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DIRECTIONAL OPERATOR DIFFERENTIABILITY OF NON-SMOOTH FUNCTIONS

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ABSTRACT. We obtain (very close) sufficient conditions and necessary conditions on the spectral measure of a self-adjoint operator A, under which any continuous function ϕ (without any additional smoothness properties) has a directional operator-derivative

$$\phi'(A)(B) := \frac{\partial}{\partial\gamma}\,\phi(A+\gamma B)_{|\gamma=0}$$

in the direction of a quite general bounded, self-adjoint operator B. Our sharpest results are in the case where B is a rank-one operator. We pay particular attention to the case where the spectral measure of A is absolutely continuous, and its additional smoothness properties compensate the lack of smoothness of the function ϕ.

KEYWORDS: *Functional calculus, rank-one perturbations, directional operator differentiability, Riesz projections, Hankel operators, Borel transform.*

MSC (2000): 47A55, 47A60, 47B15, 47B35.

1. INTRODUCTION

1.1. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$, and let $\mathcal{B}(\mathcal{H})_h := \{A \in \mathcal{B}(\mathcal{H}); A^* = A\}$ be its self-adjoint part. Let $\phi : \mathbb{R} \to \mathbb{C}$ be a continuous function. Via the usual functional calculus ϕ is extended to a function on $\mathcal{B}(\mathcal{H})_h$, denoted also by ϕ. Namely, if a self-adjoint operator A has a spectral measure E (and then $A = \int\limits_{\mathbb{R}} \lambda E(\mathrm{d}\lambda)$), then

$$\phi(A) := \int\limits_{\mathbb{R}} \phi(\lambda)\, E(\mathrm{d}\lambda).$$

It is natural to pose the following problem.

Differentiability Problem: *when is the extended function ϕ differentiable in the sense of Gateaux or Fréchet, with respect to either the uniform, the strong, or the weak operator topologies on $\mathcal{B}(\mathcal{H})$?*

Variants of this problem have been considered by many mathematicians, see for instance [1], [7], [8], [16], and [20]. It is not difficult to see that a necessary condition that the extended function ϕ is Gateaux-differentiable in $\mathcal{B}(\mathcal{H})_h$ (in the uniform operator topology) is that the scalar function ϕ must be continuously differentiable. If instead of $\mathcal{B}(\mathcal{H})_h$ we consider a commutative C^*-algebra $\mathcal{A}$, it is easily seen that the condition $\phi \in C^1$ is also sufficient. However, in [11] Farforovskaya constructed an example of function in C^1 whose extension to the self-adjoint part of $\mathcal{B}(\mathcal{H})$ is not Gateaux-differentiable.

Daletskii and Krein [8] considered the differentiability problem in the context of the self-adjoint elements in $\mathcal{B}(\mathcal{H})$. They showed that if $\phi \in C^2(\mathbb{R})$, then the extended operator function is differentiable. Moreover, they obtained a formula for the derivative of ϕ in terms of a notion of "iterated operator integrals", which they also introduced. Birman and Solomyak ([5], [6], [7]) refined this concept and introduced and developed the theory of "double operator integrals", which became the basis of all subsequent research in this area. They also found much sharper sufficient conditions for operator differentiability. For instance, scalar functions whose derivatives are Hölder continuous of some order $\alpha \in (0,1)$, or scalar functions whose derivatives have absolutely convergent Fourier series are operator differentiable. Sharper sufficient and necessary conditions, formulated in terms of Besov classes, were found by Peller [16]. Another approach in context of the double operator integral technique was proposed by Arazy, Barton and Friedman in [1] , where instead of the Fourier expansion of the scalar function ϕ a decomposition of it into Möbius functions was considered (see also [2] and [3]).

In the present paper we study the "local" version of the Differentiability Problem which deals with directional operator differentiability. Consider a pair of bounded self-adjoint operators A and B acting in a Hilbert space $\mathcal{H}$. Let us consider the family of operators

$$A_\gamma = A + \gamma B \quad (\gamma \in \mathbb{R}),$$

and let $M > 0$ be such that $\sigma(A) \subset (-M, M)$.

PROBLEM 1.1. *What spectral properties of the pair (A, B) guarantee that for every $\phi \in C[-M, M]$ the limit*

$$\lim_{\gamma \to 0} \frac{\phi(A_\gamma) - \phi(A)}{\gamma} \tag{1.1}$$

exists in the strong operator topology (or in other operator topologies)?

This "local" version of the Differentiability Problem, arises naturally in many applications. We shall show below that there are natural circumstances in which the strong operator-limit (1.1) exists for all $\phi \in C[-M, M]$ without any additional smoothness. Thus, some "smoothness" properties of the pair (A, B) compensate the lack of smoothness of $\phi \in C[-M, M]$ in this limiting procedure. Explicitly, under the assumptions that the perturbing operator B is of rank one, the spectral measure of the operator A is absolutely continuous and its density satisfies some rather mild conditions, we prove in Theorem 2.5 that the limit (1.1) exists in the uniform operator topology for any $\phi \in C[-M, M]$, and it is expressed by formula (2.38) (in which ϕ_0 is defined by (2.14) and P_+, P_- are Riesz projections in $L_2(\mathbb{R})$). It is important to notice that this formula contains the difference $P_- M_{\phi_0} P_+ - P_+ M_{\phi_0} P_-$, where M_{ϕ_0} is the operator of multiplication by ϕ_0. This difference can be expressed in terms of Hankel operators as $H_{\phi_0} - H^*_{\overline{\phi_0}}$ (where $H_\psi = P_- M_\psi P_+$ is the Hankel operator with symbol ψ, see [17]), and also in terms of the commutator $[H, M_{\phi_0}]$, where $H = \mathrm{i}(P_+ - P_-)$ is the Hilbert transform. This is not an accident. Indeed, in the paper of Peller [16] mentioned above the study of Hankel operators plays a central role in the investigation of operator differentiability. Moreover, under some preliminary conditions for the density of the spectral measure of the operator A, we also find necessary conditions for operator differentiability in the strong operator topology (see Theorems 3.3, 3.8, 4.5 and 5.2). These necessary conditions are very close to the sufficient conditions formulated in Theorem 2.5 mentioned above (see Remarks 3.9 and 4.6).

In our investigation we used a new approach, based on an expression of the operator function $\phi(A)$ via the limit of contour integrals (Proposition A1.1). We also make use of some boundary properties of Borel transform of functions (essentially, Poisson formula, see Proposition A2.1).

The paper is divided into five sections and two appendices. After this introduction, we prove in Section 2 Theorem 2.5 mentioned above. Especially, we discuss the conditions imposed in Theorem 2.5 on the density of the spectral measure of the operator A, and compare them with some known conditions (see Subsection 2.5 of Section 2 and particularly Example 2.7).

In Section 3 we obtain necessary conditions for directional operator differentiability of any continuous function, connected with a behavior of the spectral measure of the operator A near endpoints of gaps of its spectrum. We consider here the case of a general bounded self-adjoint perturbing operator B (Theorem 3.3), as well as the case of a rank-one operator B (Theorem 3.8).

Section 4 is devoted to obtain necessary conditions for the directional operator differentiability of any $\phi \in C[-M, M]$ in case B is of rank one, in terms of the behavior of the density $\widetilde{\rho}$ of the spectral measure of the operator A in interior points of its spectrum (Theorem 4.5). This theorem allows us to exhibit examples (see Example 4.7) in which directional operator differentiability in the sense of Problem 1.1 fails.

In Section 5 we link the necessary conditions for directional operator differentiability found in Sections 3 and 4 (Theorem 5.2). Finally, the appendices are devoted to establish auxiliary results. In Appendix 1 we obtain an essentially known formula for the operator function $\phi(A)$ mentioned above (Proposition A1.1), and in Appendix 2 we prove Proposition A2.1 about Borel transform of functions, which may be considered as a non-classical generalization of the well-known Sokhotskii boundary property of the integrals of Cauchy type. We add the labels "A1" and "A2" to the number of propositions and formulas in Appendices 1 and 2 respectively.

1.2. NOTATION. - $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all natural, real and complex numbers, respectively;

- $\Re z$ and $\Im z$ are the real and the imaginary parts of a number $z \in \mathbb{C}$;
- $\mathcal{O}(x_0)$ is a neighborhood of a point x_0 belonging to a topological space $\mathcal{T}$;
- $\mathbf{1}_S(x)$ $(x \in T)$ is the characteristic function of a subset S of a set T;
- $\mathrm{mes}(A)$ is the Lebesgue measure of a measurable set $A \subset \mathbb{R}$;
- $\widehat{f}$ is the Fourier transform of a function f from $L_1(\mathbb{R})$ or from $L_2(\mathbb{R})$;
- $\mathrm{supp}(f)$ is the support of a function f;
- If ρ is a measure on $\mathbb{R}$, then $\mathrm{supp}(\rho)$ denotes its support;
- M_ϕ is the operator of multiplication by a function $\phi(t)$ on $\mathbb{R}$;
- $\|\cdot\|_{\mathcal{E}}$ is the norm of a Banach space $\mathcal{E}$;
- $C[a,b]$ is the Banach algebra of continuous functions on $[a,b]$ with the supremum norm;
- $(\cdot,\cdot)_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ are the inner product and the norm in a Hilbert space $\mathcal{H}$. If it is clear what Hilbert space is meant, we shall simply write $(\cdot,\cdot)$ and $\|\cdot\|$;
- $\mathrm{span}(\mathcal{M})$ is the closure of the linear span of a subset $\mathcal{M}$ of a Hilbert space $\mathcal{H}$;
- For Banach spaces E and F, $\mathcal{B}(E,F)$ is the Banach space of all bounded linear operators from E into F;
- $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$ is the Banach algebra of bounded linear operators acting in a Hilbert space $\mathcal{H}$.

If A is a linear operator acting in a Hilbert space $\mathcal{H}$, then:

- $\sigma(A)$ and $\sigma_{\mathrm{e}}(A)$ are the spectrum and essential spectrum of A;
- $\delta(A) = \sigma(A) \setminus \sigma_{\mathrm{e}}(A)$; if A is self-adjoint, $\delta(A)$ consists of isolated eigenvalues of finite multiplicity;
- $R_\lambda(A)$ $(\lambda \notin \sigma(A))$ is the resolvent of A, i.e., $R_\lambda(A) = (A - \lambda I)^{-1}$;
- $\mathrm{tr}(A)$ is the trace of a linear operator A belonging to the trace class.

2. SUFFICIENT CONDITIONS FOR DIRECTIONAL OPERATOR DIFFERENTIABILITY

2.1. In this section we consider the case when the perturbing operator B is of rank one, that is

$$B = (\cdot,g)g \quad (g \in \mathcal{H}). \tag{2.1}$$

Like in [13] (Chapter X, Section 4, n^o 2), we shall represent the operators A and B in a more convenient form.

Let $\{E(\Delta)\}_{\Delta\in\mathcal{J}}$ be the family of spectral projections of the unperturbed operator A, where we denote by $\mathcal{J}$ the set of all intervals (open, or closed, or semi-open) of the real axis $\mathbb{R}$. Consider the subspace

$$\mathcal{H}_g = \operatorname{span}(\{E(\Delta)g\}_{\Delta\in\mathcal{J}})$$

of the space $\mathcal{H}$, where g is the vector from representation (2.1) of the operator B. As is known ([13], Chapter X, Section 4, n^o 2), $\mathcal{H}_g$ is the minimal closed subspace containing the vector g and reducing both A and $A_\gamma = A + \gamma B$ $(\gamma \in \mathbb{R})$. Moreover, for any $\gamma \in \mathbb{R}$

$$\mathcal{H}_g = \operatorname{span}(\{E_\gamma(\Delta)g\}_{\Delta\in\mathcal{J}}),$$

where $\{E_\gamma(\Delta)\}_{\Delta\in\mathcal{J}}$ is the family of spectral projections of the operator A_γ. It is evident that

$$Af = A_\gamma f, \quad \forall\, f \in \mathcal{H} \ominus \mathcal{H}_g.$$

Hence it is enough to investigate the limit (1.1) only on the subspace $\mathcal{H}_g$. Thus, without loss of generality we can assume that $\mathcal{H}_g = \mathcal{H}$, that is, g is a cyclic vector for both operators A and A_γ $(\gamma \in \mathbb{R})$.

