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GENERALIZED NEVANLINNA FUNCTIONS WITH ESSENTIALLY POSITIVE SPECTRUM

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ABSTRACT. We introduce an indefinite analogue of the so-called Stieltjes class and provide some basic results on this *indefinite Stieltjes class*. Among them: The relation between the functions $q(z)$, $zq(z)$ and $zq(z^2)$, limit properties, a distributional representation. These results generalize well known properties of functions belonging to the Stieltjes class.

KEYWORDS: *Stieltjes class, indefinite inner product, distribution.*

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1. INTRODUCTION

In the study of the vibrating string the so-called Stieltjes class $\mathcal{S}$ of functions analytic in $\mathbb{C} \setminus [0, \infty)$ having the property that $q(z)$ as well as $zq(z)$ maps the upper half plane into itself plays a prominent role. A systematic investigation of the class $\mathcal{S}$ goes back to I.S. Kac and M.G. Krein, cf. [13], [14]. For example it is proved that a function q belongs to $\mathcal{S}$ if and only if it has an analytic continuation through $\mathbb{R}^-$ and satisfies $q(x) \geqslant 0$, $x \in \mathbb{R}^-$.

Functions belonging to the Stieltjes class possess some rather remarkable properties. Let us specify a couple of them: If $q \in \mathcal{S}$, then also the function $zq(z^2)$ maps the upper half plane into itself. If $q \in \mathcal{S}$, then the limit of $q(z)$ must exist when z tends to ∞ along the negative real axis. The class $\mathcal{S}$ is closed with respect to locally uniform limits. Every function $q \in \mathcal{S}$ has an integral representation of a particular kind.

In a paper of M.G. Krein and H. Langer [20] a class $\mathcal{N}_\kappa^+$ of functions was introduced which could be viewed as an indefinite generalization of the Stieltjes class. This class occurs in the investigation of the generalized string, a string which can carry negative point masses (electric charges) and dipols, cf. [21]. However, it turned out that $\mathcal{N}_\kappa^+$ is, in a way, not the proper indefinite analogue of $\mathcal{S}$.

It is our aim to introduce a proper indefinite analogue of the Stieltjes class and to derive basic results for this class of functions. After some general discussion we will focus on the correct analogues of the above mentioned properties of $\mathcal{S}$. The exact definition of the main objects of our studies, the classes $\mathcal{N}_\kappa$ of generalized Nevanlinna functions of negative index $\kappa \in \mathbb{N} \cup \{0\}$, the class $\mathcal{N}_{<\infty}^{\mathrm{ep}}$ of essentially positive generalized Nevanlinna functions and the class $\mathcal{N}_{<\infty}^{\mathrm{sym}}$ of symmetric generalized Nevanlinna functions will be given a couple of lines below in Definition 1.1.

Let us mention that a related kind of generalization of the Stieltjes class to an indefinite setting can be found in [7], where in fact operator valued functions are considered. In another work of V. Derkach and M. Malamud, cf. [6], subclasses $\mathcal{S}^{\pm\kappa} := \{q \in \mathcal{N}_0 : z^{\pm 1}q(z) \in \mathcal{N}_\kappa\}$ have been introduced and applied to the description of certain classes of generalized resolvents of a symmetric operator in a Hilbert space. Moreover, functions of these classes have been characterized in terms of their zeros and poles as well as in terms of the parameters of their integral representation. Most intimately related to our present paper is the work [5] of V. Derkach who introduced the classes

$$\mathcal{N}_\kappa^\nu := \{q \in \mathcal{N}_\kappa : zq(z) \in \mathcal{N}_\nu\}, \tag{1.1}$$

and applied the theory of these classes to the description of the generalized resolvents of a symmetric operator in a Pontryagin space. Note that $\mathcal{N}_\kappa^0 = \mathcal{N}_\kappa^+$ and $\mathcal{N}_0^\kappa = S^{+\kappa}$.

Let us describe the content of the present paper. In Section 2 we provide some rather general statements on symmetry in reproducing kernel Pontryagin spaces. This general treatment gives a more structural view on the results of Section 3, where we deal with the class $\mathcal{N}_{<\infty}^{\mathrm{sym}}$.

In Section 4 we prove Theorem 4.1, the first main result of this paper. It states, roughly speaking, that if $q \in \mathcal{N}_{<\infty}^{\mathrm{ep}}$, then $zq(z^2) \in \mathcal{N}_{<\infty}^{\mathrm{sym}}$. This is the analogue of the first of the above mentioned properties of the Stieltjes class. We will use Theorem 4.1 to obtain Proposition 4.8 which describes the relationship of the functions $q(z)$, $zq(z)$ and $zq(z^2)$. Moreover, we shall see that Theorem 4.1 implies the appropriate analogues of the mentioned limit properties of the Stieltjes class, see Proposition 4.11 and Proposition 4.12.

The final Section 5 is devoted to the indefinite analogue of the integral representation of a function $q \in \mathcal{S}$. This representation employs a certain class of distributions which occured already in [12] and [18] where distributions are used to obtain an *integral representation* of an arbitrary generalized Nevanlinna function. In the present context the task is to single out those distributions which give rise to functions of the class $\mathcal{N}_{<\infty}^{\mathrm{ep}}$ or $\mathcal{N}_{<\infty}^{\mathrm{sym}}$, respectively. This is the content of Theorem 5.9 which can be regarded as the second main result of this paper.

The theory of generalized Nevanlinna functions is most intimately related with the theory of selfadjoint operators in Pontryagin spaces and of course the notions of symmetric and essentially positive generalized Nevanlinna functions possess their proper analogy in the operator theoretic context. However, in this paper we shall rather take the viewpoint of complex analysis and do not go into operator theoretic topics. A thorough investigation in this direction can be found in [15].

Let us recall the notion of a (matrix valued) kernel function in general, and of a generalized Nevanlinna function in particular. Let $\Omega \subseteq \mathbb{C}$ be an open set and $K : \Omega \times \Omega \to \mathbb{C}^{n\times n}$. The function $K(w,z)$ is called an *analytic hermitian kernel* (*kernel*, for short) on Ω if it is analytic in the two variables z and $\overline{w}$ and satisfies $K(w,z) = K(z,w)^*$. We say that the kernel K has κ negative squares on Ω, where κ is a nonnegative integer, if for any finite set of points $z_1, \dots, z_m \in \Omega$ and elements $f_1, \dots, f_m \in \mathbb{C}^n$ the hermitian matrix

$$((K(z_i,z_j)f_i,f_j)_{\mathbb{C}^n})_{i,j=1}^m$$

has at most κ negative eigenvalues, and if for some choice of z_i, f_i this bound is actually attained. In this case we shall write $\operatorname{ind}_- K = \kappa$. If for no $\kappa \in \mathbb{N} \cup \{0\}$ this condition is fullfilled, we write $\operatorname{ind}_- K = \infty$.

Let Ω be an open subset of $\mathbb{C}$ and let $Q : \Omega \to \mathbb{C}^{n\times n}$ be an analytic function which satisfies $Q(\overline{z}) = Q(z)^*$ whenever both z and $\overline{z}$ belong to Ω. The Nevanlinna kernel of Q is defined as

$$L_Q(w,z) := \begin{cases} \frac{Q(z)-Q(w)^*}{z-\overline{w}} & z,w \in \Omega, z \neq \overline{w}, \\ Q'(z) & z = \overline{w} \in \Omega. \end{cases} \tag{1.2}$$

Then L_Q is an analytic hermitian kernel on Ω. We say that Q is a *generalized Nevanlinna function* if $\operatorname{ind}_- L_Q < \infty$, and put $\operatorname{ind}_- Q := \operatorname{ind}_- L_Q$. Moreover, we define

$$\mathcal{N}_\kappa^{n\times n} := \{Q : \operatorname{ind}_- Q = \kappa\}, \quad \mathcal{N}_{\leqslant\kappa}^{n\times n} := \{Q : \operatorname{ind}_- Q \leqslant \kappa\},$$
$$\mathcal{N}_{<\infty}^{n\times n} := \{Q : \operatorname{ind}_- Q < \infty\}.$$

In the scalar case $n = 1$ the upper index $n \times n$ will be suppressed.

A generalized Nevanlinna function Q, which is from the start defined on some open set Ω always has an analytic continuation to $\mathbb{C} \setminus \mathbb{R}$ with possible exception of finitely many points (in fact at most $2 \operatorname{ind}_- Q$ many) which are poles, see e.g. [19]. The number $\operatorname{ind}_- Q$ does not depend on the set Ω on which Q is defined. Hence we can always think of Q as being meromorphic on $\mathbb{C} \setminus \mathbb{R}$. However, the maximal domain of analyticity of a given function Q might also contain parts of the real axis.

DEFINITION 1.1. Let $\kappa \in \mathbb{N} \cup \{0\}$. A function $Q \in \mathcal{N}_\kappa^{n\times n}$ is said to be:
(i) *symmetric*, if $Q(-z) = -Q(z)$, i.e. if Q is odd.
(ii) *essentially positive*, if Q is analytic on $\mathbb{C} \setminus [0,\infty)$ with possible exception of finitely many poles.

The subset of $\mathcal{N}_\kappa^{n\times n}$ which consists of all symmetric (essentially positive) functions will be denoted by $\mathcal{N}_\kappa^{n\times n,\mathrm{sym}}$ ($\mathcal{N}_\kappa^{n\times n,\mathrm{ep}}$, respectively).

We will freely use selfexplanatory notation like $\mathcal{N}_{\leqslant\kappa}^{\mathrm{ep}}$, $\mathcal{N}_{<\infty}^{n\times n,\,\mathrm{sym}}$ etc., which is defined correspondingly.

2. SYMMETRY IN REPRODUCING KERNEL PONTRYAGIN SPACES

Basic objects of our studies are reproducing kernel Pontryagin spaces. Let us recall the necessary definitions.

A Pontryagin space is a linear space $\mathfrak{P}$ equipped with an inner product $[\cdot,\cdot]$ such that $\mathfrak{P}$ decomposes as the orthogonal and direct sum $\mathfrak{P} = \mathfrak{P}_1[\dot{+}]\mathfrak{P}_2$ of a Hilbert space $\mathfrak{P}_1$ and a finite dimensional anti-Hilbert space $\mathfrak{P}_2$. The dimension of $\mathfrak{P}_2$ in such a decomposition is independent of the decomposition and will be called the *negative index of* $\mathfrak{P}$:

$$\operatorname{ind}_-\mathfrak{P} := \dim \mathfrak{P}_2 \,.$$

A Pontryagin space carries a unique norm-topology which is induced by the inner product. In fact, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to a point x if and only if

$$\lim_{n\to\infty}[x_n,x_n] = [x,x], \quad \lim_{n\to\infty}[x_n,y] = [x,y], \quad y \in \mathfrak{P}\,.$$

For a detailed discussion of the concept of Pontryagin spaces we refer the reader to [11] or [3].

Consider a Pontryagin space $\mathfrak{P}$ whose elements f are vector-valued analytic functions on some fixed open set $\Omega \subseteq \mathbb{C}$, $f:\Omega\to\mathbb{C}^n$, and assume that the linear operations are defined pointwise. Then, for each $w\in\Omega$, the point evaluation function

$$\chi_w : \begin{cases} \mathfrak{P} & \to \quad \mathbb{C}^n \\ f & \mapsto \quad f(w) \end{cases}$$

is a linear functional on $\mathfrak{P}$. The space $\mathfrak{P}$ is called a *reproducing kernel Pontryagin space* if for each $w\in\Omega$ the functional χ_w is continuous. The space $\mathfrak{P}$ is a reproducing kernel space if and only if there exists a kernel function, that is a function $K:\Omega\times\Omega\to\mathbb{C}^{n\times n}$ which possesses the following properties:

(i) For all $w\in\Omega$ and $x\in\mathbb{C}^n$ we have $K(w,\cdot)x\in\mathfrak{P}$.

(ii) For all $f\in\mathfrak{P}$, $w\in\Omega$ and $x\in\mathbb{C}^n$

$$[f(\cdot),K(w,\cdot)x] = x^* f(w)\,. \tag{2.1}$$

A kernel function K is uniquely determined by the properties (i) and (ii) and will be referred to as the reproducing kernel of the space $\mathfrak{P}$. It satisfies $\operatorname{ind}_- K = \operatorname{ind}_-\mathfrak{P}$.

Conversely, every kernel K on Ω with $\operatorname{ind}_- K<\infty$ gives rise to a reproducing kernel Pontryagin space $\mathfrak{P}(K)$ with $\operatorname{ind}_-\mathfrak{P}(K) = \operatorname{ind}_- K$, which has K as its reproducing kernel. In fact, $\mathfrak{P}(K)$ can be defined as completion of the linear set

$$\operatorname{span}\{K(w,\cdot)x:\ w\in\Omega, x\in\mathbb{C}^n\}$$

with respect to an inner product defined according to (2.1). For a detailed account on reproducing kernel Pontryagin spaces we refer the reader to [1].

The following elementary lemma serves as a starting point for our subsequent investigations. It models the idea of symmetry in Pontryagin spaces. This concept should not be mixed up with the notion of "symmetry with respect to an involution" as considered e.g. in [9].

