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METRICS ON THE QUANTUM HEISENBERG MANIFOLD

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ABSTRACT. Recently Rieffel has introduced the notion of compact quantum metric spaces. He has produced examples using ergodic action of compact Lie groups. In the present note we construct examples of compact quantum metric spaces on the quantum Heisenberg manifolds using ergodic action of the Heisenberg group.

KEYWORDS: *Quantum Heisenberg manifolds, compact quantum metric space.*

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1. INTRODUCTION

In noncommutative geometry, the natural way to specify a metric is by a suitable "Lipschitz seminorm". This idea was first suggested by Connes in [4], and developed further in [5]. Connes pointed out in [4] and [5] that from a Lipschitz seminorm one obtains in a simple way an ordinary metric on the state space of a C^*-algebra. A natural question in this context is when does this metric topology coincides with the weak*-topology. In his search for an answer to this question Rieffel has identified in [6],[7], and [8] a larger class of spaces, namely order unit spaces on which one can answer these questions. He has introduced the concept of Compact Quantum Metric Spaces (CQMS) in [8] as a generalization of compact metric spaces, and used this new concept for rigorous study of convergence questions of algebras much in the spirit of Gromov-Hausdorff convergence. The aim of this note is the construction of CQMS on the quantum Heisenberg manifolds (QHM). Rieffel introduced these C^*-algebras as examples of deformation quantization of Heisenberg manifolds along a Poisson bracket. These algebras have further been studied in [1], [2], [3], and [10]. Rieffel has constructed examples of CQMS in the case of C^*-dynamical systems where the dynamics is driven by ergodic action of compact Lie group. Now it is also known that the Heisenberg group acts ergodically on QHM. Using this action Weaver attempted to produce examples of CQMS out of QHM. His construction does not completely achieve that goal. Here, essentially using the technique of Rieffel in a modified way, we construct examples of CQMS from QHM.

The organization of the paper is as follows. In the next section we briefly recall the notion of QHM and the group action. Then on a suitable dense $*$-subalgebra we give a $*$-algebra norm stronger than the C^*-norm. In Section 3 we recall the definition of CQMS and construct examples out of QHM using the group action and the previously introduced norm.

2. THE QUANTUM HEISENBERG ALGEBRA

Notation: for $x \in \mathbb{R}$, $e(x)$ stands for $\mathrm{e}^{2\pi \mathrm{i}x}$.

DEFINITION 2.1. For any positive integer c, let S^c denote the space of C^∞ functions $\Phi : \mathbb{R} \times \mathbb{T} \times \mathbb{Z} \to C$ such that:

(a) $\Phi(x+k, y, p) = e(ckpy)\Phi(x, y, p)$ for all $k \in \mathbb{Z}$;

(b) for every polynomial P on $\mathbb{Z}$ and every partial differential operator $\widetilde{X} = \frac{\partial^{m+n}}{\partial x^m \partial y^n}$ on $\mathbb{R} \times \mathbb{T}$ the function $P(p)(\widetilde{X}\Phi)(x, y, p)$ is bounded on $K \times \mathbb{Z}$ for any compact subset K of $\mathbb{R} \times \mathbb{T}$.

For each $\hbar, \mu, \nu \in \mathbb{R}$, $\mu \neq 0 \neq \nu$, let $\mathcal{A}_\hbar^\infty$ denote S^c with product and involution defined by

$$(\Phi \star \Psi)(x, y, p) = \sum_q \Phi(x - \hbar(q-p)\mu, y - \hbar(q-p)\nu, q) \times \Psi(x - \hbar q\mu, y - \hbar q\nu, p - q), \tag{2.1}$$

$$\Phi^*(x, y, p) = \overline{\Phi}(x, y, -p). \tag{2.2}$$

$\pi : \mathcal{A}_\hbar^\infty \to \mathcal{B}(L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z}))$ given by

$$(\pi(\Phi)\xi)(x, y, p) = \sum_q \Phi(x - \hbar(q-2p)\mu, y - \hbar(q-2p)\nu, q)\xi(x, y, p-q) \tag{2.3}$$

gives a faithful representation of the involutive algebra $\mathcal{A}_\hbar^\infty$.

