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TWO REFORMULATIONS OF KADISON'S SIMILARITY PROBLEM

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ABSTRACT. First, we prove that Kadison's similarity problem is equivalent to a problem about the invariant operator ranges of a single operator. We construct an operator T on a separable Hilbert space such that Kadison's problem is equivalent to deciding if Dixmier's invariant operator range problem is true for each of the operators $\{T \otimes I_n\}$, where I_n denotes the identity operator on a Hilbert space of dimension n with n a countable cardinal. We prove that the answer to Dixmier's invariant operator range problem is affirmative when n is finite.

Second, using Pisier's theory of similarity and factorization degree, we prove that the answer to Kadison's problem is affirmative if and only if there exists a "universal factorization formula" of the type considered by Pisier, consisting of a particular set of scalar matrices and a set of polynomials in non-commuting variables. This formula would factor matrices over *any* C^*-algebra into products of scalar matrices and diagonal matrices, where the entries of the diagonal matrices are determined by the non-commutative polynomials.

KEYWORDS: *Completely bounded, similarity, operator range.*

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1. INTRODUCTION

It was proved by Dixmier [11] that a bounded amenable group of operators on a Hilbert space is similar to a group of unitary operators. However, L. Ehrenpreis and F. Mautner [13] gave examples showing that this result fails when the amenability assumption is removed. Thus, there exist bounded homomorphisms of groups into the group of invertible operators on a Hilbert space, that are not similar to unitary representations of the group.

The analogous problem for representations of C^*-algebras, known as Kadison's similarity problem, is still an open problem. This problem of R. Kadison [25] asks whether every bounded homomorphism of a C^*-algebra into the algebra of operators on a Hilbert space is similar to a $*$-homomorphism.

In this paper we present two reformulations of Kadison's similarity problem, one coming from invariant operator ranges and the other from Pisier's theory of similarity and factorization degree. Each reformulation reduces the problem to the study of a special case. In particular, we will first show that Kadison's problem is equivalent to a problem concerning the invariant operator ranges of a single operator. Next, we will show that the assumption that Kadison's problem is true is equivalent to the existence of certain "universal factorization formulas". These formulas are fixed expressions involving words in a free algebra and scalar matrices that yield, upon substitution of elements from *any* C^*-algebra for the variables, the factorization of a norm one matrix of elements from the C^*-algebra as a product of scalar matrices and diagonal matrices from the C^*-algebra. What is, precisely, meant by this will be made clear in later sections.

We begin with a brief history and some explanatory comments. In [20] U. Haagerup proved that every homomorphism with a cyclic vector is similar to a $*$-homomorphism. Moreover, Haagerup proved [20] that a bounded homomorphism on a C^*-algebra is similar to a $*$-homomorphism if and only if the homomorphism is completely bounded. A similar result was obtained by E. Christensen [9]. Around the same time the first author [22] proved that a bounded homomorphism is similar to a $*$-homomorphism if and only if it is a linear combination of completely positive maps, and Wittstock [48] proved that the latter condition is equivalent to the map being completely bounded. Much of this work was organized and simplified by the second author in [32].

Suppose $\mathcal{A}$ is a C^*-algebra and $\rho : \mathcal{A} \to B(H)$ is a bounded but not necessarily completely bounded homomorphism. If for each positive integer n we choose a norm-one matrix $A_n \in \mathcal{M}_n(\mathcal{A})$ such that $\|\rho_n(A_n)\| \geqslant \|\rho_n\|/2$, and let $\mathcal{A}_0$ be the C^*-algebra generated by the entries of all the A_n's, we see that $\mathcal{A}_0$ is separable and $\rho|\mathcal{A}_0$ is not completely bounded. Hence Kadison's similarity problem reduces to the case of separable C^*-algebras. In this case ρ is a direct sum of homomorphisms into separable Hilbert spaces, and we only need countably many of these summands to maintain the lack of complete boundedness. Hence we may also assume that the Hilbert space is separable. All of these facts are well-known.

E. Christensen [8] proved that the inner derivation problem for von Neumann algebras is equivalent to the question of whether every von Neumann algebra is hyperreflexive. Recently, E. Kirchberg [26] proved that these two questions are equivalent to Kadison's similarity problem.

We will be concerned with two other equivalent reformulations of Kadison's problem. The first arises from Dixmier's invariant operator range problem [10] and the second from Pisier's theory of similarity and factorization degree [40].

2. INVARIANT OPERATOR RANGES

C. Foias [17] proved that an affirmative answer to Kadison's similarity problem would imply an affirmative answer to a problem of J. Dixmier [11] concerning invariant operator ranges. Dixmier's problem asks: if the range of an operator T is invariant for every operator in a von Neumann algebra $\mathcal{M}$, then is there an operator $S \in \mathcal{M}'$ with $\text{ran}S = \text{ran}T$? It was proved by S.-C. Ong [29] that the invariant operator ranges for a C^*-algebra are the same as those of the von Neumann algebra it generates. Hence Dixmier's problem is the same for C^*-algebras as for von Neumann algebras.