LEMMA 2.1. *Let $\mathfrak{P}$ be a Pontryagin space and let $M : \mathfrak{P} \to \mathfrak{P}$ be an involutory and isometric linear mapping, i.e. assume that*

$$M \circ M = I, \quad [Mf, Mg] = [f, g], \quad f, g \in \mathfrak{P}.$$

Then $M^* = M^{-1} = M$. Put $P_{\mathrm{e}} := \frac{1}{2}(I + M)$ and $P_{\mathrm{o}} := \frac{1}{2}(I - M)$. Then P_{e} and P_{o} are orthogonal projections onto the nondegenerated and closed subspaces

$$\mathfrak{P}_{\mathrm{e}} := \operatorname{ran} P_{\mathrm{e}} = \{f \in \mathfrak{P} : \; Mf = f\},$$
$$\mathfrak{P}_{\mathrm{o}} := \operatorname{ran} P_{\mathrm{o}} = \{f \in \mathfrak{P} : \; Mf = -f\}.$$

We have $\mathfrak{P} = \mathfrak{P}_{\mathrm{e}}[\dot{+}]\mathfrak{P}_{\mathrm{o}}$. In particular, $\operatorname{ind}_- \mathfrak{P} = \operatorname{ind}_- \mathfrak{P}_{\mathrm{e}} + \operatorname{ind}_- \mathfrak{P}_{\mathrm{o}}$.

If in addition it is assumed that $\mathfrak{P}$ is a reproducing kernel space of functions on some set Ω, then also $\mathfrak{P}_{\mathrm{e}}$ and $\mathfrak{P}_{\mathrm{o}}$ have this property. The respective kernel functions $K, K_{\mathrm{e}}, K_{\mathrm{o}}$ are related by

$$K_{\mathrm{e}} = P_{\mathrm{e}}K, \quad K_{\mathrm{o}} = P_{\mathrm{o}}K.$$

In particular, $K = K_{\mathrm{e}} + K_{\mathrm{o}}$.

Proof. Let f, g be given, then

$$[Mf, g] = [Mf, M(Mg)] = [f, Mg],$$

hence $M^* = M$. Since $M \circ M = I$, we have $P_{\mathrm{e}}^2 = P_{\mathrm{e}}$ and $P_{\mathrm{o}}^2 = P_{\mathrm{o}}$. Moreover, P_{e} and P_{o} are selfadjoint. Clearly $P_{\mathrm{e}} + P_{\mathrm{o}} = I$, and

$$P_{\mathrm{e}}P_{\mathrm{o}} = P_{\mathrm{o}}P_{\mathrm{e}} = 0.$$

Thus $\mathfrak{P} = \operatorname{ran} P_{\mathrm{e}}[\dot{+}] \operatorname{ran} P_{\mathrm{o}}$. All other assertions are obvious. ∎

The situation described in this lemma often arises from analytic involutions on Ω. Let $\Omega \subseteq \mathbb{C}$ be an open set and assume that $\lambda : \Omega \to \Omega$ is an analytic involution, i.e. λ is analytic and $\lambda \circ \lambda = \mathrm{id}_\Omega$. Then we can define a linear involution M_λ on the linear space $\mathcal{O}(\Omega)^n$ of all analytic functions of Ω into $\mathbb{C}^n$ by means of composition with λ:

$$M_\lambda : \left\{ \begin{array}{ccc} \mathcal{O}(\Omega)^n & \to & \mathcal{O}(\Omega)^n \\ f & \mapsto & f \circ \lambda \end{array} \right. .$$

LEMMA 2.2. *Let $\mathfrak{P}$ be a reproducing kernel space on a set Ω, so that $\mathfrak{P} \subseteq \mathcal{O}(\Omega)^n$, and let $\lambda : \Omega \to \Omega$ be an analytic involution. In order that composition with λ induces an isometric involution on $\mathfrak{P}$, i.e. that $M_\lambda|_{\mathfrak{P}}$ maps $\mathfrak{P}$ isometrically onto itself, it is necessary and sufficient that the reproducing kernel $K : \Omega \times \Omega \to \mathbb{C}^{n \times n}$ of $\mathfrak{P}$ satisfies*

$$K \circ (\lambda \times \lambda) = K, \tag{2.2}$$

or equivalently that $K(\lambda(\cdot),\cdot) = K(\cdot,\lambda(\cdot))$. *In this case we have*

$$\mathfrak{P}_{\mathrm{e}} = \{f \in \mathfrak{P} : f \circ \lambda = f\}, \quad \mathfrak{P}_{\mathrm{o}} = \{f \in \mathfrak{P} : f \circ \lambda = -f\}. \tag{2.3}$$

Proof. Assume that $M_\lambda|_{\mathfrak{P}}$ maps $\mathfrak{P}$ isometrically onto itself. Let $v, w \in \Omega$, $x, y \in \mathbb{C}^n$, then $K(\lambda(\cdot),\lambda(\cdot))y = M_\lambda K(\lambda(\cdot),\cdot)y \in \mathfrak{P}$ and we compute

$$\begin{aligned} x^*K(\lambda(v),\lambda(w))y &= [K(\lambda(v),\lambda(\cdot))y, K(w,\cdot)x] \\ &= [M_\lambda K(\lambda(v),\lambda(\cdot))y, M_\lambda K(w,\cdot)x] \\ &= [K(\lambda(v),\lambda(\lambda(\cdot)))y, K(w,\lambda(\cdot))x] \\ &= [K(\lambda(v),\cdot)y, K(w,\lambda(\cdot))x] \\ &= \overline{y^*K(w,\lambda(\lambda(v)))x} = x^*K(v,w)y. \end{aligned} \tag{2.4}$$

Since x, y and v, w were arbitrary, this just means that the condition (2.2) holds true.

Conversely, assume that (2.2) is valid. Then we have

$$M_\lambda K(w,\cdot)x = K(w,\lambda(\cdot))x = K(\lambda(w),\cdot)x, \quad w \in \Omega, x \in \mathbb{C}^n.$$

Hence M_λ maps the linear span

$$\mathcal{L} := \{K(v,\cdot)y : v \in \Omega, y \in \mathbb{C}^n\}$$

onto itself. Moreover, by reading the formula (2.4) backwards, we see that $M_\lambda|_{\mathcal{L}}$ is isometric. It follows that $M_\lambda|_{\mathcal{L}}$ extends to an isometry of $\mathfrak{P} = \overline{\mathcal{L}}$ onto itself (cf. [1]). Since point evaluation in $\mathfrak{P}$ is continuous, this extension must actually coincide with $M_\lambda|_{\mathfrak{P}}$.

The relation (2.3) is an immediate consequence of Lemma 2.1. ∎

The spaces $\mathfrak{P}_{\mathrm{e}}$ and $\mathfrak{P}_{\mathrm{o}}$ can be abstracted from their origin as subspaces of all functions in $\mathfrak{P}$ satisfying a certain functional equation. We start with the space $\mathfrak{P}_{\mathrm{e}}$.

LEMMA 2.3. *Let* $\mathfrak{P} \subseteq \mathcal{O}(\Omega)^n$ *be a reproducing kernel space,* $\lambda : \Omega \to \Omega$ *an analytic involution and assume that* (2.2) *is fullfilled. Let* $\mu \in \mathcal{O}(\Omega)$ *be such that for every* $z_0 \in \Omega$ *we have*

$$\{z \in \Omega : \mu(z) = \mu(z_0)\} = \{z_0, \lambda(z_0)\}, \tag{2.5}$$

and put $\Omega' := \mu(\Omega)$. *Then there exists a kernel* K_+ *on* Ω' *such that*

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "K_{\mathrm{e}}"] \arrow[d, "\mu\times\mu"'] & \mathbb{C}^{n\times n} \\ \Omega' \times \Omega' \arrow[ur, dashed, "K_+"'] & \end{tikzcd}$$

We have $\mathrm{ind}_- K_+ = \mathrm{ind}_- \mathfrak{P}_{\mathrm{e}}$ *and composition with* μ *yields an isometry of* $\mathfrak{P}(K_+)$ *onto* $\mathfrak{P}_{\mathrm{e}}$

$$\Psi_+ : \left\{ \begin{array}{ccc} \mathfrak{P}(K_+) & \to & \mathfrak{P}_{\mathrm{e}} \\ f & \mapsto & f \circ \mu \end{array} \right. .$$

Proof. For every $w \in \Omega$ the function $K_{\mathrm{e}}(w,\cdot)$ belongs to $\mathfrak{P}_{\mathrm{e}}$ and hence we have $K_{\mathrm{e}} \circ (\mathrm{id} \times \lambda) = K_{\mathrm{e}}$. Since K_{e} is hermitian, $K_{\mathrm{e}} \circ (\lambda \times \mathrm{id}) = K_{\mathrm{e}}$. We conclude from (2.5) that an analytic function K_+ is well defined by the relation $K_{\mathrm{e}} = K_+ \circ (\mu \times \mu)$. Obviously K_+ is hermitian.

In order to show that the mapping $f \mapsto f \circ \mu$ has the required isometry property, let $v \in \Omega'$ and choose $w \in \Omega$ such that $\mu(w) = v$; then we have

$$K_+(v, \mu(z)) = K_+(\mu(w), \mu(z)) = K_{\mathrm{e}}(w, z)\,.$$

It follows that composition with μ maps the linear space

$$\mathcal{L}_+ := \operatorname{span}\{K_+(v,\cdot)x :\ v \in \Omega', x \in \mathbb{C}^n\}$$

onto

$$\mathcal{L}_{\mathrm{e}} := \operatorname{span}\{K_{\mathrm{e}}(w,\cdot)x :\ w \in \Omega, x \in \mathbb{C}^n\}\,.$$

In fact this mapping is isometric ($\mu(w) = v$, $\mu(w') = v'$):

$$\begin{aligned}
[K_+(v,\cdot)x, K_+(v',\cdot)y] &= y^* K_+(v,v')x = y^* K_+(\mu(w),\mu(w'))x \\
&= K_{\mathrm{e}}(w,w') = [K_{\mathrm{e}}(w,z), K_{\mathrm{e}}(w',z)] \\
&= [K_+(v,\mu(z)), K_+(v',\mu(z))]\,.
\end{aligned}$$

Hence $\operatorname{ind}_- K_+ = \operatorname{ind}_- K_{\mathrm{e}}$ and composition with μ has an extension to an isometry of $\mathfrak{P}(K_+) = \overline{\mathcal{L}_+}$ onto $\mathfrak{P}_{\mathrm{e}} = \overline{\mathcal{L}_{\mathrm{e}}}$. Since in both spaces point evaluation is continuous, this extension is just composition with μ. ∎

Let us turn our attention to $\mathfrak{P}_{\mathrm{o}}$. There the situation is a bit more complicated.

LEMMA 2.4. *Let $\mathfrak{P} \subseteq \mathcal{O}(\Omega)^n$ be a reproducing kernel space, $\lambda : \Omega \to \Omega$ an analytic involution and assume that* (2.2) *is fullfilled. Moreover, let μ and Ω' be as in previous lemma. Let $m \in \mathcal{O}(\Omega)$ be such that:*

(i) *$m \circ \lambda = -m$;*

(ii) *all zeros of m are simple and are fixed points of λ.*

Then there exists a kernel K_- on Ω' such that

$$\begin{tikzcd}
\Omega \times \Omega \arrow[rr, "\frac{1}{\overline{m(w)}m(z)}K_{\mathrm{o}}(w,z)"] \arrow[d, "\mu\times\mu"'] & & \mathbb{C}^{n\times n} \\
\Omega' \times \Omega' \arrow[urr, dashed, "K_-"'] & &
\end{tikzcd}$$

We have $\operatorname{ind}_- K_- = \operatorname{ind}_- \mathfrak{P}_{\mathrm{o}}$. *Moreover, $\mathfrak{P}(K_-)$ and $\mathfrak{P}_{\mathrm{o}}$ are isometrically isomorphic via the mapping*

$$\Psi_- : \left\{ \begin{array}{ccc} \mathfrak{P}(K_-) & \to & \mathfrak{P}_{\mathrm{o}} \\ f & \mapsto & m \cdot (f \circ \mu) \end{array} \right. .$$

Proof. For every $w \in \Omega$ the function $K_o(w,\cdot)$ belongs to $\mathfrak{P}_o$ and hence satisfies the functional equation $K_o \circ (\mathrm{id} \times \lambda) = -K_o$. Since K_o is hermitian, it follows that also $K_o \circ (\lambda \times \mathrm{id}) = -K_o$. Let z_0 be a fixed point of λ, then $K_o(w,z_0) = 0$, $w \in \Omega$. Hence $m(z)^{-1}K_o(w,z)$ is analytic on Ω. Since K_o is hermitian, we conclude that $\widehat{K}(w,z) := \overline{m(w)^{-1}}m(z)^{-1}K_o(w,z)$ is analytic in z and $\overline{w}$. Moreover, $\widehat{K} \circ (\mathrm{id} \times \lambda) = \widehat{K}$ and $\widehat{K} \circ (\lambda \times \mathrm{id}) = \widehat{K}$, and it follows from (2.5) and our assumption (i) that the relation $\overline{m(w)^{-1}}m(z)^{-1}K_o(w,z) = K_-(\mu(w),\mu(z))$ defines an analytic function K_-. Obviously K_- is hermitian.