Consider the scalar measure

$$\rho(\Delta) = (E(\Delta)g, g) \quad (\Delta \in \mathcal{J}). \tag{2.2}$$

As is known, the operator

$$f = U\widetilde{f} = \int_{-\infty}^{\infty} \widetilde{f}(\lambda)E(\mathrm{d}\lambda)g$$

maps isometrically the space

$$\widetilde{\mathcal{H}} = L_2(\mathbb{R}, \rho) \tag{2.3}$$

onto the space $\mathcal{H}$. Moreover, if we define $\widetilde{A} := U^{-1}AU$, then

$$(\widetilde{A}\widetilde{f})(\mu) = \mu\widetilde{f}(\mu) \quad (\widetilde{f} \in \widetilde{\mathcal{H}}) \tag{2.4}$$

([10], Chapter X, Section 5). Observe that the function $\widetilde{g}(\mu) = (U^{-1}g)(\mu)$ is equal to 1 almost everywhere with respect to the measure ρ. Furthermore, the operator $\widetilde{B} = U^{-1}BU$ has the form:

$$(\widetilde{B}\widetilde{f})(\mu) = (\widetilde{f}, \widetilde{g})_{\widetilde{\mathcal{H}}}\, \widetilde{g}(\mu) = \int_{-\infty}^{\infty} \widetilde{f}(t)\widetilde{g}(t)\rho(\mathrm{d}t)\widetilde{g}(\mu). \tag{2.5}$$

In the sequel we shall deal with the space $\widetilde{\mathcal{H}}$, defined by (2.3), and the operators $\widetilde{A}, \widetilde{B}$ and

$$\widetilde{A}_\gamma = \widetilde{A} + \gamma\widetilde{B}, \quad \gamma \in \mathbb{R}. \tag{2.6}$$

This is a convenient model for the family $A_\gamma = A + \gamma B, \gamma \in \mathbb{R}$. We shall call the measure ρ, defined by (2.2), the *spectral measure* of the operator $\widetilde{A}$. Notice that $\sigma(\widetilde{A}) = \mathrm{supp}(\rho)$.

In this section we study the case of an absolutely continuous spectral measure ρ of the operator $\widetilde{A}$. We shall show that, under additional assumptions on the density $\widetilde{\rho}$ of the measure ρ, the limit (1.1) exists in the uniform operator topology for any continuous function ϕ.

2.2. In the sequel we shall need a well-known property of Riesz projections. Recall that Riesz projections P_+ and P_- are the orthogonal projections in $L_2(\mathbb{R})$ on the Hardy subspaces, respectively:

$$\mathcal{H}_+ = \{f \in L_2(\mathbb{R}) : \widehat{f}(\omega) = 0 \text{ a.e. on } (-\infty, 0)\},$$

$$\mathcal{H}_- = L_2(\mathbb{R}) \ominus \mathcal{H}_+ = \{f \in L_2(\mathbb{R}) : \widehat{f}(\omega) = 0 \text{ a.e. on } (0, \infty)\}.$$

PROPOSITION 2.1. *For any $\varepsilon > 0$ the operators*

$$(P_{+,\varepsilon}h)(u) := \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{h(s)\,\mathrm{d}s}{s - u - \mathrm{i}\varepsilon}\,, \tag{2.7}$$

$$(P_{-,\varepsilon}h)(u) := -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{h(s)\,\mathrm{d}s}{s - u + \mathrm{i}\varepsilon} \tag{2.8}$$

($h \in L_2(\mathbb{R})$) are bounded in the space $L_2(\mathbb{R})$ and $\|P_{+,\varepsilon}\| = \|P_{-,\varepsilon}\| = 1$. Furthermore,

$$P_+ = \lim_{\varepsilon \downarrow 0} P_{+,\varepsilon}, \quad P_- = \lim_{\varepsilon \downarrow 0} P_{-,\varepsilon}$$

in the strong operator topology.

Proof. Indeed, let us define for $\varepsilon \in (-\infty, \infty)$ $g_\varepsilon(x) := (2\pi \mathrm{i}\,(x + \mathrm{i}\varepsilon))^{-1}$. Then

$$\widehat{g_\varepsilon}(t) = -\mathrm{sgn}(\varepsilon)\sigma(\varepsilon t)\,\mathrm{e}^{-\varepsilon t},$$

where $\sigma(a) = 1$ if $a > 0$ and $\sigma(a) = 0$ otherwise, and for $\varepsilon > 0$ we have

$$P_{+,\varepsilon}h = -h * g_\varepsilon \quad P_{-,\varepsilon}h = h * g_{-\varepsilon}\,.$$

The rest follows by standard techniques in Fourier analysis, see [19]. ∎

2.3. In what follows we shall use the following criterion of the weak convergence of multiplication operators.

PROPOSITION 2.2. *Let $\{\psi_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ be a family of functions from the class $L_\infty(\mathbb{R})$. For any $\varepsilon \in (0, \varepsilon_0)$ consider a multiplication operator M_{ψ_ε} by a function ψ_ε acting in the space $L_2(\mathbb{R})$. The family of operators $\{M_{\psi_\varepsilon}\}_{0<\varepsilon<\varepsilon_0}$ converge to a multiplication operator M_{ψ_0} by a function $\psi_0 \in L_\infty(\mathbb{R})$ as $\varepsilon \downarrow 0$ in the weak operator topology if and only if $\lim_{\varepsilon \downarrow 0} \psi_\varepsilon = \psi_0$ in the weak-* topology of $L_\infty(\mathbb{R})$, and this is equivalent to the following conditions:*

(A) *There exists* $\varepsilon_1 \in (0, \varepsilon_0)$*, such that*

$$\overline{\psi} := \sup_{0<\varepsilon<\varepsilon_1} \|\psi_\varepsilon\|_{L_\infty(\mathbb{R})} < \infty.$$

(B) *For any* $x \in \mathbb{R}$

$$\lim_{\varepsilon \downarrow 0} \int_0^x \psi_\varepsilon(t)\,\mathrm{d}t = \int_0^x \psi_0(t)\,\mathrm{d}t.$$

Proof. As is known, the weak operator convergence is equivalent to the following conditions:

(C) There exists $\varepsilon_1 \in (0, \varepsilon_0)$, such that

$$\overline{\psi} := \sup_{0<\varepsilon<\varepsilon_1} \|M_{\psi_\varepsilon}\|_{\mathcal{B}(L_2(\mathbb{R}))} < \infty;$$

(D) For some dense linear subspace D of $L_2(\mathbb{R})$,

$$\lim_{\varepsilon \downarrow 0} (M_{\psi_\varepsilon} f, g) = (M_{\psi_0} f, g) \quad \forall\, f, g \in D.$$

But condition (C) coincides with condition (A), and condition (D) is equivalent to condition (B), if we take as the set D the set of step functions with compact supports. The proposition is proven. ∎

2.4. We now return to the operators $\widetilde{A}, \widetilde{B}$ and $\widetilde{A}_\gamma$ defined by (2.4), (2.5) and (2.6) and acting in the Hilbert space $\widetilde{\mathcal{H}} = L_2(\mathbb{R}, \rho)$, where ρ is the spectral measure of the operator $\widetilde{A}$, defined by (2.2). For brevity we shall denote the inner product in $\widetilde{\mathcal{H}}$ by $(\cdot, \cdot)$. We shall use the following well known representation for the resolvent of the one-rank perturbed operator $\widetilde{A}_\gamma$ (see [4]):

$$(R_\lambda(\widetilde{A}_\gamma) - R_\lambda(\widetilde{A}))\widetilde{f}(\mu) = -\gamma \frac{\Theta(\lambda, \widetilde{f})}{D(\lambda, \gamma)(\mu - \lambda)}, \tag{2.9}$$

where

$$\Theta(\lambda, \widetilde{f}) = (R_\lambda(\widetilde{A})\widetilde{f}, \widetilde{g}) = \int_{-\infty}^{\infty} \frac{\widetilde{f}(s)\rho(\mathrm{d}s)}{s - \lambda}, \tag{2.10}$$

$$D(\lambda, \gamma) = 1 + \gamma(R_\lambda(\widetilde{A})\widetilde{g}, \widetilde{g}) = 1 + \gamma \int_{-\infty}^{\infty} \frac{\rho(\mathrm{d}s)}{s - \lambda}. \tag{2.11}$$

Recall that the function $\widetilde{g}(\mu)$ is equal to 1 almost everywhere with respect to the measure ρ.

The following result concerning a representation of $\frac{\phi(\widetilde{A}_\gamma) - \phi(\widetilde{A})}{\gamma}$ is of central importance in the sequel.

THEOREM 2.3. *Assume that the spectral measure* ρ *of the operator* $\widetilde{A}$ *is absolutely continuous, its density* $\widetilde{\rho}$ *belongs to the class* $L_\infty(\mathbb{R})$ *and has a compact support contained in an interval* $(-M, M)$ $(M > 0)$. *For any* $\phi \in C[-M, M]$ *consider the functions*

$$\psi_+(t, \varepsilon, \gamma) = \frac{\phi_0(t)}{D(t + \mathrm{i}\varepsilon, \gamma)}, \tag{2.12}$$

$$\psi_-(t, \varepsilon, \gamma) = \frac{\phi_0(t)}{D(t - \mathrm{i}\varepsilon, \gamma)}, \tag{2.13}$$

defined on $\mathbb{R}$, *where* $\varepsilon > 0$,

$$\phi_0(t) = \begin{cases} \phi(t) & t \in [-M, M], \\ 0 & t \notin [-M, M], \end{cases} \tag{2.14}$$

and the function $D(\lambda, \gamma)$ *is defined by* (2.11). *In our case*

$$D(\lambda, \gamma) = 1 + \gamma\Theta(\lambda) \quad (\gamma \in \mathbb{R}), \tag{2.15}$$

where

$$\Theta(\lambda) = \int_{-\infty}^{\infty} \frac{\widetilde{\rho}(\mu)\, \mathrm{d}\mu}{\mu - \lambda} \tag{2.16}$$

is the Borel transform of the function $\widetilde{\rho}$. *Assume that for some subset* $\mathcal{C} \subseteq C[-M, M]$ *there exists a number* $\gamma_0 > 0$ *such that for any* $\gamma \in (-\gamma_0, \gamma_0) \setminus \{0\}$ *and for any function* $\phi \in \mathcal{C}$ *there exists such a number* $\varepsilon_0 > 0$ *that the families of multiplication operators*

$$\{M_{\psi_+(\cdot,\varepsilon,\gamma)}\}_{0<\varepsilon<\varepsilon_0} \quad \text{and} \quad \{M_{\psi_-(\cdot,\varepsilon,\gamma)}\}_{0<\varepsilon<\varepsilon_0}$$

belong to $\mathcal{B}(L_2(\mathbb{R}))$ *and converge as* $\varepsilon \downarrow 0$ *in the weak operator topology to operators of multiplication*

$$M_{\psi_+(\cdot,0,\gamma)} \quad \text{and} \quad M_{\psi_-(\cdot,0,\gamma)}$$

by some functions $\psi_+(t, 0, \gamma)$ *and* $\psi_-(t, 0, \gamma)$ *belonging to* $L_\infty(\mathbb{R})$. *Then there exists a number* $\gamma_1 \in (0, \gamma_0]$ *such that for any* $\gamma \in (-\gamma_1, \gamma_1) \setminus \{0\}$ *and for any function* $\phi \in \mathcal{C}$ *the following formula is valid:*

$$\frac{\phi(\widetilde{A}_\gamma) - \phi(\widetilde{A})}{\gamma} = -2\pi\mathrm{i}\{P_- M_{\psi_+(\cdot,0,\gamma)} P_+ - P_+ M_{\psi_-(\cdot,0,\gamma)} P_-\} M_{\widetilde{\rho}}, \tag{2.17}$$

where P_+ *and* P_- *are Riesz projections in* $L_2(\mathbb{R})$.

REMARK. The right hand side of (2.17) makes sense, because the membership $\widetilde{\rho} \in L_\infty(\mathbb{R})$ implies that $M_{\widetilde{\rho}} \in \mathcal{B}(\widetilde{\mathcal{H}}, L_2(\mathbb{R}))$.

Proof. Observe that there exists a number $\gamma_2 > 0$ such that

$$\sigma(\widetilde{A_\gamma}) \subset (-M, M)$$

for any $\gamma \in (-\gamma_2, \gamma_2) \setminus \{0\}$. Let us put $\gamma_1 = \min\{\gamma_0, \gamma_2\}$. Then, making use of formula (A1.1) (Proposition A1.1) and of representation (2.9), we get for any

$\gamma \in (-\gamma_1, \gamma_1) \setminus \{0\}$, $\phi \in \mathcal{C}$ and $\widetilde{f} \in \widetilde{\mathcal{H}}$:

$$\Big(\frac{\phi(\widetilde{A}_\gamma) - \phi(\widetilde{A})}{\gamma}\widetilde{f}\Big)(\mu) = -\frac{1}{2\pi\mathrm{i}} \lim_{\varepsilon\downarrow 0} \int_{-\infty}^{\infty} \phi_0(t)\Big\{\frac{\Theta(t+\mathrm{i}\varepsilon, \widetilde{f})}{D(t+\mathrm{i}\varepsilon,\gamma)(\mu - t - \mathrm{i}\varepsilon)} - \frac{\Theta(t-\mathrm{i}\varepsilon, \widetilde{f})}{D(t-\mathrm{i}\varepsilon,\gamma)(\mu - t + \mathrm{i}\varepsilon)}\Big\}\,\mathrm{d}t, \tag{2.18}$$

where the functional $\Theta(\lambda, \cdot)$ is defined by (2.10), that is, in our case

$$\Theta(\lambda, \widetilde{f}) = \int_{-\infty}^{\infty} \frac{\widetilde{f}(s)\widetilde{\rho}(s)\,\mathrm{d}s}{s-\lambda}. \tag{2.19}$$

In view of (2.19), we can rewrite formula (2.18) in another form, making use of the operators $P_{+,\varepsilon}$ and $P_{-,\varepsilon}$ defined by (2.7) and (2.8):

$$\frac{\phi(\widetilde{A}_\gamma) - \phi(\widetilde{A})}{\gamma} = -\lim_{\varepsilon\downarrow 0} 2\pi\mathrm{i}\{P_{-,\varepsilon}M_{\psi_+(\cdot,\varepsilon,\gamma)}P_{+,\varepsilon} - P_{+,\varepsilon}M_{\psi_-(\cdot,\varepsilon,\gamma)}P_{-,\varepsilon}\}M_{\widetilde{\rho}}, \tag{2.20}$$

where the functions $\psi_+(t,\varepsilon,\gamma)$ and $\psi_-(t,\varepsilon,\gamma)$ are defined by formulas (2.12) and (2.13) respectively.