$\mathcal{A}_{\mu,\nu}^{c,\hbar}$, the norm closure of $\pi(\mathcal{A}_\hbar^\infty)$, is called the Quantum Heisenberg Manifold.

$N_\hbar$ will denote the weak closure of $\pi(\mathcal{A}_\hbar^\infty)$.

We will identify $\mathcal{A}_\hbar^\infty$ with $\pi(\mathcal{A}_\hbar^\infty)$ without any mention.

Since we are going to work with fixed parameters $c, \mu, \nu, \hbar$ we will drop them altogether and denote $\mathcal{A}_{\mu,\nu}^{c,\hbar}$ simply by $\mathcal{A}_\hbar$; here the subscript remains only to distinguish it from a general algebra.

The following theorem of Weaver gives an useful characterization of $N_\hbar$.

THEOREM 2.2 (N. Weaver). *Let* $\mathcal{H} = L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ *and* V_f, W_k, X_r *be the operators defined by*

$$\begin{aligned}(V_f\xi)(x,y,p) &= f(x,y)\xi(x,y,p),\\ (W_k\xi)(x,y,p) &= e(-ck(p^2\hbar\nu + py))\xi(x+k,y,p),\\ (X_r\xi)(x,y,p) &= \xi(x-2\hbar r\mu, y-2\hbar r\nu, p+r).\end{aligned}$$

Let $T \in \mathcal{B}(\mathcal{H})$. *Then* $T \in N_\hbar$ *if and only if* T *commutes with the operators* V_f, W_k, X_r *for all* f *in* $L^\infty(\mathbb{R} \times \mathbb{T})$, *and* $k, r \in \mathbb{Z}$.

ACTION OF THE HEISENBERG GROUP. For $\Phi \in S^c$, $(r,s,t) \in \mathbb{R}^3 \equiv G$, (as a topological space)

$$(L_{(r,s,t)}\Phi)(x,y,p) = e(p(t+cs(x-r)))\Phi(x-r,y-s,p) \tag{2.4}$$

extends to an ergodic action of the Heisenberg group on $\mathcal{A}^{c,\hbar}_{\mu,\nu}$.

THE TRACE. $\tau : \mathcal{A}^\infty_\hbar \to \mathbb{C}$, given by $\tau(\phi) = \int_0^1 \int_{\mathbb{T}} \phi(x,y,0)\mathrm{d}x\mathrm{d}y$ extends to a faithful normal tracial state on $N_\hbar$. τ is invariant under the Heisenberg group action.

DEFINITION 2.3. Let $\phi \in S^c$, then $\|\cdot\|_{\infty,\infty,1}$ is the norm defined by

$$\|\phi\|_{\infty,\infty,1} = \sum_{p\in\mathbb{Z}} \sup_{x\in\mathbb{R},y\in\mathbb{T}} |\phi(x,y,p)|.$$

PROPOSITION 2.4. $\|\cdot\|_{\infty,\infty,1}$ *is a* $*$*-algebra norm on* S^c.

Proof. Clearly the involution is an antilinear isometry in the norm $\|\cdot\|_{\infty,\infty,1}$. Let $\Phi, \Psi \in S^c$ and

$$\Phi'(p) = \sup_{x\in\mathbb{R},y\in\mathbb{T}} |\Phi(x,y,p)|, \Psi'(p) = \sup_{x\in\mathbb{R},y\in\mathbb{T}} |\Psi(x,y,p)|.$$