We consider the map $f \mapsto m \cdot (f \circ \mu)$. Let $v \in \Omega'$ and choose $w \in \Omega$ such that $\mu(w) = v$; then

$$m(z)K_-(v,\mu(z)) = m(z)K_-(\mu(w),\mu(z)) = \frac{K_o(w,z)}{\overline{m(w)}}.$$

It follows that composition with μ maps the linear space

$$\mathcal{L}_- := \operatorname{span}\{K_-(v,\cdot)x : v \in \Omega', m(\mu^{-1}(v)) \neq \{0\}, x \in \mathbb{C}^n\}$$

onto

$$\mathcal{L}_o := \operatorname{span}\{K_o(w,\cdot)x : w \in \Omega, x \in \mathbb{C}^n\}.$$

This mapping is isometric ($\mu(w) = v$, $\mu(w') = v'$):

$$\begin{aligned}
[K_-(v,\cdot)x, K_-(v',\cdot)y] &= y^*K_-(v,v')x = y^*K_-(\mu(w),\mu(w'))x \\
&= y^*\frac{K_o(\mu(w),\mu(w'))}{\overline{m(w)}m(z)}x\Big[\frac{K_o(w,\cdot)}{\overline{m(w)}}x, \frac{K_o(w',z)}{\overline{m(w')}}y\Big] \\
&= [m(z)K_-(v,\mu(z)), m(z)K_-(v',\mu(z))].
\end{aligned}$$

Thus $\mathrm{ind}_- K_- = \mathrm{ind}_- K_o$ and composition with μ extends to an isometry of $\mathfrak{P}(K_-) = \overline{\mathcal{L}_-}$ onto $\mathfrak{P}_o = \overline{\mathcal{L}_o}$. In both spaces point evaluation is continuous. Thus this extension must be equal to Ψ_-. ∎

3. SYMMETRIC NEVANLINNA FUNCTIONS

Let $Q \in \mathcal{N}_{<\infty}^{n\times n}$ and let L_Q denote the Nevanlinna kernel of Q, cf. (1.2). We are interested in the symmetry property of $\mathfrak{P}(L_Q)$ induced by the analytic involution $\lambda(z) := -z$. First of all note that, since a function $Q \in \mathcal{N}_{<\infty}^{n\times n}$ is analytic on $\mathbb{C} \setminus \mathbb{R}$ with possible exception of finitely many points, we can without loss of generality consider the kernel L_Q as a kernel being defined on some open set Ω with $\lambda(\Omega) = \Omega$. Hence the notions of Section 2 can be applied. For notational convenience we shall write $\mathfrak{P}_Q$ instead of $\mathfrak{P}(L_Q)$.

PROPOSITION 3.1. *Let $Q \in \mathcal{N}_{<\infty}^{n\times n}$. Then the mapping $M := M_\lambda : f(z) \mapsto f(-z)$ induces an isometric involution on $\mathfrak{P}_Q$ if and only if there exists a selfadjoint* *constant $a = a^* \in \mathbb{C}^{n\times n}$ such that (cf. Definition* 1.1*)*

$$a + Q \in \mathcal{N}_{<\infty}^{n\times n,\,\mathrm{sym}} .$$

Proof. First of all note that adding a selfadjoint constant to a function Q does not change the space $\mathfrak{P}_Q$. Hence, to prove the sufficiency of the given condition we may without loss of generality assume that Q is odd. Then

$$\begin{aligned} L_Q(-w,-z) &= \frac{Q(-z)-Q(-w)}{(-z)-(-w)} = \frac{-Q(z)+Q(w)}{-z+w} \\ &= \frac{Q(z)-Q(w)}{z-w} = L_Q(w,z)\,, \end{aligned}$$

i.e. the kernel relation (2.2) holds, and Lemma 2.2 yields that M is an isometry of $\mathfrak{P}_Q$ onto itself.

Assume conversely that M is an isometry of $\mathfrak{P}_Q$ onto itself. Choose $y_0 > 0$ such that Q is analytic at $\mathrm{i}y_0$ and put

$$a_0 := -\frac{Q(\mathrm{i}y_0)+Q(-\mathrm{i}y_0)}{2} = -\frac{Q(\mathrm{i}y_0)+Q(\mathrm{i}y_0)^*}{2} .$$

Then the function $Q_1 := a_0 + Q$ satisfies $Q_1(-\mathrm{i}y_0) = -Q_1(\mathrm{i}y_0)$. For the proof of necessity we may therefore assume without loss of generality that there exists a point z_0 such that $Q(-z_0) = -Q(z_0)$. From this and the validity of (2.2) we conclude that

$$\begin{aligned} \frac{-Q(-z)-Q(z_0)^*}{z-z_0} &= \frac{Q(-z)-Q(-z_0)^*}{(-z)-(-z_0)} = L_Q(-z,-z_0) \\ &= L_Q(z,z_0) = \frac{Q(z)-Q(z_0)^*}{z-z_0}\,, \end{aligned}$$

and hence that $-Q(-z) = Q(z)$ for all z. ∎

Assume that $Q \in \mathcal{N}_{<\infty}^{n\times n,\,\mathrm{sym}}$. If we make the choice $\mu(z) = z^2$ and $m(z) = z$ for the application of the Lemmata 2.3 and 2.4, we can obtain the spaces $\mathfrak{P}_+$ and $\mathfrak{P}_-$ by means of two Nevanlinna functions.

PROPOSITION 3.2. *Let $Q \in \mathcal{N}_{<\infty}^{n\times n,\,\mathrm{sym}}$ be given and define two functions Q_+, Q_- by the relations*

$$\frac{Q_+(z^2)}{z} = Q(z), \quad zQ_-(z^2) = Q(z)\,. \tag{3.1}$$

Then $Q_+, Q_- \in \mathcal{N}_{<\infty}^{n\times n}$ *and* $\operatorname{ind}_- Q_+ + \operatorname{ind}_- Q_- = \operatorname{ind}_- Q$. *We have (with the above made choice of* μ *and* m*)* $K_+ = L_{Q_+}$ *and* $K_- = L_{Q_-}$*, and the isomorphisms*

$$\begin{array}{ccc} & & \mathfrak{P}_Q \\ & & || \\ \mathfrak{P}_{Q_+} & \xrightarrow{f(z)\mapsto f(z^2)} & \mathfrak{P}_{Q,\mathrm{e}} \\ & & [\dot{+}] \\ \mathfrak{P}_{Q_-} & \xrightarrow{f(z)\mapsto z f(z^2)} & \mathfrak{P}_{Q,\mathrm{o}} \end{array}$$

Proof. First of all note that the choice of $\mu(z) = z^2$ and $m(z) = z$ is legitimate since the requirements on μ and m of the Lemmata 2.3 and 2.4 are met. Moreover, Q_+ and Q_- are well-defined since $Q(-z) = -Q(z)$. We compute $L_{Q,\mathrm{e}}$:

$$\begin{aligned} L_{Q,\mathrm{e}}(w,z) &= \frac{1}{2}\big(L_Q(w,z) + L_Q(w,-z)\big) = \frac{1}{2}\Big(\frac{Q(z)-Q(w)^*}{z-\overline{w}} + \frac{Q(-z)-Q(w)^*}{-z-\overline{w}}\Big) \\ &= \frac{1}{2}\Big(\frac{Q(z)-Q(w)^*}{z-\overline{w}} + \frac{Q(z)+Q(w)^*}{z+\overline{w}}\Big) = \frac{zQ(z)-\overline{w}Q(w)^*}{z^2-\overline{w}^2} \\ &= \frac{Q_+(z^2)-Q_+(w^2)^*}{z^2-\overline{w}^2} = L_{Q_+}(w^2,z^2)\,. \end{aligned}$$

Similarly one finds $L_{Q,\mathrm{o}}$:

$$L_{Q,\mathrm{o}}(w,z) = \frac{1}{2}\big(L_Q(w,z) - L_Q(w,-z)\big) = z\overline{w}L_{Q_-}(w^2,z^2)\,.$$

Hence $K_+ = L_{Q_+}$ and $K_- = L_{Q_-}$. The final assertions of the present proposition follow from Lemma 2.3 and Lemma 2.4. ∎

REMARK 3.3. (i) Note that the functions Q_+ and Q_- in (3.1) are related by the relation

$$Q_+(z) = zQ_-(z)\,.$$

(ii) The fact that $L_Q = L_{Q,\mathrm{e}} + L_{Q,\mathrm{o}}$ reflects in the kernel relation

$$L_Q(w,z) = L_{Q_+}(w^2,z^2) + \overline{w}zL_{Q_-}(w^2,z^2)\,, \tag{3.2}$$

which holds true by elementary calculation for every triple of analytic functions related by (3.1).

COROLLARY 3.4. *Let* $Q_- \in \mathcal{O}(\Omega)^{n\times n}$ *be given. Then the following are equivalent:*

(i) $zQ_-(z^2) \in \mathcal{N}_{<\infty}^{n\times n}$.

(ii) $Q_-(z), zQ_-(z) \in \mathcal{N}_{<\infty}^{n\times n}$.

In this case $\operatorname{ind}_- zQ_-(z^2) = \operatorname{ind}_- zQ_-(z) + \operatorname{ind}_- Q_-(z)$.

Proof. If we assume (i), we obtain from Proposition 3.2 applied to $Q(z) := zQ_-(z^2)$ that (ii) holds and that negative indices sum up. Conversely, if (ii) holds true, then the kernel relation (3.2) applied with $Q(z) = zQ_-(z^2)$, $Q_+(z) = zQ_-(z)$ shows that (i) holds. ∎

4. ESSENTIALLY POSITIVE NEVANLINNA FUNCTIONS

Theorem 4.1 below shows that the classes $\mathcal{N}_{<\infty}^{n\times n,\,\mathrm{sym}}$ and $\mathcal{N}_{<\infty}^{n\times n,\,\mathrm{ep}}$ are most intimately related. It has proved to be a powerful tool in our further investigations ([16]) and can be regarded as the first main result of this paper.

Before we come to the statement of this result let us recall that every rational function Q which satisfies $Q(\overline{z}) = Q(z)^*$ belongs to $\mathcal{N}_{<\infty}^{n\times n}$. In fact $\operatorname{ind}_- Q \leqslant n \cdot \deg Q$, where $\deg Q$ denotes the McMillan-degree of Q, see [2]. Moreover, if $r(z)$ is a scalar rational function and $Q \in \mathcal{N}_{<\infty}^{n\times n}$, then ($r^{\#}(z) := \overline{r(\overline{z})}$)

$$\widehat{Q}(z) := (r(z)r^{\#}(z))^{-1}Q(z) \in \mathcal{N}_{<\infty}^{n\times n}, \tag{4.1}$$

which can be verified using the kernel relation (cf. [4])

$$\begin{aligned} L_{\widehat{Q}}(\overline{w}, z) = {} & \frac{1}{r^{\#}(z)} \frac{Q(z) - Q(w)}{z - w} \frac{1}{r(w)} \\ & - \frac{r(z) - r(w)}{z - w} \frac{Q(z)}{r^{\#}(z) r(z) r(w)} - \frac{r^{\#}(z) - r^{\#}(w)}{z - w} \frac{Q(w)}{r^{\#}(z) r(w) r^{\#}(w)}. \end{aligned}$$

We see that in fact $\operatorname{ind}_- \widehat{Q} \leqslant \operatorname{ind}_- Q + 2n \cdot \deg r$.

THEOREM 4.1. *Assume that the function Q belongs to $\mathcal{N}_{<\infty}^{n\times n,\,\mathrm{ep}}$. Then $Q_1(z) := zQ(z^2)$ belongs to $\mathcal{N}_{<\infty}^{n\times n,\,\mathrm{sym}}$. Conversely, if Q is meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and $Q_1(z) := zQ(z^2) \in \mathcal{N}_{\kappa}^{n\times n,\,\mathrm{sym}}$, then $Q \in \mathcal{N}_{\leqslant\kappa}^{n\times n,\,\mathrm{ep}}$.*

Proof. Let $Q \in \mathcal{N}_{<\infty}^{n\times n,\,\mathrm{ep}}$. We have to show that $Q_1(z) \in \mathcal{N}_{<\infty}^{n\times n,\,\mathrm{sym}}$. First choose a rational function R such that $Q + R$ is analytic in $\mathbb{C} \setminus [0, \infty)$. If the assertion is proved for $Q + R$ it follows that

$$Q_1(z) = z(Q + R)(z^2) - zR(z^2) \in \mathcal{N}_{<\infty}^{n\times n,\,\mathrm{sym}}.$$

Hence we may assume without loss of generality that Q is analytic on $\mathbb{C} \setminus [0, \infty)$.

Recall from Proposition 2.1 of [4] that a generalized Nevanlinna function Q has an integral representation of the form

$$Q(z) = \prod_{j=1}^{s} ((z - \alpha_j)(z - \overline{\alpha_j}))^{-\rho_j} \left[(z^2 + y_0^2)^{\rho} \int_{-\infty}^{\infty} \frac{tz + y_0^2}{t - z} \, \mathrm{d}\Sigma(t) + \sum_{l=0}^{2\rho+1} B_l z^l \right] \tag{4.2}$$

with some nondecreasing and bounded $n \times n$-matrix function $\Sigma(t)$, nonnegative integers s, ρ_j, ρ, Hermitian matrices B_l, mutually different numbers $\alpha_j \in \mathbb{C}^+ \cup \mathbb{R}$ and $y_0 > 0$, $\mathrm{i}y_0 \neq \alpha_j$. This representation can be chosen such that the domain of holomorphy of Q is equal to the complement of the union of the support of $\mathrm{d}\Sigma$ and $\{\alpha_1, \overline{\alpha_1}, \dots, \alpha_s, \overline{\alpha_s}\}$. Conversely, every function represented in this way belongs to $\mathcal{N}_{<\infty}^{n\times n}$.

