} \begin{pmatrix} g_1 & \cdots & g_{n-2} & j & j \\ g_1 & \ldots & g_{n-2} & i & i \end{pmatrix} = J_{n-2}(e_{n-2})$$

for any n. Together with

$$e_3 = \sum \delta^{-2} q_i^{-2} q_j^{-2} \begin{pmatrix} g_1 & j & j \\ g_1 & i & i \end{pmatrix} = J_2^+(d_2)$$

this proves (i) and (ii). By using (†) we get that e_{2n+1} is an idempotent:

$$\begin{aligned} e_{2n+1}^2 &= \sum \delta^{-4} q_i^{-2} q_j^{-2} \begin{pmatrix} g_1 & \cdots & g_{2n-1} & j & j \\ g_1 & \cdots & g_{2n-1} & i & i \end{pmatrix} q_I^{-2} q_J^{-2} \begin{pmatrix} G_1 & \cdots & G_{2n-1} & J & J \\ G_1 & \cdots & G_{2n-1} & I & I \end{pmatrix} \\ &= \sum_{j \sim g_{2n-1}} \delta^{-4} q_j^{-4} q_i^{-2} q_J^{-2} \begin{pmatrix} g_1 & \cdots & g_{2n-1} & J & J \\ g_1 & \cdots & g_{2n-1} & i & i \end{pmatrix} = e_{2n+1}. \end{aligned}$$

By using (‡) we get that e_{2n+2} is an idempotent:

$$\begin{aligned} e_{2n+2}^2 &= \sum q_i^2 q_j^2 \begin{pmatrix} g_1 & \cdots & g_{2n} & j & j \\ g_1 & \cdots & g_{2n} & i & i \end{pmatrix} q_I^2 q_J^2 \begin{pmatrix} G_1 & \cdots & G_{2n} & J & J \\ G_1 & \cdots & G_{2n} & I & I \end{pmatrix} \\ &= \sum q_i^2 q_J^2 q_j^4 \begin{pmatrix} g_1 & \cdots & g_{2n} & J & J \\ g_1 & \cdots & g_{2n} & i & i \end{pmatrix} = e_{2n+2}. \end{aligned}$$

The first Jones relation can be verified as follows:

$$\begin{aligned} e_3 \widetilde{I}_3(e_2) e_3 &= \sum \delta^{-4} q_i^{-2} q_k^{-2} \begin{pmatrix} K & k & k \\ K & i & i \end{pmatrix} \sum q_h^2 q_k^2 \begin{pmatrix} h & h & l \\ k & k & l \end{pmatrix} \sum q_h^{-2} q_j^{-2} \begin{pmatrix} H & j & j \\ H & h & h \end{pmatrix} \\ &= \sum \delta^{-4} q_i^{-2} q_l^{-2} q_l^2 q_l^2 q_l^{-2} q_j^{-2} \begin{pmatrix} l & j & j \\ l & i & i \end{pmatrix} = \delta^{-2} e_3. \end{aligned}$$

The other relation is proved in a similar way:

$$\begin{aligned} \widetilde{I}_3(e_2) e_3 \widetilde{I}_3(e_2) &= \sum \delta^{-2} q_k^2 q_i^2 \begin{pmatrix} k & k & K \\ i & i & K \end{pmatrix} \sum q_h^{-2} q_k^{-2} \begin{pmatrix} l & h & h \\ l & k & k \end{pmatrix} \sum q_h^2 q_j^2 \begin{pmatrix} j & j & H \\ h & h & H \end{pmatrix} \\ &= \sum \delta^{-2} q_l^2 q_i^2 q_l^{-2} q_l^{-2} q_l^2 q_j^2 \begin{pmatrix} j & j & l \\ i & i & l \end{pmatrix} = \delta^{-2} \widetilde{I}_3(e_2). \end{aligned}$$

By applying inclusions and shifts we get all Jones relations. ∎

Together with $J_n^+ J_n^- = J_n$ Lemma 3.4 shows that the elements e_n belong to the sequence of algebras obtained by going south-east starting from the algebra $A^{\otimes 2}$ in the first row of (I).

In other words, the Jones projections live at the same places as they do in standard λ-lattices axiomatized by Popa in [24].

In next four lemmas we prove that (I) together with the Jones projections and the bimodule maps in (E) satisfies Popa's axioms, namely the Jones formulae (1.1.2), the Pimsner-Popa formulae (1.3.3″), the commuting square condition (1.1.1) and the commutation relations (2.1.1) in [24]. The Diagram (I) is not a standard λ-lattice in general, because the bimodule maps in (E) are not conditional expectations with respect to some trace.

LEMMA 3.5. *The following equalities hold:*

$$e_{n+2}(I_{n+2}(x))e_{n+2} = (I_{n+2}I_{n+1}E_{n+1}(x))e_{n+2},$$
$$\delta^2(I_{n+2}E_{n+2}(ye_{n+2}))e_{n+2} = ye_{n+2},$$

for any $x \in A^{\otimes n+1}$ and $y \in A^{\otimes n+2}$.

Proof. The first formula follows from the following computation:

$$\begin{aligned}
&e_{n+2}I_{n+2}\begin{pmatrix} j_1 & \dots & j_{n+1} \\ i_1 & \dots & i_{n+1}\end{pmatrix} e_{n+2} \\
&= \sum \delta^{-1\pm1}(q_iq_j)^{\pm2}\begin{pmatrix} g_1 & \dots & g_n & j & j \\ g_1 & \dots & g_n & i & i\end{pmatrix}\begin{pmatrix} j_1 & \dots & j_{n+1} & l \\ i_1 & \dots & i_{n+1} & l\end{pmatrix} e_{n+2} \\
&= \sum \delta^{-1\pm1}(q_iq_{i_{n+1}})^{\pm2}\begin{pmatrix} j_1 & \dots & j_n & j_{n+1} & i_{n+1} \\ i_1 & \dots & i_n & i & i\end{pmatrix} e_{n+2} \\
&= \sum \delta^{-2\pm2}(q_iq_{i_{n+1}}q_Iq_j)^{\pm2}\begin{pmatrix} j_1 & \dots & j_n & j_{n+1} & i_{n+1} \\ i_1 & \dots & i_n & i & i\end{pmatrix}\begin{pmatrix} g_1 & \dots & g_n & j & j \\ g_1 & \dots & g_n & I & I\end{pmatrix} \\
&= \sum \delta^{-2\pm2}(q_iq_{i_{n+1}}^2q_j)^{\pm2}\delta_{i_{n+1}j_{n+1}}\begin{pmatrix} j_1 & \dots & j_n & j & j \\ i_1 & \dots & i_n & i & i\end{pmatrix} \\
&= \sum \delta^{-2\pm2}(q_iq_j)^{\pm2}q_{i_{n+1}}^{\pm4}\delta_{i_{n+1}j_{n+1}}\begin{pmatrix} j_1 & \dots & j_n & l_1 & l_2 \\ i_1 & \dots & i_n & l_1 & l_2\end{pmatrix}\begin{pmatrix} g_1 & \dots & g_n & j & j \\ g_1 & \dots & g_n & i & i\end{pmatrix} \\
&= \left(I_{n+2}I_{n+1}E_{n+1}\begin{pmatrix} j_1 & \dots & j_{n+1} \\ i_1 & \dots & i_{n+1}\end{pmatrix}\right) e_{n+2}.
\end{aligned}$$