For $p \in \mathbb{Z}$

$$\begin{aligned}&|(\Phi \star \Psi)(x,y,p)|\\ &\quad\leqslant \sum_q |\Phi(x-\hbar(q-p)\mu, y-\hbar(q-p)\nu, q)| \times |\Psi(x-\hbar q\mu, y-\hbar q\nu, p-q)|\\ &\quad\leqslant \sum_q \Phi'(q)\Psi'(p-q).\end{aligned}$$

Therefore,

$$\sum_p \sup_{x\in\mathbb{R},y\in\mathbb{T}} |(\Phi\star\Psi)(x,y,p)| \leqslant \sum_p\sum_q \Phi'(q)\Psi'(p-q) = \|\Phi\|_{\infty,\infty,1}\cdot\|\Psi\|_{\infty,\infty,1}.$$

This proves that $\|\cdot\|_{\infty,\infty,1}$ is an algebra norm. ∎

PROPOSITION 2.5. *The topology given by* $\|\cdot\|_{\infty,\infty,1}$ *is stronger than the topology given by the* C^**-norm coming from* $\mathcal{A}_\hbar$.

Proof. It suffices to show for $\phi \in S^c$, $\|\phi\| \leqslant \|\phi\|_{\infty,\infty,1}$. Let $\phi' : \mathbb{Z} \to \mathbb{R}_+$ be given by $\phi'(n) = \sup_{x\in\mathbb{R}, y\in\mathbb{T}} |\phi(x,y,n)|$. Then for $\xi \in L^2(\mathbb{R}\times\mathbb{T}\times\mathbb{Z})$ we have,

$$|(\phi\xi)(x,y,p)| \leqslant (\phi' \star |\xi(x,y,\cdot)|)(p),$$

where $\star$ denotes convolution on $\mathbb{Z}$ and $|\xi(x,y,\cdot)|$ is the function $p \mapsto |\xi(x,y,p)|$. By Young's inequality

$$\|(\phi\xi)(x,y,\cdot)\|_{l_2} \leqslant \|\phi' \star |\xi(x,y,\cdot)|\|_{l_2} \leqslant \|\phi'\|_{l_1}\|\xi(x,y,\cdot)\|_{l_2}.$$

Therefore, $\|\phi\| \leqslant \|\phi\|_{\infty,\infty,1}$, since $\|\phi\|_{\infty,\infty,1} = \|\phi'\|_{l_1}$. ∎

REMARK 2.6. Let $S^c_{\infty,\infty,1}$ be the space of all functions $\psi : \mathbb{R}\times\mathbb{T}\times\mathbb{Z} \to \mathbb{C}$ satisfying the following three conditions:

(i) ψ is measurable;

(ii) $\psi_p = \sup_{x\in\mathbb{R}, y\in\mathbb{T}} |\psi(x,y,p)|$ is an l_1 sequence;

(iii) $\psi(x+k,y,p) = e(ckyp)\psi(x,y,p)$ for all $k \in \mathbb{Z}$.

Then, for $\phi \in S^c_{\infty,\infty,1}$, arguments similar to Proposition 2.5 will show $\pi(\phi)$ defined by (2.3) gives a bounded operator on $L^2(\mathbb{R}\times\mathbb{T}\times\mathbb{Z})$. In fact Theorem 2.2 shows $\pi(\phi)$ belongs to $N_\hbar$.

3. COMPACT QUANTUM METRIC SPACE: THE EXAMPLE ON QHM

We recall some of the definitions from [9].

DEFINITION 3.1. An order unit space is a real partially ordered vector space A with a distinguished element e, the order unit satisfying:

(i) (Order Unit property) For each $a \in A$ there is an $r \in \mathbb{R}$ such that $a \leqslant re$.

(ii) (The Archimedean property) If $a \in A$ and if $a \leqslant re$ for all $r \in \mathbb{R}$ with $r > 0$, then $a \leqslant 0$.