Since we assume that the function Q under consideration is analytic in $\mathbb{C} \setminus [0, \infty)$, we can choose $y_0 = 1$ and we know that $\alpha_j \in [0, \infty)$. The same argument as in the first paragraph of this proof shows that we may assume that $(d\Sigma)(\{0\}) = 0$. Let us recall that (cf. [20])

$$(1+z^2)^\rho \frac{tz+1}{t-z} = \Big(\frac{1}{t-z} - (t+z)\sum_{k=1}^{\rho+1}\frac{(1+z^2)^{k-1}}{(1+t^2)^k}\Big)(1+t^2)^{\rho+1} + z(1+z^2)^\rho .$$

The function

$$q(z) := \int_0^\infty \frac{tz+1}{t-z}\, d\Sigma(t)$$

belongs to $\mathcal{N}_0^{n\times n}$. It suffices to prove that $zq(z^2) \in \mathcal{N}_{<\infty}^{n\times n}$, since the assertion $zQ(z^2) \in \mathcal{N}_{<\infty}^{n\times n}$ will then follow from (4.1). The fact that $zQ(z^2)$ is odd is anyway obvious.

In view of Corollary 3.4 it is enough to show that $zq(z) \in \mathcal{N}_{<\infty}^{n\times n}$. This, however, is immediate from the identity

$$z\frac{tz+1}{t-z} = (1+z^2)\frac{t}{t-z} - 1 .$$

We proceed to the proof of the converse part. Let $Q_1 \in \mathcal{N}_\kappa^{n\times n,\,\mathrm{sym}}$, then by Proposition 3.2 we have $Q \in \mathcal{N}_{\leqslant\kappa}^{n\times n}$. Since Q_1 is meromorphic in $\mathbb{C}^+$, the function Q is meromorphic in $\mathbb{C}\setminus[0,\infty)$. Moreover, its nonreal poles correspond to the nonreal poles of Q_1 which lie off the imaginary axis, and its poles on the negative real half axis correspond to those on the imaginary axis. Alltogether, there can exist only finitely many poles in $\mathbb{C}\setminus[0,\infty)$. ∎

As a first consequence we shall formulate a connection with the classes $\mathcal{N}_\kappa^\nu$, cf. (1.1).

COROLLARY 4.2. *We have*

$$\mathcal{N}_\kappa^{\mathrm{ep}} = \bigcup_{\nu\in\mathbb{N}\cup\{0\}} \mathcal{N}_\kappa^\nu .$$

Let $Q \in \mathcal{N}_{<\infty}^{n\times n,\,\mathrm{ep}}$. By putting together the formulas of the proof of Theorem 4.1 we obtain an estimate for the number of negative squares of $zQ(z^2)$.

REMARK 4.3. The estimate given in the corollary below is very rough, however, the only thing of importance is to see that the negative index of $zQ(z^2)$ is bounded by a value which depends only on n, $\kappa(Q)$ and $\gamma(Q)$.

COROLLARY 4.4. *For $Q \in \mathcal{N}_{<\infty}^{n\times n,\,\mathrm{ep}}$ put $\kappa(Q) := \mathrm{ind}_- Q$ and let $\gamma(Q)$ denote the number of poles of Q in $\mathbb{C}\setminus[0,\infty)$ counted according to their multiplicities. Then we have*

$$\mathrm{ind}_-\, zQ(z^2) \leqslant l(\kappa(Q),\gamma(Q),n) .$$

Proof. With the notation of Theorem 4.1 we obtain

$$zQ(z^2) = \frac{(z^4+1)^\rho}{\prod\limits_{j=1}^{s}((z^2-\alpha_j)(z^2-\overline{\alpha_j}))^{\rho_j}} zq(z^2) + \frac{\sum\limits_{l=0}^{2\rho+1} B_l z^{2l+1}}{\prod\limits_{j=1}^{s}((z^2-\alpha_j)(z^2-\overline{\alpha_j}))^{\rho_j}} + \\ + \frac{(z^4+1)^\rho}{\prod\limits_{j=1}^{s}((z^2-\alpha_j)(z^2-\overline{\alpha_j}))^{\rho_j}} (-\Sigma(\{0\}) - zR(z^2)) \,.$$

Thereby the numbers ρ, ρ_j in (4.2) satisfy (cf. [4])

$$\rho \sum_{j=1}^{s} \rho_j \leqslant \kappa(Q) \,.$$

Moreover,

$$zq(z^2) = p(z) + (1+z^2) \int\limits_{-\infty}^{\infty} \frac{tz+1}{t-z} \,\mathrm{d}\widehat{\Sigma}(t)$$

where p is a polynomial of degree at most 3. The asserted estimate follows from the discussion on counting negative squares in the beginning of the present section. In fact we can choose $l(\kappa(Q), \gamma(Q), n) = 2n \cdot (2\gamma(Q) + 10\kappa(Q) + 5)$. ∎

EXAMPLE 4.5. Let us consider the particular case that $q \in \mathcal{N}_0$ is of the form

$$q(z) = \sum_{k=1}^{N} \frac{-\alpha_k}{t_k + z} - \frac{\alpha_0}{z} + \alpha + \beta z + \int\limits_{0+}^{\infty} \frac{\mathrm{d}\sigma(t)}{t-z}$$

where $t_k, \alpha_k > 0$, $\alpha_0, \beta \geqslant 0$, $\alpha \in \mathbb{R}$, and

$$\int\limits_{0+}^{\infty} \frac{\mathrm{d}\sigma(t)}{1+|t|} < \infty \,.$$

Then

$$zq(z^2) = \sum_{k=1}^{N} \frac{-\alpha_k}{2} \Big[\frac{1}{z + \mathrm{i}\sqrt{t_k}} + \frac{1}{z - \mathrm{i}\sqrt{t_k}}\Big] - \frac{\alpha_0}{z} + \int\limits_{0+}^{\infty} \frac{z}{t-z^2} \mathrm{d}\sigma(t) + \alpha z + \beta z^3 \,.$$

The integral term can be written as

$$\frac{1}{2}\int\limits_{0+}^{\infty} \Big[\Big(\frac{1}{u-z} - \frac{u}{1+u^2}\Big) - \Big(\frac{1}{u+z} - \frac{u}{1+u^2}\Big)\Big] \mathrm{d}\sigma(u^2) = \frac{1}{2}\int\limits_{-\infty}^{\infty} \Big(\frac{1}{v-z} - \frac{v}{1+v^2}\Big) \mathrm{d}\tau(v),$$

where $d\tau(u) = d\sigma(u^2)$ on the positive half axis and $d\tau(u) = d\tau(-u)$. This measure satisfies

$$\int_{-\infty}^{\infty} \frac{d\tau(v)}{1+v^2} < \infty ,$$

and, therefore, the integral term belongs to $\mathcal{N}_0$. We conclude that

$$zq(z^2) \in \mathcal{N}_\kappa ,$$

where

$$\kappa = N + \begin{cases} 0 & \beta = 0, \alpha \geqslant 0, \\ 1 & (\beta = 0, \alpha < 0) \text{ or } \beta > 0, \\ 2 & \beta < 0. \end{cases}$$

For the sake of simplicity we shall restrict ourselves for the rest of this paper to the scalar case $n = 1$.

The first task is to characterize those functions which might appear as q_+ (or q_-) in Proposition 3.2. In order to give an answer, we need one more lemma.

LEMMA 4.6. *Let $q \in \mathcal{N}_{<\infty}$ and assume that q is meromorphic on $\mathbb{R}^-$, i.e. can be considered as an analytic mapping of $\mathbb{C} \setminus [0, \infty)$ into the Riemann sphere $\mathbb{S}^2$. Then outside of a sufficiently large disk the poles and zeros of q are real, simple and interlace.*

The same assertion holds true when we consider functions q meromorphic on $\mathbb{R}^+$.

Proof. According to [8] we can write q as

$$q(z) = r(z) \cdot q_1(z) , \tag{4.3}$$

with some rational function r of the form

$$r(z) = \frac{\prod_{i=1}^{n_1} (z - \alpha_i)(z - \overline{\alpha_i})}{\prod_{i=1}^{n_2} (z - \beta_i)(z - \overline{\beta_i})} , \tag{4.4}$$

and a function $q_1 \in \mathcal{N}_0$ which again is meromorphic on $\mathbb{R}^-$.

Choose R such that all poles and zeros of $r(z)$ lie inside the disk with radius R. Then outside this disk the poles and zeros of q coincide with those of q_1. Since in every point $t_0 \in \mathbb{R}^-$ of analyticity of q_1 we have $q_1'(t_0) > 0$, between two poles of q_1 there must lie exactly one zero, i.e. the poles and zeros of q_1 interlace. ∎

Note that $\mathcal{N}_\kappa$ and $\mathcal{N}_\kappa^{\text{sym}}$ are closed with respect to the transformation $q \mapsto -\frac{1}{q}$. With the aid of the above lemma we obtain the same statement for $\mathcal{N}_\kappa^{\text{ep}}$.

COROLLARY 4.7. *We have $q \in \mathcal{N}_\kappa^{\text{ep}}$ if and only if $-\frac{1}{q} \in \mathcal{N}_\kappa^{\text{ep}}$.*

Proof. Assume that $q \in \mathcal{N}_\kappa^{\text{ep}}$. Then $-\frac{1}{q} \in \mathcal{N}_\kappa$. The poles of $-\frac{1}{q}$ located on $\mathbb{R}^-$ correspond to the zeros of q. By the above lemma q can have only finitely many zeros located on the negative half axis. ∎

PROPOSITION 4.8. *The following assertions are equivalent:*

(i) $q(z) \in \mathcal{N}_{<\infty}^{\mathrm{ep}}$.

(ii) $zq(z) \in \mathcal{N}_{<\infty}^{\mathrm{ep}}$.

(iii) $zq(z^2) \in \mathcal{N}_{<\infty}^{\mathrm{sym}}$.

In this case we have

$$\operatorname{ind}_- zq(z^2) = \operatorname{ind}_- q(z) + \operatorname{ind}_- zq(z)\,. \tag{4.5}$$

The condition (iii) *can be substituted by* $zq(z^2) \in \mathcal{N}_{<\infty}$, *since symmetry is anyway obvious.*

Proof. The equivalence of (i) and (iii) is just the statement of Theorem 4.1. By taking inverses we obtain the equivalence of (ii) and (iii): For if we put $\widehat{q}(z) := -(zq(z))^{-1}$, then $z\widehat{q}(z^2) = -(zq(z^2))^{-1}$. The validity of (4.5) was already proved in Proposition 3.2. ∎

An inductive application of the above proposition yields:

COROLLARY 4.9. *We have* $q \in \mathcal{N}_{<\infty}^{\mathrm{ep}}$ *if and only if*

$$z^k q(z) \in \mathcal{N}_{<\infty} \quad \text{for all } k \in \mathbb{Z}\,.$$

REMARK 4.10. From Proposition 4.8 and Corollary 3.4 it is obvious how the classes $\mathcal{N}_\kappa^{\mathrm{ep}}$ are related with the classes $\mathcal{N}_\kappa^+$: Recall that a function q is said to belong to $\mathcal{N}_\kappa^+$ if $q(z) \in \mathcal{N}_\kappa$ and $zq(z) \in \mathcal{N}_0$. We conclude that $\mathcal{N}_\kappa^+ \subseteq \mathcal{N}_\kappa^{\mathrm{ep}}$. This inclusion is also evident from the integral representation ([20], Satz 3.8) for functions of the class $\mathcal{N}_\kappa^+$.

An interesting consequence of (4.5) is that $q \in \mathcal{N}_\kappa^+$ if and only if $q \in \mathcal{N}_\kappa$ and $zq(z^2) \in \mathcal{N}_\kappa$.

The class $\mathcal{N}_0^+$ is nothing else but the Stieltjes class $\mathcal{S}$, cf. [14]. For an elaborate discussion of the connection of the function triple $q(z), zq(z), zq(z^2)$ with the theory of strings see also [17].

We are going to exploit Theorem 4.1 to obtain some more information on functions of the class $\mathcal{N}_{<\infty}^{\mathrm{ep}}$. First we bring a result dealing with the asymptotics of a generalized Nevanlinna function. It generalizes a property of functions of the Stieltjes class which goes back to the original definition of $\mathcal{S}$, compare (iii) of Proposition 4.11 below and [14]. After that we will deal with limits of sequences of generalized Nevanlinna functions.

Any function $q \in \mathcal{N}_{<\infty}$ can be considered as an analytic mapping of $\mathbb{C} \setminus \mathbb{R}$ into the Riemann sphere $\mathbb{S}^2$. The maximal domain of analyticity Ω of a given function $q \in \mathcal{N}_{<\infty}$ considered as mapping into $\mathbb{S}^2$ can be strictly larger than $\mathbb{C} \setminus \mathbb{R}$. For example $\Omega \supseteq \mathbb{C} \setminus [0,\infty)$ whenever $q \in \mathcal{N}_{<\infty}^{\mathrm{ep}}$.

We will always consider the Riemann sphere $\mathbb{S}^2$ as a metric space endowed with the spherical metric χ and of $\mathbb{C}$ as embedded in $\mathbb{S}^2$ by means of the stereographical projection. Thereby χ should be normalized so that $\chi(0,\infty) = 1$, $\chi(1,-1) = 1$, $\chi(1,\infty) = \frac{1}{\sqrt{2}}$, etc.

We shall establish some limit properties of generalized Nevanlinna functions, in particular of symmetric and essentially positive ones. This result can be seen in a fairly straightforward manner, however, for the convenience of the reader we shall include its proof.