The right term in the second formula is given in terms of a basis by:

$$\begin{aligned}
\begin{pmatrix} j_1 & \dots & j_{n+2} \\ i_1 & \dots & i_{n+2}\end{pmatrix} e_{n+2} &= \sum \delta^{-1\pm1}(q_iq_j)^{\pm2}\begin{pmatrix} j_1 & \dots & j_{n+2} \\ i_1 & \dots & i_{n+2}\end{pmatrix}\begin{pmatrix} g_1 & \dots & g_n & j & j \\ g_1 & \dots & g_n & i & i\end{pmatrix} \\
&= \sum \delta^{-1\pm1}(q_{j_{n+1}}q_j)^{\pm2}\delta_{j_{n+1}j_{n+2}}\begin{pmatrix} j_1 & \dots & j_n & j & j \\ i_1 & \dots & i_n & i_{n+1} & i_{n+2}\end{pmatrix}.
\end{aligned}$$

Thus the left term is given by the following formula:

$$\begin{aligned}
&\delta^2 I_{n+2}E_{n+2}\left(\begin{pmatrix} j_1 & \dots & j_{n+2} \\ i_1 & \dots & i_{n+2}\end{pmatrix} e_{n+2}\right) e_{n+2} \\
&= \sum \delta^2\delta^{-1\mp1}\delta^{-1\pm1}(q_{j_{n+1}}q_{i_{n+2}})^{\pm2}\delta_{j_{n+1}j_{n+2}}q_{i_{n+2}}^{\mp4}\begin{pmatrix} j_1 & \dots & j_n & i_{n+2} & l \\ i_1 & \dots & i_n & i_{n+1} & l\end{pmatrix} e_{n+2} \\
&= \sum \delta^{-1\pm1}(q_{j_{n+1}}q_{i_{n+2}}^{-1}q_iq_j)^{\pm2}\delta_{j_{n+1}j_{n+2}}\begin{pmatrix} j_1 & \dots & j_n & i_{n+2} & l \\ i_1 & \dots & i_n & i_{n+1} & l\end{pmatrix}\begin{pmatrix} g_1 & \dots & g_n & j & j \\ g_1 & \dots & g_n & i & i\end{pmatrix} \\
&= \sum \delta^{-1\pm1}(q_{j_{n+1}}q_{i_{n+2}}^{-1}q_{i_{n+2}}q_j)^{\pm2}\delta_{j_{n+1}j_{n+2}}\begin{pmatrix} j_1 & \dots & j_n & j & j \\ i_1 & \dots & i_n & i_{n+1} & i_{n+2}\end{pmatrix}.
\end{aligned}$$

By cancelling $q_{i_{n+2}}^{-1}q_{i_{n+2}}$ this is equal to the right term. ∎

LEMMA 3.6. *The following equalities hold:*

$$f_{n+2}(J^+_{n+1}(x))f_{n+2} = (J^+_{n+1}J^-_{n+1}E^-_{n+1}(x))f_{n+2}\,,$$
$$\delta^2(J^+_{n+1}E^+_{n+1}(yf_{n+2}))f_{n+2} = yf_{n+2}\,,$$

for any $x \in A \otimes A^{\otimes n}$ and $y \in A^{\otimes n+2}$, with $f_{n+2} = I_{n+2}I_{n+1}\dots I_4 I_3(e_2)$.

Proof. The element f_{n+2} is given by

$$f_{n+2} = \sum q_i^2 q_j^2 \begin{pmatrix} j & j & g_3 & \cdots & g_{n+2} \\ i & i & g_3 & \dots & g_{n+2} \end{pmatrix}.$$

The first formula follows from the following computation:

$$\begin{aligned}
f_{n+2}J^+_{n+1} &\left(\begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_{n+2} \\ i_3 & \dots & i_{n+2} \end{pmatrix} \right) f_{n+2} \\
&= \sum q_i^2 q_j^2 \begin{pmatrix} j & j & g_3 & \cdots & g_{n+2} \\ i & i & g_3 & \dots & g_{n+2} \end{pmatrix} \begin{pmatrix} h & j_2 & \cdots & j_{n+2} \\ h & i_2 & \dots & i_{n+2} \end{pmatrix} f_{n+2} \\
&= \sum q_i^2 q_{i_2}^2 \begin{pmatrix} i_2 & j_2 & j_3 & \cdots & j_{n+2} \\ i & i & i_3 & \dots & i_{n+2} \end{pmatrix} f_{n+2} \\
&= \sum q_i^2 q_{i_2}^2 q_I^2 q_j^2 \begin{pmatrix} i_2 & j_2 & j_3 & \cdots & j_{n+2} \\ i & i & i_3 & \dots & i_{n+2} \end{pmatrix} \begin{pmatrix} j & j & g_3 & \cdots & g_{n+2} \\ I & I & g_3 & \dots & g_{n+2} \end{pmatrix} \\
&= \sum q_i^2 q_{i_2}^4 q_j^2 \delta_{i_2 j_2} \begin{pmatrix} j & j & j_3 & \cdots & j_{n+2} \\ i & i & i_3 & \dots & i_{n+2} \end{pmatrix} \\
&= \sum q_i^2 q_{i_2}^4 q_j^2 \delta_{i_2 j_2} \begin{pmatrix} l_1 & l_2 & j_3 & \cdots & j_{n+2} \\ l_1 & l_2 & i_3 & \dots & i_{n+2} \end{pmatrix} \begin{pmatrix} j & j & g_3 & \cdots & g_{n+2} \\ i & i & g_3 & \dots & g_{n+2} \end{pmatrix} \\
&= \left(J^+_{n+1}J^-_{n+1}E^-_{n+1} \left(\begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_{n+2} \\ i_3 & \dots & i_{n+2} \end{pmatrix} \right) \right) f_{n+2}\,.
\end{aligned}$$

The right term in the second formula is given in terms of a basis by

$$\begin{aligned}
\begin{pmatrix} j_1 & \cdots & j_{n+2} \\ i_1 & \dots & i_{n+2} \end{pmatrix} f_{n+2} &= \sum q_i^2 q_j^2 \begin{pmatrix} j_1 & \cdots & j_{n+2} \\ i_1 & \dots & i_{n+2} \end{pmatrix} \begin{pmatrix} j & j & g_3 & \cdots & g_{n+2} \\ i & i & g_3 & \dots & g_{n+2} \end{pmatrix} \\
&= \sum q_{j_1}^2 q_j^2 \delta_{j_1 j_2} \begin{pmatrix} j & j & j_3 & \cdots & j_{n+2} \\ i_1 & i_2 & i_3 & \dots & i_{n+2} \end{pmatrix}.
\end{aligned}$$

Thus the left term is given by the following formula:

$$\begin{aligned}
\delta^2 J^+_{n+1}E^+_{n+1} &\left(\begin{pmatrix} j_1 & \cdots & j_{n+2} \\ i_1 & \dots & i_{n+2} \end{pmatrix} f_{n+2} \right) f_{n+2} \\
&= \sum q_{j_1}^2 q_{i_1}^2 \delta_{j_1 j_2} q_{i_1}^{-4} \begin{pmatrix} h & i_1 & j_3 & \cdots & j_{n+2} \\ h & i_2 & i_3 & \dots & i_{n+2} \end{pmatrix} f_{n+2}
\end{aligned}$$

$$= \sum q_{j_1}^2 q_{i_1}^2 q_i^2 q_j^2 \delta_{j_1 j_2} q_{i_1}^{-4} \begin{pmatrix} h & i_1 & j_3 & \dots & j_{n+2} \\ h & i_2 & i_3 & \dots & i_{n+2} \end{pmatrix} \begin{pmatrix} j & j & g_3 & \dots & g_{n+2} \\ i & i & g_3 & \dots & g_{n+2} \end{pmatrix}$$
$$= \sum q_{j_1}^2 q_{i_1}^4 q_j^2 \delta_{j_1 j_2} q_{i_1}^{-4} \begin{pmatrix} j & j & j_3 & \dots & j_{n+2} \\ i_1 & i_2 & i_3 & \dots & i_{n+2} \end{pmatrix}.$$

By cancelling $q_{i_1}^4 q_{i_1}^{-4}$ this is equal to the right term. ∎

LEMMA 3.7. *The following equalities hold:*

$$d_{n+2}(J_{n+2}^-(x))d_{n+2} = (J_{n+2}^- J_n^+ E_n^+(x))d_{n+2}\,,$$
$$\delta^2 (J_{n+2}^- E_{n+2}^-(y d_{n+2}))d_{n+2} = y d_{n+2}\,,$$

for any $x \in A^{\otimes n+1}$ and $y \in A \otimes A^{\otimes n+1}$, with $d_{n+2} = (\mathrm{id} \otimes I_{n+1} I_n \dots I_3 I_2)(d_2)$.

