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ON THE MAXIMALITY OF SUBDIAGONAL ALGEBRAS

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ABSTRACT. We consider Arveson's problem on the maximality of subdiagonal algebras. We prove that a subdiagonal algebra is maximal if it is invariant under the modular group of a faithful normal state which is preserved by the conditional expectation associated with the subdiagonal algebra.

KEYWORDS: *Subdiagonal algebra, maximality, modular group, conditional expectation.*

MSC (2000): 46L10, 47D25.

1. INTRODUCTION

Let $\mathcal{M}$ be a von Neumann algebra. Let $\mathcal{E}$ be a normal faithful conditional expectation from $\mathcal{M}$ onto a von Neumann subalgebra $\mathcal{D}$ of $\mathcal{M}$. A σ-weakly closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ is called a *subdiagonal algebra* in $\mathcal{M}$ with respect to $\mathcal{E}$ if the following conditions are satisfied:

(i) $\mathcal{A} + \mathcal{A}^*$ is σ-weakly dense in $\mathcal{M}$;
(ii) $\mathcal{E}$ is multiplicative on $\mathcal{A}$;
(iii) $\mathcal{A} \cap \mathcal{A}^* = \mathcal{D}$, where $\mathcal{A}^* = \{x^* : x \in \mathcal{A}\}$.

$\mathcal{D}$ is then called the *diagonal* of $\mathcal{A}$.

This notion was introduced by Arveson in [1] with the perspective to give a unified theory of non-selfadjoint operator algebras, including the algebra of bounded analytic matrix valued (or more generally, operator valued) functions and nest algebras. One fundamental result proved in [1] is an inner-outer type factorization, which extends significantly the previous inner-outer factorization for analytic matrix valued functions obtained independently by Helson-Lowdenslager [6] and Wiener-Masani [16] (see also [5]). This theorem was further generalized and studied in many related contexts (see [9] and for more references therein). On the other hand, as the well-known Szëgo inner-outer factorization in the theory of the classical Hardy spaces, this factorization is central for the development of the non-commutative Hardy space theory (cf. [12], [13], [14]).

In all these works, and in fact since the creation of the theory of subdiagonal algebras by Averson, a certain maximality assumption has always played a preeminent role. Recall that a subdiagonal algebra $\mathcal{A}$ with respect to $\mathcal{E}$ is said to be *maximal* if $\mathcal{A}$ is properly contained in no larger subdiagonal algebra with respect to $\mathcal{E}$. It was proved in [1] that any subdiagonal algebra $\mathcal{A}$ is contained in a unique maximal subdiagonal algebra, denoted by $\mathcal{A}_{\max}$, which is described by

$$\mathcal{A}_{\max} = \{x \in \mathcal{M} : \mathcal{E}(axb) = 0, \forall\, a \in \mathcal{A}, \forall\, b \in \mathcal{A}_0\},$$

where

$$\mathcal{A}_0 = \{a \in \mathcal{A} : \mathcal{E}(a) = 0\}.$$

Many known examples of subdiagonal algebras are maximal. A long standing open problem raised by Arveson in [1] is that whether every subdiagonal algebra is automatically maximal.

Only more than two decades later that Exel [3] gave a partial solution for this problem: if there is a normal faithful tracial state τ on $\mathcal{M}$ such that $\tau \circ \mathcal{E} = \tau$ (in this case $\mathcal{A}$ is called a *finite subdiagonal algebra*), then $\mathcal{A}$ is maximal. In fact, Exel's arguments show a little bit more, namely, that every subdiagonal algebra of a finite von Neumann algebra is automatically maximal. By the way, we recall another problem posed in [1], still unsolved too, is that whether a subdiagonal algebra of a finite and σ-finite von Neumann algebra is a finite subdiagonal algebra.

Very recently, Ji, Ohwada and Saito proved in [8] that if $\mathcal{A}$ is a maximal subdiagonal algebra in a σ-finite von Neumann algebra $\mathcal{M}$ with respect to $\mathcal{E}$, then $\mathcal{A}$ is invariant under the modular automorphism group σ_t^{φ} of every $\mathcal{E}$-invariant normal faithful state φ on $\mathcal{M}$. Recall that φ is $\mathcal{E}$-invariant if $\varphi \circ \mathcal{E} = \varphi$. They then asked whether the converse is true. Let us explicitly state this question as follows (see Question 2.7 in [8]).