G. Pisier ([38], Theorem 10.5) proves that the answer to Kadison's problem is affirmative if and only if the answer to Dixmier's problem is affirmative, thus showing the converse to Foias' result.

The precise connection between these two problems is stated below. We present a slightly different proof of the fact that Dixmier's problem implies Kadison's problem than appears in Pisier's book.

PROPOSITION 2.1. *If $\mathcal{A} \subset B(H)$ is a C^*-algebra, such that every bounded homomorphism of $\mathcal{A}$ is similar to a $*$-homomorphism and the range of T is invariant for $\mathcal{A}$, then there is $S \in \mathcal{A}'$ with* $\text{ran}S = \text{ran}T$. *Conversely, if $\mathcal{A}$ is a C^*-algebra such that for every $*$-homomorphism $\tau : \mathcal{A} \to B(H)$ and for every operator $T \in B(H)$ whose operator range is invariant for $\tau(\mathcal{A})''$, there is an operator $S \in \tau(\mathcal{A})''$ with* $\text{ran}S = \text{ran}T$, *then every bounded homomorphism of $\mathcal{A}$ into $B(H)$ is similar to a $*$-homomorphism. Consequently, Kadison's similarity problem is equivalent to Dixmier's operator range problem.*

Proof. For a proof of the first statement see [17] or Theorem 10.5 of [38].

For the converse, suppose that Dixmier's question has an affirmative answer for $\mathcal{A}$, and suppose that $\tau : \mathcal{A} \to B(H)$ is a bounded homomorphism. It was proved by E. Christensen [9] that there is a positive operator W with $\ker W = 0$ and a $*$-homomorphism $\sigma : \mathcal{A} \to B(H)$ such that, for every $x \in \mathcal{A}$,

$$\pi(x)W = W\tau(x).$$

It follows that $\text{ran}W$ is invariant for $\pi(\mathcal{A})$. Since Dixmier's problem has an affirmative solution, there is an operator $V \in \pi(\mathcal{A})'$ such that $\text{ran}V = \text{ran}W$. It follows from the closed graph theorem that $D = V^{-1}W$ is a bounded invertible operator whose inverse is $W^{-1}V$. However, for each $x \in \mathcal{A}$, we have

$$D^{-1}\pi(x)D = W^{-1}V\pi(x)V^{-1}W = W^{-1}\pi(x)VV^{-1}W = W^{-1}\pi(x)W = \tau(x).$$

Hence, τ is similar to a $*$-homomorphism. ∎

COROLLARY 2.2. *Kadison's similarity problem is equivalent to the assertion that for every separable C^*-subalgebra $\mathcal{A}$ of $B(\ell^2)$ and every positive injective operator T such that* $\text{ran}T$ *is invariant for every operator in $\mathcal{A}$, the map $\rho(A) = T^{-1}AT$ is completely bounded.*

We now use ideas of E.A. Nordgren, M. Radjabalipour, H. Radjavi and P. Rosenthal [28] to reduce the condition in the preceding corollary to a single operator T.

THEOREM 2.3. *Suppose* $H = \ell^2 \oplus \ell^2 \oplus \cdots$ *and* $D = 1 \oplus (1/2) \oplus (1/4) \oplus \cdots \in B(H)$. *Then Kadison's similarity problem is equivalent to the assertion that, for every separable* C^**-subalgebra of* $B(H)$ *leaving* $\operatorname{ran} D$ *invariant, the map* $\tau(A) = D^{-1}AD$ *is completely bounded.*

Proof. Assume that Kadison's similarity problem has a negative answer. By the preceding corollary, there is a $T \in B(\ell^2)$, $T \geqslant 0$, $\ker T = 0$ and a separable unital C^*-subalgebra $\mathcal{A}$ of $B(\ell^2)$ that leaves $\operatorname{ran} T$ invariant such that the map $\rho(A) = T^{-1}AT$ is not completely bounded. Since ρ is unaffected when T is replaced by $T/\|T\|$, there is no harm in assuming that $\|T\| = 1$.

Suppose $\pi : \mathcal{A} \to B(\ell^2)$ is a faithful unital representation of $\mathcal{A}$. If we replace $\mathcal{A}$ with $\{A \oplus \pi(A) \oplus \pi(A) \oplus \cdots : A \in \mathcal{A}\}$ and replace T with $T \oplus 1 \oplus (1/2) \oplus (1/4) \oplus \cdots$, we still get that the induced mapping ρ is not completely bounded. Hence we can assume that each of the numbers $1/2^n$, $n = 0, 1, 2, 3, \ldots$, is an eigenvalue of T with infinite multiplicity. For each $n \in \mathbb{N}$, let P_n be the spectral projection of T corresponding to the set $(1/2^n, 1/2^{n-1}]$. Since $\|T\| = 1$, $T \geqslant 0$ and $\ker T = 0$, we have $1 = P_1 + P_2 + \cdots$. Clearly, each P_n has infinite rank. Let $D = \sum_{n \in \mathbb{N}} (1/2^n) P_n$. Then $D \leqslant T \leqslant 2D$. It follows that $\operatorname{ran} T = \operatorname{ran} D$ and the map $\tau(A) = D^{-1}AD$ is also not completely bounded.