Proof. The element d_{n+2} is given by

$$d_{n+2} = \sum \delta^{-2} q_i^{-2} q_j^{-2} \begin{pmatrix} j \\ i \end{pmatrix} \otimes \begin{pmatrix} j & g_3 & \dots & g_{n+2} \\ i & g_3 & \dots & g_{n+2} \end{pmatrix}.$$

The first formula follows from the following computation:

$$d_{n+2} J_{n+2}^- \begin{pmatrix} j_2 & \dots & j_{n+2} \\ i_2 & \dots & i_{n+2} \end{pmatrix} d_{n+2}$$
$$= \sum \delta^{-2} q_i^{-2} q_j^{-2} \left(\begin{pmatrix} g \\ g \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} \otimes \begin{pmatrix} j & g_3 & \dots & g_{n+2} \\ i & g_3 & \dots & g_{n+2} \end{pmatrix} \begin{pmatrix} j_2 & \dots & j_{n+2} \\ i_2 & \dots & i_{n+2} \end{pmatrix} \right) d_{n+2}$$
$$= \sum \delta^{-2} q_i^{-2} q_{i_2}^{-2} \left(\begin{pmatrix} i_2 \\ i \end{pmatrix} \otimes \begin{pmatrix} j_2 & j_3 & \dots & j_{n+2} \\ i & i_3 & \dots & i_{n+2} \end{pmatrix} \right) d_{n+2}$$
$$= \sum \delta^{-4} q_i^{-2} q_{i_2}^{-2} q_I^{-2} q_j^{-2} \begin{pmatrix} i_2 \\ i \end{pmatrix} \begin{pmatrix} j \\ I \end{pmatrix} \otimes \begin{pmatrix} j_2 & j_3 & \dots & j_{n+2} \\ i & i_3 & \dots & i_{n+2} \end{pmatrix} \begin{pmatrix} j & g_3 & \dots & g_{n+2} \\ I & g_3 & \dots & g_{n+2} \end{pmatrix}$$
$$= \sum \delta^{-4} q_i^{-2} q_{i_2}^{-4} q_j^{-2} \begin{pmatrix} j \\ i \end{pmatrix} \otimes \begin{pmatrix} j & j_3 & \dots & j_{n+2} \\ i & i_3 & \dots & i_{n+2} \end{pmatrix}$$
$$= \sum \delta^{-4} q_i^{-2} q_{i_2}^{-4} q_j^{-2} \begin{pmatrix} l_1 \\ l_1 \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} \otimes \begin{pmatrix} l_2 & j_3 & \dots & j_{n+2} \\ l_2 & i_3 & \dots & i_{n+2} \end{pmatrix} \begin{pmatrix} j & g_3 & \dots & g_{n+2} \\ i & g_3 & \dots & g_{n+2} \end{pmatrix}$$
$$= \left(J_{n+2}^- J_n^+ E_n^+ \begin{pmatrix} j_2 & \dots & j_{n+2} \\ i_2 & \dots & i_{n+2} \end{pmatrix} \right) d_{n+2}\,.$$

The right term in the second formula is given in terms of a basis by

$$\left(\begin{pmatrix} j_1 \\ i_1 \end{pmatrix} \otimes \begin{pmatrix} j_2 & \dots & j_{n+2} \\ i_2 & \dots & i_{n+2} \end{pmatrix} \right) d_{n+2}$$
$$= \sum \delta^{-2} q_i^{-2} q_j^{-2} \begin{pmatrix} j_1 \\ i_1 \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} \otimes \begin{pmatrix} j_2 & \dots & j_{n+2} \\ i_2 & \dots & i_{n+2} \end{pmatrix} \begin{pmatrix} j & g_3 & \dots & g_{n+2} \\ i & g_3 & \dots & g_{n+2} \end{pmatrix}$$
$$= \sum \delta^{-2} q_{j_1}^{-2} q_j^{-2} \delta_{j_1 j_2} \begin{pmatrix} j \\ i_1 \end{pmatrix} \otimes \begin{pmatrix} j & j_3 & \dots & j_{n+2} \\ i_2 & i_3 & \dots & i_{n+2} \end{pmatrix}.$$

Thus the left term is given by the following formula:

$$\delta^2 J_{n+2}^- E_{n+2}^- \left(\left(\left(\begin{pmatrix} j_1 \\ i_1 \end{pmatrix} \otimes \begin{pmatrix} j_2 & \cdots & j_{n+2} \\ i_2 & \dots & i_{n+2} \end{pmatrix} \right) d_{n+2} \right) d_{n+2} \right.$$

$$= \sum q_{j_1}^{-2} q_{i_1}^{-2} \delta_{j_1 j_2} q_{i_1}^4 \left(\begin{pmatrix} g \\ g \end{pmatrix} \otimes \begin{pmatrix} i_1 & j_3 & \cdots & j_{n+2} \\ i_2 & i_3 & \dots & i_{n+2} \end{pmatrix} \right) d_{n+2}$$

$$= \sum \delta^{-2} q_{j_1}^{-2} q_{i_1}^{-2} \delta_{j_1 j_2} q_{i_1}^4 q_i^{-2} q_j^{-2} \begin{pmatrix} g \\ g \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} \otimes \begin{pmatrix} i_1 & j_3 & \cdots & j_{n+2} \\ i_2 & i_3 & \dots & i_{n+2} \end{pmatrix} \begin{pmatrix} j & g_3 & \cdots & g_{n+2} \\ i & g_3 & \dots & g_{n+2} \end{pmatrix}$$

$$= \sum \delta^{-2} q_{j_1}^{-2} q_{i_1}^{-4} \delta_{j_1 j_2} q_{i_1}^4 q_j^{-2} \begin{pmatrix} j \\ i_1 \end{pmatrix} \otimes \begin{pmatrix} j & j_3 & \cdots & j_{n+2} \\ i_2 & i_3 & \dots & i_{n+2} \end{pmatrix}.$$

By cancelling $q_{i_1}^{-4} q_{i_1}^4$ this is equal to the right term. ∎

LEMMA 3.8. (i) *The diagram obtained from* (I) *by replacing its vertical rows with the vertical rows of* (E) *commutes.*

(ii) *For any rectangular subdiagram of* (I) *having* $\mathbb{C}$ *in the south-west corner, the algebra in the north-west corner commutes with the algebra in the south-east corner.*

Proof. (i) Let (IE) be this Diagram. The commutation in its first row follow from

$$\begin{array}{ccc} \begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \dots & i_n \end{pmatrix} & \xrightarrow{I_{n+1}} & \sum \begin{pmatrix} j_1 & \cdots & j_n & l \\ i_1 & \dots & i_n & l \end{pmatrix} \\ \downarrow E_{n-1}^+ & & \downarrow E_n^+ \\ \delta_{i_1 j_1} \delta^{-2} q_{i_1}^{-4} \begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_n \\ i_3 & \dots & i_n \end{pmatrix} & \xrightarrow{\mathrm{id} \otimes I_{n-1}} & \sum \delta_{i_1 j_1} \delta^{-2} q_{i_1}^{-4} \begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_n & l \\ i_3 & \dots & i_n & l \end{pmatrix}. \end{array}$$

The commutation in the second row of (IE) follow from

$$\begin{array}{ccc} \begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} & \xrightarrow{\mathrm{id} \otimes I_n} & \sum \begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_{n+1} & l \\ i_3 & \dots & i_{n+1} & l \end{pmatrix} \\ \downarrow E_n^- & & \downarrow E_{n+1}^- \\ \delta_{i_2 j_2} q_{i_2}^4 \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} & \xrightarrow{I_n} & \sum \delta_{i_2 j_2} q_{i_2}^4 \begin{pmatrix} j_3 & \cdots & j_{n+1} & l \\ i_3 & \dots & i_{n+1} & l \end{pmatrix}. \end{array}$$

Both Diagrams (I) and (E) being 2-periodic on the vertical, (IE) is 2-periodic as well on the vertical, so it commutes.

