

EXCISION AND A THEOREM OF POPA

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ABSTRACT. We give an elementary C^* -algebraic proof of a result of Sorin Popa which is of fundamental importance to Elliott's classification program.

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MSC (2000): 46L05.

1. INTRODUCTION

Throughout most of the 1990's, work on the finite case of Elliott's Classification Program focused on inductive limits of homogeneous C^* -algebras. Gradually it was noticed that such inductive limits enjoy a certain internal, finite dimensional approximation property that is a useful tool in classification (see, for example, Theorem 2.21 and Corollary 2.24 of [5]). In 1997 Sorin Popa noticed that his previous work on injective von Neumann algebras could be used to show that certain quasidiagonal C^* -algebras (cf. [3], [11]) enjoy a very similar internal, finite dimensional approximation property. These developments set the stage for Huaxin Lin's definition and subsequent classification of "tracially AF" algebras (cf. [8]). This work is so exciting because Popa's result only assumes quasidiagonality and deduces an approximation property which looks similar to the approximation property which is sufficient for classification. In other words, one can now imagine that the mild hypothesis of quasidiagonality would actually imply an AH inductive limit decomposition! (Of course, other hypotheses such as nuclearity and simplicity will also be needed.)

While Popa's result is certainly known to many experts it seems that relatively few have worked through the proof. This may be partly due to the fact that the main technical ingredient in the proof (cf. Theorem 2.3 of [10]) relies on some non-trivial II_1 -factor theory and a bit of free probability. In this note we will give a completely C^* -algebraic proof of Popa's result. Our hope is that the classification community will find this proof easier to understand. However, our real goal

is simply to expose and advertise the beauty and simplicity of Popa's ideas. Indeed, the language of this note may differ from Popa's, but the main techniques and estimates involved come directly from Popa's work.

In honor of Popa's contribution we make the following definition.

DEFINITION 1.1. Let A be a simple, unital C^* -algebra. Then A is called a *Popa algebra* if for every finite set $\mathfrak{F} \subset A$ and $\varepsilon > 0$ there exists a non-zero finite dimensional subalgebra $B \subset A$ with unit e such that:

- (i) $\|[e, x]\| < \varepsilon$ for all $x \in \mathfrak{F}$; and
- (ii) $e\mathfrak{F}e \subset^\varepsilon B$ (i.e. for every $x \in \mathfrak{F}$ there exists $b \in B$ with $\|exe - b\| < \varepsilon$).

The main result in [10] can then be stated as follows.

THEOREM 1.2. *Let A be simple, unital, quasidiagonal (cf. [3], [11]) and have real rank zero (cf. [2]). Then A is a Popa algebra.*

Actually, Popa gets away with less than real rank zero, however, in the classification program this is the main case of interest and hence we will only prove the result in this case.

Before getting into any details we wish to point out that the key technical results required in this note can be traced back to the fundamental work of Glimm. Though these things are now well known, we state below Glimm's work which we will need.

LEMMA 1.3 ([7], Lemma 1.10). *For every $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if A is a C^* -algebra, $p \in A$ is a projection and $\{w_1, \dots, w_n\} \subset A$ is a set such that $\|w_j^*w_i - \delta_{i,j}p\| < \delta$ ($\delta_{i,j}$ is the Kronecker delta function) then there exist actual partial isometries $\{v_1, \dots, v_n\} \subset A$ such that $v_j^*v_i = \delta_{i,j}p$ and, moreover, $\|w_i - v_i\| < \varepsilon$ for $1 \leq i \leq n$.*

This result simply states that a finite set of elements which are almost partial isometries with common support and orthogonal ranges can be perturbed to actual partial isometries with common support and orthogonal ranges. This is not exactly the content of Lemma 1.10 in [7], but Glimm's proof is easily adapted to prove the statement above.

The other result of Glimm which we will need comes from his work on the noncommutative Stone-Weierstrass problem.

THEOREM 1.4 ([6], Theorem 2). *Let $A \subset B(H)$ be a C^* -algebra which acts irreducibly (i.e. $A'' = B(H)$) and which contains no non-zero compact operators. Then the pure states on A are weak- $*$ dense in the state space of A .*

2. EXCISION OF STATES

The main technical idea in the present paper is *excision of states*. This idea is already present in Popa's paper as Theorem 2.3 of [10] is really a result about

excising factorial traces. However, in this section we will observe that general facts about excision easily give us the tools we need to prove Popa's result.

DEFINITION 2.1. Let A be a C^* -algebra and $\phi \in S(A)$ be a state on A . We say that ϕ can be excised if there exists a net of positive, norm one elements $h_\lambda \in A$ such that $\|h_\lambda^{1/2} a h_\lambda^{1/2} - \phi(a) h_\lambda\| \rightarrow 0$ for all $a \in A$. If the h_λ 's can be taken to be projections then we say ϕ can be excised by projections.

The definitive result concerning excision is due to Akemann, Anderson and Pedersen: A state can be excised if and only if it belongs to the weak- $*$ closure of the pure states (cf. [1]). When combined with Glimm's results on density of pure states we arrive at the following result (which is far from the most general but sufficient for our purposes).