Using unitary equivalence, we obtain a separable C^*-subalgebra $\mathcal{A}$ of $B(H)$ leaving invariant the range of the operator $D = 1 \oplus (1/2) \oplus (1/4) \oplus \cdots$ such that the map $\tau(A) = D^{-1}AD$ is not completely bounded. ∎

REMARK 2.4. Note that the mapping $\tau(A) = D^{-1}AD$ is the Schur product of A with the matrix (2^{j-i}). In [19] J. Froelich and B. Mathes give an example of a linear subspace $\mathcal{S}$ of $B(H)$ and a matrix M such that the map φ on $\mathcal{S}$ defined as the Schur product with M is bounded on $\mathcal{S}$, but not completely bounded. If we let $\mathcal{A} = \left\{ \begin{pmatrix} \lambda & S \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C}, S \in \mathcal{S} \right\}$, then $\rho \begin{pmatrix} \lambda & S \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & \varphi(S) \\ 0 & \lambda \end{pmatrix}$ is the Schur product with the matrix $\begin{pmatrix} I & M \\ 0 & I \end{pmatrix}$, and ρ is bounded but not completely bounded. Moreover, ρ is a unital algebra homomorphism.

Now let us suppose M is a separable Hilbert space, $H = M \oplus M \oplus \cdots$, $D = \operatorname{diag}(1, 1/2, 1/4, \ldots)$, let

$$\mathcal{D} = \{T \in B(H) : D^{-1}TD \in B(H)\},$$

and let

$$\mathcal{R} = \{D^{-1}TD : T \in \mathcal{D}\}.$$

It is clear that $\mathcal{R} = \{S \in B(H) : DSD^{-1} \in B(H)\} = \mathcal{D}^*$, and $\mathcal{D}$ is the set of all operators leaving ran$D$ invariant. Define $\tau : \mathcal{D} \to \mathcal{R}$ by $\tau(T) = D^{-1}TD$, and define $\tau^{-1} : \mathcal{R} \to \mathcal{D}$ by $\tau^{-1}(S) = DSD^{-1}$.

For each $n \in \mathbb{Z}$, define $\rho_n : B(H) \to B(H)$ by

$$\rho_n(A_{ij}) = (B_{ij}),$$

where $B_{ij} = 0$ if $j \neq i + n$ and $B_{ij} = A_{ij}$ if $j = i + n$. In other words ρ_n annihilates all the diagonals of an operator matrix except the n^{th} one, and leaves the n^{th} diagonal alone. We define $\sigma_n : B(H) \to B(H)$ by

$$\sigma_n(T) = \sum_{|k| \leqslant n} \rho_n.$$

The following proposition contains some of the basic results we need.

PROPOSITION 2.5. *Suppose $\mathcal{S} \subset \mathcal{D}$ and $\mathcal{T} \subset \mathcal{R}$ are norm-closed linear subspaces. The following are true:*

(i) *$\mathcal{R}$ and $\mathcal{D}$ are unital algebras, and τ, τ^{-1} are unital algebra homomorphisms.*

(ii) *$\tau|\mathcal{S}$ and $\tau^{-1}|\mathcal{T}$ are continuous.*

(iii) *$\|\rho_n|\mathcal{S}\| \leqslant \|\tau|\mathcal{S}\|/2^{|n|}$ and $\|\rho_n|\mathcal{S}\|_{\mathrm{cb}} \leqslant \|\tau|\mathcal{S}\|_{\mathrm{cb}}/2^{|n|}$ for $n \leqslant 0$.*

(iv) *$\|\rho_n|\mathcal{T}\| \leqslant \|\tau^{-1}|\mathcal{T}\|/2^n$ and $\|\rho_n|\mathcal{T}\|_{\mathrm{cb}} \leqslant \|\tau^{-1}|\mathcal{T}\|_{\mathrm{cb}}/2^n$ for $n \geqslant 0$.*

(v) *If $\mathcal{S}^* \subset \mathcal{D}$, then $\|\rho_n|\mathcal{S}\| \leqslant \|\tau|\mathcal{S}\|/2^{|n|}$ and $\|\rho_n|\mathcal{S}\|_{\mathrm{cb}} \leqslant \|\tau|\mathcal{S}\|_{\mathrm{cb}}/2^{|n|}$ for $n \in \mathbb{Z}$.*