(ii) Let $a < b$ be the ranks of the lines of (I) containing the north and south vertices of the rectangle. For b odd the rectangle is of the form

$$\begin{array}{ccc} A^{\otimes 2s} & \xrightarrow{I} & A^{\otimes 2s+k} \\ \uparrow & & \uparrow J \\ \mathbb{C} & \longrightarrow & A^{\otimes k} \end{array}, \qquad \begin{array}{ccc} A \otimes A^{\otimes 2s} & \xrightarrow{\mathrm{id} \otimes I} & A \otimes A^{\otimes 2s+k} \\ \uparrow & & \uparrow J_{2s+k+1}^- J \\ \mathbb{C} & \longrightarrow & A^{\otimes k} \end{array},$$

depending on the parity of a, with I and J given by

$$I = I_{2s+k} I_{2s+k-1} \dots I_{2s+2} I_{2s+1}, \quad J = J_{2s+k-1} J_{2s+k-3} \cdots J_{k+3} J_{k+1},$$

The corresponding images are:

$$\mathrm{Im}(I)=\left\{\sum\lambda\begin{pmatrix}j_1&\dots&j_{2s}&l_1&\dots&l_k\\ i_1&\dots&i_{2s}&l_1&\dots&l_k\end{pmatrix}\right\},$$

$$\mathrm{Im}(J)=\left\{\sum\lambda\begin{pmatrix}g_1&\dots&g_{2s}&j_1&\dots&j_k\\ g_1&\dots&g_{2s}&i_1&\dots&i_k\end{pmatrix}\right\},$$

and $\mathrm{Im}(\mathrm{id}\otimes I)=1\otimes\mathrm{Im}(I)$ and $\mathrm{Im}(J^-_{2s+k+1}J)=1\otimes\mathrm{Im}(J)$, so commutation is clear. For b even the rectangle is of the form

$$\begin{array}{ccc} A^{\otimes 2s+1} & \xrightarrow{I} & A^{\otimes 2s+k+2}\\ \uparrow & & \uparrow J\\ \mathbb{C} & \longrightarrow & A\otimes A^{\otimes k} \end{array},\qquad \begin{array}{ccc} A\otimes A^{\otimes 2s+1} & \xrightarrow{\mathrm{id}\otimes I} & A\otimes A^{\otimes 2s+k+2}\\ \uparrow & & \uparrow J^-_{2s+k+3}J\\ \mathbb{C} & \longrightarrow & A\otimes A^{\otimes k} \end{array},$$

depending on the parity of a, with I and J given by

$$I=I_{2s+k+1}I_{2s+k}\dots I_{2s+3}I_{2s+2},\quad J=J_{2s+k+1}J_{2s+k-1}\dots J_{k+5}J_{k+3}J^+_{k+1},$$

The corresponding images are

$$\mathrm{Im}(I)=\left\{\sum\lambda\begin{pmatrix}j_1&\dots&j_{2s}&j_{2s+1}&l_1&\dots&l_{k+1}\\ i_1&\dots&i_{2s}&i_{2s+1}&l_1&\dots&l_{k+1}\end{pmatrix}\right\},$$

$$\mathrm{Im}(J)=\left\{\sum\lambda\begin{pmatrix}g_1&\dots&g_{2s}&h&j_2&\dots&j_{k+2}\\ g_1&\dots&g_{2s}&h&i_2&\dots&i_{k+2}\end{pmatrix}\right\},$$

and $\mathrm{Im}(\mathrm{id}\otimes I)=1\otimes\mathrm{Im}(I)$ and $\mathrm{Im}(J^-_{2s+k+3}J)=1\otimes\mathrm{Im}(J)$, so commutation is clear. ∎

Define a linear form ψ on A by

$$\psi\begin{pmatrix}j\\ i\end{pmatrix}=\delta_{ij}p_i^4$$

where p_i are the following positive numbers:

$$p_i=\delta^{-\frac{1}{2}}q_i^{-1}\Big(\sum_{l\sim i}q_l^4\Big)^{\frac{1}{4}}.$$

By using (†) and (‡) we get

$$\sum p_i^4=\sum_{i\sim l}\delta^{-2}q_i^{-4}q_l^4=\sum q_l^4=1.$$

This Formula will be called (‡) for p_i's. There is also a corresponding (†) formula:

$$\sum_{i\sim k}p_i^{-4}=\sum_{i\sim k}\delta^2q_i^4\Big(\sum_{l\sim k}q_l^4\Big)^{-1}=\delta^2.$$

Next lemma shows that the linear forms φ_n define a filtered linear form φ_∞, which fails to commute globally with vertical maps in (E) because φ and ψ are not equal in general. In fact $\varphi=\psi$ if and only if $q_i=p_i$ and the above formula for the numbers p_i shows that this happens if and only if φ is the Perron-Frobenius trace.

LEMMA 3.9. *Consider the Diagram* (I) *in Lemma* 3.1.

(i) *The linear maps* φ_n *in Proposition* 3.1 *define a filtered unital linear form* φ_∞ *on the sequence of algebras in the first row of* (I).

(ii) *The restriction of* φ_∞ *to the algebra* $A \otimes A^{\otimes n-1}$ *in the second row is* $\psi \otimes \varphi_{n-1}$.

(iii) *The diagram of restrictions of* φ_∞ *is 2-periodic on the vertical.*

(iv) *The restrictions of* φ_∞ *commute with the horizontal maps in* (E).

(v) *We have* $\varphi_{n-1}E_n^- E_n^+ = \psi_2 E_{2n}$*, where* $E_{2n} = E_3 E_4 \dots E_{n-1} E_n$.

(vi) *We have* $(\psi \otimes \varphi_{n-1})E_n^+ = \psi_2 I_2 E_{1n}$*, where* $E_{1n} = E_2 E_{2n}$.