PROPOSITION 4.11. *Let $q \in \mathcal{N}_{<\infty}$.*
(i) *For each $\delta > 0$ the limit*

$$\lim_{\substack{z\to\infty\\ \delta\leqslant\arg z\leqslant\pi-\delta}} \frac{1}{z}q(z) \tag{4.6}$$

exists as an element of $\mathbb{S}^2$ and belongs to $\mathbb{R}\cup\{\infty\}$.
(ii) *If $q \in \mathcal{N}_{<\infty}^{\mathrm{sym}}$, then for all but finitely many $y > 0$ we have $q(\mathrm{i}y) \in \mathrm{i}\mathbb{R}$. The limit*

$$\lim_{y\to+\infty} \frac{1}{\mathrm{i}y}q(\mathrm{i}y) \tag{4.7}$$

exists in the two-point compactification $\mathbb{R}\cup\{\pm\infty\}$ of $\mathbb{R}$.
(iii) *If $q \in \mathcal{N}_{<\infty}^{\mathrm{ep}}$, then for each $\delta > 0$ the limit*

$$\lim_{\substack{z\to\infty\\ \delta\leqslant\arg z\leqslant 2\pi-\delta}} q(z)$$

exists as an element of $\mathbb{S}^2$ and belongs to $\mathbb{R}\cup\{\infty\}$. Moreover, for all but finitely many $x < 0$ we have $q(x) \in \mathbb{R}$, and

$$\lim_{x\to-\infty} q(x)$$

exists in $\mathbb{R}\cup\{\pm\infty\}$.

Proof. (i) Consider a function $q_1 \in \mathcal{N}_0$. It follows from its integral representation

$$q_1(z) = a + bz + \int_{-\infty}^{\infty} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\,\mathrm{d}\mu(t)$$

that the limit $\lim\limits_{\substack{z\to\infty\\ \delta<\arg z<\pi-\delta}} z^n q_1(z)$ equals:

$$\begin{cases} \infty & n \geqslant 2,\\ -\int_{-\infty}^{\infty}\mathrm{d}\mu(t) \in (-\infty,0] & n=1, b=0, \int_{-\infty}^{\infty}\mathrm{d}\mu(t) < \infty, a = \int_{-\infty}^{\infty}\frac{t}{1+t^2}\mathrm{d}\mu(t),\\ \infty & n=1 \text{ and we are not in the above case},\\ b\in[0,+\infty) & n=-1,\\ 0 & n\leqslant -2. \end{cases} \tag{4.8}$$

Let $q \in \mathcal{N}_{<\infty}$ be given and consider the factorization $q = r\cdot q_1$ as in (4.3). Since $r \sim z^{2(n_1-n_2)}$, $z\to\infty$, we have

$$\frac{1}{z}q(z) \sim z^{2(n_1-n_2)-1}q_1(z),\ z\to\infty\,,$$

and the assertion follows from (4.8).

(ii) Since q is odd we have for $y > 0$ with $q(\mathrm{i}y) \neq \infty$ that

$$\overline{q(\mathrm{i}y)} = q(\overline{\mathrm{i}y}) = q(-\mathrm{i}y) = -q(iy)\,.$$

Hence $q(\mathrm{i}y) \in \mathrm{i}\mathbb{R}$ for such y.

If the limit (4.6) belongs to $\mathbb{C}$ then by the above said it must belong to $\mathbb{R}$. Assume that the limit (4.6) is equal to ∞, i.e. $|(\mathrm{i}y)^{-1}q(\mathrm{i}y)| \to \infty$ for $y \to +\infty$. Choose $C > 0$ such that for all $y \geqslant C$ we have $q(\mathrm{i}y) \neq \infty$ and $\chi((\mathrm{i}y)^{-1}q(\mathrm{i}y), \infty) \leqslant \frac{1}{\sqrt{2}}$. Then $(\mathrm{i}y)^{-1}q(\mathrm{i}y)$ is a continuous function of $[C, \infty)$ into $(-\infty, 1] \cup [1, \infty) \subseteq \mathbb{S}^2$ and, therefore, cannot change its sign. This shows that the limit (4.7) exists in $\mathbb{R} \cup \{\pm\infty\}$.

(iii) With $\widehat{q}(z) := zq(z^2)$ we have $\widehat{q} \in \mathcal{N}^{\mathrm{sym}}_{<\infty}$. Taking square roots maps angles $\delta < \arg z < 2\pi - \delta$ onto angles $\frac{\delta}{2} < \arg z < \pi - \frac{\delta}{2}$ and maps $\mathbb{R}^-$ to $\mathrm{i}\mathbb{R}^+$. Hence the assertion of (iii) follows from the already proved statements (i) and (ii) applied to $\widehat{q}$. ∎

Let us make the notion of convergence of a sequence of generalized Nevanlinna functions more precise. We provide the set $\mathcal{O}(\mathbb{C} \setminus \mathbb{R}, \mathbb{S}^2)$ of all analytic functions of $\mathbb{C} \setminus \mathbb{R}$ into $\mathbb{S}^2$ with the compact-open topology, that is to say with the topology of uniform convergence on compact subsets of $\mathbb{C} \setminus \mathbb{R}$. The set $\mathcal{N}_{<\infty}$ is always assumed to carry the subspace topology of $\mathcal{O}(\mathbb{C} \setminus \mathbb{R}, \mathbb{S}^2)$. Note that $\mathcal{N}_{<\infty}$ is not closed in $\mathcal{O}(\mathbb{C} \setminus \mathbb{R}, \mathbb{S}^2)$. By the theorem of Mittag-Leffler we in fact have

$$\overline{\mathcal{N}_{<\infty}} = \{f \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R}, \mathbb{S}^2) : f(\overline{z}) = \overline{f(z)}\}.$$

As in Corollary 4.4 we denote for $q \in \mathcal{N}^{\mathrm{ep}}_{<\infty}$ by $\gamma(q)$ the total number of poles of q in $\mathbb{C} \setminus [0, \infty)$.

PROPOSITION 4.12. *Let $q_n \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R}, \mathbb{S}^2)$, $n \in \mathbb{N}$, and assume that we have $\lim_{n\to\infty} q_n = q$ in $\mathcal{O}(\mathbb{C} \setminus \mathbb{R}, \mathbb{S}^2)$.*

(i) *If $q_n \in \mathcal{N}_{\leqslant\kappa}$ for all $n \in \mathbb{N}$, then $q \in \mathcal{N}_{\leqslant\kappa}$.*

(ii) *If $q_n \in \mathcal{N}^{\mathrm{sym}}_{\leqslant\kappa}$ for all $n \in \mathbb{N}$, then $q \in \mathcal{N}^{\mathrm{sym}}_{\leqslant\kappa}$.*

(iii) *If $q_n \in \mathcal{N}^{\mathrm{ep}}_{\leqslant\kappa}$ for all $n \in \mathbb{N}$ and*

$$\sup_{n\in\mathbb{N}} \gamma(q_n) < \infty\,, \tag{4.9}$$

then $q \in \mathcal{N}^{\mathrm{ep}}_{\leqslant\kappa}$.

Proof. (i) Let $K \subseteq \mathbb{C} \setminus \mathbb{R}$ be a compact subset with nonempty interior such that $\infty \notin q(K)$. For sufficiently large $n \in \mathbb{N}$ we have $\infty \notin q_n(K)$, and since convergence in $\mathcal{O}(\mathbb{C} \setminus \mathbb{R}, \mathbb{S}^2)$ implies pointwise convergence it follows that

$$\lim_{n\to\infty} L_{q_n}(w, z) = L_q(w, z), \quad w, z \in K\,.$$

This implies $\operatorname{ind}_- q \leqslant \sup_{n\in\mathbb{N}} \operatorname{ind}_- q_n$.

(ii) This statement is obvious from the already proved part (i) and the fact that pointwise convergence preserves the property of being odd.

(iii) Showing (iii) amounts to proving that q belongs to $\mathcal{N}^{\mathrm{ep}}_{<\infty}$ since by (i) we already know that $q \in \mathcal{N}_{\leqslant\kappa}$. To this end consider the sequence

$$\widehat{q}_n(z) := zq_n(z^2), \quad n \in \mathbb{N}.$$

By Theorem 4.1 we have $\widehat{q}_n \in \mathcal{N}^{\mathrm{sym}}_{<\infty}$ and by Corollary 4.4 our assumption (4.9) guarantees that

$$\widehat{\kappa} := \sup_{n\in\mathbb{N}} \operatorname{ind}_- \widehat{q}_n < \infty .$$

Choose a compact subset K of $\mathbb{C}^+$ with nonempty interior such that $\infty \notin q(K)$, and denote by $\widehat{K} \subset \mathbb{C}^+$ the image of K under the proper branch of the square root map. Then we have $\widehat{q}_n(z) \to \widehat{q}(z) := zq(z^2)$, $z \in \widehat{K}$. As we saw in the proof of (i) this implies $\widehat{q} \in \mathcal{N}_{\leqslant\widehat{\kappa}}$ and hence $q \in \mathcal{N}^{\mathrm{ep}}_{<\infty}$. ▮

Let us remark that the condition (4.9) in the assertion (iii) above is essential and cannot be dropped.

5. DISTRIBUTIONS ASSOCIATED TO SYMMETRIC AND ESSENTIALLY POSITIVE NEVANLINNA FUNCTIONS

In [12] and [18] it was shown that a generalized Nevanlinna function admits a representation similar to the integral representation

$$q(z) = a + bz + \int_{-\infty}^{\infty} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)(1+t^2)\,\mathrm{d}\mu$$

of an ordinary Nevanlinna function $q \in \mathcal{N}_0$. Thereby the measure μ has to be replaced by a certain distribution ϕ.

In our context a natural question arises: Which kind of distributions correspond in this representation to symmetric or essentially positive generalized Nevanlinna functions ?

It is the aim of this section to answer this question. However, first we would like to properly introduce the notion of distributions on $\overline{\mathbb{R}}$ and give a couple of useful lemmata. Our standard reference concerning the theory of distributions is [23].

5.1. DISTRIBUTIONS ON $\overline{\mathbb{R}}$. We consider the one-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of the real numbers as a C^∞-manifold in the usual way by making use of the two charts

$$\gamma_0 : \left\{ \begin{array}{ccc} \mathbb{R} & \to & \overline{\mathbb{R}} \\ t & \mapsto & t \end{array} \right. , \quad \gamma_1 : \left\{ \begin{array}{ccc} \mathbb{R} & \to & \overline{\mathbb{R}} \\ t & \mapsto & \frac{1}{t} \end{array} \right.$$

where we have put $\frac{1}{0} := \infty$. Similarly the unit circle in the plane $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is considered as a C^∞-manifold in the usual way. We fix a diffeomorphism γ from $\mathbb{T}$ to $\overline{\mathbb{R}}$, let us choose the fractional linear transformation

$$\gamma : \begin{cases} \mathbb{T} & \to & \overline{\mathbb{R}} \\ w & \mapsto & \mathrm{i}\frac{1-w}{1+w} \end{cases} . \tag{5.1}$$

Clearly then

$$\widetilde{\gamma} : \begin{cases} C^\infty(\overline{\mathbb{R}}) & \to & C^\infty(\mathbb{T}) \\ f & \mapsto & f \circ \gamma \end{cases}$$

is a bijection.

On $C^\infty(\mathbb{T})$ we have the topology of test functions, which is the F-space topology induced by the family of seminorms ($n \in \mathbb{N}$)

$$\widetilde{p}_n : \begin{cases} C^\infty(\mathbb{T}) & \to & \mathbb{R} \\ f & \mapsto & \max\{|f^{(n)}(w)| : w \in \mathbb{T}\} \end{cases} .$$

On the space $C^\infty(\overline{\mathbb{R}})$ we define a topology by the requirement that $\widetilde{\gamma}$ is a homeomorphism. This turns $C^\infty(\overline{\mathbb{R}})$ into a F-space, the topology being induced by the family of seminorms

$$p_n : \begin{cases} C^\infty(\overline{\mathbb{R}}) & \to & \mathbb{R} \\ f & \mapsto & \max\{|(f \circ \gamma)^{(n)}(w)| : w \in \mathbb{T}\} \end{cases} .$$

The space of distributions on $\overline{\mathbb{R}}$ is defined to be the dual $C^\infty(\overline{\mathbb{R}})'$.

We will often make use of the classical theory of distributions on $\mathbb{R}$ by employing the following localization principle. For $K \subseteq \mathbb{R}$ compact let $\mathcal{D}_K$ be the F-space of all C^∞-functions on $\mathbb{R}$ whose support lies in K endowed with the topology induced by the family of seminorms

$$q_n(f) := \max\{|f^{(n)}(t)| : t \in K\}, \quad n \in \mathbb{N} .$$

The natural embedding $\iota : \mathcal{D}_K \to C^\infty(\overline{\mathbb{R}})$,

$$(\iota f)(t) := \begin{cases} f(t) & t \in \mathbb{R}, \\ 0 & t = \infty, \end{cases}$$

maps $\mathcal{D}_K$ bijectively onto the closed subspace $\{g \in C^\infty(\overline{\mathbb{R}}) : \operatorname{supp} g \subseteq K\}$ of $C^\infty(\overline{\mathbb{R}})$.

Let $f \in \mathcal{D}_K$. Since γ as well as each derivative $\gamma^{(n)}$ is bounded on $\gamma^{-1}(K)$, we find constants C_n such that

$$|(f \circ \gamma)^{(n)}(w)| \leqslant C_n \cdot \max_{0 \leqslant k \leqslant n} \sup\{|f^{(k)}(t)| : t \in K\}, \quad w \in \mathbb{T} .$$

Thus for all $f \in \mathcal{D}_K$

$$p_n(\iota f) \leqslant C_n \cdot \max_{0 \leqslant k \leqslant n} q_k(f)$$

and hence ι is continuous. By the open mapping theorem it is a homeomorphism. We can hence identify $\mathcal{D}_K$ with the subspace $\iota\mathcal{D}_K$ of $C^\infty(\overline{\mathbb{R}})$.