(vi) *If $\mathcal{T}^* \subset \mathcal{R}$, then $\|\rho_n|\mathcal{T}\| \leqslant \|\tau^{-1}|\mathcal{T}\|/2^{|n|}$ and $\|\rho_n|\mathcal{T}\|_{\mathrm{cb}} \leqslant \|\tau^{-1}|\mathcal{T}\|_{\mathrm{cb}}/2^{|n|}$ for $n \in \mathbb{Z}$.*

(vii) *If $\dim M = d < \infty$, then $\|\rho_n\|_{\mathrm{cb}} \leqslant d\|\rho_n\|$ for $n \in \mathbb{Z}$.*

Proof. (i) This is obvious.

(ii) These follow from the closed graph theorem.

First note that, for every $A \in \mathcal{S}$, $B \in \mathcal{T}$, and every $n \in \mathbb{Z}$, $\rho_n(\tau(A)) = 2^n\rho_n(A)$, $\rho_n(\tau(A^*))^* = 2^n\rho_{-n}(A)$, and $\rho_n(\tau^{-1}(B)) = 2^{-n}\rho_n(B)$. From this fact, statements (iii), (iv), (v) and (vi) are immediate consequences.

To prove statement (vii) note that if $\phi : \mathcal{S} \to \mathcal{M}_d(\mathbb{C})$ is a linear map, then $\|\phi\|_{\mathrm{cb}} \leqslant d\|\phi\|$. Also the cb-norm of a direct sum of maps is the supremum of the cb-norms of the summands. It follows that if $\dim M = d$, then $\|\rho_n|\mathcal{W}\|_{\mathrm{cb}} \leqslant d\|\rho_n|\mathcal{W}\|$ for any linear subspace $\mathcal{W}$ of $B(H)$. ∎

When M is finite-dimensional, we will characterize the C^*-subalgebras of $\mathcal{D}$. In particular, it is known that Kadison's similarity problem has an affirmative answer for such algebras.

THEOREM 2.6. *If $\dim M = d < \infty$, and $\mathcal{A}$ is a unital C^*-algebra contained in $\mathcal{D}$, then there is a number N such that every irreducible representation of $\mathcal{A}$ is at most N-dimensional.*

Proof. We know from the proof of the preceding theorem that there is an n_0 so that $\sigma_{n_0} : \mathcal{A} \to \sigma_{n_0}(\mathcal{A})$ is a cb-isomorphism. Define a map $\alpha : \sigma_{n_0}(B(H)) \to$

$\sum_{|i-j|\leqslant n_0}^{\oplus} B(M)$ by

$$\alpha(A)(i,j) = A_{ij}.$$

Since α is a direct sum of complete contractions, it is clear that it is completely bounded. It is also clear that α is surjective. Moreover, for $-n_0 \leqslant k \leqslant n_0$, the map β_k that sends $\alpha(A)$ to $\rho_k(A)$ is a complete contraction, and since $\alpha^{-1} = \sum_{|k|\leqslant n_0} \beta_k$, we see that α^{-1} is completely bounded. Hence $\mathcal{A}$ is cb-isomorphic to a subspace of $\sum_{|i-j|\leqslant n_0}^{\oplus} B(M)$. Since M is finite-dimensional, every bounded map of an operator space into $\sum_{|i-j|\leqslant n_0}^{\oplus} B(M)$ is completely bounded. Therefore, every bounded map from an operator space into $\mathcal{A}$ is completely bounded. By Theorem 1.3 of [46], every C^*-algebra with this property must satisfy the conclusion of this theorem. ∎

In the case where $\dim M = \aleph_0$, the ideas in the preceding two theorems still give some information.

THEOREM 2.7. *Suppose* $\dim M = \aleph_0$ *and* $1 \in \mathcal{A} \subset \mathcal{D}$ *is a* C^**-algebra. The following are equivalent.*

(i) $\tau|\mathcal{A}$ *is completely bounded.*

(ii) *There is a positive integer* n_0 *such that* $\sigma_{n_0} : \tau(\mathcal{A}) \to \sigma_{n_0}(\tau(\mathcal{A}))$ *is a* cb-*isomorphism.*