QUESTION. *Let $\mathcal{A}$ be a subdiagonal algebra of a σ-finite von Neumann algebra $\mathcal{M}$ with respect to $\mathcal{E}$. Assume that $\mathcal{A}$ is σ_t^{φ}-invariant (i.e., $\sigma_t^{\varphi}(\mathcal{A}) \subset \mathcal{A}$, $\forall\, t \in \mathbb{R}$) for every $\mathcal{E}$-invariant normal faithful state φ on $\mathcal{M}$. Is $\mathcal{A}$ maximal?*

The aim of this note is to answer this question in the affirmative. Below is our main result.

THEOREM 1.1. *Let $\mathcal{M}$ be a σ-finite von Neumann algebra and $\mathcal{E}$ a normal faithful conditional expectation from $\mathcal{M}$ onto a von Neumann subalgebra $\mathcal{D} \subset \mathcal{M}$. Let $\mathcal{A}$ be a subdiagonal algebra of $\mathcal{M}$ with respect to $\mathcal{E}$. If there is a normal faithful state φ on $\mathcal{M}$ such that $\mathcal{E}$ commutes with σ_t^{φ} (i.e., $\sigma_t^{\varphi} \circ \mathcal{E} = \mathcal{E} \circ \sigma_t^{\varphi}$ for all $t \in \mathbb{R}$) and $\mathcal{A}$ is σ_t^{φ}-invariant, then $\mathcal{A}$ is maximal.*

REMARK. It is classical that if φ is $\mathcal{E}$-invariant, then $\mathcal{E}$ and σ_t^{φ} commute ([2], 1.4.3).

The remainder of the note is essentially devoted to the proof of the theorem above. Our strategy is to reduce the present situation to that of finite von Neumann algebras, and then to use Exel's theorem quoted previously. The key ingredient of this reduction is an unpublished important result of Haagerup. It roughly says that every von Neumann algebra can be embedded, in an appropriate way, into a large von Neumann algebra, which is a kind of inductive limit of some nice finite von Neumann subalgebras. In the next section, we will recall this reduction theorem of Haagerup and the construction of these nicely disposed subalgebras. The proof of the above theorem will be given in Section 3. Section 4 contains a generalization to weights instead of states.

2. HAAGERUP'S REDUCTION THEOREM

In this section we recall an important unpublished theorem due to Haagerup [4]. It states that any von Neumann algebra can be embedded, as the image of a normal faithful conditional expectation, into a large von Neumann algebra which is generated by an increasing family of finite subalgebras, each of which is the image of a normal conditional expectation. Haagerup's original intention is to approximate his non-commutative L^p-spaces based on type III von Neumann algebras by those constructed from a trace. This approximation theorem on Haagerup non-commutative L^p-spaces is very important in non-commutative analysis. In many situations, it permits to consider only non-commutative L^p-spaces associated with traces. We refer to [10] for more recent applications of Haagerup's reduction theorem to non-commutative martingale and ergodic theories. Note that [10] also contains a reproduction of Haagerup's unpublished manuscript [4].

The main tool of Haagerup's construction is crossed products. Our references for crossed products are [11] and [15]. Throughout, G will denote the discrete subgroup $\bigcup_{n\geqslant 1} 2^{-n}\mathbb{Z}$ of $\mathbb{R}$. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H and φ a normal faithful state on $\mathcal{M}$. We consider the crossed product $\mathcal{M} \rtimes_{\sigma^\varphi} G$ of $\mathcal{M}$ by G with respect to σ^φ. In the sequel, we will denote this crossed product by $\mathcal{R}$. Recall that $\mathcal{R}$ is a von Neumann algebra on $\ell^2(G, H)$ generated by the operators $\pi(x), x \in \mathcal{M}$ and $\lambda(t), t \in G$, which are defined by

$$(\pi(x)\xi)(s) = \sigma^\varphi_{-s}(x)\xi(s), \quad (\lambda(t)\xi)(s) = \xi(s-t), s \in G, \xi \in \ell^2(G, H).$$