It is often practical to exchange the roles of the points $0, \infty \in \overline{\mathbb{R}}$. The mapping $t \mapsto t^{-1}$ (where we have put $0^{-1} := \infty$, $\infty^{-1} := 0$) is a diffeomorphism of $\overline{\mathbb{R}}$ onto itself, hence gives rise to a bijection

$$\mathrm{Inv} : \left\{ \begin{array}{rcl} C^\infty(\overline{\mathbb{R}}) & \to & C^\infty(\overline{\mathbb{R}}) \\ f(t) & \mapsto & f(\frac{1}{t}) \end{array} \right. .$$

Since we have

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\gamma"] \arrow[d, "\xi"'] & \overline{\mathbb{R}} \arrow[d, "t\mapsto \frac{1}{t}"] \\ \mathbb{T} \arrow[r, "\gamma"'] & \overline{\mathbb{R}} \end{tikzcd}$$

with $\xi(w) := -\overline{w}$, we find

$$\begin{tikzcd} C^\infty(\mathbb{T}) & C^\infty(\overline{\mathbb{R}}) \arrow[l, "\widetilde{\gamma}"'] \\ C^\infty(\mathbb{T}) \arrow[u, "\widetilde{\xi}"] & C^\infty(\overline{\mathbb{R}}) \arrow[l, "\widetilde{\gamma}"] \arrow[u, "\mathrm{Inv}"'] \end{tikzcd}$$

where $\widetilde{\xi}(f) := f \circ \xi$. The map $\widetilde{\xi}$ is an automorphism with respect to the topology of test functions on $C^\infty(\mathbb{T})$, thus also Inv is an automorphism of $C^\infty(\overline{\mathbb{R}})$.

As shall be explained in the sequel one can associate to each element $\phi \in C^\infty(\overline{\mathbb{R}})'$ an analytic function $q : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$.

Let $z_0 = x_0 + iy_0 \in \mathbb{C}^+$ be fixed and consider the functions ($z \in \mathbb{C} \setminus \mathbb{R}$)

$$\beta_z(t) := \begin{cases} \left(\frac{1}{t-z} - \frac{t-x_0}{|t-z_0|^2}\right)|t-z_0|^2 & t \in \mathbb{R}, \\ z - x_0 & t = \infty. \end{cases}$$

Obviously,

$$\beta_z(t) = \Big(\frac{1}{t-z} - \frac{t-x_0}{(t-x_0)^2 + y_0^2}\Big)\big((t-x_0)^2 + y_0^2\big), \quad t \in \mathbb{R}\,. \tag{5.2}$$

A short computation shows that for $w \in \mathbb{T}$

$$(\beta_z \circ \gamma)(w) = \frac{-(\mathrm{i} + x_0)zw + (\mathrm{i} - x_0)z + (\mathrm{i}x_0 + |z_0|^2)w + (|z_0|^2 - \mathrm{i}x_0)}{-zw - z - \mathrm{i}w + \mathrm{i}}\,. \tag{5.3}$$

The right hand side of (5.3) defines a function $\alpha(z,w)$ which is analytic on $\mathbb{C}^2 \setminus \mathfrak{C}$, where

$$\mathfrak{C} := \{(z,w) \in \mathbb{C}^2 :\ \mathrm{i}(1-w) = z(1+w)\}\,.$$

Since $((\mathbb{C} \setminus \mathbb{R}) \times \mathbb{T}) \cap \mathfrak{C} = \emptyset$ and $(\beta_z \circ \gamma)(w) = \alpha(z,w)|_{(\mathbb{C}\setminus\mathbb{R})\times\mathbb{T}}$, each function $\beta_z \circ \gamma$ belongs in particular to $C^\infty(\mathbb{T})$. Thus $\beta_z \in C^\infty(\overline{\mathbb{R}})$.

LEMMA 5.1. *Let $\phi \in C^\infty(\overline{\mathbb{R}})'$ and define*

$$q(z) := \phi(\beta_z), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Then q is analytic on $\mathbb{C} \setminus \mathbb{R}$ and admits an analytic continuation to $\mathbb{C} \setminus \operatorname{supp} \phi$.

Proof. From the Cauchy integral representation of $\alpha(z,w)$ it follows that, whenever $O_1, O_2 \subseteq \mathbb{C}$ are open sets such that $(O_1 \times O_2) \cap \mathfrak{C} = \emptyset$ and $n \in \mathbb{N} \cup \{0\}$, the limit relation

$$\lim_{\zeta \to z} \frac{\partial^n}{\partial w^n} \frac{\alpha(\zeta, w) - \alpha(z,w)}{\zeta - w} = \frac{\partial^{n+1}}{\partial w^n \partial z} \alpha(z,w) \tag{5.4}$$

holds locally uniformly on $O_1 \times O_2$.

Assume first that $z_1 \in \mathbb{C} \setminus \mathbb{R}$. Then $(\{z_1\} \times \mathbb{T}) \cap \mathfrak{C} = \emptyset$, and hence we can choose open sets O_1, O_2 such that $z_1 \in O_1$, $\mathbb{T} \subseteq O_2$, $(O_1 \times O_2) \cap \mathfrak{C} = \emptyset$. Then (5.4) shows that $\beta_z \circ \gamma$ is analytic for $z \in O_1$ in the topology of $C^\infty(\mathbb{T})$. Hence β_z is analytic at z_1 in the topology of $C^\infty(\overline{\mathbb{R}})$.

Assume next that $z_1 \in \mathbb{R} \setminus \operatorname{supp} \phi$. Let $\widetilde{\phi} \in C^\infty(\mathbb{T})'$ be defined as $\widetilde{\phi} := \phi \circ \widetilde{\gamma}^{-1}$, so that we have

$$q(z) = \widetilde{\phi}(\beta_z \circ \gamma), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Choose $O_1, O_2 \subseteq \mathbb{C}$ open such that $z_1 \in O_1$, $\operatorname{supp} \widetilde{\phi} \subseteq O_2$ and $(O_1 \times O_2) \cap \mathfrak{C} = \emptyset$. This choice is possible since $z_1 \notin \operatorname{supp} \phi$ and hence $(\{z_1\} \times \operatorname{supp} \widetilde{\phi}) \cap \mathfrak{C} = \emptyset$. Next choose a partition of unity $\chi_0, \chi_1 \in C^\infty(\mathbb{T})$ subordinate to $O_2 \cap \mathbb{T}, \mathbb{T} \setminus \operatorname{supp} \widetilde{\phi}$, and consider for each z

$$\chi_0(w)\alpha(z,w)$$

as a function of $w \in \mathbb{T}$. If $z \in O_1$, then $(\{z\} \times \operatorname{supp} \chi_0) \cap \mathfrak{C} = \emptyset$, hence for all such z this function belongs to $C^\infty(\mathbb{T})$. Moreover, by (5.4), it depends analytically on $z \in O_1$ in the topology of $C^\infty(\mathbb{T})$. Thus also

$$q_1(z) := \widetilde{\phi}(\chi_0(w)\alpha(z,w))$$

is analytic on O_1. However, if $z \in O_1 \setminus \mathbb{R}$ and $w \in \mathbb{T}$, we have $\alpha(z,w) = (\beta_z \circ \gamma)(w)$, and hence

$$\chi_0(w)\alpha(z,w) = (\beta_z \circ \gamma)(w) - \chi_1(w)(\beta_z \circ \gamma)(w).$$

Since $\operatorname{supp} \widetilde{\phi} \cap \operatorname{supp} \chi_1 = \emptyset$,

$$q_1(z) = \widetilde{\phi}(\chi_0(w)\alpha(z,w)) = \widetilde{\phi}(\beta_z \circ \gamma) = q(z), \quad z \in O_1 \setminus \mathbb{R}.$$

We found an analytic extension of q across an interval containing z_1. ∎

5.2. THE CLASS $\mathcal{F}(\overline{\mathbb{R}})$. REPRESENTATION OF GENERALIZED NEVANLINNA FUNCTIONS. Recall that measures can be considered as distributions. If μ is a complex Borel measure on $\overline{\mathbb{R}}$, the functional

$$f \mapsto \int_{\overline{\mathbb{R}}} f \, d\mu$$

belongs to $C^\infty(\overline{\mathbb{R}})'$. This identification has a local version:

DEFINITION 5.2. Let M be an open subset of $\overline{\mathbb{R}}$ and let μ be a positive Borel measure on M with $\mu(K) < \infty$ for all compact $K \subseteq M$. Moreover, let $\phi \in C^\infty(\overline{\mathbb{R}})'$. We say that ϕ equals μ on M, $\phi =_M \mu$, if

$$\phi(f) = \int_M f \, d\mu, \quad f \in C^\infty(\overline{\mathbb{R}}), \ \operatorname{supp} f \subseteq M \, .$$

Note that hereby the measure μ is not assumed to satisfy $\mu(M) < \infty$.

A distribution ϕ is called real, if it takes real values on real-valued test functions.

The following definition was given in [18]. In the case of distributions with compact support in $\mathbb{R}$ it goes back to [12].

DEFINITION 5.3. Let $\phi \in C^\infty(\overline{\mathbb{R}})'$. We write $\phi \in \mathcal{F}(\overline{\mathbb{R}})$, if ϕ is real and if there exists a finite set $s(\phi) \subseteq \overline{\mathbb{R}}$ and a positive Borel measure μ on $\overline{\mathbb{R}} \setminus s(\phi)$ with $\mu(K) < \infty$ for all compact $K \subseteq \overline{\mathbb{R}} \setminus s(\phi)$, such that $\phi =_{\overline{\mathbb{R}} \setminus s(\phi)} \mu$.

From the same sources let us recall the following representation of generalized Nevanlinna functions. Denote by $\mathbb{C}(z)$ the space of all complex rational functions.

PROPOSITION 5.4. *Let $z_0 \in \mathbb{C}^+$, $r \in \mathbb{C}(z)$, $r = r^\#$, and $\phi \in \mathcal{F}(\overline{\mathbb{R}})$. Then the function*

$$q(z) := r(z) + \phi(\beta_z) \tag{5.5}$$

belongs to $\mathcal{N}_{<\infty}$. Conversely, if $q \in \mathcal{N}_{<\infty}$ and $z_0 \in \mathbb{C}^+$ is fixed, then there exists a unique function $r \in \mathbb{C}(z)$ analytic on $\mathbb{R}$ with $r = r^\#$, $r = O(1)$ at ∞, and a unique distribution $\phi \in \mathcal{F}(\overline{\mathbb{R}})$, such that (5.5) *holds.*

Let $\phi \in \mathcal{F}(\overline{\mathbb{R}})$, and let $s(\phi)$ and μ be as in Definition 5.3. The measure μ can be recovered from the function $q(z) = \phi(\beta_z)$ by means of the Stieltjes inversion formula.

LEMMA 5.5. *Let $\phi \in C^\infty(\overline{\mathbb{R}})'$ be real, fix $z_0 \in \mathbb{C}^+$ and put $q(z) := \phi(\beta_z)$. Assume that on some interval (a_0, b_0) the distribution ϕ coincides with a measure μ in the sense of Definition* 5.2 *and let ν be the measure defined by* $d\nu(t) = |t - z_0|^2 \, d\mu(t)$.

Then for all $a, b \in (a_0, b_0)$, $a < b$, we have

$$\frac{1}{\pi} \lim_{y \searrow 0} \int_a^b \operatorname{Im} q(x + \mathrm{i}y)\, \mathrm{d}x = \nu((a,b)) + \frac{\nu(\{a\}) + \nu(\{b\})}{2}\,.$$

Proof. Choose a partition of unity $\chi_0, \chi_1 \in C^\infty(\overline{\mathbb{R}})$ subordinate to $\overline{\mathbb{R}} \setminus [a,b]$, (a_0, b_0). Since $\phi =_{(a_0,b_0)} \mu$, we have

$$\phi(f) = (\chi_0 \phi)(f) + \int_{(a_0,b_0)} \chi_1 f \,\mathrm{d}\mu\,. \tag{5.6}$$

Since $[a,b] \cap \operatorname{supp} \chi_0 = \emptyset$, we can choose $a', b' \in \mathbb{R}$ such that

$$[a,b] \subseteq (a', b') \subseteq [a', b'] \subseteq \overline{\mathbb{R}} \setminus \operatorname{supp} \chi_0\,.$$

Then we have $\chi_1|_{[a',b']} = 1$, and the second summand in (5.6) can be written as

$$\int_{(a_0,b_0)} \chi_1 f \,\mathrm{d}\mu = \int_{[a',b']} f \,\mathrm{d}\mu + \int_{(a_0,b_0)\setminus[a',b']} \chi_1 f \,\mathrm{d}\mu\,. \tag{5.7}$$

Applying the relations (5.6) and (5.7) to the function $f = \beta_z$ yields

$$q(z) = (\chi_0\phi)(\beta_z) + \int_{[a',b']} \beta_z \,\mathrm{d}\mu + \int_{(a_0,b_0)\setminus[a',b']} \chi_1 \beta_z \,\mathrm{d}\mu\,.$$

Since $\operatorname{supp}(\chi_0\phi) \cap [a,b] = \emptyset$ the first summand has an analytic continuation to a neighbourhood of $(\mathbb{C} \setminus \mathbb{R}) \cup [a,b]$. Since ϕ is real, it takes real values on $[a,b]$. As $[a,b] \subseteq (a',b')$ the last summand as well has an analytic continuation to a neighbourhood of $(\mathbb{C} \setminus \mathbb{R}) \cup [a,b]$. Clearly, it also takes real values on $[a,b]$. This implies ($z = x + \mathrm{i}y$)

$$\lim_{y \searrow 0} \operatorname{Im} (\chi_0\phi)(\beta_z) = \lim_{y \searrow 0} \operatorname{Im} \int_{(a_0,b_0)\setminus[a',b']} \chi_1 \beta_z \,\mathrm{d}\mu = 0 \tag{5.8}$$

uniformly for $x \in [a,b]$.