Proof. (i) $\Rightarrow$ (ii). Suppose $\|\tau|\mathcal{A}\|_{\text{cb}} = C < \infty$. It follows from the proof of part (v) of Proposition 2.5 that, for each $n \in \mathbb{Z}$,

$$\|\rho_n|\mathcal{A}\|_{\text{cb}} \leqslant \frac{C}{2^{|n|}}.$$

Since $\sum_{n\in\mathbb{Z}} \|\rho_n|\mathcal{A}\|_{\text{cb}} < \infty$, it follows that $\sum_{n\in\mathbb{Z}} \rho_n|\mathcal{A}$ converges in the cb-norm to the identity mapping $\text{id}_{\mathcal{A}}$ on $\mathcal{A}$. It follows that, for some $n_0 \in \mathbb{N}$,

$$\|\text{id}_{\mathcal{A}} - \sigma_{n_0}|\mathcal{A}\|_{\text{cb}} < 1.$$

It follows that $\sigma_{n_0}|\mathcal{A}$ is a cb-isomorphism from $\mathcal{A}$ to $\sigma_{n_0}(\mathcal{A})$. It is clear that $\tau|\sigma_{n_0}(\mathcal{A})$ is a cb-isomorphism from $\sigma_{n_0}(\mathcal{A})$ to $\tau(\sigma_{n_0}(\mathcal{A})) = \sigma_{n_0}(\tau(\mathcal{A}))$. Hence taking inverse maps we get a cb-isomorphism from $\sigma_{n_0}(\tau(\mathcal{A}))$ to $\mathcal{A}$, and when we follow this map with the cb-map τ we get that the inverse of the map $\sigma_{n_0}|\tau(\mathcal{A})$ is a cb-map. Since $\sigma_{n_0}|\tau(\mathcal{A})$ is a cb-map, we have (ii) is true.

(ii) $\Rightarrow$ (i). This follows from the fact that if (ii) is true, then

$$\tau = (\sigma_{n_0}|\tau(\mathcal{A}))^{-1} \circ (\tau|\sigma_{n_0}(\mathcal{A})) \circ (\sigma_{n_0}|\mathcal{A})$$

is the composition of cb-maps. ∎

3. CONSEQUENCES OF PISIER'S FACTORIZATION DEGREE

Fix a C^*-algebra $\mathcal{A}$ and assume that every bounded homomorphism of $\mathcal{A}$ into $B(H)$ is similar to a $*$-homomorphism for every Hilbert space H. That is, assume that the answer to Kadison's similarity problem is affirmative for $\mathcal{A}$. Pisier [38] has proven that this is equivalent to the existence of an integer, d, called the *factorization degree* of $\mathcal{A}$ and a constant K, such that the following holds:

Given an n and any $(a_{i,j}) \in M_n(\mathcal{A})$ with $\|(a_{i,j})\| < 1$, there exist scalar matrices, $C_1, \dots, C_{d+1}$ of appropriate sizes with $\|C_1\| \cdots \|C_{d+1}\| < K$ and diagonal matrices, $D_1, \dots, D_d$, whose entries come from the unit ball of $\mathcal{A}$, such that $(a_{i,j}) = C_1 D_1 \cdots C_d D_d C_{d+1}$.

Note that in this formulation, the scalar matrices and the diagonal matrices depend on the particular element $(a_{i,j})$, and d and K depend on the particular algebra.

A further consequence of the result of G. Pisier [40] is that if the answer to Kadison's similarity problem is assumed to be affirmative, then there exist d and K that are independent of the particular C^*-algebra (see also [38], [33]). In this section, we will prove that if the answer is assumed to be affirmative, then the scalar matrices appearing in the above formula can not only be chosen independent of the particular element but also independent of the particular algebra. Moreover, in this case we will show that there are fixed polynomials in $2n^2$ non-commuting variables so that the entries of the diagonal matrices are given by evaluating these polynomials at $a_{1,1}, \dots, a_{n,n}, a_{1,1}^*, \dots, a_{n,n}^*$.

Thus, these scalar matrices and non-commuting polynomials serve to give a *universal factorization formula* that allows one to factor matrices over an arbitrary C^*-algebra. Conversely, if one had such a formula, then it is easy to show that any bounded homomorphism τ of any C^*-algebra is completely bounded with $\|\tau\|_{\mathrm{cb}} \leqslant K\|\tau\|^d$ and so the answer to Kadison's similarity problem is affirmative.

Consequently, we see that Kadison's problem is equivalent to the existence of these universal factorization formulas.

We begin by constructing some universal C^*-algebras. To this end let $\mathcal{F}_n$ denote the free, unital $*$-algebra with $2n^2$ generators, which we label $x_{1,1}, \dots, x_{n,n}$, $x_{1,1}^*, \dots, x_{n,n}^*$. We shall often refer to an element of $\mathcal{F}_n$ as a *non-commuting $*$-polynomial in n^2 variables.* Given any C^*-algebra $\mathcal{A}$ and n^2 elements of $\mathcal{A}$, $a_{1,1}, \dots, a_{n,n}$ there exists a unique $*$-homomorphism π of $\mathcal{F}_n$ into $\mathcal{A}$ given by setting $\pi(x_{i,j}) = a_{i,j}$. We call such a $*$-homomorphism π *admissible* provided that $\|(a_{i,j})\| \leqslant 1$, where the norm is taken in $M_n(\mathcal{A})$. We endow $\mathcal{F}_n$ with a (pre)C^*-seminorm by setting $\|p\| = \sup\{\|\pi(p)\| : \pi \text{ admissible}\}$ for $p \in \mathcal{F}_n$.