Note that π is a normal faithful representation of $\mathcal{M}$ on $\ell^2(G, H)$. Thus we will identify $\pi(\mathcal{M})$ and $\mathcal{M}$ whenever possible. The operators $\pi(x)$ and $\lambda(t)$ satisfy the following commutation relation:

$$\lambda(t)\pi(x)\lambda(t)^* = \pi(\sigma^\varphi_t(x)), \quad t \in G, x \in \mathcal{M}. \tag{2.1}$$

Let $\widehat{\varphi}$ be the dual weight of φ on $\mathcal{R}$. Then $\widehat{\varphi}$ is again a normal faithful state on $\mathcal{R}$ uniquely determined by

$$\widehat{\varphi}(\lambda(t)x) = \begin{cases} \varphi(x) & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}, \quad x \in \mathcal{M}, t \in G. \tag{2.2}$$

In particular, $\widehat{\varphi}\big|_{\mathcal{M}} = \varphi$. The modular automorphism group of $\widehat{\varphi}$ is uniquely determined by

$$\sigma_t^{\widehat{\varphi}}(x) = \sigma_t^{\varphi}(x), \quad \sigma_t^{\widehat{\varphi}}(\lambda(s)) = \lambda(s), \quad x \in \mathcal{M},\ t, s \in G. \tag{2.3}$$

Consequently, $\sigma_t^{\widehat{\varphi}}\big|_{\mathcal{M}} = \sigma_t^{\varphi}$, and so $\sigma_t^{\widehat{\varphi}}(\mathcal{M}) = \mathcal{M}$ for all $t \in \mathbb{R}$. It also follows that

$$\sigma_t^{\widehat{\varphi}}(x) = \lambda(t)x\lambda(t)^*, \quad x \in \mathcal{R}, t \in G. \tag{2.4}$$

It is classical that there is a unique normal faithful conditional expectation Φ from $\mathcal{R}$ onto $\mathcal{M}$ determined by

$$\Phi(\lambda(t)x) = \begin{cases} x & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}, \quad x \in \mathcal{M}, t \in G. \tag{2.5}$$

By (2.2), (2.3) and (2.5), we deduce that

$$\widehat{\varphi} \circ \Phi = \widehat{\varphi} \quad \text{and} \quad \sigma_t^{\widehat{\varphi}} \circ \Phi = \Phi \circ \sigma_t^{\widehat{\varphi}}, \quad t \in \mathbb{R}. \tag{2.6}$$

With these notations, Haagerup's reduction theorem asserts that there is an increasing sequence $(\mathcal{R}_n)_{n\geqslant 1}$ of von Neumann subalgebras of $\mathcal{R}$ with the following properties:

(i) each $\mathcal{R}_n$ is finite;

(ii) $\bigcup_{n\geqslant 1} \mathcal{R}_n$ is σ-weakly dense in $\mathcal{R}$;

(iii) for every $n \geqslant 1$ there is a normal faithful conditional expectation Φ_n from $\mathcal{R}$ onto $\mathcal{R}_n$ such that:

$$\widehat{\varphi} \circ \Phi_n = \widehat{\varphi}, \quad \sigma_t^{\widehat{\varphi}} \circ \Phi_n = \Phi_n \circ \sigma_t^{\widehat{\varphi}}, \quad \Phi_n \circ \Phi_{n+1} = \Phi_n, \quad n \geqslant 1, t \in \mathbb{R}. \tag{2.7}$$

Note that a normal conditional expectation satisfying the first equality in (2.7) is unique. Since $\Phi_n \circ \Phi_{n+1}$ is also a conditional expectation under which $\widehat{\varphi}$ is invariant, this uniqueness implies $\Phi_n \circ \Phi_{n+1} = \Phi_n$, that is, the third equality in (2.7) is a consequence of the first. Note that the second equality is also a consequence of the first by Connes' classical result already quoted before.

In Haagerup's construction, $\mathcal{R}_n$ is the centralizer of a normal faithful state φ_n on $\mathcal{R}$ such that its modular automorphism group $\sigma_t^{\varphi_n}$ is periodic of period 2^{-n}. In the sequel, we will need the precise form of φ_n. Thus let us briefly recall this construction.