THEOREM 2.2. *If A is simple, unital and infinite dimensional then every state on A can be excised.*

Proof. Since A is simple, unital and infinite dimensional there exists a (necessarily faithful) irreducible representation whose image (necessarily) contains no non-zero compact operators. Hence the pure states on A are dense in the state space of A . Hence every state can be excised. ■

When there are enough projections around we can even excise by projections.

LEMMA 2.3. *Assume A has real rank zero and $\phi \in S(A)$ can be excised. Then ϕ can be excised by projections.*

Proof. Let h_λ be a net of positive, norm one elements in the C^* -algebra A such that $\|h_\lambda^{1/2} a h_\lambda^{1/2} - \phi(a) h_\lambda\| \rightarrow 0$ for all $a \in A$. Since A has real rank zero we may assume that each h_λ has finite spectrum and hence we can write

$$h_\lambda = \sum_{i=1}^{k(\lambda)} \alpha_i^{(\lambda)} Q_i^{(\lambda)},$$

where $\{Q_i^{(\lambda)}\}$ are orthogonal projections and $1 = \alpha_1^{(\lambda)} > \alpha_2^{(\lambda)} > \dots > \alpha_{k(\lambda)}^{(\lambda)} > 0$. We claim that the projections $Q_1^{(\lambda)}$ will excise ϕ . Indeed, we have the following inequality:

$$\begin{aligned} \|Q_1^{(\lambda)} a Q_1^{(\lambda)} - \phi(a) Q_1^{(\lambda)}\| &= \|Q_1^{(\lambda)} (h_\lambda^{1/2} a h_\lambda^{1/2} - \phi(a) h_\lambda) Q_1^{(\lambda)}\| \\ &\leq \|h_\lambda^{1/2} a h_\lambda^{1/2} - \phi(a) h_\lambda\|. \quad \blacksquare \end{aligned}$$

COROLLARY 2.4. *If A is simple, unital, infinite dimensional and has real rank zero then every state on A can be excised by projections.*

3. POPA'S LOCAL QUANTIZATION TECHNIQUE

In [9] Popa gave a new proof of Connes' uniqueness theorem for the injective II_1 -factor. The main idea in the proof is what Popa calls "local quantization" and this technique has since been exploited with great success. In this section we explain how Popa's technique can be used to "excise" certain finite dimensional, unital, completely positive maps.

The main idea is as follows. We begin with an algebra A and a state $\phi \in S(A)$ which can be excised by projections. Let $L^2(A, \phi)$ be the GNS Hilbert space and $P \in B(L^2(A, \phi))$ be a finite rank projection. Then $PB(L^2(A, \phi))P$ is just a finite dimensional matrix algebra and we define a unital, completely positive map $\Phi : A \rightarrow PB(L^2(A, \phi))P$ by $a \mapsto P\pi_\phi(a)P$, where π_ϕ is the GNS representation. Popa's local quantization technique essentially says that in this situation Φ can be excised. Note that if a state ϕ can be excised by projections $\{p_\lambda\}$ then we can formulate excision as follows: There exists a net of $*$ -monomorphisms $\rho_\lambda : \mathbb{C} \hookrightarrow A$ (i.e. $\alpha \mapsto \alpha p_\lambda$) such that

$$\|\rho_\lambda(1)a\rho_\lambda(1) - \rho_\lambda(\phi(a))\| \rightarrow 0,$$

for all $a \in A$.

If A is a C^* -algebra and $\phi \in S(A)$ is a state on A then for each element $a \in A$ we will let $\hat{a} \in L^2(A, \phi)$ denote the corresponding vector.

THEOREM 3.1 (Popa's Local Quantization). *Let $\phi \in S(A)$ be a state which can be excised by projections. Let $\{y_i\}_{i=1}^m \subset A$ be such that $\phi(y_j^*y_i) = \delta_{i,j}$ (i.e. in $L^2(A, \phi)$, $\{\hat{y}_i\}_{i=1}^m$ is an orthonormal set of vectors) and $P \in B(L^2(A, \phi))$ be the orthogonal projection onto the span of $\{\hat{y}_i\}_{i=1}^m$. Then there exists a net of $*$ -monomorphisms $\rho_\lambda : P\pi_\phi(a)P \hookrightarrow A$ such that*

$$\|\rho_\lambda(P)a\rho_\lambda(P) - \rho_\lambda(\Phi(a))\| \rightarrow 0,$$

for all $a \in A$.

Moreover, for each λ we have the following commutator inequality for every unitary element $u \in A$:

$$\|[u, \rho_\lambda(P)]\|^2 \leq \|[P, \pi_\phi(u)]\|^2 + 2\|\rho_\lambda(P)u\rho_\lambda(P) - \rho_\lambda(\Phi(u))\|.$$

Proof. Let $\mathfrak{F} \subset A$ be a finite set of norm one elements which, for convenience, contains the unit of A , and $\varepsilon > 0$ be arbitrary. Evidently it suffices to show that there exists a $*$ -monomorphism $\rho : P\pi_\phi(a)P \hookrightarrow A$ such that $\|\rho(P)a\rho(P) - \rho(\Phi(a))\| < \varepsilon$, for all $a \in \mathfrak{F}$.