It is most likely the case that the above formula actually defines a norm on $\mathcal{F}_n$. We let $\mathcal{B}_n$ denote the C^*-algebra that one obtains by completing $\mathcal{F}_n$ in this C^*-seminorm. Clearly, $\mathcal{B}_n$ is the *universal C^*-algebra for the entries of a $n \times n$ contraction*.

Henceforth, when we wish to assume that the answer to Kadison's similarity problem is affirmative, we will simply say "*assume* KSP". As remarked above in this case we will let d and K denote the universal similarity degree and constant, respectively, that apply to all C^*-algebras.

We begin with a lemma that provides a useful estimate.

LEMMA 3.1. *Let $\mathcal{W}$ be an operator space and let $W \in M_n(\mathcal{W})$. Then we may factor $W = C_1 D C_2$ where C_1, C_2 are scalar matrices with $\|C_1\| \|C_2\| \leqslant n$ and a diagonal matrix $D \in M_{n^2}(\mathcal{W})$ whose diagonal entries are the entries of W, with $\|D\| \leqslant \|W\|$.*

Proof. Let C_1 be the $n \times n^2$ matrix whose i-th row is all 0's except for the $1 + (i-1)n$ through in entries which are 1's, let C_2 be the $n^2 \times n$ matrix whose j-th column is 0's except for the $j + kn$ entries which are 1's for $0 \leqslant k \leqslant (n-1)$, and let D be the diagonal matrix with $w_{i,j}$ for the $[j + (i-1)n]$-th diagonal entry. Since $\|C_1\| = \|C_2\| = \sqrt{n}$, the result follows. ∎

THEOREM 3.2. *Assume* KSP *and let (d,K) be the universal constants. Then for each natural number n, there exist scalar matrices $C_1, \dots, C_{d+1}$, of appropriate sizes, with $\|C_1\| \cdots \|C_{d+1}\| \leqslant (K+1)$ and diagonal matrices $D_1, \dots, D_d$, whose entries are from the unit ball of $\mathcal{F}_n$ such that the $n \times n$ matrix*

$$(x_{i,j}) = C_1 D_1 \cdots C_d D_d C_{d+1}$$

where $x_{ij}, 1 \leqslant i, j \leqslant n$, denote the generators of $\mathcal{F}_n$.

Proof. Set $X = (x_{i,j})$. Since $\|X\| = 1$ in $M_n(\mathcal{B}_n)$, given any $\varepsilon > 0$ there exists such a factorization of X where the norm of each scalar matrix is at most $(K+\varepsilon)^{1/(d+1)}$ and the entries of the diagonal matrices all come from the unit ball of $\mathcal{B}_n$. Because $\mathcal{F}_n$ is dense in $\mathcal{B}_n$ we may choose diagonal matrices $E_1, \dots, E_d$ whose entries are from the unit ball of $\mathcal{F}_n$ and satisfy $\|D_i - E_i\| < \varepsilon$ for all i.

Set $Y = C_1 E_1 \cdots C_d E_d C_{d+1}$, so that $Y \in M_n(\mathcal{F}_n)$ and $\|X - Y\| < d(K+\varepsilon)\varepsilon$. Since $W = (X - Y) \in M_n(\mathcal{F}_n)$, by the lemma, there exist scalar matrices, C_1', C_2' with $\|C_1'\| = \|C_2'\| \leqslant \sqrt{n\|W\|}$ and a diagonal matrix E_1' with entries in the unit ball of $\mathcal{F}_n$ such that $W = C_1' E_1' C_2'$.

Hence,

$$X = Y + W = (C_1, C_1')(E_1 \oplus E_1')(C_2 \oplus I)(E_2 \oplus I) \cdots (C_d \oplus I)(E_d \oplus I)(C_{d+1}, C'),$$

where the I's denote identity matrices of the appropriate sizes. Since ε was arbitrary, we may choose it sufficiently small that the products of the norms of the scalar matrices appearing in this product are at most $K+1$ and the result follows. ∎

REMARK 3.3. Given any C^*-algebra $\mathcal{A}$ and any matrix $(a_{i,j})$ in the unit ball of $M_n(\mathcal{A})$, there is a $*$-homomorphism $\pi : \mathcal{B}_n \to \mathcal{A}$ with $\pi(x_{i,j}) = a_{i,j}$. Applying π to the factorization given by the above theorem, we see that

$$(a_{i,j}) = C_1 \pi(D_1) \cdots C_d \pi(D_d) C_{d+1},$$

where $\pi(D_i)$ is the diagonal matrix obtained by applying π to each entry of D_i. Since the effect of π on elements in $\mathcal{F}_n$ is merely to substitute the $a_{i,j}$'s for the $x_{i,j}$'s and $a_{i,j}^*$'s for the $x_{i,j}^*$'s, we see that the non-commutative polynomials and scalar matrices appearing in the above theorem actually give a *universal factorization formula* that holds for the unit ball of $M_n(\mathcal{A})$ for any C^*-algebra $\mathcal{A}$.