We have

$$\operatorname{Im} \beta_z(t) = \frac{\operatorname{Im} z}{|t-z|^2} \cdot |t - z_0|^2\,,$$

and hence

$$\operatorname{Im} \int_{[a',b']} \beta_z \,\mathrm{d}\mu = \operatorname{Im} z \int_{[a',b']} \frac{\mathrm{d}\nu(t)}{(t-x)^2 + y^2}\,.$$

The Stieltjes inversion formula (cf. Section 5.4 of [22]) gives

$$\lim_{y \searrow 0} \int_a^b \Big(\frac{1}{\pi} \operatorname{Im} \int_{[a',b']} \beta_z \,\mathrm{d}\mu \Big)\, \mathrm{d}x = \nu((a,b)) + \frac{\nu(\{a\}) + \nu(\{b\})}{2}\,.$$

By (5.8)

$$\frac{1}{\pi}\lim_{y\searrow 0}\int_a^b \operatorname{Im} q(x+\mathrm{i}y)\,\mathrm{d}x = \lim_{y\searrow 0}\int_a^b \Big(\frac{1}{\pi}\operatorname{Im}\int_{[a',b']}\beta_z\,\mathrm{d}\mu\Big)\,\mathrm{d}x\,. \qquad ∎$$

It is interesting to note that for distributions of the class $\mathcal{F}(\overline{\mathbb{R}})$ also a converse of Lemma 5.1 holds. A proof of this fact different to the one given below could be obtained by combining the methods of Proposition 3.1 in [18] and [10]. We prefer to stick to a more elementary method, also because it gives a stronger result which is of good use in the sequel.

LEMMA 5.6. *Let $\phi \in C^\infty(\overline{\mathbb{R}})'$ be real, fix $z_0 \in \mathbb{C}^+$ and set $q(z) := \phi(\beta_z)$. Moreover, let $a_0, b_0 \in \mathbb{R}$, $a_0 < b_0$, and $s \in (a_0, b_0)$. Assume that $\phi =_{(a_0,b_0)\setminus\{s\}} \mu$ for some measure μ and that q has an analytic continuation to $(\mathbb{C}\setminus\mathbb{R})\cup(a_0,b_0)$. Then $(a_0,b_0)\cap\operatorname{supp}\phi = \emptyset$.*

Proof. Since ϕ is real, the function q takes real values on (a,b). An application of the Stieltjes inversion formula to the intervals (a_0,s) and (s,b_0) yields $\mu = 0$. Thus $\operatorname{supp}\phi\cap(a_0,b_0)\subseteq\{s\}$. Choose $a_1, b_1\in\mathbb{R}$ such that

$$\{s\}\subseteq(a_1,b_1)\subseteq[a_1,b_1]\subseteq(a_0,b_0)\,.$$

Consider the distribution $\widetilde{\phi} := \iota'\phi \in \mathcal{D}_{[a_1,b_1]}$ where $\iota' : C^\infty(\overline{\mathbb{R}})' \to \mathcal{D}'_{[a_1,b_1]}$ is the dual of the canonical embedding of $\mathcal{D}_{[a_1,b_1]}$ into $C^\infty(\overline{\mathbb{R}})$. Then $\operatorname{supp}\widetilde{\phi}\subseteq\{s\}$ and thus

$$\widetilde{\phi} = \sum_{k\leqslant n} c_k\delta_s^{(k)}\,,$$

where δ_s denotes the evaluation functional $\delta_s(f) = f(s)$, cf. [23].

Choose $a', b'\in\mathbb{R}$ such that

$$\{s\}\subseteq(a',b')\subseteq[a',b']\subseteq(a_1,b_1)$$

and let $\chi_0,\chi_1\in C^\infty(\overline{\mathbb{R}})$ be a partition of unity subordinate to $(a_1,b_1), \overline{\mathbb{R}}\setminus[a',b']$. Then

$$q(z) = \phi(\chi_1\beta_z) + \phi(\chi_0\beta_z)\,.$$

The first summand is analytic on $[a',b']$ since $\operatorname{supp}\chi_1\cap[a',b'] = \emptyset$. As $\operatorname{supp}\chi_0\subseteq(a_1,b_1)$, the second summand computes as

$$\phi(\chi_0\beta_z) = \widetilde{\phi}(\chi_0\beta_z|_{[a_1,b_1]}) = \sum_{k\leqslant n} c_k(\chi_0\beta_z)^{(k)}(s) = \sum_{k\leqslant n} c_k\beta_z^{(k)}(s)\,.$$

By (5.2) the function $\beta_z^{(k)}(s)$ has a pole of order $k+1$ at s. Hence $\phi(\chi_0\beta_z)$ has a pole at s unless all numbers c_k vanish. Since q as well as $\phi(\chi_1\beta_z)$ is analytic at s, we must have $c_k = 0$, $k = 0,\dots,n$, and hence $\operatorname{supp}\phi\cap(a_0,b_0) = \emptyset$. ∎

COROLLARY 5.7. *Let $\phi\in\mathcal{F}(\overline{\mathbb{R}})$ and $q(z) = \phi(\beta_z)$. Then the maximal domain of analyticity of q in $\mathbb{C}$ is equal to $\mathbb{C}\setminus\operatorname{supp}\phi$.*

5.3. DISTRIBUTIONS ASSOCIATED TO SYMMETRIC AND ESSENTIALLY POSITIVE NEVANLINNA FUNCTIONS.

DEFINITION 5.8. Denote by $C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}$ the set of all odd functions in $C^\infty(\overline{\mathbb{R}})$. Note that $f(0) = f(\infty) = 0$ whenever $f \in C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}$. Let us introduce the following notation:

$$\mathcal{F}^{\mathrm{ep}} := \{\phi \in \mathcal{F}(\overline{\mathbb{R}}) : \operatorname{supp}\phi \subseteq [0,\infty) \cup \{\infty\}\},$$
$$\mathcal{F}^{\mathrm{sym}} := \{\phi \in \mathcal{F}(\overline{\mathbb{R}}) : \ker\phi \supseteq C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}\}.$$

Clearly, $\mathcal{F}^{\mathrm{ep}}$ and $\mathcal{F}^{\mathrm{sym}}$ are linear subspaces of $C^\infty(\overline{\mathbb{R}})'$. Moreover, a distribution ϕ belongs to $\mathcal{F}^{\mathrm{sym}}$ if and only if $\phi(f(t)) = \phi(f(-t))$ for all $f \in C^\infty(\overline{\mathbb{R}})$.

The following theorem is the second main result of this paper.

THEOREM 5.9. *Let a function q be given. Then:*

(i) *$q \in \mathcal{N}^{\mathrm{ep}}_{<\infty}$ if and only if it can be represented as $q(z) = r(z) + \phi(\beta_z)$ where $r \in \mathbb{C}(z)$, $r = r^\#$, the point z_0 used for the definition of β_z belongs to $\mathbb{C}^+$, and $\phi \in \mathcal{F}^{\mathrm{ep}}$. In this case r can be chosen analytic on $[0,\infty)$ and such that $r = O(1)$ at ∞.*

(ii) *$q \in \mathcal{N}^{\mathrm{sym}}_{<\infty}$ if and only if it can be represented as $q(z) = r(z) + \phi(\beta_z)$ where $r \in \mathbb{C}(z)$ is odd, $r = r^\#$, the point z_0 used for the definition of β_z belongs to $\mathrm{i}\mathbb{R}^+$, and $\phi \in \mathcal{F}^{\mathrm{sym}}$. In this case r can be chosen analytic on $\mathbb{R}$ and such that $r = o(1)$ at ∞.*

Proof. (of Theorem 5.9, (i)*)* Assume that q is represented as in (i). By Lemma 5.1, the function $\phi(\beta_z)$ is analytic on $\mathbb{C} \setminus [0,\infty)$ and by Proposition 5.4 it belongs to $\mathcal{N}_{<\infty}$. Alltogether we see that $q \in \mathcal{N}^{\mathrm{ep}}_{<\infty}$.

Conversely, let $q \in \mathcal{N}^{\mathrm{ep}}_{<\infty}$ be given, and let $r_1(z)$ be the unique rational function with $r_1 = o(1)$ at ∞ which is analytic on $[0,\infty)$, and is such that $q - r_1$ is analytic on $\mathbb{C} \setminus [0,\infty)$. Then $q_1 := q - r_1$ belongs to $\mathcal{N}^{\mathrm{ep}}_{<\infty}$ and can be represented according to Proposition 5.4 as $q_1(z) = r(z) + \phi(\beta_z)$. Since q_1 is analytic on $\mathbb{C} \setminus [0,\infty)$ in this representation r is constant and, by Corollary 5.7, $\phi \in \mathcal{F}^{\mathrm{ep}}$. ∎

The relation between the classes $\mathcal{N}^{\mathrm{sym}}_{<\infty}$ and $\mathcal{F}^{\mathrm{sym}}$ is similar. However, the proof of this fact is not so straightforward and will be carried out in several steps.

Proof. (of Theorem 5.9, (ii), 1^{st} *part)* Assume that q is represented as in (ii). Since $z_0 = \mathrm{i}y_0$, the functions β_z can be rewritten as

$$\beta_z(t) = \Big(\frac{1}{t-z} - \frac{t}{t^2+y_0^2}\Big)(t^2+y_0^2) = \frac{y_0^2 + tz}{t-z},$$

and hence satisfy $\beta_z(t) = -\beta_{-z}(-t)$. We obtain

$$\begin{aligned} q(-z) &= r(-z) + \phi(\beta_{-z}(t)) = -r(z) + \phi(-\beta_z(-t)) \\ &= -r(z) - \phi(\beta_z(t)) = -q(z). \quad ∎ \end{aligned}$$

Note that we have in fact proved the following more general statement.

COROLLARY 5.10. *Assume that* $z_0 = \mathrm{i}y_0 \in \mathrm{i}\mathbb{R}^+$. *Let* $\phi \in C^\infty(\overline{\mathbb{R}})'$ *and assume that* $\ker\phi \supseteq C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}$. *Then the function* $\phi(\beta_z)$ *is odd.*

The proof of the converse part of Theorem 5.9 is based on the following two statements.

LEMMA 5.11. *Let* $K \subseteq [0,\infty)$ *be compact, and let* $\lambda \in C(\mathbb{R})$ *be such that* $\lambda(x) + (-1)^{n+1}\lambda(-x) = 0$, $x \in K$. *Then the functional defined on* $C^\infty(\overline{\mathbb{R}})$ *as*

$$\phi(f) = (-1)^n \int\limits_{K\cup -K} \lambda(x) f^{(n)}(x)\,\mathrm{d}x$$

is continuous and satisfies $\ker\phi \supseteq C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}$.

Proof. The set $K \cup -K$ is compact, hence has finite Lebesgue measure. Moreover, the function λ is bounded on $K \cup -K$. Let γ be as in (5.1). Since γ^{-1} and all of its derivatives are bounded on $K \cup -K$, it follows that with appropriate constants C, C_n', C_n''

$$\begin{aligned}
|\phi(f)| &\leqslant C \sup_{x\in K\cup -K} |\lambda(x)| \cdot \sup\{|f^{(n)}(t)| : t \in K \cup -K\} \\
&\leqslant C \sup_{x\in K\cup -K} |\lambda(x)| \cdot C_n' \cdot \max_{0\leqslant k\leqslant n} \sup\{|(f\circ\gamma)^{(n)}(w)| : w \in \gamma^{-1}(K\cup -K)\} \\
&\leqslant C_n'' \max_{0\leqslant k\leqslant n} p_n(f).
\end{aligned}$$

We see that ϕ is a continuous functional on $C^\infty(\overline{\mathbb{R}})$.

Let $f \in C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}$, then $f^{(n)}(-x) = (-1)^{n+1} f(x)$ and

$$\begin{aligned}
(-1)^n\phi(f) &= \int\limits_K \lambda(x) f^{(n)}(x)\,\mathrm{d}x + \int\limits_{-K} \lambda(x) f^{(n)}(x)\,\mathrm{d}x \\
&= \int\limits_K \lambda(x) f^{(n)}(x)\,\mathrm{d}x + \int\limits_K \lambda(-x) f^{(n)}(-x)\,\mathrm{d}x \\
&= \int\limits_K [\lambda(x) + (-1)^{n+1}\lambda(-x)] f^{(n)}(x)\,\mathrm{d}x = 0. \qquad \blacksquare
\end{aligned}$$

PROPOSITION 5.12. *Let* $\phi \in C^\infty(\overline{\mathbb{R}})'$ *be real and let* S *be a finite subset of* $\overline{\mathbb{R}}$. *Assume that* $\ker\phi \supseteq \{f \in C^\infty(\overline{\mathbb{R}})^{\mathrm{od}} : \operatorname{supp} f \cap S = \emptyset\}$ *and that* $q(z) = \phi(\beta_z)$ *is odd (where we have chosen* $z_0 = \mathrm{i}y_0 \in \mathrm{i}\mathbb{R}^+$*). Then* $\ker\phi \supseteq C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}$.