Since ϕ can be excised by projections, we can, for any $\delta > 0$, find a projection $p \in A$ such that $\|p(y_j^*xy_i)p - \phi(y_j^*xy_i)p\| < \delta$ for all $x \in \mathfrak{F}$ and $1 \leq i, j \leq m$. In particular, note that $\|(y_j p)^*(y_i p) - \delta_{i,j}p\| < \delta$ for all $1 \leq i, j \leq m$. In other words, $\{y_i p\}_{i=1}^m$ is almost a set of partial isometries with orthogonal ranges and common support p .

Thus we can perturb the $y_i p$'s to honest partial isometries $\{v_i\}$ such that $v_j^* v_i = \delta_{i,j} p$ and $\|v_i - y_i p\| < \varepsilon/4m^2$ (hence δ above is a number which depends on m and ε , however, we also assume that $\delta < \varepsilon/2m^2$). Hence if we define $f_{i,j} = v_i v_j^*$ then $\{f_{i,j}\}$ is a set of matrix units for a $m \times m$ -matrix algebra. Moreover, notice that if we let $q = \sum f_{i,i}$ be the unit of this matrix algebra then cutting an element $x \in \mathfrak{F}$ we have

$$\begin{aligned} qxq &= \left(\sum_{i=1}^m f_{i,i} \right) x \left(\sum_{j=1}^m f_{j,j} \right) = \sum_{i,j=1}^m v_i v_i^* x v_j v_j^* \approx \sum_{i,j=1}^m v_i (y_i p)^* x (y_j p) v_j^* \\ &\approx \sum_{i,j=1}^m v_i (\phi(y_i^* x y_j) p) v_j^* = \sum_{i,j=1}^m \phi(y_i^* x y_j) f_{i,j}. \end{aligned}$$

In the two approximations above the norm difference is less than $\varepsilon/2$ and hence the triangle inequality implies that

$$\left\| qxq - \sum_{i,j=1}^m \phi(y_i^* x y_j) f_{i,j} \right\| < \varepsilon.$$

The only thing left to notice is that the matrix of $P\pi_\phi(x)P$ with respect to the orthonormal basis $\{\hat{y}_i\}_{i=1}^m$ is just $(\langle \pi_\phi(x) \hat{y}_j, \hat{y}_i \rangle)_{i,j} = (\phi(y_i^* x y_j))_{i,j}$. Hence we can identify $PB(L^2(A, \phi))P$ with the matrix algebra $C^*(\{f_{i,j}\})$ in such a way that $P\pi_\phi(x)P \mapsto \sum_{i,j=1}^m \phi(y_i^* x y_j) f_{i,j}$, for all $x \in A$ and the proof of the first part of the theorem is complete.

Finally, if x is a unitary, the commutator estimate goes as follows:

$$\begin{aligned} \|[q, x]\|^2 &= \|qx - qxq + qxq - xq\|^2 = \|qxq^\perp - q^\perp xq\|^2 \\ &= \max\{\|q^\perp x^* q\|^2, \|q^\perp xq\|^2\} = \max\{\|qxq^\perp x^* q\|, \|qx^* q^\perp xq\|\} \\ &= \max\{\|q - qxqx^* q\|, \|q - qx^* qxq\|\} \\ &\leq \max\{\|q - \rho(\Phi(x)\Phi(x^*))\|, \|q - \rho(\Phi(x^*)\Phi(x))\|\} \\ &\quad + 2\|qxq - \rho(\Phi(x))\| \\ &= \max\{\|P - P\pi_\phi(x^*)P\pi_\phi(x)P\|, \|P - P\pi_\phi(x)P\pi_\phi(x^*)P\|\} \\ &\quad + 2\|qxq - \rho(\Phi(x))\| \\ &= \|[P, \pi_\phi(x)]\|^2 + 2\|qxq - \rho(\Phi(x))\|. \quad \blacksquare \end{aligned}$$

We stated the theorem above only for projections onto finite dimensional subspaces which are actually spanned by vectors coming from A . However, a simple density argument carries this technique over to arbitrary finite rank projections.

We are now ready for the harvest.

THEOREM 3.2 ([10], Theorem 3.2). *Let A be simple, unital, quasidiagonal and have real rank zero. Then A is a Popa algebra.*

Proof. Let ϕ be any state on A . Since A is simple and unital we have that π_ϕ is faithful and its image contains no non-zero compact operators. Since A is quasidiagonal we can apply Voiculescu's Theorem to construct a sequence of non-zero finite rank projections $P_n \in B(L^2(A, \phi))$ such that $\|[P_n, \pi_\phi(a)]\| \rightarrow 0$ for all $a \in A$.

By Corollary 2.4, ϕ can be excised by projections and hence we can apply Popa's local quantization technique (using the commutator estimates) to conclude that A is a Popa algebra. ■

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