Recall that, by Proposition A1.1, the limit of the family of operators in the right hand side of (2.20) exists in the uniform operator topology. Thus, in order to prove the formula (2.17), it is enough to show that in the weak operator topology this limit is equal to the operator in the right hand side of (2.17). To this end consider the following representation for $\widetilde{f}, \ \widetilde{g} \in \widetilde{\mathcal{H}}$:

$$((P_+M_{\psi_-(\cdot,0,\gamma)}P_- - P_{+,\varepsilon}M_{\psi_-(\cdot,\varepsilon,\gamma)}P_{-,\varepsilon})M_{\widetilde{\rho}}\widetilde{f}, \quad \widetilde{g}) = J_{1,\varepsilon}(\widetilde{f},\widetilde{g}) + J_{2,\varepsilon}(\widetilde{f},\widetilde{g}) + J_{3,\varepsilon}(\widetilde{f},\widetilde{g}), \tag{2.21}$$

where

$$J_{1,\varepsilon}(\widetilde{f},\widetilde{g}) = ((M_{\psi_-(\cdot,0,\gamma)} - M_{\psi_-(\cdot,\varepsilon,\gamma)})P_-M_{\widetilde{\rho}}\widetilde{f}, \quad P_+M_{\widetilde{\rho}}\widetilde{g})_{L_2(\mathbb{R})}, \tag{2.22}$$

$$J_{2,\varepsilon}(\widetilde{f},\widetilde{g}) = (M_{\psi_-(\cdot,\varepsilon,\gamma)}P_-M_{\widetilde{\rho}}\widetilde{f}, \quad (P_+ - P_{+,\varepsilon})M_{\widetilde{\rho}}\widetilde{g})_{L_2(\mathbb{R})}, \tag{2.23}$$

$$J_{3,\varepsilon}(\widetilde{f},\widetilde{g}) = (M_{\psi_-(\cdot,\varepsilon,\gamma)}(P_- - P_{-,\varepsilon})M_{\widetilde{\rho}}\widetilde{f}, \quad P_{+,\varepsilon}M_{\widetilde{\rho}}\widetilde{g})_{L_2(\mathbb{R})}. \tag{2.24}$$

By the assumption of the theorem, the family of operators $\{M_{\psi_-(\cdot,\varepsilon,\gamma)}\}_{0<\varepsilon<\varepsilon_0}$ converges to $M_{\psi_-(\cdot,0,\gamma)}$ as $\varepsilon \downarrow 0$ in the weak operator topology. Therefore, in view of (2.22),

$$\lim_{\varepsilon\downarrow 0} J_{1,\varepsilon}(\widetilde{f},\widetilde{g}) = 0. \tag{2.25}$$

Furthermore, by the criterion of the weak convergence of multiplication operators, given in Proposition 2.2, there exist $\overline{M} > 0$ and $\varepsilon_1 \in (0, \varepsilon_0)$, such that for any $\varepsilon \in (0, \varepsilon_1)$

$$\|M_{\psi_-(\cdot,\varepsilon,\gamma)}\|_{\mathcal{B}(L_2(\mathbb{R}))} = \|\psi_-(\cdot,\varepsilon,\gamma)\|_{L_\infty(\mathbb{R})} \leqslant \overline{M}. \tag{2.26}$$

The latter fact, definition (2.23) and Proposition 2.1 imply that

$$\lim_{\varepsilon\downarrow 0} J_{2,\varepsilon}(\widetilde{f},\widetilde{g}) = 0. \tag{2.27}$$

From definition (2.24), estimate (2.26) and Proposition 2.1 we get:

$$\lim_{\varepsilon\downarrow 0} J_{3,\varepsilon}(\widetilde{f},\widetilde{g}) = 0. \tag{2.28}$$

The representation (2.21) and the limiting relations (2.25), (2.27) and (2.28) imply that for any $\widetilde{f},\widetilde{g} \in \widetilde{\mathcal{H}}$

$$\lim_{\varepsilon\downarrow 0}(P_{+,\varepsilon}M_{\psi_-(\cdot,\varepsilon,\gamma)}P_{-,\varepsilon}M_{\widetilde{\rho}}\widetilde{f},\quad \widetilde{g}) = (P_+M_{\psi_-(\cdot,0,\gamma)}P_-M_{\widetilde{\rho}}\widetilde{f},\quad \widetilde{g}).$$

In the analogous manner we prove that for any $\widetilde{f},\widetilde{g} \in \widetilde{\mathcal{H}}$

$$\lim_{\varepsilon\downarrow 0}(P_{-,\varepsilon}M_{\psi_-(\cdot,\varepsilon,\gamma)}P_{+,\varepsilon}M_{\widetilde{\rho}}\widetilde{f},\quad \widetilde{g}) = (P_-M_{\psi_-(\cdot,0,\gamma)}P_+M_{\widetilde{\rho}}\widetilde{f},\quad \widetilde{g}).$$

These circumstances mean that the family of operators in the right hand side of (2.20) converges to the operator in the right hand side of (2.17) in the weak operator topology. The theorem is proven. ∎

The following concrete realization of Theorem 2.3 is valid.

THEOREM 2.4. *Assume that the spectral measure ρ of the operator $\widetilde{A}$ is absolutely continuous, its density $\widetilde{\rho}$ belongs to the class $L_\infty(\mathbb{R})$ and has a compact support contained in an interval $(-M,M)$ $(M>0)$. Furthermore, assume that*

$$\mathcal{R}(t) = \int_0^\infty \frac{|\widetilde{\rho}(t+s)-\widetilde{\rho}(t-s)|}{s}\,\mathrm{d}s < \infty \quad \textit{for almost all } t \in (-M,M), \tag{2.29}$$

$$\mathcal{R} \in L_\infty(-M,M). \tag{2.30}$$

Then for every small enough $\gamma \neq 0$ and for any function $\phi \in C[-M,M]$ the representation (2.17) *is valid with the $L_\infty(\mathbb{R})$ functions*

$$\psi_-(t,0,\gamma) = \frac{\phi_0(t)}{D(t-\mathrm{i}0,\gamma)} \quad \textit{and} \quad \psi_+(t,0,\gamma) = \frac{\phi_0(t)}{D(t+\mathrm{i}0,\gamma)}, \tag{2.31}$$

where the function $\phi_0(t)$ is defined by (2.14),

$$D(t+\mathrm{i}0,\gamma) = 1+\gamma\Theta(t+\mathrm{i}0), \quad D(t-\mathrm{i}0,\gamma) = 1+\gamma\Theta(t-\mathrm{i}0), \tag{2.32}$$

$$\Theta(t+\mathrm{i}0) = \int_0^\infty \frac{\widetilde{\rho}(t+s)-\widetilde{\rho}(t-s)}{s}\,\mathrm{d}s + \mathrm{i}\pi\widetilde{\rho}(t), \tag{2.33}$$

$$\Theta(t-\mathrm{i}0) = \int_0^\infty \frac{\widetilde{\rho}(t+s)-\widetilde{\rho}(t-s)}{s}\,\mathrm{d}s - \mathrm{i}\pi\widetilde{\rho}(t). \tag{2.34}$$

Proof. Taking into account definitions (2.15) and (2.16), we obtain from assertion (ii) of Proposition A2.1 that there exists $\gamma_0 > 0$, such that for every $\gamma \in (-\gamma_0, \gamma_0)$, $t \in \mathbb{R}$ and $\varepsilon > 0$

$$|D(t + \mathrm{i}\varepsilon, \gamma)| = |1 + \gamma\Theta(t + \mathrm{i}\varepsilon)| \geqslant \frac{1}{2}, \tag{2.35}$$

$$|D(t - \mathrm{i}\varepsilon, \gamma)| = |1 + \gamma\Theta(t - \mathrm{i}\varepsilon)| \geqslant \frac{1}{2}. \tag{2.36}$$

Take a function $\phi \in C[-M, M]$. In view of the above estimates and definitions (2.12), (2.13) and (2.14), for any $\gamma \in (-\gamma_0, \gamma_0)$ the families of functions

$$\{\psi_+(t, \varepsilon, \gamma)\}_{\varepsilon>0} \quad \text{and} \quad \{\psi_-(t, \varepsilon, \gamma)\}_{\varepsilon>0} \tag{2.37}$$

satisfy condition (A) of Proposition 2.2 with $\overline{\psi} = 2\|\phi\|_{C[-M,M]}$. Furthermore, assertion (i) of Proposition A2.1 yields the limiting relations

$$\lim_{\varepsilon\downarrow 0} \Theta(t + \mathrm{i}\varepsilon) = \Theta(t + \mathrm{i}0), \quad \lim_{\varepsilon\downarrow 0} \Theta(t - \mathrm{i}\varepsilon) = \Theta(t - \mathrm{i}0)$$

for almost all $t \in \mathbb{R}$, where $\Theta(t + \mathrm{i}0)$ and $\Theta(t - \mathrm{i}0)$ are defined by (2.33) and (2.34). Thus, definitions (2.12)–(2.15) and estimates (2.35) and (2.36) imply that, if $\gamma \in (-\gamma_0, \gamma_0)$, the following limiting relations are valid for almost all $t \in \mathbb{R}$:

$$\lim_{\varepsilon\downarrow 0} \psi_+(t, \varepsilon, \gamma) = \psi_+(t, 0, \gamma) \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} \psi_-(t, \varepsilon, \gamma) = \psi_-(t, 0, \gamma),$$

where $\psi_+(t, 0, \gamma)$ and $\psi_-(t, 0, \gamma)$ are defined by (2.31), (2.32) and (2.14). Hence, by Lebesgue's Theorem, also condition (B) of Proposition 2.2 is satisfied for the families (2.37). Thus, by Proposition 2.2, for any $\gamma \in (-\gamma_0, \gamma_0)$

$$\lim_{\varepsilon\downarrow 0} M_{\psi_+(\cdot,\varepsilon,\gamma)} = M_{\psi_+(\cdot,0,\gamma)} \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} M_{\psi_-(\cdot,\varepsilon,\gamma)} = M_{\psi_-(\cdot,0,\gamma)}$$

in the weak operator topology on $\mathcal{B}(L_2(\mathbb{R}))$. So, the conditions of Theorem 2.3 are satisfied if we take there $\mathcal{C} = C[-M, M]$. Hence the desired assertion is valid. ∎

We now turn to the main theorem of this section.

THEOREM 2.5. *Assume that the spectral measure ρ of the operator $\widetilde{A}$ is absolutely continuous, its density $\widetilde{\rho}$ belongs to the class $L_\infty(\mathbb{R})$ and has a compact support contained in an interval $(-M, M)$ $(M > 0)$. If conditions* (2.29) *and* (2.30) *are satisfied, then for any function $\phi \in C[-M, M]$ the limit*

$$\lim_{\gamma\to 0} \frac{\phi(\widetilde{A}_\gamma) - \phi(\widetilde{A})}{\gamma}$$

exists in the uniform operator topology and it is equal to

$$2\pi\mathrm{i}\{P_+ M_{\phi_0} P_- - P_- M_{\phi_0} P_+\} M_{\widetilde{\rho}}, \tag{2.38}$$

where P_+ and P_- are Riesz projections in $L_2(\mathbb{R})$ and the function $\phi_0(t)$ is defined by (2.14).

Proof. The assertion of the theorem immediately follows from (2.17) and (2.31), and the fact that, in view of (2.32) and Corollary A2.2, the families of functions

$$\left\{\frac{1}{D(t+\mathrm{i}0,\gamma)}\right\}_{\gamma\in\mathbb{R}} \quad \text{and} \quad \left\{\frac{1}{D(t-\mathrm{i}0,\gamma)}\right\}_{\gamma\in\mathbb{R}}$$

tend to 1 as $\gamma \to 0$ in the $L_\infty(\mathbb{R})$-norm. ∎

REMARK 2.6. It is easy to show that formula (2.38) can be written also in the form:

$$\lim_{\gamma\to 0}\frac{\phi(\widetilde{A}_\gamma)-\phi(\widetilde{A})}{\gamma} = \pi(HM_{\phi_0}-M_{\phi_0}H)M_{\widetilde{\rho}} = \pi[H,M_{\phi_0}]M_{\widetilde{\rho}},$$

where $H = \mathrm{i}(P_+ - P_-)$ is Hilbert transform in $L_2(\mathbb{R})$.