Proof. (i) It is enough to verify the equality $\varphi_{n-1}E_n = \varphi_n$:

$$\begin{aligned}\varphi_{n-1}E_n \begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \dots & i_n \end{pmatrix} &= \delta_{i_n j_n}\delta^{-1\mp 1} q_{i_n}^{\mp 4} \varphi_{n-1} \begin{pmatrix} j_1 & \cdots & j_{n-1} \\ i_1 & \dots & i_{n-1} \end{pmatrix} \\ &= \delta_{i_n j_n}\delta^{-1\mp 1+\frac12 \pm \frac12 - n+1} q_{i_n}^{\mp 4}\delta_{(i_1\dots i_{n-1})(j_1\dots j_{n-1})} q^4_{(i_1\dots i_{n-1})} \\ &= \delta_{(i_1\dots i_n)(j_1\dots j_n)}\delta^{\frac12\mp\frac12 - n} q^4_{(i_1\dots i_n)} = \varphi_n \begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \dots & i_n \end{pmatrix}.\end{aligned}$$

(ii) is verified as follows:

$$\begin{aligned}&\varphi_{n+1}J_n^+ \left(\begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} \right) \\ &\qquad = \sum \varphi_{n+1} \begin{pmatrix} h & j_2 & \cdots & j_{n+1} \\ h & i_2 & \dots & i_{n+1} \end{pmatrix} \\ &\qquad = \sum_{h\sim i_2} \delta^{\frac12 \pm \frac12 - n-1}\delta_{(i_2\dots i_{n+1})(j_2\dots j_{n+1})} q_h^4 q_{i_2}^{-4} q^4_{(i_3\dots i_{n+1})} \\ &\qquad = \delta_{i_2 j_2} p_{i_2}^4 \delta^{\frac12\pm\frac12 - n+1}\delta_{(i_3\dots i_{n+1})(j_3\dots j_{n+1})} q^4_{(i_3\dots i_{n+1})} \\ &\qquad = (\psi\otimes\varphi_{n-1}) \left(\begin{pmatrix} j_2 \\ i_2 \end{pmatrix} \otimes \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} \right).\end{aligned}$$

(iii) It is enough to show that the restrictions of φ_∞ to the algebras in the third row are the linear forms φ_n. This follows from the equality $J_n^+ J_n^- = J_n$ in Lemma 3.1 and from the Formulae (†) and (‡):

$$\begin{aligned}\varphi_{n+1}J_n \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} &= \varphi_{n+1} \begin{pmatrix} h & g & j_3 & \cdots & j_{n+1} \\ h & g & i_3 & \dots & i_{n+1} \end{pmatrix} \\ &= \sum_{g\sim h} q_h^4 q_g^{-4}\delta^{-2}\varphi_{n-1} \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix} \\ &= \sum q_h^4 \varphi_{n-1} \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix}\end{aligned}$$

$$= \varphi_{n-1} \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \dots & i_{n+1} \end{pmatrix}.$$

(iv) From proof of (i) we know that the restrictions of φ_∞ to the algebras in the first row commute with the bimodule morphisms. By tensoring everything to the left with id we get the assertion for the second row. By vertical 2-periodicity of everything this extends to the whole diagram.

(v) The map E_{2n} is given by the following formula:

$$E_{2n}\begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \ldots & i_n \end{pmatrix} = \begin{pmatrix} j_1 & j_2 \\ i_1 & i_2 \end{pmatrix} \varphi_{n-2}\begin{pmatrix} j_3 & \cdots & j_n \\ i_3 & \ldots & i_n \end{pmatrix}.$$

The map $\varphi_{n-1}E_n^- E_n^+$ is given by

$$\varphi_{n-1}E_n^- E_n^+\begin{pmatrix} j_1 & \cdots & j_{n+1} \\ i_1 & \ldots & i_{n+1} \end{pmatrix} = \delta_{i_1j_1}\delta_{i_2j_2}\delta^{-2}q_{i_1}^{-4}q_{i_2}^4\varphi_{n-1}\begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix}$$

and together with the definition of ψ_2 before Proposition 3.1 this proves (v).

(vi) From the defining formulae of ψ and E_n^+ we get

$$\begin{aligned}(\psi\otimes\varphi_{n-1})E_n^+\begin{pmatrix} j_1 & \cdots & j_{n+1} \\ i_1 & \ldots & i_{n+1} \end{pmatrix} &= \delta_{i_1j_1}\delta^{-2}q_{i_1}^{-4}\psi\begin{pmatrix} j_2 \\ i_2 \end{pmatrix}\varphi_{n-1}\begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix} \\ &= \delta_{i_1j_1}\delta_{i_2j_2}\delta^{-2}q_{i_1}^{-4}p_{i_2}^4\varphi_{n-1}\begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix}.\end{aligned}$$

From $i_1 \sim i_2$ and from the definition of numbers p_i we get

$$p_{i_1}^4 q_{i_1}^4 = \delta^{-2}\sum_{l\sim i_1} q_l^4 = \delta^{-2}\sum_{l\sim i_2} q_l^4 = p_{i_2}^4 q_{i_2}^4 .$$

By replacing in the above formula $q_{i_1}^{-4}p_{i_2}^4$ by $q_{i_2}^{-4}p_{i_1}^4$ we get

$$(\psi\otimes\varphi_{n-1})E_n^+\begin{pmatrix} j_1 & \cdots & j_{n+1} \\ i_1 & \ldots & i_{n+1} \end{pmatrix} = \delta_{i_1j_1}\delta_{i_2j_2}\delta^{-2}p_{i_1}^4 q_{i_2}^{-4}\varphi_{n-1}\begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \ldots & i_{n+1} \end{pmatrix}.$$

From the definition of numbers p_i we get

$$\psi_2 I_2\begin{pmatrix} j_1 \\ i_1 \end{pmatrix} = \sum \psi_2\begin{pmatrix} j_1 & l \\ i_1 & l \end{pmatrix} = \sum_{l\sim i_1}\delta_{i_1j_1}\delta^{-2}q_{i_1}^{-4}q_l^4 = \delta_{i_1j_1}p_{i_1}^4.$$

The map E_{1n} is given by

$$E_{1n}\begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \ldots & i_n \end{pmatrix} = \delta_{i_2j_2}\delta^{-2}q_{i_2}^{-4}\begin{pmatrix} j_1 \\ i_1 \end{pmatrix}\varphi_{n-2}\begin{pmatrix} j_3 & \cdots & j_n \\ i_3 & \ldots & i_n \end{pmatrix}$$

so the composition $\psi_2 I_2 E_{1n}$ is given by the same formula as $(\psi\otimes\varphi_{n-1})E_n^+$. ∎

Proof of Proposition 3.1. Let $Q_n \subset A^{\otimes n}$ be a sequence of C^*-algebras satisfying conditions (i), (ii), (iii) in Proposition 3.1. Define $R_n = E_n^+(Q_{n+1})$ and consider the following Diagram ($\star$):

$$\begin{array}{ccccccccc}
\mathbb{C} & \subset & Q_1 & \subset & Q_2 & \subset & Q_3 & \subset & \cdots \\
 & & \cup & & \cup & & \cup & & \\
 & & \mathbb{C} & \subset & R_1 & \subset & R_2 & \subset & \cdots \\
 & & & & \cup & & \cup & & \\
 & & & & \mathbb{C} & \subset & Q_1 & \subset & \cdots \\
 & & & & & & \cup & & \\
 & & & & & & \cdots & & \cdots
\end{array}$$

We claim that this is a subsystem of C^*-algebras of the Diagram (I) in Lemma 3.3.

First, the map E_n^+ being an involutive bimodule morphism, R_n is a C^*-algebra. The other thing is to verify that all inclusions in the statement make sense. The assumption $I_n(Q_{n-1}) \subset Q_n$ in (1) justifies the inclusions in the first row. The bimodule property of E_n^+ shows that R_n is included in Q_{n+1} via J_n^+, so the first row of vertical inclusions is the good one as well. The commuting square property in Lemma 3.9 (i) justifies the second row of horizontal inclusions. From condition $J_n(Q_{n-1}) \subset Q_{n+1}$ in (1) and from $J_n^+ J_n^- = J_n$ in Lemma 3.3 and $E_n^+ J_n^+ = \mathrm{id}$ in Lemma 3.4 we get

$$J_n^-(Q_{n-1}) = E_n^+ J_n^+ J_n^-(Q_{n-1}) = E_n^+ J_n(Q_{n-1}) \subset E_n^-(Q_{n+1}) = R_n$$

so the second row of vertical inclusions is the good one. By vertical 2-periodicity we get that ($\star$) is a subsystem of (I).