Proof. Put $\widehat{S} := S \cup -S$. Since partitions of unity subordinate to open sets O_j with $O_j = -O_j$ can be chosen to consist of even functions, it suffices to prove that every point $s \in S$ has a neighbourhood (a_1, b_1) such that $\{f \in C^\infty(\overline{\mathbb{R}})^{\mathrm{od}} : \operatorname{supp} f \subseteq (a_1,b_1) \cup (-b_1,-a_1)\} \subseteq \ker\phi$.

First consider a point $s \in \widehat{S}$, $s > 0$. Choose $a_1, b_1 \in \mathbb{R}$ such that

$$[a_1, b_1] \subseteq \mathbb{R}^+, \quad \widehat{S} \cap [a_1, b_1] = \{s\}\,.$$

Put $A_1 := [a_1, b_1] \cup [-b_1, -a_1]$.

Consider $\iota' : C^\infty(\overline{\mathbb{R}})' \to \mathcal{D}'_{A_1}$. Since A_1 is compact, there exists $\lambda \in C(\mathbb{R})$, $\operatorname{supp} \lambda \subseteq A_1$, and $n \in \mathbb{N} \cup \{0\}$ such that

$$\phi(\iota f) = (\iota' \phi)(f) = (-1)^n \int_{A_1} \lambda(x) f^{(n)}(x)\, \mathrm{d}x, \quad f \in \mathcal{D}_{A_1}.$$

Since ϕ is real, the function λ is real valued.

Put $I_\mathrm{l} := (a_1, s)$, $I_\mathrm{r} := (s, b_1)$. For $f \in C^\infty(\mathbb{R}^+)$, $\operatorname{supp} f \subseteq I_\mathrm{l}$, consider its odd continuation

$$\widetilde{f}(x) := \begin{cases} f(x) & x > 0, \\ -f(-x) & x < 0, \\ 0 & x = 0, \infty. \end{cases}$$

Then $\widetilde{f}$ belongs to $C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}$ and $\operatorname{supp} \widetilde{f} \cap \widehat{S} = \emptyset$. Hence, $\phi(\widetilde{f}) = 0$ and we compute

$$\begin{aligned} 0 = \phi(\widetilde{f}) &= (-1)^n \int_{A_1} \lambda(x) \widetilde{f}^{(n)}(x)\, \mathrm{d}x \\ &= (-1)^n \int_{I_\mathrm{l}} [\lambda(x) + (-1)^{n+1} \lambda(-x)] f^{(n)}(x)\, \mathrm{d}x. \end{aligned}$$

Since f was arbitrary, it follows that

$$\lambda(x) + (-1)^{n+1} \lambda(-x) = r_1(x), \quad x \in I_\mathrm{l},$$

with some polynomial r_1 of degree at most $n-1$. The same argument applies with I_r instead of I_l and we find a polynomial r_2 whose degree does not exceed $n-1$ such that

$$\lambda(x) + (-1)^{n+1} \lambda(-x) = r_2(x), \quad x \in I_\mathrm{r}.$$

Define a function $\widetilde{\lambda}$ on A_1 by

$$\widetilde{\lambda}(x) := \begin{cases} \lambda(x) & x \in [a_1, b_1], \\ \lambda(x) + (-1)^n r_1(-x) & x \in [-s, -a_1], \\ \lambda(x) + (-1)^n r_2(-x) & x \in [-b_1, -s]. \end{cases}$$

Since $r_1(s) = r_2(s)$, this function is continuous on A_1. Moreover, $\widetilde{\lambda}$ satisfies by its definition $\widetilde{\lambda}(x) + (-1)^{n+1} \widetilde{\lambda}(-x) = 0$, $x \in A_1$. Let $\widetilde{\phi}$ be the distribution defined by means of Lemma 5.11 applied to $\widetilde{\lambda}$ and put $\psi := \phi - \widetilde{\phi}$. By our assumption on q and Corollary 5.10 the function

$$Q(z) := \psi(\beta_z)$$

is odd. By the definition of $\widetilde{\lambda}$ we have $(A_1 \setminus \{-s\}) \cap \operatorname{supp} \psi = \emptyset$. In fact,

$$\psi(f) = \int_{-s}^{-a_1} [r_1(-x) - r_2(-x)] f^{(n)}\, \mathrm{d}x, \quad \operatorname{supp} f \subseteq A_1.$$

Since $(a_1, b_1) \cap \operatorname{supp} \psi = \emptyset$, the function Q admits an analytic continuation to $(\mathbb{C} \setminus \mathbb{R}) \cup (a_1, b_1)$, and by symmetry also to $(\mathbb{C} \setminus \mathbb{R}) \cup (-b_1, -a_1)$. Moreover, since $(-I_\mathrm{l} \cup -I_\mathrm{r}) \cap \operatorname{supp} \psi = \emptyset$, we have $\psi =_{(-I_\mathrm{l} \cup -I_\mathrm{r})} 0$. Now Lemma 5.6 implies that $(-b_1, -a_1) \cap \operatorname{supp} \psi = \emptyset$. Hence, for all f with $\operatorname{supp} f \subseteq (a_1, b_1) \cup (-b_1, -a_1)$ we must have $\phi(f) = \widetilde{\phi}(f)$. In case f is odd, this shows that $\phi(f) = 0$.

The points 0 and ∞ play a somewhat different role. Assume that $0 \in \widehat{S}$. Choose $0 < b_2 < b_1 < b_0$ such that $(-b_0, b_0) \cap \widehat{S} = \{0\}$. Similar as above we find $n \in \mathbb{N}$ and $\lambda \in C((-b_0, b_0))$ such that

$$\phi(f) = (-1)^n \int\limits_{(-b_0, b_0)} \lambda(x) f^{(n)} \, \mathrm{d}x, \quad \operatorname{supp} f \subseteq [-b_1, b_1] \,.$$

Also in the same way as in the previous step of this proof we find a polynomial r of degree at most $n-1$ such that

$$\lambda(x) + (-1)^{n+1} \lambda(-x) = r(x), \quad x \in (0, b_1) \,.$$

Define $\widetilde{\lambda} \in C([-b_1, b_1])$ by

$$\widetilde{\lambda}(x) := \begin{cases} \lambda(x) & x \in [0, b_1], \\ \lambda(x) + (-1)^n r(-x) & x \in [-b_1, 0], \end{cases}$$

and let $\widetilde{\phi}$ be as in Lemma 5.11 applied to $\widetilde{\lambda}$, so that $C^\infty(\overline{\mathbb{R}})^{\mathrm{od}} \subseteq \ker \widetilde{\phi}$ and

$$\widetilde{\phi}(f) = (-1)^n \int\limits_{[-b_1, b_1]} \widetilde{\lambda}(x) f^{(n)}(x) \, \mathrm{d}x \,.$$

With $\psi := \phi - \widetilde{\phi}$ the function $Q(z) := \psi(\beta_z)$ is odd. Since $\operatorname{supp} \psi \cap ((-b_1, 0) \cup (0, b_1)) = \emptyset$, the function Q is analytic on $(\mathbb{C} \setminus \mathbb{R}) \cup ((-b_1, 0) \cup (0, b_1))$, i.e. 0 is an isolated singularity of Q.

For f with $\operatorname{supp} f \subseteq (-b_1, b_1)$ we obtain by integration by parts that

$$\psi(f) = (-1)^n \int_0^{b_1} r(x) f^{(n)} \, \mathrm{d}x = \Big(\sum_{k=0}^{n-1} \rho_k \delta_0^{(k)} \Big)(f)$$

where ρ_k are appropriate constants and δ_0 denotes the point evaluation functional at 0.

Choose an even partition of unity $\chi_0, \chi_1 \in C^\infty(\overline{\mathbb{R}})$ which is subordinate to $(-b_1, b_1), \overline{\mathbb{R}} \setminus [-b_2, b_2]$. Then

$$Q(z) = (\chi_1 \psi)(\beta_z) + \psi(\chi_0 \beta_z) \,.$$

The first summand is analytic at 0, the second one computes as

$$\psi(\chi_0 \beta_z) = \sum_{k=0}^{n-1} \rho_k \frac{\partial^k}{\partial x^k} \beta_z(x)|_{x=0} \,.$$

Since

$$\beta_z(x) = \frac{y_0^2 + xz}{x - z}$$

we see that $\frac{\partial^k}{\partial x^k}\beta_z(x)|_{x=0}$ has a pole of order $k+1$ at 0. As Q is odd we obtain that $\rho_1 = \rho_3 = \cdots = 0$.

This, however, implies that whenever $f \in C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}$, $\operatorname{supp} f \subseteq (-b_1, b_1)$, we must have $\psi(f) = 0$ and thus also $\phi(f) = 0$.

So far we have shown that every $\phi \in C^\infty(\overline{\mathbb{R}})'$ subject to the conditions of the present proposition annulates all $f \in C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}$ with $\operatorname{supp} F \subseteq \mathbb{R}$ compact.

Finally let us consider the case that $\infty \in \widehat{S}$. This case could be treated similar as the case $0 \in \widehat{S}$. However, we prefer to reduce it to the already proved statement.

We start with a simple observation. The function $\phi(\beta_z)$ is odd if and only if

$$\beta_z(t) + \beta_{-z}(t) \in \ker\phi\,,$$

where z ranges in some open set. A computation gives

$$\beta_z(t) + \beta_{-z}(t) = 2(y_0^2 + z^2)\frac{t}{(t-z)(t+z)}\,, \tag{5.9}$$

hence $\phi(\beta_z)$ is odd if and only if

$$\frac{t}{(t-z)(t+z)} \in \ker\phi$$

for z in some open set.

We see that, if $\widehat{\beta}_z$ is defined by use of $\widehat{z}_0 = \mathrm{i}\widehat{y}_0 \in \mathrm{i}\mathbb{R}^+$ instead of $z_0 = \mathrm{i}y_0$, the function $\phi(\beta_z)$ is odd if and only if $\phi(\widehat{\beta}_z)$ has this property.

We shall apply the automorphism Inv'. A computation gives

$$\beta_z\Big(\frac{1}{t}\Big) = -y_0^2\widehat{\beta}_{\frac{1}{z}}(t)$$

where $\widehat{\beta}_z$ is defined with $\widehat{y}_0 = y_0^{-1}$. It follows that

$$(\mathrm{Inv}'\,\phi)(\beta_z(t)) = \phi\Big(\beta_z\Big(\frac{1}{t}\Big)\Big) = -y_0^2\phi(\widehat{\beta}_{\frac{1}{z}}(t))\,.$$

By the above considerations $\phi(\beta_z)$ is odd if and only if $\phi(\widehat{\beta}_{\frac{1}{z}})$ has this property. Hence, with ϕ also $\mathrm{Inv}'\,\phi$ satisfies all hypothesis of the present proposition and we conclude that $(\mathrm{Inv}'\,\phi)(f) = 0$ whenever $f \in C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}$, $\operatorname{supp} f \subseteq \mathbb{R}$ compact. Alltogether this shows that $\phi(f) = 0$ for all $f \in C^\infty(\overline{\mathbb{R}})^{\mathrm{od}}$, $\operatorname{supp} f \subseteq \overline{\mathbb{R}} \setminus \{0\}$. ∎

We now obtain:

Proof. (of Theorem 5.9, (ii), *converse part)* Assume that $q \in \mathcal{N}_{<\infty}^{\mathrm{sym}}$ and let r_1 be the rational function which is analytic on $\mathbb{R}$, $r_1 = o(1)$ at ∞, and such that $q - r_1$ is analytic on $\mathbb{C} \setminus \mathbb{R}$. Since q is odd, also r_1 has this property. Thus we may assume without loss of generality that q is analytic on $\mathbb{C}\setminus\mathbb{R}$. Then the representation (5.5), where we chose $z_0 = \mathrm{i}y_o \in \mathbb{R}^+$, is of the form

$$q(z) = \alpha + \phi(\beta_z)$$

with some $\alpha \in \mathbb{R}$. By (5.9) we have $\beta_{\mathrm{i}y_0}(t) + \beta_{-\mathrm{i}y_0}(t) = 0$. Moreover,

$$\overline{\beta_{\mathrm{i}y_0}(t)} = \overline{\Big(\frac{y_0^2 + t\mathrm{i}y_0}{t-\mathrm{i}y_0}\Big)} = \beta_{-\mathrm{i}y_0}(t).$$

Alltogether it follows that

$$2\operatorname{Re}\phi(\beta_{\mathrm{i}y_0}) = \phi(\beta_{\mathrm{i}y_0}) + \overline{\phi(\beta_{\mathrm{i}y_0})} = \phi(\beta_{\mathrm{i}y_0}) + \phi(\overline{\beta_{\mathrm{i}y_0}}) = \phi(\beta_{\mathrm{i}y_0}) + \phi(\beta_{-\mathrm{i}y_0}) = 0.$$

Since q is odd $\overline{q(\mathrm{i}y_0)} = q(-\mathrm{i}y_0) = -q(\mathrm{i}y_0)$, and we find

$$0 = \operatorname{Re} q(\mathrm{i}y_0) = \alpha + \operatorname{Re}\phi(\beta_{\mathrm{i}y_0}) = \alpha.$$

Hence it suffices to consider functions of the form $\phi(\beta_z)$ with $\phi \in \mathcal{F}(\overline{\mathbb{R}})$.

Let the measure μ be as in Definition 5.3, so that $\phi =_{\mathbb{R}\setminus s(\phi)} \mu$. Since q is odd the Stieltjes inversion formula Lemma 5.5 shows that $\mu(E) = \mu(-E)$ for all Borel sets E with $E \cap s(\phi) = (-E)\cap s(\phi) = \emptyset$. Thus ϕ satisfies the hypothesis of Proposition 5.12 and we conclude that $\phi \in \mathcal{F}^{\mathrm{sym}}$. ∎

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