2.5. In this subsection we shall discuss conditions (2.29) and (2.30) of Theorems 2.4 and 2.5. It is evident that these conditions follow from the following ones:

$$\widetilde{\mathcal{R}}(t) = \int_{-\infty}^{\infty}\frac{|\widetilde{\rho}(t+s)-\widetilde{\rho}(t)|}{|s|}\,\mathrm{d}s < \infty \quad \text{a.e. in } (-M,M), \tag{2.39}$$

$$\widetilde{\mathcal{R}} \in L_\infty(-M,M) \tag{2.40}$$

(as above, we assume that $\mathrm{supp}(\widetilde{\rho}) \subset (-M,M)$). The latter conditions are well known in the theory of integrals of Cauchy type ([12], Chapter I, Section 5, n^{o}5.2). Under these conditions and some additional assumptions the classical Sokhotskii formulas ([12], Chapter I, Section 4, n^{o}4.2) are valid for the boundary values of the integrals of Cauchy type. The formulas (2.33) and (2.34) can be considered as non-classical generalizations of Sokhotskii formulas (see Proposition A2.1). Observe that any function $\widetilde{\rho}$ having a compact support and belonging to Hölder class $\mathrm{Lip}_\alpha(\mathbb{R})$ with $\alpha \in (0,1)$, satisfies conditions (2.39) and (2.40) (hence, also conditions (2.29) and (2.30)).

Observe also that Hölder condition (and also the more general conditions (2.39) and (2.40)) forbid very steep up- and down-slopes of the graph of a function at each point. Also, conditions (2.29) and (2.30) forbid such slopes in the intervals of monotony of the function, because they are equivalent to conditions (2.39) and (2.40) in such intervals. But conditions (2.29) and (2.30)) permit arbitrarily acute "cusps" of the graph (up- or down-directed), because in the integral of (2.29) the values of the function $\widetilde{\rho}$ at each pair of points $t+s$ and $t-s$ may be almost equal, and thus annihilate each other. Therefore the up- and down-slopes of these "cusps" may be arbitrarily steep. This means that conditions (2.29) and (2.30) are essentially milder than (2.39) and (2.40).

The following example confirms the above qualitative arguments.

EXAMPLE 2.7. Consider a real-valued function $\widetilde{\rho}$ having the properties:

(a) $\widetilde{\rho} \in C(\mathbb{R})$;

(b) $\operatorname{supp}(\widetilde{\rho}) = [-1, 1]$;
(c) $\widetilde{\rho}$ is even, that is $\widetilde{\rho}(-t) = \widetilde{\rho}(t)$ for any $t \in \mathbb{R}$;
(d) the function $\widetilde{\rho}$ is continuously differentiable in $\mathbb{R} \setminus \{0\}$;
(e) $\widetilde{\rho}$ is increasing in the interval $(0, \frac{1}{2})$;
(f) $\widetilde{\rho}$ is concave in $(0, \frac{1}{2})$;
(g) the behavior of the function $\widetilde{\rho}$ as $t \to 0$ is defined by the condition:

$$\int_0^{1/2} \frac{\widetilde{\rho}(\mu) - \widetilde{\rho}(0)}{\mu} \, d\mu = \infty.$$

We shall show that the function $\widetilde{\rho}$ satisfies conditions (2.29) and (2.30) with $M > 1$, but it does not satisfy condition (2.40).

Indeed, taking into account conditions (a), (b) and (d), it is not difficult to show that

$$\sup_{t \in (-M,M)} \int_{1/4}^{\infty} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s} \, ds < \infty,$$
$$\sup_{t \in (-M,M) \setminus [-\frac{1}{4}, \frac{1}{4}]} \int_0^{1/4} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s} \, ds < \infty.$$

Thus, in order to prove that conditions (2.29) and (2.30) are satisfied, it is enough to show that

$$\sup_{t \in [-\frac{1}{4}, \frac{1}{4}]} \int_0^{1/4} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s} \, ds < \infty. \tag{2.41}$$

Assume that $t \in [0, \frac{1}{4}]$. Consider the representation:

$$\int_0^{1/4} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s} \, ds = I_1(t) + I_2(t), \tag{2.42}$$

where

$$I_1(t) = \int_0^{t} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s} \, ds \quad \text{and} \quad I_2(t) = \int_t^{1/4} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s} \, ds.$$

Taking into account conditions (c), (d), (e), (f) and the fact that the derivative of a concave function is non-increasing, we get:

$$I_1(t) = \int_0^t \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s}\,\mathrm{d}s \leqslant 2\int_0^t \widetilde{\rho}'(t-s)\,\mathrm{d}s = 2(\widetilde{\rho}(t) - \widetilde{\rho}(0)),$$

$$I_2(t) = \int_t^{1/4} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(s-t)}{s}\,\mathrm{d}s \leqslant 2\int_t^{1/4} \widetilde{\rho}'(s-t)\,\mathrm{d}s = 2\Big(\widetilde{\rho}\Big(\frac{1}{4} - t\Big) - \widetilde{\rho}(0)\Big).$$

The latter estimates and representation (2.42) imply that

$$\sup_{t\in[0,\frac{1}{4}]} \int_0^{1/4} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s}\,\mathrm{d}s < \infty.$$

In the analogous manner we prove that

$$\sup_{t\in[-\frac{1}{4},0)} \int_0^{1/4} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s}\,\mathrm{d}s < \infty.$$

So, we have proved property (2.41), that is, conditions (2.29) and (2.30) are satisfied for the function $\widetilde{\rho}$.

Observe that, in view of conditions (a), (b) and (d), condition (2.39) is satisfied for the function $\widetilde{\rho}$. It remains to show that condition (2.40) fails. Consider the function $\widetilde{\mathcal{R}}(t)$ defined in (2.39). Taking into account condition (e), we have for $t > 0$:

$$\widetilde{\mathcal{R}}(t) \geqslant \int_0^{1/4} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t)}{s}\,\mathrm{d}s. \tag{2.43}$$

In view of condition (f), the function

$$\Gamma(t) = \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t)}{s}$$

is non-increasing in the interval $(0, \frac{1}{4})$ for any fixed $s \in (0, \frac{1}{4})$. Thus, by the Monotone Convergence Theorem ([18], Part One, Chapter 4) and conditions (e) and (g), we have:

$$\lim_{t\downarrow 0} \int_0^{1/4} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t)}{s}\,\mathrm{d}s = \int_0^{1/4} \frac{\widetilde{\rho}(s) - \widetilde{\rho}(0)}{s}\,\mathrm{d}s = \infty.$$

Applying estimate (2.43), we obtain $\lim\limits_{t\downarrow 0} \widetilde{\mathcal{R}}(t) = \infty$. Therefore $\widetilde{\mathcal{R}} \notin L_\infty(-M, M))$, that is, condition (2.40) is not satisfied for the function $\widetilde{\rho}$.

Notice that condition (g) is satisfied, if, for instance, the function $\widetilde{\rho}$ has the form in a neighborhood $(-\delta,\delta)$ of the point $t=0$:

$$\widetilde{\rho}(t)=\Big\{\prod_{k=1}^{m-1}\ln^{[k]}\Big(\frac{1}{|t|}\Big)\Big(\ln^{[m]}\Big(\frac{1}{|t|}\Big)\Big)^{\alpha}\Big\}^{-1}+C,\ \text{if } t\in(-\delta,\delta)\setminus\{0\},\quad \text{and } \widetilde{\rho}(0)=C, \tag{2.44}$$

where $\alpha\in(0,1)$, $m\in\mathbb{N}$, $C\in\mathbb{R}$ and

$$\ln^{[1]}u=\ln u,\quad \ln^{[k+1]}u=\ln(\ln^{[k]}u).$$

Here the number $\delta>0$ is chosen small enough, such that the function $\widetilde{\rho}(t)$ is continuous, increasing and concave in $[0,\delta)$.

3. NECESSARY CONDITIONS CONNECTED WITH GAPS OF $\sigma(A)$

In this section we shall obtain some necessary conditions for directional operator differentiability of any continuous function. These conditions will be imposed on the behavior of the spectral measure of the unperturbed operator near endpoints of gaps of its spectrum.

3.1. We shall first consider a general situation. Let A and B be bounded self-adjoint operators acting in a Hilbert space $\mathcal{H}$ and M be such a positive number that $\sigma(A)\subset(-M,M)$. Denote by $\Sigma(\cdot,\gamma)$ the bounded linear operator acting from the space $C[-M,M]$ into the space $\mathcal{B}(\mathcal{H})$ and defined by

$$\Sigma(\phi,\gamma)=\frac{\phi(A_\gamma)-\phi(A)}{\gamma},$$

where $A_\gamma=A+\gamma B$ and $\gamma\in\mathbb{R}$ $(\gamma\neq0)$. Observe that the operator $\Sigma(\cdot,\gamma)$ is defined for every small enough $\gamma\neq0$, for which $\sigma(A_\gamma)\subset(-M,M)$. We shall be based on the following

LEMMA 3.1. *If the following condition is satisfied:*

$$\limsup_{\gamma\to0}\|\Sigma(\cdot,\gamma)\|_{\mathcal{B}(C[-M,M],\mathcal{B}(\mathcal{H}))}=\infty, \tag{3.1}$$

then there exists a function $\phi\in C[-M,M]$, such that the limit (1.1) *does not exist in the strong operator topology.*

Proof. By Banach-Steinhaus Theorem, condition (3.1) implies that there exists a function $\phi\in C[-M,M]$, such that

$$\limsup_{\gamma\to0}\|\Sigma(\phi,\gamma)\|_{\mathcal{B}(\mathcal{H})}=\infty.$$

Making use again of Banach-Steinhaus Theorem, we get the assertion of the lemma. ∎

REMARK 3.2. In Lemma 3.1, the space $C[-M, M]$ can be complex or real.

The main theorem of this subsection is following:

THEOREM 3.3. *If the limit* (1.1) *exists in the strong operator topology for any* $\phi \in C[-M, M]$, *then*

$$\exists\, \gamma_0 > 0, \quad \forall\, \gamma \in (-\gamma_0, \gamma_0): \quad \sigma(A_\gamma) \subseteq \sigma(A). \tag{3.2}$$

Proof. Assume that the condition of the theorem is satisfied, but property (3.2) is not valid. Then there exists a sequence of numbers $\gamma_k \in \mathbb{R} \setminus \{0\}$ and a sequence of points $\lambda_k \in \sigma(A_{\gamma_k})$, such that

$$\lim_{k\to\infty} \gamma_k = 0 \tag{3.3}$$

and $\lambda_k \in (-M, M) \setminus \sigma(A)$. Therefore there exists a sequence of closed intervals of the form $\Delta_k = [\lambda_k - \delta_k, \lambda_k + \delta_k]$ ($\delta_k > 0$) such that $\Delta_k \subset (-M, M)$ and

$$\Delta_k \cap \sigma(A) = \emptyset \quad \forall\, k \in \mathbb{N}. \tag{3.4}$$

Denote $\widetilde{\Delta}_k = \left[\lambda_k - \frac{\delta_k}{2}, \lambda_k + \frac{\delta_k}{2}\right]$. Consider a sequence of real-valued functions ϕ_k in $C[-M, M]$ having the properties for any $k \in \mathbb{N}$:

(a) $\operatorname{supp}(\phi_k) \subseteq \Delta_k$;
(b) $\phi_k(t) \geqslant 0 \quad \forall\, t \in \Delta_k$;
(c) $\phi_k(t) = 1 \quad \forall\, t \in \widetilde{\Delta}_k$;
(d) $\|\phi_k\|_{C[-M,M]} = 1$.