Condition (2) and the definition of φ_∞ show that the restrictions of φ_∞ to the algebras in ($\star$) are traces. We prove now that ($\star$), together with φ_∞ and with the Jones projections in Lemma 3.4 is a standard λ-lattice, with $\lambda = \delta^2$.

For, it is enough to verify the commutation of φ_∞ with the maps in (E), cf. discussion preceding Lemma 3.5. Commutation with horizontal maps follows from Lemma 3.9 (iv). By using vertical 2-periodicity of everything, it remains to prove that the restrictions of φ_∞ commute with the maps of the form E_n^+ and E_n^- in the first two rows of vertical maps of (E).

From Lemma 3.9 (iv) we get $\varphi_{n+1} = \varphi_2 E_{2n} = \varphi E_{1n}$. Lemma 3.9 (ii) shows that the commutation of φ_∞ with E_n^+ is equivalent to the equality $(\psi \otimes \varphi_{n-1})E_n^+ = \varphi_{n+1}$. This follows from Lemma 3.9 (vi), from $\varphi_{n+1} = \varphi E_{1n}$ and from condition (3). The formulae for restrictions φ_∞ in Lemma 3.9 (i), (ii), (iii) show that their commutation with E_n^- is equivalent to the equality $\varphi_{n-1}E_n^- = \psi \otimes \varphi_{n-1}$. This must be true on the image of E_n^+, so is equivalent to the equality $\varphi_{n-1}E_n^- E_n^+ = (\psi \otimes \varphi_{n-1})E_n^+$ obtained by composing with E_n^+. This follows from Lemma 3.9 (v), from $\varphi_{n+1} = \varphi_2 E_{2n}$ and from condition (3).

Thus ($\star$) is a standard λ-lattice with $\lambda = \delta^2$. The *bubbling* construction of Jones in [16] applies and proves Proposition 3.1. ∎

4. THE PLANAR ALGEBRA OF A COACTION — TWISTED CASE

Let H be a Hopf $*$-algebra as in Section 1. Associated to any complex number z is a multiplicative functional $f_z : H \to \mathbb{C}$ such that the following equalities hold (Theorem 5.6 in [29]):

(f1) $f_0 = \varepsilon$ and $(f_z \otimes f_t)\Delta = f_{z+t}$ for any z, t.
(f2) $S^2 = (f_1 \otimes \mathrm{id} \otimes f_{-1})\Delta^{(2)}$, where $\Delta^{(2)} = (\mathrm{id} \otimes \Delta)\Delta$.
(f3) $f_z S = f_{-z}$ and $f_z * = \overline{f}_{-\bar{z}}$ for any z.
(f4) $\sigma = (f_1 \otimes \mathrm{id} \otimes f_1)\Delta^{(2)}$ satisfies $h(ab) = h(b\sigma(a))$ for any a, b.

Let A be a finite dimensional C^*-algebra. Write A as a direct sum of complex matrix algebras and let Tr be the trace of A which on matrix subalgebras is the usual trace of matrices.

LEMMA 4.1. *If $v : A \to A \otimes H$ is a coaction there exists a unique $Q \in A$ satisfying the following conditions:*
(i) $\mathrm{ad}(Q) = (\mathrm{id} \otimes f_{\frac{1}{4}})v$;
(ii) $Q > 0$;
(iii) $\mathrm{Tr}(Q^4) = 1$;
(iv) *the numbers* $\mathrm{Tr}(B^{-4})$ *with B matrix block of Q are all equal.*

Proof. For any real number z consider the linear map $\rho_z = (\mathrm{id} \otimes f_z)v$. Since both f_z and v are multiplicative, this is an automorphism of the complex algebra A. By applying (f1) we get $\rho_0 = \mathrm{id}$ and

$$\begin{aligned}\rho_{z+t} &= (\mathrm{id} \otimes f_z \otimes f_t)(\mathrm{id} \otimes \Delta)v = (\mathrm{id} \otimes f_z \otimes f_t)(v \otimes \mathrm{id})v \\ &= (\mathrm{id} \otimes f_z)v(\mathrm{id} \otimes f_t)v = \rho_z\rho_t.\end{aligned}$$

This shows that ρ_z has n-th roots for any n. But ρ_z must leave invariant the center $Z(A)$ of A, so its restriction is an automorphism of $Z(A)$ having n-th roots for any n. This is not possible if the restriction is not the identity. Since ρ_z preserves the central minimal idempotents of A, it has to preserve the matrix blocks. But on matrix algebras automorphisms are inner, so ρ_z is inner. Choose $Q_z \in A$ such that ρ_z is equal to $\mathrm{ad}(Q_z) = Q_z \cdot Q_z^{-1}$. By using the second equality in (f3) with z real we get

$$\rho_{-z} = (\mathrm{id} \otimes \overline{f}_z)(\mathrm{id} \otimes *)v = *(\mathrm{id} \otimes f_z)(* \otimes *)v = *(\mathrm{id} \otimes f_z)v* = *\rho_z*$$

and together with $\rho_z\rho_{-z} = \rho_0 = \mathrm{id}$ this gives $\rho_z * \rho_z * = \mathrm{id}$. On the other hand

$$\rho_z * \rho_z * (a) = Q_z(Q_z a^* Q_z^{-1})^* Q_z^{-1} = \mathrm{ad}((Q_z^* Q_z^{-1})^{-1})(a)$$

so $Q_z^* Q_z^{-1}$ is in $Z(A)$. Let B be a matrix block of Q_z and let λ be a complex number such that $B^* = \lambda B$. By applying $*$ we get $B = \overline{\lambda} B^*$ and by combining these two formulae we get $B = \lambda\overline{\lambda}B$, so λ is of modulus one. Choose a half root $\lambda^{\frac{1}{2}}$ of λ and let $B' = \lambda^{\frac{1}{2}} B$. Then

$$(B')^* = \overline{\lambda}^{\frac{1}{2}} B^* = \lambda^{-\frac{1}{2}} \lambda B = \lambda^{\frac{1}{2}} B = B'.$$

Rescale in this way all blocks of Q_z such that they become self-adjoint. We have $\rho_z = \mathrm{ad}(Q_z)$ and $Q_z = Q_z^*$ for any z. Let $Q = Q_{\frac{1}{8}}^2$. Then Q is positive and

$$\mathrm{ad}(Q) = \mathrm{ad}(Q_{\frac{1}{8}})^2 = \rho_{\frac{1}{8}}^2 = \rho_{\frac{1}{4}} = (\mathrm{id} \otimes f_{\frac{1}{4}})v.$$

We can rescale all blocks of Q such that (iv) holds, then rescale Q such that (iii) holds.

For the converse, if Q' is another element satisfying all conditions in the statement then (i) shows that $Q'Q^{-1}$ is central, so if $Q = (B_i)$ is a decomposition of Q then Q' must be of the form $(\lambda_i B_i)$. From (ii) we get that $\lambda_i > 0$, then from (iv) we get that the λ_i's are equal, and finally from (iii) we get that they are all equal to 1. ∎

Choose a system of matrix units $X \subset A$ such that the element $Q \in A$ in Lemma 4.1 is diagonal, with eigenvalues q_i. Let δ be the square root of the numbers in Lemma 4.1 (iv). Then $\varphi = \mathrm{Tr}(Q^4 .)$ is a δ-form (see Section 3).

By using boxes instead of discs as in Jones' paper [16], we say that a tangle in $\mathcal{P}$ is *vertical* if it can be isotoped to a tangle all whose strings are parallel to the y-axis.