REMARK 3.2. The following prescription defines a norm on an order unit space:

$$\|a\| = \inf\{r \in \mathbb{R} : -re \leqslant a \leqslant re\}.$$

DEFINITION 3.3. By a state of an order unit space (A,e) we mean a $\mu \in A'$, the dual of $(A, \|\cdot\|)$ such that $\mu(e) = 1 = \|\mu\|'$. Here $\|\cdot\|'$ stands for the dual norm on A'. The collection of states on (A,e) is denoted by $S(A)$.

REMARK 3.4. States are automatically positive.

EXAMPLE 3.5. The motivating example of the above concept is the real subspace of selfadjoint elements in a C^*-algebra with the order structure inherited from the C^*-algebra.

DEFINITION 3.6. Let (A, e) be an order unit space. By a Lip norm on A we mean a seminorm L, on A such that:

(i) for $a \in A$, we have $L(a) = 0$ if and only if $a \in \mathbb{R}e$;

(ii) the topology on $S(A)$ coming from the metric

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : L(a) \leqslant 1\}$$

is the w^*-topology.

DEFINITION 3.7. A compact quantum metric space is a pair (A, L) consisting of an order unit space A and a Lip norm L defined on it.

The following theorem of Rieffel will be of crucial importance.

THEOREM 3.8 (Theorem 4.5 of [9]). *Let L be a seminorm on the order unit space A such that $L(a) = 0$ if and only if $a \in \mathbb{R}e$. Then ρ_L gives $S(A)$ the w^*-topology exactly if*

(i) *(A, L) has finite radius, i.e, $\rho_L(\mu, \nu) \leqslant C$ for all $\mu, \nu \in S(A)$ for some constant C, and*

(ii) *$\mathcal{B}_1 = \{a : L(a) \leqslant 1,$ and $\|a\| \leqslant 1\}$ is totally bounded in A for $\| \cdot \|$.*

GENERAL SCHEME OF CONSTRUCTION. Let (A, G, α) be a C^* dynamical system with G an n dimensional Lie group acting ergodically. Let $A^\infty = \{a \in A : g \mapsto \alpha_g(a)$ is smooth $\}$. Then any $X \in \mathrm{Lie}(G)$, the Lie algebra of G, induces a derivation $\delta_X : A^\infty \to A^\infty$. Let $X_1, \ldots, X_n$ be a basis of $\mathrm{Lie}(G)$. $L(a) = \bigvee_{i=1}^{n} \|\delta_{X_i}(a)\|_n$, should be a good candidate for a Lip norm. Here $\| \cdot \|_n$ stands for an algebra norm on A not necessarily the norm coming from the algebra. This is essentially Rieffel's construction; the only modification is he considers the case where $\| \cdot \|_n$ is the algebra norm. Here the problem of construction of Lip norms reduces to construction of the norm $\| \cdot \|_n$ such that L so defined becomes a Lip norm.

ILLUSTRATION IN THE CONTEXT OF QUANTUM HEISENBERG MANIFOLDS. Let

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

be the canonical basis of the Lie algebra of the Heisenberg group. Then the associated derivations are given by

$$\begin{aligned} \delta_1(\phi)(x, y, p) &= -\frac{\partial \phi}{\partial x}(x, y, p), \\ \delta_2(\phi)(x, y, p) &= 2\pi \mathrm{i} c p x \phi(x, y, p) - \frac{\partial \phi}{\partial y}(x, y, p), \\ \delta_3(\phi)(x, y, p) &= 2\pi \mathrm{i} p \phi(x, y, p). \end{aligned}$$

Taking $\|\cdot\|_n$ as $\|\cdot\|_{\infty,\infty,1}$, we get a seminorm $L : S^c_{\text{s.a.}} \to \mathbb{R}_+$ explicitly given by

$$L(\phi) = \bigvee_1^3 \|\delta_i(\phi)\|_{\infty,\infty,1}.$$

Notation: Henceforth A will stand for $S^c_{\text{s.a.}}$.