Let $E(\Delta)$ and $E_\gamma(\Delta)$ be the families of spectral projections of the operators A and A_γ respectively. Making use of the relation (3.4) and property (a) of the functions ϕ_k, we have:

$$\frac{\phi_k(A_{\gamma_k}) - \phi_k(A)}{\gamma_k} = \frac{1}{\gamma_k}\Big\{ \int_{\Delta_k} \phi_k(t) E_{\gamma_k}(\mathrm{d}t) - \int_{\Delta_k} \phi_k(t) E(\mathrm{d}t) \Big\} = \frac{1}{\gamma_k} \int_{\Delta_k} \phi_k(t) E_{\gamma_k}(\mathrm{d}t).$$

Take a sequence of vectors $f_k \in \mathcal{H}$, such that $\|f_k\| = 1$ and $f_k \in E_{\gamma_k}(\widetilde{\Delta}_k)(\mathcal{H})$. Then, taking into account properties (b) and (c) of the functions ϕ_k, we get:

$$\begin{aligned} \Big\| \frac{\phi_k(A_{\gamma_k}) - \phi_k(A)}{\gamma_k} \Big\|_{\mathcal{B}(\mathcal{H})} &\geqslant \Big| \Big(\frac{\phi_k(A_{\gamma_k}) - \phi_k(A)}{\gamma_k} f_k, f_k \Big) \Big| \\ &\geqslant \frac{1}{|\gamma_k|} \int_{\widetilde{\Delta}_k} \phi_k(t) (E_{\gamma_k}(\mathrm{d}t) f_k, f_k) = \frac{1}{|\gamma_k|}. \end{aligned}$$

Hence, in view of (3.3),

$$\lim_{k\to\infty} \Big\| \frac{\phi_k(A_{\gamma_k}) - \phi_k(A)}{\gamma_k} \Big\|_{\mathcal{B}(\mathcal{H})} = \infty.$$

In view of property (d) of functions ϕ_k, the latter fact means that the limiting relation (3.1) is valid. Hence Lemma 3.1 implies that there exists a function $\phi \in C[-M, M]$, such that the limit (1.1) does not exist in the strong operator topology. The latter contradicts the condition of the theorem. So, our assumption that property (3.2) fails, is not true. ∎

REMARK 3.4. It is evident that, in view of Remark 3.2, it is enough to assume in the formulation of Theorem 3.3 that the limit (1.1) exists in the strong operator topology for any real valued function $\phi \in C[-M, M]$.

3.2. If the perturbing operator B is compact, we can formulate a stronger necessary condition for the directional operator differentiability of any continuous function. It is based on the following

PROPOSITION 3.5. *If the operator B is compact and property* (3.2) *is valid, then*

$$\exists\, \gamma_0 > 0, \quad \forall\, \gamma \in (-\gamma_0, \gamma_0): \quad \sigma(A_\gamma) = \sigma(A). \tag{3.5}$$

Proof. By Weyl Theorem ([13], Chapter IV, Section 5, n^o 6, Theorem 5.35),

$$\sigma_e(A_\gamma) = \sigma_e(A). \tag{3.6}$$

Then, in view of (3.2), it is enough to prove that

$$\delta(A) \subseteq \delta(A_\gamma) \quad \forall \gamma \in (-\gamma_0, \gamma_0). \tag{3.7}$$

Take $\lambda_0 \in \delta(A)$. Then, in view of (3.6) and (3.2), there exists a neighborhood $(\lambda_0 - \delta, \lambda_0 + \delta)$ of the point λ_0, such that $((\lambda_0 - \delta, \lambda_0 + \delta) \setminus \{\lambda_0\}) \cap \sigma(A_\gamma) = \emptyset$ for any $\gamma \in (-\gamma_0, \gamma_0)$. Take $\sigma \in (0, \delta)$. Since the function

$$T(\gamma) = -\frac{1}{2\pi i} \operatorname{tr}\Big(\oint_{|\lambda| = \sigma} R_\lambda(A_\gamma)\, d\lambda \Big)$$

is continuous and takes only non-negative integer values and $T(0) > 0$, then $T(\gamma) > 0$ for any $\gamma \in (-\gamma_0, \gamma_0)$. This means that for these values of γ the point λ_0 belongs to $\delta(A_\gamma)$. Property (3.7) is proven, hence the proposition is proven too. ∎

Theorem 3.3 and Proposition 3.5 imply

COROLLARY 3.6. *If the operator B is compact and the limit* (1.1) *exists in the strong operator topology for all $\phi \in C[-M, M]$, then property* (3.5) *holds for the spectra $\sigma(A)$ and $\sigma(A_\gamma)$.*

3.3. In the case if the perturbing operator B is of rank one, the above necessary conditions acquire a more constructive form. We return to the situation described in Subsection 2.1 of Section 2, that is we consider the unperturbed operator $\widetilde{A}$ of the form (2.4), the perturbing operator $\widetilde{B}$ of the form (2.5) and the operator $\widetilde{A}_\gamma$ of the form $\widetilde{A}_\gamma = \widetilde{A} + \gamma \widetilde{B}$ $(\gamma \in \mathbb{R})$. All these operators act in the space $\widetilde{\mathcal{H}} = L_2(\mathbb{R}, \rho)$. We have called ρ the spectral measure of $\widetilde{A}$. We assume that $\widetilde{A} \in \mathcal{B}(\widetilde{\mathcal{H}})$, hence $\sigma(\widetilde{A}) = \operatorname{supp}(\rho)$ is a compact set. We turn to the following

PROPOSITION 3.7. *Let $\{(a_j, b_j)\}$ be the gaps of the spectrum $\sigma(\widetilde{A})$ of the operator $\widetilde{A}$, that is*

$$\mathbb{R} \setminus \operatorname{supp}(\rho) = (a_1, b_1) \cup (a_2, b_2) \cup \bigcup_{j \in \mathcal{N} \setminus \{1,2\}} (a_j, b_j),$$

where $a_1 = -\infty$, $b_2 = \infty$ and either $\mathcal{N} = \{1, 2, \dots, N\}$ $(N \geqslant 2)$, or $\mathcal{N} = \mathbb{N}$. Then the property

$$\exists\, \gamma_0 > 0, \quad \forall\, \gamma \in (-\gamma_0, \gamma_0): \quad \sigma(\widetilde{A}_\gamma) \subseteq \sigma(\widetilde{A}) \tag{3.8}$$

of the spectra of the operators $\widetilde{A}$ and $\widetilde{A}_\gamma$ is valid if and only if

$$\sup_{\lambda \in \mathbb{R} \setminus \operatorname{supp}(\rho)} \Big| \int_{-\infty}^{\infty} \frac{\rho(\mathrm{d}s)}{s - \lambda} \Big| < \infty. \tag{3.9}$$

That is, the Borel transform $\Theta(\lambda) := \int_{\mathbb{R}} (t - \lambda)^{-1} \rho(\mathrm{d}t)$ of ρ is bounded in $\mathbb{R} \setminus \operatorname{supp}(\rho)$. Furthermore, property (3.9) *implies that*

$$\forall\, k \in \mathcal{N} \setminus \{1\}: \quad \int_{-\infty}^{a_k} \frac{\rho(\mathrm{d}s)}{a_k - s} < \infty, \tag{3.10}$$

$$\forall\, k \in \mathcal{N} \setminus \{2\}: \quad \int_{b_k}^{\infty} \frac{\rho(\mathrm{d}s)}{s - b_k} < \infty. \tag{3.11}$$

Proof. As is known, $\sigma(\widetilde{A}_\gamma) \setminus \sigma(\widetilde{A})$ is the set of points $\lambda \in \mathbb{R} \setminus \operatorname{supp}(\rho)$ satisfying the equation

$$\int_{-\infty}^{\infty} \frac{\rho(\mathrm{d}s)}{s - \lambda} = -\frac{1}{\gamma}$$

(see [4]). On the other hand, the function $\Theta(\lambda) = \int_{-\infty}^{\infty} \frac{\rho(\mathrm{d}s)}{s - \lambda}$ increases in each gap (a_k, b_k) of $\operatorname{supp}(\rho)$, because

$$\frac{\mathrm{d}\Theta(\lambda)}{\mathrm{d}\lambda} = \int_{-\infty}^{\infty} \frac{\rho(\mathrm{d}s)}{(s - \lambda)^2}.$$

Furthermore, since $\operatorname{supp}(\rho)$ is compact, then

$$\lim_{|\lambda| \to \infty} \int_{-\infty}^{\infty} \frac{\rho(\mathrm{d}s)}{s - \lambda} = 0.$$

Thus, it is evident that property (3.8) is valid if and only if there exists $\overline{\theta} > 0$, such that

$$\lim_{\lambda \downarrow a_k} \int_{-\infty}^{\infty} \frac{\rho(\mathrm{d}s)}{s-\lambda} \geqslant -\overline{\theta} \quad \text{for any } k \in \mathcal{N} \setminus \{1\},$$

$$\lim_{\lambda \uparrow b_k} \int_{-\infty}^{\infty} \frac{\rho(\mathrm{d}s)}{s-\lambda} \leqslant \overline{\theta} \qquad \text{for any } k \in \mathcal{N} \setminus \{2\}.$$

These circumstances imply the equivalence of (3.8) and (3.9).

We now turn to the proof of the second assertion of the proposition. Assume that $k \in \mathcal{N} \setminus \{1\}$ and $\lambda \in (a_k, b_k)$. Then from the equality

$$\int_{a_k}^{\infty} \frac{\rho(\mathrm{d}s)}{s-\lambda} = \int_{b_k}^{\infty} \frac{\rho(\mathrm{d}s)}{s-\lambda}$$

and the fact that $\operatorname{supp}(\rho) \subseteq [b_1, a_2]$, we get the existence of the following limit:

$$\lim_{\lambda \downarrow a_k} \int_{a_k}^{\infty} \frac{\rho(\mathrm{d}s)}{s-\lambda} = \int_{b_k}^{\infty} \frac{\rho(\mathrm{d}s)}{s-a_k}.$$

In this situation, the Monotone Convergence Theorem ([18], Part One, Chapter 4) and property (3.9) imply that

$$\int_{-\infty}^{a_k} \frac{\rho(\mathrm{d}s)}{a_k - s} = \lim_{\lambda \downarrow a_k} \int_{-\infty}^{a_k} \frac{\rho(\mathrm{d}s)}{\lambda - s} < \infty.$$

That is, property (3.10) is valid. Property (3.11) is proved analogously. ∎

Theorem 3.3 and Proposition 3.7 imply the following result.

THEOREM 3.8. *Let $\widetilde{A}, \widetilde{B}, \widetilde{A}_\gamma$ and ρ be as above. Let $M > 0$ be such, that* $\operatorname{supp}(\rho) \subset (-M, M)$. *If for any function $\phi \in C[-M, M]$ the limit*

$$\lim_{\gamma \to 0} \frac{\phi(\widetilde{A}_\gamma) - \phi(\widetilde{A})}{\gamma}$$

exists in the strong operator topology, then the spectral measure ρ of the operator $\widetilde{A}$ has property (3.9).

REMARK 3.9. It is easy to check that if the spectral measure ρ of the operator $\widetilde{A}$ is absolutely continuous with a density $\widetilde{\rho}$, then condition (3.9) can be written in the form:

$$\sup_{t \in \mathbb{R} \setminus \operatorname{supp}(\rho)} \Big(\Big| \int_0^{\infty} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s} \, \mathrm{d}s \Big| \Big) < \infty.$$

This means that the sufficient conditions for the directional operator differentiability, found in Theorem 2.5, are close in some sense to necessary ones in the gaps of the spectrum of the unperturbed operator.

The following consequence of Theorem 3.8 is valid:

COROLLARY 3.10. *If the conditions of Theorem* 3.8 *are satisfied, then* $\delta(\widetilde{A}) = \emptyset$, *that is* $\sigma(\widetilde{A}) = \sigma_{\mathrm{e}}(\widetilde{A})$. *The same holds for the operator* A.

Proof. Assume that $\lambda_0 \in \delta(\widetilde{A})$, that is, λ_0 is an isolated eigenvalue of $\widetilde{A}$ (equivalently, λ_0 is an isolated point of $\operatorname{supp}(\rho)$). This implies that in some punctured neighborhood $\mathcal{O}(\lambda_0) \setminus \{\lambda_0\}$ the function $\Theta(\lambda) = \int_{-\infty}^{\infty} \frac{\rho(\mathrm{d}s)}{s-\lambda}$ is analytic and the point $\lambda = \lambda_0$ is a pole of this function. Hence $\lim_{\lambda \uparrow \lambda_0} \Theta(\lambda) = \infty$, that is, condition (3.9) is not satisfied. This fact and Theorem 3.8 imply the desired assertion. ∎

4. NECESSARY CONDITIONS CONNECTED WITH THE INTERIOR OF $\sigma(A)$

In this section we shall consider only the case of a rank-one perturbation. Recall that the measure ρ, the space $\widetilde{\mathcal{H}}$ and the operators $\widetilde{A}, \widetilde{B}$ and $\widetilde{A}_\gamma$ have been defined in Subsection 2.1 of Section 2. We shall obtain necessary conditions for the directional operator differentiability, imposed on the spectral measure ρ of the operator $\widetilde{A}$ in intervals contained in its spectrum. We assume the spectral measure ρ to be absolutely continuous and, furthermore, we impose on its density $\widetilde{\rho}$ some preliminary conditions. These necessary conditions are close to sufficient ones, formulated in Theorem 2.5 (see Remark 4.6 below).

4.1. Before formulating the main result of this section, we need two lemmas and a proposition.

LEMMA 4.1. *Assume that a real-valued function* $\widetilde{\rho} \in L_1(\mathbb{R})$, *with a compact support, has bounded variation in a closed interval* $[c, d]$ $(c < d)$. *Then*

$$\int_0^{\infty} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s} \, \mathrm{d}s \; < \; \infty \tag{4.1}$$

is valid for almost all $t \in (c, d)$.