For a von Neumann algebra $\mathcal{N}$ and a normal faithful state ψ on $\mathcal{N}$ we denote, as usual, by $\mathcal{Z}(\mathcal{N})$ the center of $\mathcal{N}$ and by $\mathcal{N}_\psi$ the centralizer of ψ in $\mathcal{N}$. Recall that $\mathcal{N}_\psi$ is the algebra of the fixed points of σ_t^{ψ}. By (2.4), $\lambda(t) \in \mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$ for all $t \in G$. For any given $n \in \mathbb{N}$, by functional calculus, there is $b_n \in \mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$ such that

$$0 \leqslant b_n \leqslant 2\pi \quad \text{and} \quad e^{ib_n} = \lambda(2^{-n}).$$

Set $a_n = 2^n b_n$. Then again $a_n \in \mathcal{Z}(\mathcal{R}_{\widehat{\varphi}}), n \geqslant 1$. The desired state φ_n is defined as

$$\varphi_n(x) = \frac{1}{\widehat{\varphi}(e^{-a_n})}\widehat{\varphi}(e^{-a_n}x), \quad x \in \mathcal{R}, n \geqslant 1. \tag{2.8}$$

Since $a_n \in \mathcal{R}_{\widehat{\varphi}}$,

$$\sigma_t^{\varphi_n}(x) = e^{-ita_n}\sigma_t^{\widehat{\varphi}}(x)e^{ita_n}, \quad x \in \mathcal{R}, t \in \mathbb{R}, n \geqslant 1. \tag{2.9}$$

Then by (2.4) and the definition of a_n, $\sigma_t^{\varphi_n}$ is 2^{-n}-periodic. Let $\mathcal{R}_n = \mathcal{R}_{\varphi_n}$. Then $\varphi_n|_{\mathcal{R}_n}$ is a normal faithful tracial state on $\mathcal{R}_n$, and so $\mathcal{R}_n$ is a finite von Neumann subalgebra of $\mathcal{R}$.

Define $\Phi_n : \mathcal{R} \to \mathcal{R}_n$ by

$$\Phi_n(x) = 2^n \int_0^{2^{-n}} \sigma_t^{\varphi_n}(x)\mathrm{d}t, \quad x \in \mathcal{R}.$$

By the 2^{-n}-periodicity of $\sigma_t^{\varphi_n}$, we have

$$\Phi_n(x) = \int_0^1 \sigma_t^{\varphi_n}(x)\mathrm{d}t, \quad x \in \mathcal{R}. \tag{2.10}$$

Then it is routine to check that Φ_n is a normal faithful conditional expectation satisfying (2.7). Hence to prove Haagerup's reduction theorem mentioned above it remains to show that $(\mathcal{R}_n)$ is increasing and the union of the $\mathcal{R}_n's$ is σ-weakly dense in $\mathcal{R}$. We refer the reader to [4] and [10] for more details.

3. THE PROOF

This section is devoted to the proof of Theorem 1.1. Throughout this section, $\mathcal{M}, \mathcal{D}, \mathcal{E}, \mathcal{A}$ and φ will be fixed as in that theorem. $\mathcal{R}$ will be the crossed product $\mathcal{M} \rtimes_{\sigma^\varphi} G$ as in the last section, and we will keep all notations introduced there. The idea of the proof is to first lift $\mathcal{A}$ to a subdiagonal algebra in $\mathcal{R}$, then compress the latter to a subdiagonal algebra in $\mathcal{R}_n$ by the conditional expectation Φ_n, and finally come back to $\mathcal{A}$ by passing to limit as $n \to \infty$.

For easy later reference let us state the commutation assumption on $\mathcal{E}$ and σ_t^φ as follows

$$\sigma_t^\varphi \circ \mathcal{E} = \mathcal{E} \circ \sigma_t^\varphi, \quad t \in \mathbb{R}. \tag{3.1}$$

This implies that $\mathcal{D}$ is σ_t^φ-invariant and $\sigma_t^\varphi|_{\mathcal{D}}$ is exactly the modular automorphism group of $\varphi|_{\mathcal{D}}$. Consequently, we do not need to distinguish φ and $\varphi|_{\mathcal{D}}$, σ_t^φ

and $\sigma_t^{\varphi}\big|_{\mathcal{D}}$, respectively. Now let $\mathcal{S} = \mathcal{D} \rtimes_{\sigma^{\varphi}} G$. Then $\mathcal{S}$ is naturally identified as a von Neumann subalgebra of $\mathcal{R}$, generated by all operators $\pi(x)$, $x \in \mathcal{D}$ and $\lambda(t)$, $t \in G$. The dual weight of $\varphi\big|_{\mathcal{D}}$ on $\mathcal{S}$ is equal to $\widehat{\varphi}\big|_{\mathcal{S}}$. Again, we will denote this restriction by the same symbol $\widehat{\varphi}$. It is not hard to extend $\mathcal{E}$ to a normal faithful conditional expectation $\widehat{\mathcal{E}}$ from $\mathcal{R}$ onto $\mathcal{S}$, which is uniquely determined by

$$\widehat{\mathcal{E}}(\lambda(t)x) = \lambda(t)\,\mathcal{E}(x), \quad x \in \mathcal{M}, t \in G. \tag{3.2}$$