PROPOSITION 3.9. *For all $\mu, \nu \in S(A)$,*

$$\rho_L(\mu,\nu) = \sup\{|\mu(a) - \nu(a)| : L(a) \leqslant 1\} \leqslant 6.$$

Proof. For $\phi \in S^c$ let us define $\phi^{(0)} \in S^c$ as follows,

$$\phi^{(0)}(x,y,p) = \begin{cases} \phi(x,y,p) & \text{for } p = 0, \\ 0 & \text{for } p \neq 0. \end{cases}$$

Clearly $\delta_i(\phi)^{(0)} = \delta_i(\phi^{(0)})$ for $i = 1,2,3$ and $\|\phi^{(0)}\|_{\infty,\infty,1} \leqslant \|\phi\|_{\infty,\infty,1}$. Therefore, if we further assume that $L(\phi) \leqslant 1$ then we have the following chain of implications,

$$\begin{aligned} L(\phi) \leqslant 1 &\Longleftrightarrow \|\delta_i(\phi)\|_{\infty,\infty,1} \leqslant 1, \quad \text{for } i = 1,2,3; \\ &\Longrightarrow \|\delta_i(\phi^{(0)})\|_{\infty,\infty,1} \leqslant 1, \quad \text{for } i = 1,2,3; \\ &\Longleftrightarrow L(\phi^{(0)}) \leqslant 1. \end{aligned}$$

Let $\phi \in S^c$ be such that $L(\phi) \leqslant 1$ and consequently we have $L(\phi^{(0)}) \leqslant 1$. Let $f_3(p) = |2\pi p\phi(x,y,p)|$, then from $L(\phi) \leqslant 1$ it follows that $\sum_p f_3(p) \leqslant 1$. Now,

(i) $\|\phi - \phi^{(0)}\| \leqslant \|\phi - \phi^{(0)}\|_{\infty,\infty,1} \leqslant \sum_{p\neq 0} \frac{f_3(p)}{2\pi|p|} \leqslant \sum_p f_3(p) \leqslant 1$,

(ii) $\|\phi^{(0)} - L_{(r,s,0)}\phi^{(0)}\|_{\infty,\infty,1} \leqslant 2$, and

(iii) we also have $\int_0^1 \int_{\mathbb{T}} L_{(r,s,0)}(\phi^{(0)}) \mathrm{d}r\mathrm{d}s = \tau(\phi^{(0)})I$.

Using these three we get

$$\begin{aligned} |\mu(\phi) - \tau(\phi^{(0)})| &\leqslant |\mu(\phi) - \mu(\phi^{(0)})| + |\mu(\phi^{(0)}) - \mu(\tau(\phi^{(0)})I)| \\ &\leqslant \|\phi - \phi^{(0)}\| + \int_0^1 \int_0^1 |\mu(\phi^{(0)}) - \mu(L_{(r,s,0)}(\phi^{(0)})| \mathrm{d}r\mathrm{d}s \leqslant 3. \end{aligned}$$

This completes the proof. ∎

PROPOSITION 3.10. *L as defined above is a Lip norm.*

Proof. Since the action is ergodic and $\|\cdot\|_{\infty,\infty,1}$ is a norm it follows that $L(\phi) = 0$ if and only if ϕ is a constant multiple of identity. The previous proposition gives finite radius of (A, L). Therefore, by Theorem 3.8, it suffices to show that every sequence $\{\phi_n\}_{n\geqslant 1}$ in $\mathcal{B}_1 = \{\phi : L(\phi) \leqslant 1, \text{ and } \|\phi\| \leqslant 1\}$ admits a subsequence convergent in the norm coming from the C^*-algebra.