Proof. Let Σ be the set of all points $t \in (c, d)$, for which property (4.1) is valid. Let $\widetilde{\Sigma}$ be the set of all points $t \in (c, d)$, for which

$$\int_0^{\infty} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t)| + |\widetilde{\rho}(t) - \widetilde{\rho}(t-s)|}{s} \, \mathrm{d}s \; < \; \infty.$$

It is evident that $\Sigma \supseteq \widetilde{\Sigma}$. Let $T = (c,d) \setminus \widetilde{\Sigma}$. Clearly, it is enough to prove that $\mathrm{mes}(T) = 0$. If $t \in T$, then either

$$\int_0^\infty \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t)|}{s}\,\mathrm{d}s = \infty, \quad \text{or} \quad \int_0^\infty \frac{|\widetilde{\rho}(t) - \widetilde{\rho}(t-s)|}{s}\,\mathrm{d}s = \infty,$$

or both. Therefore, either there exists a sequence $s_n^+ \downarrow 0$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \frac{|\widetilde{\rho}(t+s_n^+) - \widetilde{\rho}(t)|}{s_n^+} = \infty,$$

or, there exists a sequence $s_n^- \downarrow 0$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \frac{|\widetilde{\rho}(t-s_n^-) - \widetilde{\rho}(t)|}{s_n^-} = \infty.$$

In these circumstances it follows that the set T is contained in the set $\widetilde{T}$ of all points $t \in (c,d)$, at which the derivative $\widetilde{\rho}'(t)$ does not exist. Thus, by a known property of functions of bounded variation, $\mathrm{mes}(\widetilde{T}) = 0$. Hence $\mathrm{mes}(T) = 0$. The lemma is proven. ∎

LEMMA 4.2. *Assume that $\widetilde{\rho} \in C(\mathbb{R})$ has support in an interval $(-M, M)$ $(M > 0)$ and has bounded variation. Assume also that $(a,b) \subset (-M,M)$ and that $\widetilde{\rho}(t) > 0$ in (a,b). Let $\phi \in C[-M,M]$ have support in (a,b), and consider the following functions:*

$$\text{(4.2)} \quad \psi_+(t,\varepsilon,\gamma) = \frac{\phi_0(t)}{D(t+\mathrm{i}\varepsilon,\gamma)}, \quad \psi_-(t,\varepsilon,\gamma) = \frac{\phi_0(t)}{D(t-\mathrm{i}\varepsilon,\gamma)} \quad (\varepsilon > 0,\ \gamma \in \mathbb{R}),$$

where the function $D(\lambda,\gamma)$ is defined by (2.15) *and* (2.16), *and the function ϕ_0 is defined by* (2.14). *Then, for any fixed $\gamma \neq 0$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ the functions* (4.2) *belong to $L_\infty(\mathbb{R})$ in the variable t, and*

$$\lim_{\varepsilon\downarrow 0} \phi_+(t,\varepsilon,\gamma) = \phi_+(t,0,\gamma) \quad \textit{and} \quad \lim_{\varepsilon\downarrow 0} \phi_-(t,\varepsilon,\gamma) = \phi_-(t,0,\gamma)$$

in the weak- topology of $L_\infty(\mathbb{R})$, where*

$$\text{(4.3)} \qquad \psi_+(t,0,\gamma) = \frac{\phi_0(t)}{D(t+\mathrm{i}0,\gamma)} \quad \textit{and} \quad \psi_-(t,0,\gamma) = \frac{\phi_0(t)}{D(t-\mathrm{i}0,\gamma)},$$

and the functions $D(t+\mathrm{i}0,\gamma)$ and $D(t-\mathrm{i}0,\gamma)$ are defined by formulas (2.32), (2.33) *and* (2.34). *Thus, the functions* (4.3) *belong to $L_\infty(\mathbb{R})$ and the families of multiplication operators*

$$\{M_{\psi_+(\cdot,\varepsilon,\gamma)}\}_{0<\varepsilon<\varepsilon_0} \quad \textit{and} \quad \{M_{\psi_-(\cdot,\varepsilon,\gamma)}\}_{0<\varepsilon<\varepsilon_0},$$

acting in $L_2(\mathbb{R})$, converge as $\varepsilon \downarrow 0$ in the weak operator topology to the multiplication operators $M_{\psi_+(\cdot,0,\gamma)}$ and $M_{\psi_-(\cdot,0,\gamma)}$ respectively.

Proof. We shall use the criterion for weak operator convergence of multiplication operators given in Proposition 2.2. Let $[\alpha,\beta]$ be a closed interval for which

$$\text{(4.4)} \qquad \mathrm{supp}(\phi) \subseteq [\alpha,\beta] \subset (a,b).$$

In view of (2.15) and (2.16), we have:

$$\Im D(t+\mathrm{i}\varepsilon,\gamma)=\gamma\Im\Theta(t+\mathrm{i}\varepsilon,\gamma)=\gamma\varepsilon\int\limits_{-M}^{M}\frac{\widetilde{\rho}(\mu)\,\mathrm{d}\mu}{(\mu-t)^2+\varepsilon^2}.$$

Hence, by the continuity of the function $\widetilde{\rho}$ and a well-known property of the Poisson kernel

$$\mathcal{P}_\varepsilon(s)=\frac{\varepsilon}{\pi(s^2+\varepsilon^2)},$$

the family of functions $\{\Im D(t+\mathrm{i}\varepsilon,\gamma)\}_{\varepsilon>0}$ converges uniformly as $\varepsilon\downarrow 0$ to the function $\gamma\pi\widetilde{\rho}$ in the interval $[\alpha,\beta]$. Since $\widetilde{\rho}(t)>0$ in $[\alpha,\beta]$ and $\gamma\neq 0$, we obtain:

$$\exists\ \varepsilon_0>0,\qquad \inf_{(t,\varepsilon)\in[\alpha,\beta]\times(0,\varepsilon_0)}|D(t+\mathrm{i}\varepsilon,\gamma)|>0. \tag{4.5}$$

Thus, in view of definition (4.2) of the function $\psi_+(t,\varepsilon,\gamma)$, the inclusion (4.4) and definition (2.14), we have:

$$\sup_{(t,\varepsilon)\in\mathbb{R}\times(0,\varepsilon_0)}|\psi_+(t,\varepsilon,\gamma)|<\infty. \tag{4.6}$$

That is, condition (A) of Proposition 2.2 is satisfied for the family $\{\psi_+(t,\varepsilon,\gamma)\}$ $(0<\varepsilon<\varepsilon_0)$.

On the other hand, by Lemma 4.1 and assertion (i) of Proposition A2.1, $\lim\limits_{\varepsilon\downarrow 0}\Theta(t+\mathrm{i}\varepsilon)=\Theta(t+\mathrm{i}0)$ for almost all $t\in\mathbb{R}$, where $\Theta(t+\mathrm{i}0)$ is defined by (2.33). Thus, in view of (4.2), (4.4), (4.5), (2.14) and (2.15), we get that for almost all $t\in\mathbb{R}$:

$$\lim_{\varepsilon\downarrow 0}\psi_+(t,\varepsilon,\gamma)=\psi_+(t,0,\gamma),$$

where $\psi_+(t,0,\gamma)$ is defined by (4.3). This limiting relation, property (4.6), and Lebesgue's Theorem imply that also condition (B) of Proposition 2.2 holds for the family $\{\psi_+(t,\varepsilon,\gamma)\}$ $(\varepsilon\in(0,\varepsilon_0))$. So, we have proved the assertion of the lemma for this family.

The family $\{\psi_-(t,\varepsilon,\gamma)\}$ is treated analogously. ∎

We now turn to the following

PROPOSITION 4.3. *Assume that the spectral measure ρ of the operator $\widetilde{A}$ is absolutely continuous, its density $\widetilde{\rho}$ is continuous on $\mathbb{R}$, its support is contained in an interval $(-M,M)$ $(M>0)$ and $\widetilde{\rho}$ has bounded variation. Furthermore, assume that $\widetilde{\rho}(t)>0$ in some interval (a,b) contained in $(-M,M)$. Then there exists a number $\gamma_0>0$ such that for any $\gamma\in(-\gamma_0,\gamma_0)\setminus\{0\}$ and for any function $\phi\in C[-M,M]$ with* $\mathrm{supp}(\phi)\subset(a,b)$ *the representation* (2.17) *is valid for $(\phi(\widetilde{A}_\gamma)-\phi(\widetilde{A}))/\gamma$, where the functions $\psi_+(t,0,\gamma)$ and $\psi_-(t,0,\gamma)$ are defined via formulas* (4.3), (2.32), (2.33), (2.34) *and* (2.14), *and they belong to $L_\infty(\mathbb{R})$.*

Proof. By Lemma 4.2, the conditions of Theorem 2.3 are satisfied if we take there $\mathcal{C} = \{\phi \in C[-M, M] : \operatorname{supp}(\phi) \subset (a, b))\}$. Hence the desired assertion is valid. ∎

In what follows we shall need the following result from summability theory (see [15], Chapter X, Section 1). A sequence $\{\Phi_n(t, x)\}_{n=0}^{\infty}$ of functions on the square $[a, b] \times [a, b]$ is said to be a (summability) *kernel*, if

(1) for every $x \in [a, b]$ the functions $t \mapsto \Phi_n(t, x)$ are in $L^1[a, b]$;

(2) for all choices of α, β and x such that $a \leqslant \alpha < x < \beta \leqslant b$,

$$\lim_{n\to\infty} \int_{\alpha}^{\beta} \Phi_n(t, x)\, \mathrm{d}t = 1.$$

Let $\Psi(t, x)$ and $\Phi(t, x)$ be functions on $[a, b] \times [a, b]$. $\Psi(t, x)$ is said to be a *convex majorant* of $\Phi(t, x)$ if

(1) $|\Phi(t, x)| \leqslant \Psi(t, x)\ \forall\, t, x \in [a, b]$;

(2) for every $x \in [a, x]$ the map $t \mapsto \Psi(t, x)$ is non-decreasing in $[a, x]$ and non-increasing in $[x, b]$.

The following result is due to D.K. Faddeyev (see Chapter X, Section 2, Theorem 3 of [15]).

THEOREM 4.4. *Let $\{\Phi_n(t, x)\}_{n=0}^{\infty}$ be a kernel on $[a, b] \times [a, b]$. Assume that $\Psi_n(t, x)$ is a convex majorant of $\Phi_n(t, x)$ for every n, and that*

$$\int_a^b \Psi_n(t, x)\, \mathrm{d}t < K(x) < \infty \quad \forall x \in [a, b],\ \forall n \in \mathbb{N}.$$

Then, for every $f \in L_1(a, b)$ and every Lebesgue point x of f,

$$\lim_{n\to\infty} \int_a^b f(t)\Phi_n(t, x)\, \mathrm{d}t = f(x).$$

We now turn to the main theorem of this section.

THEOREM 4.5. *Assume that the spectral measure ρ of the operator $\widetilde{A}$ is absolutely continuous, its density $\widetilde{\rho}$ is continuous on $\mathbb{R}$, its support is contained in an interval $(-M, M)$ $(M > 0)$ and $\widetilde{\rho}$ has bounded variation. Furthermore, assume that $\widetilde{\rho}(t) > 0$ in some interval (a, b) and the limit*

$$\lim_{\gamma\to 0} \frac{\phi(\widetilde{A}_\gamma) - \phi(\widetilde{A})}{\gamma} \tag{4.7}$$

exists in the strong operator topology for any function $\phi \in C[-M, M]$. Then the integral

$$S(t) := \int_0^{\infty} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s}\, \mathrm{d}s \tag{4.8}$$

exists for almost all $t \in \mathbb{R}$, and for any closed interval $[\alpha, \beta] \subset (a, b)$ $(\alpha < \beta)$ the function $S|_{(\alpha,\beta)}$ belongs to the class $L_\infty(\alpha, \beta)$.

REMARK. It is plain that the assumption that the strong operator limit (4.7) exists for all $\phi \in C[-M, M]$ is equivalent to its existence for all *real valued* $\phi \in C[-M, M]$.

Proof. The first assertion of the theorem follows from Lemma 4.1 and assertion (i) of Proposition A2.1.