The reader is referred to [10] for details and for more extensions of this type. By (2.4), (3.1) and (3.2), we deduce

$$\sigma_t^{\widehat{\varphi}} \circ \widehat{\mathcal{E}} = \widehat{\mathcal{E}} \circ \sigma_t^{\widehat{\varphi}}, \quad t \in G. \tag{3.3}$$

On the other hand, using (2.9), (3.3) and the fact that $a_n \in \mathcal{S}$ and $\widehat{\mathcal{E}}$ is a conditional expectation with respect to $\mathcal{S}$, we get

$$\sigma_t^{\varphi_n} \circ \widehat{\mathcal{E}} = \widehat{\mathcal{E}} \circ \sigma_t^{\varphi_n}, \quad t \in \mathbb{R}, n \geqslant 1. \tag{3.4}$$

Hence by the definition (2.10) of the conditional expectation $\Phi_n : \mathcal{R} \to \mathcal{R}_n$, we deduce

$$\Phi_n \circ \widehat{\mathcal{E}} = \widehat{\mathcal{E}} \circ \Phi_n, \quad n \geqslant 1. \tag{3.5}$$

In particular, $\mathcal{R}_n$ and $\mathcal{S}$ are respectively $\widehat{\mathcal{E}}$-invariant and Φ_n-invariant.

Now let $\mathcal{S}_n = \mathcal{S}_{\varphi_n}, n \geqslant 1$. Then clearly, $\mathcal{S}_n = \mathcal{R}_n \cap \mathcal{S}$ for every $n \geqslant 1$. Also note that $\Phi_n\big|_{\mathcal{S}}$ and $\widehat{\mathcal{E}}\big|_{\mathcal{R}_n}$ are normal faithful conditional expectations from $\mathcal{S}$ onto $\mathcal{S}_n$, respectively, from $\mathcal{R}_n$ onto $\mathcal{S}_n$. $(\mathcal{S}_n)_{n\geqslant 1}$ and $(\Phi_n\big|_{\mathcal{S}})_{n\geqslant 1}$ are the increasing sequences of von Neumann subalgebras of $\mathcal{S}$ and respectively the sequence of the corresponding conditional expectations given by Haagerup's construction presented in the last section relative to $(\mathcal{D}, \varphi\big|_{\mathcal{D}})$ instead of $(\mathcal{M}, \varphi)$. Again, we will denote these restriction mappings by the same symbols as the mappings theirselves when no confusion can occur.

Since $\mathcal{A}$ is σ_t^{φ}-invariant, by (2.1), the family of all linear combinations on $\lambda(t)\,\pi(x)$, $t \in G$, $x \in \mathcal{A}$, is a $*$-subalgebra of $\mathcal{R}$. Let $\widehat{\mathcal{A}}$ be its σ-weakly closure in $\mathcal{R}$ and $\mathcal{A}_n = \widehat{\mathcal{A}} \cap \mathcal{R}_n$. The following lemmas show that $\widehat{\mathcal{A}}$ (respectively $\mathcal{A}_n$) is a subdiagonal algebra with respect to $\widehat{\mathcal{E}}$ (respectively $\widehat{\mathcal{E}}\big|_{\mathcal{R}_n}$).