Consider the planar algebra $P(A)$ associated to the bipartite graph of the inclusion $\mathbb{C} \subset A$, with Perron-Frobenius spin vector. See Jones [14].

THEOREM 4.2. *Let $v : A \to A \otimes H$ be a coaction. Assume that v preserves the linear form $\varphi = \mathrm{Tr}(Q^4 \cdot)$ with $Q \in A$ given by Lemma 4.1. There exists a unique C^*-planar algebra structure $Q(v)$ on the sequence of spaces of fixed points of v_n such that*

(i) *For any vertical tangle $T \in \mathcal{P}$ the multilinear map of $Q(v)$ associated to T is the restriction of the multilinear map of $P(A)$ associated to T.*

(ii) *The Jones projections are given by*

$$e_n = \sum \delta^{-1 \pm 1} q_i^{\pm 2} q_j^{\pm 2} \begin{pmatrix} g_1 & \cdots & g_{n-2} & j & j \\ g_1 & \cdots & g_{n-2} & i & i \end{pmatrix}.$$

This C^*-planar algebra is spherical and of modulus δ.

Proof. Uniqueness follows from the fact that $\mathcal{P}$ is generated by vertical tangles and Jones projections (see Section 2). It remains to verify conditions (i), (ii), (iii) in Proposition 3.1.

(i) By using Lemma 2.2 and Lemma 2.3, it is enough to check the modularity condition in Lemma 2.3. The formula of θ in Section 2 gives

$$\theta \begin{pmatrix} j \\ i \end{pmatrix} = q_i^4 q_j^{-4} \begin{pmatrix} j \\ i \end{pmatrix} = \mathrm{ad}(Q)^4 \begin{pmatrix} j \\ i \end{pmatrix} = (\mathrm{id} \otimes f_1) v \begin{pmatrix} j \\ i \end{pmatrix}.$$

By using twice the axiom for coactions $(\mathrm{id}\otimes\Delta)v=(v\otimes\mathrm{id})v$ we get

$$(\mathrm{id}\otimes\Delta^{(2)})v=(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(v\otimes\mathrm{id})v=(v\otimes\mathrm{id}\otimes\mathrm{id})(v\otimes\mathrm{id})v.$$

By using (f4) we get that the modularity condition is satisfied:

$$\begin{aligned}(\mathrm{id}\otimes\sigma)v&=(\mathrm{id}\otimes f_1\otimes\mathrm{id}\otimes f_1)(v\otimes\mathrm{id}\otimes\mathrm{id})(v\otimes\mathrm{id})v\\&=(((f_1\otimes\mathrm{id})v)\otimes\mathrm{id})v(\mathrm{id}\otimes f_1)v=(\theta\otimes\mathrm{id})v\theta\,.\end{aligned}$$

(ii) This will follow from $\theta_n=(\mathrm{id}\otimes f_1)v_n$. In terms of the basis, the map $(\mathrm{id}\otimes f_1)v_n$ is

$$\begin{aligned}(\mathrm{id}\otimes f_1)v_n\begin{pmatrix}j_1&\dots&j_n\\i_1&\dots&i_n\end{pmatrix}&=\sum\begin{pmatrix}l_1&\dots&l_n\\k_1&\dots&k_n\end{pmatrix}q^{-1}_{(k_1\dots k_n)}q_{(i_1\dots i_n)}q_{(j_1\dots j_n)}q^{-1}_{(l_1\dots l_n)}\\&\qquad\cdot\ f_1V_n\begin{pmatrix}l_1&\dots&l_n&j_1&\dots&j_n\\k_1&\dots&k_n&i_1&\dots&i_n\end{pmatrix},\end{aligned}$$

so condition $\theta_n=(\mathrm{id}\otimes f_1)v_n$ is equivalent to the following Condition (f1) for V_n:

$$f_1V_n\begin{pmatrix}l_1&\dots&l_n&j_1&\dots&j_n\\k_1&\dots&k_n&i_1&\dots&i_n\end{pmatrix}=\delta_{\left(\begin{smallmatrix}j_1&\dots&j_n\\i_1&\dots&i_n\end{smallmatrix}\right)\left(\begin{smallmatrix}l_1&\dots&l_n\\k_1&\dots&k_n\end{smallmatrix}\right)}\ q^4_{(i_1\dots i_n)}q^{-4}_{(j_1\dots j_n)}\,.$$

Since θ_1 is the modular map of φ, this is true for $n=1$:

$$f_1V\begin{pmatrix}l&j\\k&i\end{pmatrix}=\delta_{ki}\delta_{jl}\ q_i^4q_j^{-4}.$$

Since f_1 is multiplicative, from (f1) for V we get (f1) for V_2:

$$\begin{aligned}f_1V_2\begin{pmatrix}l_1&l_2&j_1&j_2\\k_1&k_2&i_1&i_2\end{pmatrix}&=f_1\begin{pmatrix}k_2&i_2\\k_1&i_1\end{pmatrix}f_1\begin{pmatrix}l_1&j_1\\l_2&j_2\end{pmatrix}\\&=\delta_{k_1i_1}\delta_{k_2i_2}q^4_{i_1}q^{-4}_{i_2}\delta_{l_2j_2}\delta_{l_1j_1}q^4_{j_2}q^{-4}_{j_1}\\&=\delta_{(k_1k_2)(i_1i_2)}\delta_{(l_1l_2)(j_1j_2)}q^4_{(i_1i_2)}q^{-4}_{(j_1j_2)}\,.\end{aligned}$$

Since f_1 is multiplicative, from (f1) for V we get (f1) for V_3:

$$\begin{aligned}f_1V_3\begin{pmatrix}l_1&l_2&l_3&j_1&j_2&j_3\\k_1&k_2&k_3&i_1&i_2&i_3\end{pmatrix}&=f_1V\begin{pmatrix}k_2&i_2\\k_1&i_1\end{pmatrix}f_1V\begin{pmatrix}l_3&j_3\\k_3&i_3\end{pmatrix}f_1V\begin{pmatrix}l_1&j_1\\l_2&j_2\end{pmatrix}\\&=\delta_{k_1i_1}\delta_{k_2i_2}q^4_{i_1}q^{-4}_{i_2}\delta_{k_3i_3}\delta_{l_3j_3}q^4_{i_3}q^{-4}_{j_3}\delta_{l_2j_2}\delta_{l_1j_1}q^4_{j_2}q^{-4}_{j_1}\\&=\delta_{(k_1k_2k_3)(i_1i_2i_3)}\delta_{(l_1l_2l_3)(j_1j_2j_3)}q^4_{(i_1i_2i_3)}q^{-4}_{(j_1j_2j_3)}\,.\end{aligned}$$

The proof is similar for arbitrary n.