Let

$$f_{1,n}(p) = \sup_{x\in\mathbb{R},y\in\mathbb{T}} \left|\frac{\partial\phi_n}{\partial x}(x,y,p)\right|,$$

$$f_{2,n}(p) = \sup_{x\in\mathbb{R},y\in\mathbb{T}} \left|2\pi i c p x \phi_n(x,y,p) - \frac{\partial\phi_n}{\partial y}(x,y,p)\right|,$$

$$f_{3,n}(p) = \sup_{x\in\mathbb{R},y\in\mathbb{T}} |2\pi i p \phi_n(x,y,p)|,$$

$L(\phi_n) \leqslant 1$ is equivalent with $\sum_p f_{i,n}(p) \leqslant 1$ for $i = 1,2,3$. Then

$$\sup_{|x|\leqslant 2, y\in\mathbb{T}} \left|\frac{\partial\phi_n}{\partial y}(x,y,p)\right| \leqslant 4\pi c f_{3,n}(p) + f_{2,n}(p) \leqslant 1 + 4\pi c.$$

Now by Arzela-Ascoli theorem there exists $\phi : \mathbb{R}\times\mathbb{T}\times\mathbb{Z} \to \mathbb{C}$ such that

$$\sup_{|x|\leqslant 2, y\in\mathbb{T}} |\phi_n(x,y,p) - \phi(x,y,p)| \to 0, \quad \text{for each } p \in \mathbb{Z}.$$

Clearly ϕ satisfies the periodicity condition.

Claim:

$$\sum_p \sup_{x,y} |\phi(x,y,p)| < \infty.$$

Proof of Claim: Suppose not, then for any $N \in \mathbb{N}, \exists p_1, \ldots, p_k > N$ such that $\sum_i \sup_{x,y} |\phi(x,y,p_i)| > 2$. Therefore by taking n sufficiently large we get

$$\sum_{|p|\geqslant N} \sup_{x,y} |\phi_n(x,y,p)| \geqslant \sum_i \sup_{x,y} |\phi(x,y,p_i)| - \frac{1}{2} > \frac{3}{2}.$$

On the other hand note that,

$$\sum_{|p|\geqslant N} \sup_{x,y} |\phi_n(x,y,p)| = \sum_{|p|\geqslant N} \frac{f_{3,n}(p)}{p} \leqslant \frac{1}{N}\sum_p f_{3,n}(p) = \frac{1}{N}.$$

This leads to a contradiction.

Note that ϕ defines a bounded operator by Remark 2.6. Therefore in view of Proposition 2.5 it is enough to show that ϕ_n converges to ϕ in $\|\cdot\|_{\infty,\infty,1}$ norm. For $N \in \mathbb{N}$ let

$$\phi_{|p|\leqslant N}(x,y,p) = \begin{cases} \phi(x,y,p) & \text{for } |p| \leqslant N, \\ 0 & \text{for } |p| > N. \end{cases}$$

Let $\varepsilon > 0$ be given. Choose N such that

(i) $\|\phi - \phi_{|p|\leqslant N}\|_{\infty,\infty,1} \leqslant \varepsilon$, and

(ii) $\frac{1}{N} \leqslant \varepsilon$.

Then by Definition 3.1 $\|\phi_n - \phi_{n,|p|\leqslant N}\|_{\infty,\infty,1} \leqslant \varepsilon, \forall n$. Now choose m such that for $m \leqslant n$, $\|\phi_{n,|p|\leqslant N} - \phi_{|p|\leqslant N}\|_{\infty,\infty,1} \leqslant \varepsilon$. Therefore $\forall\, n \geqslant m,\ \|\phi_n - \phi\|_{\infty,\infty,1} \leqslant 3\varepsilon$. ∎

THEOREM 3.11. *$((A, I), L)$ is a compact quantum metric space.*

Proof. Follows from the previous two propositions. ∎

REMARK 3.12. Let M be the seminorm given by $M(\phi) = \bigvee_{1}^{3} \|\delta_i(\phi)\|$. Certainly one would like M to have the Lip norm property. The fact that L constructed here is a Lip norm is a weaker property. Because by the comparison lemma of [7], M is a Lipnorm along with Proposition 2.5 implies L is so.

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