LEMMA 3.1. *$\widehat{\mathcal{A}}$ is a subdiagonal algebra of $\mathcal{R}$ with respect to $\widehat{\mathcal{E}}$.*

Proof. We first prove that $\widehat{\mathcal{A}} + \widehat{\mathcal{A}}^*$ is σ-weakly dense in $\mathcal{R}$. For this it suffices to show that for any $t \in G$ and $x \in \mathcal{M}$, $\lambda(t)\,\pi(x)$ is the limit of elements in $\widehat{\mathcal{A}} + \widehat{\mathcal{A}}^*$. Since $\mathcal{A} + \mathcal{A}^*$ is σ-weakly dense in $\mathcal{M}$, there are $a_i, b_i \in \mathcal{A}$ such that

$$x = \lim_i (a_i + b_i^*) \quad \sigma\text{-weakly}.$$

Since π is normal,

$$\pi(x) = \lim_i (\pi(a_i) + \pi(b_i)^*) \quad \sigma\text{-weakly}.$$

Therefore,

$$\lambda(t)\pi(x) = \lim_i (\lambda(t)\pi(a_i) + \lambda(t)\pi(b_i)^*) \quad \sigma\text{-weakly.}$$

This is the desired limit.

Next we show that $\widehat{\mathcal{E}}$ is multiplicative on $\widehat{\mathcal{A}}$. To this end we note that by (3.2), for any $s, t \in G$ and $x, y \in \mathcal{A}$

$$\begin{aligned}\widehat{\mathcal{E}}(\lambda(s)\pi(x)\pi(y)\lambda(t)) &= \lambda(s)\pi(\mathcal{E}(xy))\lambda(t) \\ &= \lambda(s)\pi(\mathcal{E}(x)\mathcal{E}(y))\lambda(t) \\ &= \widehat{\mathcal{E}}(\lambda(s)\pi(x))\widehat{\mathcal{E}}(\pi(y)\lambda(t)),\end{aligned}$$

where we have used the multiplicativity of $\mathcal{E}$ on $\mathcal{A}$. Then the linearity and normality of $\widehat{\mathcal{E}}$ imply the multiplicativity of $\widehat{\mathcal{E}}$ on $\widehat{\mathcal{A}}$.

Thus it remains to show $\widehat{\mathcal{A}} \cap \widehat{\mathcal{A}}^* = \mathcal{S}$. To this end, we will use the matrix representation $(x_{s,t})_{s,t\in G}$ of an element $x \in B(\ell^2(G, H))$ in the natural basis of $\ell^2(G)$. It is well-known that $x \in \mathcal{R}$ if and only if there is a function $X : G \to \mathcal{M}$ such that

$$x_{s,t} = \sigma^{\varphi}_{-s}(X(s\,t^{-1})), \quad s, t \in G.$$

(cf. Section 22.1 of [15]). Clearly, this function X is unique. Now we claim that if $x \in \widehat{\mathcal{A}}$, then $X(t) \in \mathcal{A}$ for all $t \in G$. Indeed, this is clear if $x = \lambda(t_0)\,\pi(x_0)$ for some $t_0 \in G$ and $x_0 \in \mathcal{A}$. It then follows that the claim is true if x is a linear combination of $\lambda(t)\,\pi(y), t \in G, y \in \mathcal{A}$. For a general $x \in \widehat{\mathcal{A}}$, there is a net $\{x_i\}$ of linear combinations on $\lambda(t)\,\pi(y)$, $t \in G$, $y \in \mathcal{A}$, such that

$$x = \lim_i x_i \quad \sigma\text{-weakly.}$$

If X_i denotes the function corresponding to x_i, then clearly

$$X(t) = \lim_i X_i(t) \quad \sigma\text{-weakly}, t \in G.$$

Hence by the σ-weak closedness of $\mathcal{A}$, we conclude that $X(t) \in \mathcal{A}$ for all $t \in G$, proving our claim.

Similarly, if $x \in \widehat{\mathcal{A}}^*$, then $X(t) \in \mathcal{A}^*$ for all $t \in G$. Now let $x \in \widehat{\mathcal{A}} \cap \widehat{\mathcal{A}}^*$. Then $X(t) \in \mathcal{A} \cap \mathcal{A}^* = \mathcal{D}$ for all $t \in G$. Therefore, $x \in \mathcal{S}$, and so $\widehat{\mathcal{A}} \cap \widehat{\mathcal{A}}^* \subset \mathcal{S}$. The converse inclusion is trivial. Thus $\widehat{\mathcal{A}} \cap \widehat{\mathcal{A}}^* = \mathcal{S}$. Therefore $\widehat{\mathcal{A}}$ is a subdiagonal algebra with respect to $\widehat{\mathcal{E}}$. ∎