(iii) The coaction v_2 is given by the formula (see Section 1)

$$v_2\begin{pmatrix}j_1&j_2\\i_1&i_2\end{pmatrix}=\sum\begin{pmatrix}l_1&l_2\\k_1&k_2\end{pmatrix}\otimes q^{-1}_{k_1}q_{k_2}q_{i_1}q^{-1}_{i_2}q_{j_1}q^{-1}_{j_2}q^{-1}_{l_1}q_{l_2}V\begin{pmatrix}k_2&i_2\\k_1&i_1\end{pmatrix}V\begin{pmatrix}l_1&j_1\\l_2&j_2\end{pmatrix}.$$

Consider the following matrix:

$$W=\sum\begin{pmatrix}i_1&i_2\\k_1&k_2\end{pmatrix}\otimes q^{-1}_{k_1}q_{k_2}q_{i_1}q^{-1}_{i_2}V\begin{pmatrix}k_2&i_2\\k_1&i_1\end{pmatrix}.$$

By using $(*)$ we get that the matrix W^* is given by

$$W^* = \sum \begin{pmatrix} k_1 & k_2 \\ i_1 & i_2 \end{pmatrix} \otimes q_{k_1}^{-1} q_{k_2} q_{i_1} q_{i_2}^{-1} V \begin{pmatrix} k_1 & i_1 \\ k_2 & i_2 \end{pmatrix}$$
$$= \sum \begin{pmatrix} l_1 & l_2 \\ j_1 & j_2 \end{pmatrix} \otimes q_{l_1}^{-1} q_{l_2} q_{j_1} q_{j_2}^{-1} V \begin{pmatrix} l_1 & j_1 \\ l_2 & j_2 \end{pmatrix}.$$

The linear map $x \mapsto W(x \otimes 1)W^*$ is given by

$$W\left(\begin{pmatrix} j_1 & j_2 \\ i_1 & i_2 \end{pmatrix} \otimes 1\right) W^* = \left(\sum \begin{pmatrix} I_1 & I_2 \\ k_1 & k_2 \end{pmatrix} \otimes q_{k_1}^{-1} q_{k_2} q_{I_1} q_{I_2}^{-1} V \begin{pmatrix} k_2 & I_2 \\ k_1 & I_1 \end{pmatrix}\right)$$
$$\cdot \left(\begin{pmatrix} j_1 & j_2 \\ i_1 & i_2 \end{pmatrix} \otimes 1\right)\left(\sum \begin{pmatrix} l_1 & l_2 \\ J_1 & J_2 \end{pmatrix} \otimes q_{l_1}^{-1} q_{l_2} q_{J_1} q_{J_2}^{-1} V \begin{pmatrix} l_1 & J_1 \\ l_2 & J_2 \end{pmatrix}\right)$$

so we have $v_2(x) = W(x \otimes 1)W^*$ for any x. By using (S) we get

$$(\mathrm{id} \otimes S)W = \sum \begin{pmatrix} i_1 & i_2 \\ k_1 & k_2 \end{pmatrix} \otimes q_{k_1}^{-1} q_{k_2} q_{i_1} q_{i_2}^{-1} SV \begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix}$$
$$= \sum \begin{pmatrix} i_1 & i_2 \\ k_1 & k_2 \end{pmatrix} \otimes q_{k_1}^{-1} q_{k_2} q_{i_1} q_{i_2}^{-1} q_{k_1}^{2} q_{i_1}^{-2} q_{i_2}^{2} q_{k_2}^{-2} V \begin{pmatrix} i_1 & k_1 \\ i_2 & k_2 \end{pmatrix}$$
$$= \sum \begin{pmatrix} i_1 & i_2 \\ k_1 & k_2 \end{pmatrix} \otimes q_{k_1} q_{k_2}^{-1} q_{i_1}^{-1} q_{i_2} V \begin{pmatrix} i_1 & k_1 \\ i_2 & k_2 \end{pmatrix}$$
$$= \sum \begin{pmatrix} k_1 & k_2 \\ i_1 & i_2 \end{pmatrix} \otimes q_{i_1} q_{i_2}^{-1} q_{k_1}^{-1} q_{k_2} V \begin{pmatrix} k_1 & i_1 \\ k_2 & i_2 \end{pmatrix} = W^*.$$

By using (ε) and (Δ) we get

$$(\mathrm{id} \otimes \varepsilon)W = \sum \begin{pmatrix} i_1 & i_2 \\ k_1 & k_2 \end{pmatrix} q_{k_1}^{-1} q_{k_2} q_{i_1} q_{i_2}^{-1} \delta_{k_1 i_1} \delta_{k_2 i_2} = \sum \begin{pmatrix} k_1 & k_2 \\ k_1 & k_2 \end{pmatrix} = 1_2,$$
$$(\mathrm{id} \otimes \Delta)W = \sum \begin{pmatrix} i_1 & i_2 \\ k_1 & k_2 \end{pmatrix} \otimes q_{k_1}^{-1} q_{k_2} q_{i_1} q_{i_2}^{-1} V \begin{pmatrix} k_2 & h \\ k_1 & g \end{pmatrix} \otimes V \begin{pmatrix} h & i_2 \\ g & i_1 \end{pmatrix}.$$

On the other hand, $W_{12}W_{13}$ is given by

$$W_{12}W_{13} = \sum \begin{pmatrix} g & h \\ k_1 & k_2 \end{pmatrix}\begin{pmatrix} i_1 & i_2 \\ g & h \end{pmatrix} \otimes q_{k_1}^{-1} q_{k_2} q_g q_h^{-1} V \begin{pmatrix} k_2 & h \\ k_1 & g \end{pmatrix} \otimes q_g^{-1} q_h q_{i_1} q_{i_2}^{-1} V \begin{pmatrix} h & i_2 \\ g & i_1 \end{pmatrix}$$

and this is equal to $(\mathrm{id} \otimes \Delta)W$. Thus W is a unitary corepresentation and $v_2 = \mathrm{ad}(W)$. Consider the matrix

$$Q_W = (\mathrm{id} \otimes f_{\frac{1}{2}})W = \sum \begin{pmatrix} i_1 & i_2 \\ k_1 & k_2 \end{pmatrix} q_{k_1}^{-1} q_{k_2} q_{i_1} q_{i_2}^{-1} f_{\frac{1}{2}} V \begin{pmatrix} k_2 & i_2 \\ k_1 & i_1 \end{pmatrix}.$$

By using (f1) for V we can compute Q_W:

$$Q_W = \sum \begin{pmatrix} i_1 & i_2 \\ k_1 & k_2 \end{pmatrix} q_{k_1}^{-1} q_{k_2} q_{i_1} q_{i_2}^{-1} \delta_{k_1 i_1} \delta_{k_2 i_2} q_{i_1}^{2} q_{i_2}^{-2} = \sum \begin{pmatrix} i_1 & i_2 \\ i_1 & i_2 \end{pmatrix} q_{i_1}^{2} q_{i_2}^{-2}.$$

From the formulae (f1)–(f4) and from cosemisimplicity of H we get that the equality $\mathrm{Tr}(Q_W^2 \cdot) = \mathrm{Tr}(Q_W^{-2} \cdot)$ holds on $\mathrm{End}(W)$ (Lemma 1.1 in [1]). On the other hand, the above formula of Q_W shows that $\mathrm{Tr}(Q_W^2\cdot)$ is φ_2 and $\mathrm{Tr}(Q_W^{-2}\cdot)$ is ψ_2. Together with the equality $\mathrm{End}(W) = Q_2(v)$ coming from $v_2 = \mathrm{ad}(W)$, this shows that $\varphi_2 = \psi_2$ on $Q_2(v)$. ∎

A natural question now is about how to verify the assumptions of Theorem 4.2, namely that v preserves the linear form $\varphi = \mathrm{Tr}(Q^4\cdot)$ with $Q \in A$ given by Lemma 4.1.

(1) In the $S^2 = \mathrm{id}$ case we have $Q = 1$ and the condition is satisfied. This is not very interesting, because Theorem 4.2 is weaker anyway than Theorem 2.4 in this case.

(2) In the $S^2 \neq \mathrm{id}$ case there are basically two examples. First is the case of adjoint coactions, where the condition is satisfied. This follows from the explicit formulae of Woronowicz in [29], and the whole thing is discussed in detail in [1]. The other example is with the universal coactions in [3], and once again, one can check that the condition is satisfied.

We do not know if there is a simpler characterisation of coactions v for which $Q(v)$ is a planar algebra. In case there is one, getting it from what we do in this paper is probably a purely Hopf C^*-algebraic problem, with no planar topology involved. This possible remaining problem is to be added to those mentioned at the end of the introduction.

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