By decomposing the above factorization into homogeneous terms, one can gain further insight into KSP. If $\|(a_{i,j})\| \leqslant 1$, then for any $0 \leqslant t \leqslant 1$, we have that $\|(ta_{i,j})\| \leqslant 1$ and consequently there exist $*$-homomorphisms, $\pi_t : \mathcal{B}_n \to \mathcal{B}_n$, defined by $\pi_t(x_{i,j}) = tx_{i,j}$, for $0 \leqslant t \leqslant 1$. If $p_j \in \mathcal{F}_n$ is a homogeneous polynomial of degree j, then it is easy to see that $\pi_t(p_j) = t^j p_j$.

While it is probably the case that the inclusion of $\mathcal{F}_n$ into $\mathcal{B}_n$ is one-to-one, that is, that the seminorm is actually a norm, the following result will be sufficient for our needs.

LEMMA 3.4. *Let $p \in \mathcal{F}_n$ be a non-commutative polynomial and let $p = \sum_{j=0}^{m} p_j$ be the decomposition of p into homogeneous terms of degree j. If $\|p\| = 0$, then $\|p_j\| = 0$ for every j.*

Proof. Applying the homomorphisms π_t, we see that $\left\| \sum_{j=0}^{m} t^j p_j \right\| = 0$, for all $t, 0 \leqslant t \leqslant 1$. Setting $t = 0$, yields $\|p_0\| = 0$, and hence $\left\| \sum_{j=1}^{m} t^{j-1} p_j \right\| = 0$, from which it follows that $\|p_1\| = 0$.

The result follows by repeating this argument. ∎

Now let $(x_{i,j}) = C_1 D_1 \cdots C_d D_d C_{d+1}$ be the factorization obtained in the above theorem. Decomposing each of the entries of the diagonal matrices D_i into a sum of homogeneous terms, we may write $D_i = \sum_j H_{i,j}$ where $H_{i,j}$ is a diagonal matrix each of whose terms is homogeneous of degree j. Applying π_t, we find

$$\begin{aligned}(tx_{i,j}) = (\pi_t(x_{i,j})) &= C_1 \pi_t(D_1) \cdots C_d \pi_t(D_d) C_{d+1} \\ &= \sum_{j_1,\dots,j_d} t^{j_1+\cdots+j_d} C_1 H_{1,j_1} \cdots C_d H_{d,j_d} C_{d+1}.\end{aligned}$$

By the above lemma, we may equate like powers of t in the above expression. Equating the terms of degree 0, yields

$$C_1 H_{1,0} C_2 \cdots C_d H_{d,0} C_{d+1} = 0.$$

Considering the terms of degree 1, leads to

$$(x_{i,j}) = \sum_{j_1+\cdots+j_d=1} C_1 H_{1,j_1} \cdots C_d H_{d,j_d} C_{d+1}.$$

Since $j_1 + \cdots + j_d = 1$ implies that each j_k is either 0 or 1, we have that this latter sum has exactly d terms and

$$(x_{i,j}) = C_1 H_{1,1} C_2 H_{2,0} \cdots C_d H_{d,0} C_{d+1} + \cdots + C_1 H_{1,0} C_2 \cdots C_d H_{d,1} C_{d+1}.$$

Note that each $H_{i,0}$ is a scalar matrix times the identity of $\mathcal{B}_n$ and so each term in this last sum is actually of the form $A_i H_{i,1} B_i$ where A_i and B_i are scalar matrices. Moreover, since $H_{i,0} = \pi_0(D_i)$, we have that $\|H_{i,0}\| \leqslant 1$ and consequently, $\|A_i\|\|B_i\| \leqslant \|C_1\| \cdots \|C_{d+1}\| \leqslant (K+1)$.

Thus, we have $(x_{i,j}) = A_1 H_{1,1} B_1 + \cdots + A_d H_{d,1} B_d$.

We let $\Lambda_n = \max\{\|H_{1,1}\|, \ldots, \|H_{d,1}\|\}$. Also, let $\mathcal{X}_n$ denote the n^2 dimensional subspace of $\mathcal{B}_n$ spanned by the $x_{i,j}$'s and let $\mathcal{W}_n$ denote the $2n^2$ dimensional subspace of $\mathcal{B}_n$ spanned by these elements and their adjoints.

We assume that the reader is familiar with the concept of the *maximal operator space of a normed space V*, denoted $\mathrm{MAX}(V)$. See [34] or [33] for a discussion of this concept.