LEMMA 3.2. *Every $\mathcal{A}_n$ is a finite subdiagonal algebra in $\mathcal{R}_n$ with respect to $\widehat{\mathcal{E}}\big|_{\mathcal{R}_n}$.*

Proof. Since $\widehat{\mathcal{E}}$ is multiplicative on $\widehat{\mathcal{A}}$, $\widehat{\mathcal{E}}\big|_{\mathcal{R}_n}$ is multiplicative on $\mathcal{A}_n$. On the other hand,

$$\mathcal{A}_n \cap \mathcal{A}_n^* = \widehat{\mathcal{A}} \cap \widehat{\mathcal{A}}^* \cap \mathcal{R}_n = \mathcal{S} \cap \mathcal{R}_n = \mathcal{S}_n.$$

Thus it remains to show the σ-weak density of $\mathcal{A}_n + \mathcal{A}_n^*$ in $\mathcal{R}_n$. Let $x \in \mathcal{R}_n$. Since $\widehat{\mathcal{A}} + \widehat{\mathcal{A}}^*$ is σ-weakly dense in $\mathcal{R}$, there are $a_i, b_i \in \widehat{\mathcal{A}}$ such that

$$x = \lim_i (a_i + b_i^*) \quad \sigma\text{-weakly}.$$

Then by the normality of Φ_n, we have

$$x = \Phi_n(x) = \lim_i (\Phi_n(a_i) + \Phi_n(b_i)^*) \quad \sigma\text{-weakly}.$$

However, by (2.9), (2.10) and the assumption that $\mathcal{A}$ is σ_t^{φ}-invariant, we easily deduce that $\widehat{\mathcal{A}}$ is Φ_n-invariant for all $n \geqslant 1$. Hence, $\Phi_n(a_i), \Phi_n(b_i) \in \widehat{\mathcal{A}} \cap \mathcal{R}_n = \mathcal{A}_n$. It follows that $\mathcal{A}_n + \mathcal{A}_n^*$ is σ-weakly dense in $\mathcal{R}_n$. Thus $\mathcal{A}_n$ is a subdiagonal algebra with respect to $\widehat{\mathcal{E}}|_{\mathcal{R}_n}$. Note that as a by-product we have also proved $\mathcal{A}_n = \Phi_n(\mathcal{R}_n)$. ∎

We recall that if $\mathcal{A}$ is a subdiagonal algebra in $\mathcal{M}$ with respect to $\mathcal{E}$, then the maximal subdiagonal algebra containing $\mathcal{A}$ is

$$\mathcal{A}_{\max} = \{x \in \mathcal{M} : \mathcal{E}(\mathcal{A}x\mathcal{A}_0) = \mathcal{E}(\mathcal{A}_0 x\mathcal{A}) = 0\}.$$

LEMMA 3.3. *$\widehat{\mathcal{A}}$ is maximal.*

Proof. We must show $(\widehat{\mathcal{A}})_{\max} = \widehat{\mathcal{A}}$. Let $x \in (\widehat{\mathcal{A}})_{\max}$. Set $x_n = \Phi_n(x), n \geqslant 1$. We claim that $x_n \in (\mathcal{A}_n)_{\max}$. Indeed, let $a, b \in \mathcal{A}_n$ with $\widehat{\mathcal{E}}(b) = 0$. Then $a, b \in \mathcal{A} \cap \mathcal{R}_n$. Since Φ_n is a conditional expectation with respect to $\mathcal{R}_n$, by (3.5), we have

$$\widehat{\mathcal{E}}(ax_nb) = \widehat{\mathcal{E}}(a\Phi_n(x)b) = \widehat{\mathcal{E}}(\Phi_n(axb)) = \Phi_n(\widehat{\mathcal{E}}(axb)) = 0.$$

This yields our claim. However, by Lemma 3.2 and Exel's theorem, $\mathcal{A}_n$ is maximal. Hence $x_n \in \widehat{\mathcal{A}}$ for $n \geqslant 1$. On the other hand, (2.7) implies that $x_n \to x$ σ-weakly. Since $\widehat{\mathcal{A}}$ is σ-weakly closed, we conclude that $x \in \widehat{\mathcal{A}}$. Therefore, $\widehat{\mathcal{A}}$ is maximal. ∎

Finally, we are ready to prove our main theorem.