Let us prove the second assertion. Consider the Borel transform $\Theta(\lambda)$ of the function $\widetilde{\rho}$ defined by (2.16). By Lemma 4.1 and assertion (i) of Proposition A2.1, the limits $\Theta(t + \mathrm{i}0) = \lim_{\varepsilon\downarrow 0} \Theta(t + \mathrm{i}\varepsilon)$ and $\Theta(t - \mathrm{i}0) = \lim_{\varepsilon\downarrow 0} \Theta(t - \mathrm{i}\varepsilon)$ exist for almost all $t \in \mathbb{R}$, the functions $\Theta(t + \mathrm{i}0)$ and $\Theta(t - \mathrm{i}\varepsilon)$ belong to $L_2(\mathbb{R})$ and formulas (2.33) and (2.34) are valid for them. Hence, in view of the fact that $\widetilde{\rho}$ is continuous and compactly supported, the function $S(t)$ (defined by (4.8)) belongs to $L_2(\mathbb{R})$ as well. Assume, on the contrary, that the second assertion of the theorem does not hold, that is there exists a closed interval $[\alpha, \beta] \subset (a, b)$ $(\alpha < \beta)$ such that

$$S|_{(\alpha,\beta)} \notin L_\infty(\alpha, \beta). \tag{4.9}$$

Our aim is to prove that, under the above assumption, there exists a function $\phi \in C[-M, M]$ such that the limit (4.7) does not exist in the strong operator topology. In view of Lemma 3.1, it is enough to prove that

$$\limsup_{\gamma\to 0} \|\Sigma(\cdot, \gamma)\|_{\mathcal{B}(C[-M,M],\mathcal{B}(\widetilde{\mathcal{H}}))} = \infty, \tag{4.10}$$

where $\Sigma(\cdot, \gamma)$ is the bounded linear operator acting from $C[-M, M]$ into $\mathcal{B}(\widetilde{\mathcal{H}})$ $(\widetilde{\mathcal{H}} = L_2(\mathbb{R}, \rho))$, defined by the expression:

$$\Sigma(\phi, \gamma) = \frac{\phi(\widetilde{A}_\gamma) - \phi(\widetilde{A})}{\gamma}. \tag{4.11}$$

Notice that, since $\mathrm{supp}(\widetilde{\rho}) \subset (-M, M)$ and $\widetilde{\rho}(t) > 0$ in (a, b), the interval (a, b) is contained in $(-M, M)$. By Proposition 4.3, there exists a number $\gamma_0 > 0$ such that for any $\gamma \in (-\gamma_0, \gamma_0) \setminus \{0\}$ and for any function $\phi \in C[-M, M]$ with $\mathrm{supp}(\phi) \subset (a, b)$ the representation (2.17) is valid for $\Sigma(\phi, \gamma)$, where the functions $\psi_+(t, 0, \gamma)$ and $\psi_-(t, 0, \gamma)$ are defined via formulas (4.3), (2.32), (2.33), (2.34) and (2.14), and they belong to $L_\infty(\mathbb{R})$. Hence we have:

$$\Sigma(\phi, \gamma) = 2\pi\mathrm{i}\{(H_{\overline{\psi_-(\cdot,0,\gamma)}})^* - H_{\psi_+(\cdot,0,\gamma)}\}M_{\widetilde{\rho}}, \quad \text{if } \mathrm{supp}(\phi) \subset (a, b). \tag{4.12}$$

Here H_ξ denotes the Hankel operator on $L_2(\mathbb{R})$ with a symbol $\xi(t)$ (see [17]), that is

$$H_\xi = P_- M_\xi P_+,$$

and P_+ and P_- are Riesz projections in $L_2(\mathbb{R})$. Observe that, in view of this definition, the operators

$$H_{\psi_+(\cdot,0,\gamma)} \quad \text{and} \quad (H_{\overline{\psi_-(\cdot,0,\gamma)}})^*$$

are disjointly supported, i.e. have orthogonal ranges and co-kernels. Hence, in view of (4.12),

$$\text{(4.13)}\qquad \|\Sigma(\cdot,\gamma)\|_{\mathcal{B}(C[-M,M],\mathcal{B}(\widetilde{\mathcal{H}}))} \geqslant 2\pi \sup\{\max(\|H_{\psi_+(\cdot,0,\gamma)}\, M_{\widetilde{\rho}}\|_{\mathcal{B}(\widetilde{\mathcal{H}})}, \|M_{\widetilde{\rho}}\, H_{\overline{\psi_-(\cdot,0,\gamma)}}\|_{\mathcal{B}(\widetilde{\mathcal{H}})}); \|\phi\|_{C[-M,M]} \leqslant 1,\ \mathrm{supp}(\phi) \subset (a,b)\},$$

where $\psi_+(t,0,\gamma)$ and $\psi_-(t,0,\gamma)$ are connected with $\phi(t)$ by formulas (4.3) and (2.14). Thus, in order to prove (4.10), it is enough to show that

$$\text{(4.14)}\qquad \limsup_{\gamma\to 0}(\sup\{\|H_{\psi_+(\cdot,0,\gamma)}M_{\widetilde{\rho}}\|_{\mathcal{B}(\widetilde{\mathcal{H}})}; \|\phi\|_{C[-M,M]} \leqslant 1, \mathrm{supp}(\phi) \subset (a,b)\}) \;=\; \infty.$$

Consider a closed interval $[\alpha,\beta] \subset (a,b)$ $(\alpha < \beta)$, for which the assumption (4.9) holds, and recall that $\widetilde{\mathcal{H}} = L_2(\mathbb{R},\rho)$. We have:

$$\begin{aligned}\text{(4.15)}\qquad &\|H_{\psi_+(\cdot,0,\gamma)}M_{\widetilde{\rho}}\|_{\mathcal{B}(\widetilde{\mathcal{H}})}\\ &= \sup\{|(H_{\psi_+(\cdot,0,\gamma)}M_{\widetilde{\rho}}\widetilde{f},\ \widetilde{g})_{\widetilde{\mathcal{H}}}|; \widetilde{f},\widetilde{g} \in \widetilde{\mathcal{H}} \text{ and } \|\widetilde{f}\|_{\widetilde{\mathcal{H}}}, \|\widetilde{g}\|_{\widetilde{\mathcal{H}}} \leqslant 1\}\\ &= \sup\{|(M_{\psi_+(\cdot,0,\gamma)}P_+M_{\widetilde{\rho}}\widetilde{f},\ \ P_-M_{\widetilde{\rho}}\widetilde{g})_{L_2(\mathbb{R})}|; f,g \in \widetilde{\mathcal{H}} \text{ and } \|\widetilde{f}\|_{\widetilde{\mathcal{H}}}, \|\widetilde{g}\|_{\widetilde{\mathcal{H}}} \leqslant 1\}\\ &\geqslant \sup\Big\{\Big|\int_{-M}^{M} \psi_+(t,0,\gamma)(P_+f)(t)\overline{(P_-g)(t)}\,\mathrm{d}t\Big|; f,g \in B_{1,q}\Big\},\end{aligned}$$

where

$$B_{1,q} = \{f \in L_2(\mathbb{R}) : \mathrm{supp}(f) \subseteq [\alpha,\beta],\ \|f\|_{L_2(\mathbb{R},q)} \leqslant 1\},$$

$$\|f\|_{L_2(\mathbb{R},q)} = \Big(\int_{-\infty}^{\infty} |f(t)|^2 q(t)\,\mathrm{d}t\Big)^{1/2},$$

$$q(t) = \begin{cases} \frac{1}{\widetilde{\rho}(t)} & t \in [\alpha,\beta],\\ 0 & t \notin [\alpha,\beta].\end{cases}$$

Observe that, since $\widetilde{\rho}$ is continuous and $\widetilde{\rho}(t) > 0$ in $[\alpha,\beta]$,

$$\overline{q} = \sup_{t\in\mathbb{R}} q(t) < \infty.$$

Then the estimate

$$\|f\|_{L_2(\mathbb{R},q)} \leqslant \sqrt{\overline{q}}\|f\|_{L_2(\mathbb{R})}$$

implies that

$$\text{(4.16)}\qquad B_{\frac{1}{\sqrt{\overline{q}}}} \subseteq B_{1,q},$$

where

$$B_r = \{f \in L_2(\mathbb{R}) : \operatorname{supp}(f) \subseteq [\alpha, \beta],\ \|f\|_{L_2(\mathbb{R})} \leqslant r\}. \tag{4.17}$$

Taking into account the inclusion (4.16), we can proceed with estimate (4.15) in the following manner:

$$\|H_{\psi_+(\cdot,0,\gamma)} M_{\widetilde{\rho}}\|_{\mathcal{B}(\widetilde{\mathcal{H}})} \geqslant \frac{1}{\bar{q}} \sup_{f,g \in B_1} \Big| \int_{-M}^{M} \psi_+(t,0,\gamma)(P_+f)(t)\overline{(P_-g)(t)}\,\mathrm{d}t \Big|. \tag{4.18}$$

Let $[\widetilde{\alpha}, \widetilde{\beta}]$ be a closed interval, such that $[\alpha, \beta] \subset (\widetilde{\alpha}, \widetilde{\beta})$ and $[\widetilde{\alpha}, \widetilde{\beta}] \subset (a, b)$. Consider a function $\kappa \in C(\mathbb{R})$ having the properties:

$$\kappa(t) = \begin{cases} 1 & t \in [\alpha, \beta], \\ 0 & t \notin [\widetilde{\alpha}, \widetilde{\beta}], \end{cases} \tag{4.19}$$

and $0 \leqslant \kappa(t) \leqslant 1$ for any $t \in \mathbb{R}$. It is clearly enough to prove (4.14) where ϕ is restricted to have the form $\phi(t) = \psi(t)\kappa(t)$ $(t \in [-M, M])$ for $\psi \in C[-M, M]$. In view of estimate (4.18) and definitions (4.3) and (2.14), in order to prove (4.14) for the class of functions $\phi(t) = \psi(t)\kappa(t)$, it is enough to find a sequence of real numbers γ_k, such that $\gamma_k \neq 0$ for any $k \in \mathbb{N}$,

$$\lim_{k \to \infty} \gamma_k = 0, \tag{4.20}$$

$$\lim_{k \to \infty} \Big(\sup \Big\{ \Big| \int_{-M}^{M} \psi(t)\Phi(t, \gamma_k)(P_+f)(t)\overline{(P_-g)(t)}\,\mathrm{d}t \Big|; \tag{4.21}$$

$$\|\psi\|_{C[-M,M]} \leqslant 1,\ f, g \in B_1 \Big\} \Big) = \infty, \tag{4.22}$$

where B_1 is defined by (4.17) with $r = 1$,

$$\Phi(t, \gamma) = \frac{\kappa(t)}{D(t + \mathrm{i}0, \gamma)}, \tag{4.23}$$

$$D(t + \mathrm{i}0, \gamma) = 1 + \gamma \Theta(t + \mathrm{i}0). \tag{4.24}$$

Notice that, in view of definition (4.23), the inclusion $\operatorname{supp}(\kappa) \subset (a, b)$ and Lemma 4.2, the function $\Phi(t, \gamma)$ belongs to $L_\infty(\mathbb{R})$ for each $\gamma \neq 0$. Recall that the function $\Theta(t + \mathrm{i}0)$ satisfies (2.33), that is

$$\Theta(t + \mathrm{i}0) = S(t) + \mathrm{i}\pi\widetilde{\rho}(t). \tag{4.25}$$

Also, recall that the function $S(t)$ is defined by (4.8). Since the function S belongs to $L_2(\mathbb{R})$, it is measurable. For every $k \in \mathbb{N}$ we define the set

$$A_k = \{t \in (\alpha, \beta) : |S(t)| \geqslant k\}. \tag{4.26}$$

Observe that, since the assumption (4.9) holds for the interval $[\alpha, \beta]$, the sets A_k have positive measures. Recall that $\Theta(\cdot + \mathrm{i}0) \in L_2(\mathbb{R})$. Thus, for each $k \in \mathbb{N}$ there exists a point $t_k \in A_k$, which is a Lebesgue point of the function $\Theta(t + \mathrm{i}0)$. In view of (4.25) and the continuity of the function $\widetilde{\rho}$, each t_k is a Lebesgue point also for the function $S(t)$. Hence, in particular, $|S(t_k)| < \infty$. On the other hand, in view of (4.26), we have:

$$|S(t_k)| \geqslant k \quad \forall\, k \in \mathbb{N}.$$

Hence, if we put

$$\gamma_k = -\frac{1}{S(t_k)}, \tag{4.27}$$

then the limiting relation (4.20) holds for γ_k, and $\gamma_k \neq 0$ for all $k \in \mathbb{N}$. Furthermore, in view of (4.23),(4.24), (4.25), (4.19), (4.27) and the membership $t_k \in (\alpha, \beta)$, we have the estimate:

$$|\Phi(t_k, \gamma_k)| \;=\; \frac{1}{\pi|\gamma_k|\widetilde{\rho}(t_k)} \;\geqslant\; \frac{1}{\pi|\gamma_k|\overline{\rho}}\,, \tag{4.28}$$

where $\overline{\rho} = \max\limits_{t\in[\alpha,\beta]} \widetilde{\rho}(t)$. We also see that $|\Phi(t_k, \gamma_k)| < \infty$. Using (4.19) (4.23), (4.24) and (4.25), we have the estimate:

$$|\Phi(t, \gamma_k) - \Phi(t_k, \gamma_k)| \leqslant \frac{|\Theta(t + \mathrm{i}0) - \Theta(t_k + \mathrm{i}0)|}{\pi^2 m^2 |\gamma_k|},$$

where t belongs to a neighborhood of t_k contained in $[\alpha, \beta]$, and

$$m = \min_{t\in[\alpha,\beta]} \widetilde{\rho}(t).$$

Since each t_k is a Lebesgue point of the function $\Theta(t + \mathrm{i}0)$, the latter estimate implies that it is a Lebesgue's point of the function $\Phi(t, \gamma_k)$ as well.