LEMMA 3.5. *Assume that* KSP *is true, let d,K be the universal constants and let Λ_n be the constants obtained above. Then for every n, we have that*

$$\|(x_{i,j})\|_{\mathrm{MAX}(\mathcal{W}_n)} \leqslant d(K+1)\Lambda_n.$$

Proof. By the characterization of the MAX norms given in [34], we have that

$$\|(x_{i,j})\|_{\mathrm{MAX}(\mathcal{W}_n)} \leqslant \|A_1\|\|B_1\|\Lambda_n + \cdots + \|A_d\|\|B_d\|\Lambda_n \leqslant d(K+1)\Lambda_n. \quad \blacksquare$$

Let $\mathcal{T}_n$ denote the $n \times n$ matrices equipped with the trace class norm and let $e_{i,j}$ denote the standard basis.

LEMMA 3.6. *For every n, the map $\phi : \mathcal{X}_n \to \mathcal{T}_n$ given by $\phi(x_{i,j}) = e_{i,j}$, is an isometry.*

Proof. Assume that $\left\|\sum a_{i,j} x_{i,j}\right\| \leqslant 1$, and let $(b_{i,j})$ be in the unit ball of M_n. Since there is a $*$-homomorphism sending $x_{i,j}$ to the complex number $b_{i,j}$, we have that $\left|\sum a_{i,j} b_{i,j}\right| \leqslant 1$. Thus, $\left\|\sum a_{i,j} e_{i,j}\right\| \leqslant 1$ in $\mathcal{T}_n$ and ϕ is contractive.

To prove the converse, recall that the matrices of trace norm at most one are the convex hull of the matrices of the form $(\alpha_i \beta_j)$ where $v = (\alpha_1, \ldots, \alpha_n)$ and $w = (\beta_1, \ldots, \beta_n)$ are vectors in the unit ball of Hilbert space. Thus, to prove that ϕ^{-1} is contractive it is enough to observe that $\left\|\sum \alpha_i x_{i,j} \beta_j\right\| \leqslant 1$, since this quantity is the product $v(x_{i,j})w^{\mathrm{t}}$. $\blacksquare$

LEMMA 3.7. *For all n,*

$$\|(x_{i,j})\|_{\mathrm{MAX}(\mathcal{W}_n)} = \|(x_{i,j})\|_{\mathrm{MAX}(\mathcal{X}_n)} = n.$$

Proof. By the defining properties of the MAX operator space structure, whenever $X \subset W$, it follows that $\|(x_{i,j})\|_{\mathrm{MAX}(W)} \leqslant \|(x_{i,j})\|_{\mathrm{MAX}(X)}$.

By Lemma 3.1, it follows that for any $n \times n$ matrix $\|(x_{i,j})\|_{\mathrm{MAX}(X)} \leqslant n$.

Thus,

$$\|(x_{i,j})\|_{\mathrm{MAX}(\mathcal{W}_n)} \leqslant \|(x_{i,j})\|_{\mathrm{MAX}(\mathcal{X}_n)} \leqslant n.$$

Now given a contractive linear map $\psi : \mathcal{X}_n \to B(H)$ with $\psi(x_{i,j}) = T_{i,j}$, we obtain a contractive linear map $\Psi : \mathcal{W}_n \to B(H \oplus H)$ by setting $\Psi(x_{i,j}) = \begin{pmatrix} 0 & T_{i,j} \\ 0 & 0 \end{pmatrix}$ and $\Psi(x^*_{i,j}) = \begin{pmatrix} 0 & 0 \\ T^*_{i,j} & 0 \end{pmatrix}$. Using this construction it readily follows that

$$\|(x_{i,j})\|_{\mathrm{MAX}(\mathcal{W}_n)} = \|(x_{i,j})\|_{\mathrm{MAX}(\mathcal{X}_n)}.$$

Finally, if we let $E_{i,j}$ denote the standard matrix units for M_n, then it is readily checked that the map defined on $\mathcal{X}_n$ by sending $x_{i,j}$ to $E_{i,j}$ is a contraction, since the trace norm dominates the operator norm. But $\|(E_{i,j})\| = n$.

Thus, $\|(x_{i,j})\|_{\mathrm{MAX}(\mathcal{X}_n)} \geqslant n$, and the result follows. ∎

Combining the above results leads to the following theorem, giving a lower bound on the rate of growth of the constants Λ_n.

THEOREM 3.8. *Assume that* KSP *is true, let d,K be the universal constants and let Λ_n be the constants obtained above. Then for every n, we have that $n \leqslant d(K+1)\Lambda_n$.*

Recall that the constant Λ_n is the norm of the homogeneous term of degree 1 of a non-commutative polynomial whose norm is at most 1. Thus, if one could obtain estimates on the norms of these terms of degree 1 that grew at a rate less than n, one could prove that KSP is false.

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