Proof of Theorem 1.1. Applying the preceding discussion to $\mathcal{A}_{\max}$ in the place of $\mathcal{A}$, we get a subdiagonal algebra $\widehat{\mathcal{A}_{\max}}$ of $\mathcal{R}$ with respect to $\widehat{\mathcal{E}}$. Since $\mathcal{A} \subset \mathcal{A}_{\max}$, $\widehat{\mathcal{A}} \subset \widehat{\mathcal{A}_{\max}}$. However, by Lemma 3.3, $\widehat{\mathcal{A}}$ is maximal. Hence $\widehat{\mathcal{A}} = \widehat{\mathcal{A}_{\max}}$. Consequently, for any $x \in \mathcal{A}_{\max}, \pi(x) \in \widehat{\mathcal{A}_{\max}} = \widehat{\mathcal{A}}$. Then necessarily, $x \in \mathcal{A}$. Thus $\mathcal{A} = \mathcal{A}_{\max}$, and so $\mathcal{A}$ is maximal. ∎

4. A GENERALIZATION

It is not clear to the author at the time of this writing whether the state φ in Theorem 1.1 can be replaced by a semifinite normal faithful weight (keeping all other assumptions). The author is able to prove this only for normal faithful weights whose restrictions to $\mathcal{D}$ are strictly semifinite. Recall that a weight φ on $\mathcal{M}$ is said to be strictly semifinite if there is a family $\{\psi_j\}_{j\in J}$ of normal positive functionals whose supports are pairwise disjoint and such that

$$\varphi = \sum_{j\in J} \psi_j.$$

This is equivalent to saying that φ is semifinite on the centralizer $\mathcal{M}_\varphi$. Our main theorem can be extended to weights as follows.

THEOREM 4.1. *Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{E}$ a normal faithful conditional expectation from $\mathcal{M}$ onto a von Neumann subalgebra $\mathcal{D} \subset \mathcal{M}$. Let $\mathcal{A}$ be a subdiagonal algebra of $\mathcal{M}$ with respect to $\mathcal{E}$. If there is a normal faithful weight φ on $\mathcal{M}$ such that $\varphi|_{\mathcal{D}}$ is strictly semifinite on $\mathcal{D}$, $\mathcal{E}$ commutes with σ_t^φ and $\mathcal{A}$ is σ_t^φ-invariant, then $\mathcal{A}$ is maximal.*

As a corollary, we get the following generalization of Exel's theorem to the semifinite case. See [7] for a related result.

COROLLARY 4.2. *Let $\mathcal{A}$ be a subdiagonal algebra of $\mathcal{M}$ with respect to $\mathcal{E}$. If there is a normal semifinite faithful trace τ on $\mathcal{M}$ such that τ is semifinite on $\mathcal{D}$, then $\mathcal{A}$ is maximal.*

The proof of Theorem 4.1 above can be reduced to the state case via a standard way. Indeed, let φ be a weight as in the theorem and consider again the crossed product $\mathcal{R} = \mathcal{M} \rtimes_{\sigma^\varphi} G$. Using the strict semifiniteness and the construction in Section 2, one can prove that there is an increasing family $\{\mathcal{R}_i\}_{i\in I}$ of w^*-closed $*$-subalgebras of $\mathcal{R}$ satisfying the following properties:

(i) each $\mathcal{R}_i$ is finite and σ-finite;
(ii) the union of all $\mathcal{R}_i$ is w^*-dense in $\mathcal{R}$;
(iii) the identity p_i of $\mathcal{R}_i$ belongs to $\mathcal{R}_{\widehat{\varphi}}$;
(iv) there is a normal conditional expectation Φ_i from $\mathcal{R}$ onto $\mathcal{R}_i$ such that

$$\widehat{\varphi} \circ \Phi_i = p_i \widehat{\varphi} p_i \quad \text{and} \quad \sigma_t^{\widehat{\varphi}} \circ \Phi_i = \Phi_i \circ \sigma_t^{\widehat{\varphi}}, \quad t \in \mathbb{R}, i \in I;$$

(v) for all $i, j \in I$ with $i \leqslant j$,

$$\Phi_i \circ \Phi_j = \Phi_j \circ \Phi_i = \Phi_i.$$

We refer to [10] for more details. Then repeating the arguments in Section 3, we can prove Theorem 4.1. We omit all details.

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