In order to prove that (4.22) holds, let us choose the functions f and g in the following manner:

$$f(t) = f_k(t) := \frac{\mathrm{e}^{\mathrm{i}\tau_k t}\Delta\Big(\frac{t-t_k}{\sigma_k}\Big)}{\Big\|\Delta\Big(\frac{t-t_k}{\sigma_k}\Big)\Big\|_{L_2(\mathbb{R})}} = \sqrt{\frac{3}{2\sigma_k}}\mathrm{e}^{\mathrm{i}\tau_k t}\Delta\Big(\frac{t-t_k}{\sigma_k}\Big), \tag{4.29}$$

$$g(t) = g_k(t) := \overline{f_k(t)}, \tag{4.30}$$

where

$$\Delta(x) = \begin{cases} 1 - |x| & |x| \leqslant 1, \\ 0 & |x| > 1, \end{cases} \tag{4.31}$$

and the sequences $\{\sigma_k\}$ and $\{\tau_k\}$ satisfy the conditions: $\sigma_k > 0$, $\tau_k > 0$ for any $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \sigma_k = 0 \tag{4.32}$$

and $\lim\limits_{k\to\infty} \tau_k = \infty$. In the sequel some additional conditions will be imposed on the sequences $\{\sigma_k\}$ and $\{\tau_k\}$ and a relationship between them will be specified. The straightforward calculation yields the following formula for the Fourier transform of the function f_k:

$$\widehat{f_k}(\omega) = \sqrt{\frac{3\sigma_k}{2}}\widehat{\Delta}(\sigma_k(\omega - \tau_k))e^{-i(\omega-\tau_k)t_k}, \tag{4.33}$$

where

$$\widehat{\Delta}(\omega) = 2\sqrt{\frac{2}{\pi}}\frac{\sin^2\left(\frac{\omega}{2}\right)}{\omega^2}. \tag{4.34}$$

On the other hand, by the definition of Riesz projections, we get:

$$\widehat{P_+ f_k}(\omega) = \mathbf{1}_{[0,\infty)}(\omega)\widehat{f_k}(\omega) \tag{4.35}$$

and

$$\widehat{P_- g_k}(\omega) = \mathbf{1}_{(-\infty,0]}(\omega)\widehat{g}_k(\omega).$$

From these equalities, definition (4.30) and the property $\widehat{\overline{h}}(\omega) = \overline{\widehat{h}(-\omega)}$ of the Fourier transform, we obtain that the Fourier transform of the function $\overline{P_- g_k}$ coincides with the Fourier transform of the function $P_+ f_k$. Hence

$$\overline{P_- g_k} = P_+ f_k. \tag{4.36}$$

Observe that, in view of definitions (4.29)–(4.31) and the membership $t_k \in (\alpha, \beta)$, we can choose the sequence $\{\sigma_k\}$ such that both the limiting relation (4.32) and the following inclusion

$$\forall\, k \in \mathbb{N} \quad \mathrm{supp}(f_k) = \mathrm{supp}(g_k) = [t_k - \sigma_k, t_k + \sigma_k] \subset (\alpha, \beta) \tag{4.37}$$

hold. Furthermore, we see from (4.29) and (4.30) that $\|f_k\|_{L_2(\mathbb{R})} = \|g_k\|_{L_2(\mathbb{R})} = 1$. Thus $f_k, g_k \in B_1$, where B_1 is defined by (4.17) with $r = 1$. Then, taking into account (4.36), we have for $\psi \in C[-M, M]$:

$$\sup\left\{\left|\int_{-M}^{M} \psi(t)\Phi(t,\gamma_k)(P_+ f)(t)\overline{(P_- g)(t)}\,dt\right|;\ f, g \in B_1\right\}$$
$$\geqslant \left|\int_{-M}^{M} \psi(t)\Phi(t,\gamma_k)((P_+ f_k)(t))^2\,dt\right|. \tag{4.38}$$

Making use of (4.29), let us represent the latter integral in the following manner:

$$\int_{-M}^{M} \psi(t)\Phi(t,\gamma_k)((P_+ f_k)(t))^2\,dt = I_1(\psi, t_k, \gamma_k, \sigma_k) + I_2(\psi, \gamma_k), \tag{4.39}$$

where

$$I_1(\psi, x, \gamma_k, \sigma) = \int_{-M}^{M} \psi(t)\Phi(t, \gamma_k)e^{2i\tau_k t}\delta(t - x, \sigma)\,dt \quad (\sigma > 0), \tag{4.40}$$

$$\delta(t, \sigma) = \frac{\left(\Delta\left(\frac{t}{\sigma}\right)\right)^2}{\left\|\Delta\left(\frac{t}{\sigma}\right)\right\|^2_{L_2(\mathbb{R})}}, \tag{4.41}$$

$$I_2(\psi, \gamma_k) = \int_{-M}^{M} \psi(t)\Phi(t, \gamma_k)(((P_+ f_k)(t))^2 - (f_k(t))^2)\,dt. \tag{4.42}$$

Let us put

$$\psi(t) = \psi_k(t) := e^{-2i\tau_k t}. \tag{4.43}$$

Then we get from (4.40):

$$I_1(\psi_k, x, \gamma_k, \sigma) = \int_{-M}^{M} \Phi(t, \gamma_k)\delta(t - x, \sigma)\,dt. \tag{4.44}$$

Observe that, in view of (4.41) and (4.31), for any fixed $k \in \mathbb{N}$ the kernel $\delta(t - x, \sigma)$ of the integral operator $I_1(\psi_k, x, \gamma_k, \sigma)$ is a summability kernel (as $\sigma \downarrow 0$) and $\int_{-\infty}^{\infty} \delta(t, \sigma)\,dt = 1$. Furthermore, it is non-negative, it is non-increasing for $t \geqslant x$ and it is non-decreasing for $t \leqslant x$. Recall that for any $k \in \mathbb{N}$ the function $\Phi(t, \gamma_k)$ belongs to $L_\infty(\mathbb{R})$ and the point t_k is a Lebesgue point of $\Phi(t, \gamma_k)$. Thus, the conditions of Faddeyev Theorem 4.4 are satisfied for the integral operator (4.44), if we take the function $\delta(t - x, \sigma)$ as a convex majorant for itself. Then we have for any fixed $k \in \mathbb{N}$:

$$\lim_{\sigma \downarrow 0} I_1(\psi_k, t_k, \gamma_k, \sigma) = \Phi(t_k, \gamma_k).$$

Hence, in view of estimate (4.28) and the limiting relation (4.20), we can choose the sequence $\{\sigma_k\}$ so that the limiting relation (4.32), the inclusion (4.37) and the following limiting relation

$$\lim_{k \to \infty} |I_1(\psi_k, \gamma_k, \sigma_k)| = \infty \tag{4.45}$$

all hold. Let us estimate the integral $I_2(\psi_k, \gamma_k)$ (defined by (4.42)) using the choice (4.43) of the function ψ_k and formulas (4.33), (4.34) and (4.35):

$$\begin{aligned}
|I_2(\psi_k, \gamma_k)| &\leqslant I(\Phi, \gamma_k)(\|P_+ f_k\|_{L_\infty(\mathbb{R})} + \|f_k\|_{L_\infty(\mathbb{R})})\|f_k - P_+ f_k\|_{L_\infty(\mathbb{R})} \\
&\leqslant \frac{1}{2\pi} I(\Phi, \gamma_k)(\|\widehat{P_+ f_k}\|_{L_1(\mathbb{R})} + \|\widehat{f}_k\|_{L_1(\mathbb{R})})\|\widehat{f}_k - \widehat{P_+ f_k}\|_{L_1(\mathbb{R})} \\
&\leqslant \frac{6\sigma_k}{\pi^2} I(\Phi, \gamma_k) \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{\sigma_k(\omega-\tau_k)}{2}\right)}{(\sigma_k(\omega-\tau_k))^2}\, \mathrm{d}\omega \int_{-\infty}^{0} \frac{\sin^2\left(\frac{\sigma_k(\omega-\tau_k)}{2}\right)}{(\sigma_k(\omega-\tau_k))^2}\, \mathrm{d}\omega \\
&\leqslant \frac{6}{\pi^2} I(\Phi, \gamma_k) \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{s}{2}\right)}{s^2}\, \mathrm{d}s \int_{-\infty}^{0} \frac{\mathrm{d}\omega}{(\sigma_k(\omega-\tau_k))^2} \\
&= \frac{6}{\pi^2\sigma_k^2\tau_k} I(\Phi, \gamma_k) \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{s}{2}\right)}{s^2}\, \mathrm{d}s,
\end{aligned}$$

where

$$I(\Phi, \gamma) = \int_{-M}^{M} |\Phi(t, \gamma)|\, \mathrm{d}t.$$

If we choose the sequence $\{\tau_k\}$ such that $\lim_{k\to\infty} \tau_k = \infty$ and

$$\lim_{k\to\infty} \frac{I(\Phi, \gamma_k)}{\sigma_k^2 \tau_k} = 0,$$

then we get from the latter estimate that

$$\lim_{k\to\infty} I_2(\psi_k, \gamma_k) = 0. \tag{4.46}$$

Taking into account that, in view of (4.43), $\|\psi_k\|_{C[-M,M]} = 1$, we obtain from (4.38), (4.39), (4.45) and (4.46) that the desired limiting relation (4.22) is valid. The theorem is proven. ∎

REMARK 4.6. If, along with the conditions of Theorem 4.5, the density $\widetilde{\rho}$ is a monotone function in the interval (a, b), then the assertion

$$S|_{(\alpha,\beta)} \in L_\infty(\alpha, \beta), \quad \text{if } [\alpha, \beta] \subset (a, b)$$

of this theorem is equivalent to condition (2.30) of Theorems 2.4 and 2.5, in which the function $\mathcal{R}(t)$ is defined by (2.29) and the numbers $-M$ and M are replaced by α and β respectively. This means that the necessary conditions of the directional operator differentiability given by Theorem 4.5 are close to the sufficient conditions given by Theorem 2.5.

4.2. In the following example we apply Theorem 4.5 in order to establish if there exists a continuous function, for which the directional operator differentiability fails.

EXAMPLE 4.7. Consider an absolute continuous spectral measure ρ of the operator $\widetilde{A}$ with a density $\widetilde{\rho}$ having the properties:
(a) $\widetilde{\rho} \in C(\mathbb{R})$;
(b) $\mathrm{supp}(\widetilde{\rho}) = [-1, 1]$;
(c) $\widetilde{\rho}(t) > 0$ for any $t \in (-1, 1)$;
(d) the function $\widetilde{\rho}$ is continuously differentiable in $\mathbb{R} \setminus \{0\}$;
(e) $\widetilde{\rho}$ is increasing in the interval $(-\frac{1}{2}, \frac{1}{2})$;
(f) $\widetilde{\rho}$ is concave in $(0, \frac{1}{2})$;
(g) the behavior of the function $\widetilde{\rho}$ as $t \downarrow 0$ is defined by the condition:

$$\int_0^{1/2} \frac{\widetilde{\rho}(\mu) - \widetilde{\rho}(0)}{\mu} \, d\mu = \infty.$$

Observe that condition (g) is satisfied, if, for instance, the function $\widetilde{\rho}$ has the form (2.44) in a suitably small semi-interval $(0, \delta)$ with $C > 0$ and $\alpha \in (0, 1)$. It is evident that the function $\widetilde{\rho}$ has bounded variation in $[-1, 1]$. Our aim is to show that

$$S \notin L_\infty(-\frac{1}{2}, \frac{1}{2}), \tag{4.47}$$

where the function S is defined by (4.8), that is the necessary condition of the directional operator differentiability, given by Theorem 4.5, is not satisfied. This means that in the situation of this example there exists a function

$$\phi \in C[-M, M] \ (M > 1),$$

such that the limit

$$\lim_{\gamma \to 0} \frac{\phi(\widetilde{A}_\gamma) - \phi(\widetilde{A})}{\gamma}$$

does not exist in the strong operator topology.

Indeed, it is evident that

$$\sup_{t \in [-\frac{1}{4}, \frac{1}{4}]} \Big| \int_{1/4}^{\infty} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s} \, ds \Big| < \infty.$$

Hence, in order to prove (4.47), it is enough to show that

$$\lim_{t \downarrow 0} \int_0^{1/4} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s} \, ds = \infty. \tag{4.48}$$

In view of condition (e), we have for any $t \in (0, \frac{1}{4})$:

$$\int_0^{1/4} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s}\,\mathrm{d}s \geqslant \int_0^{1/4} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t)}{s}\,\mathrm{d}s. \tag{4.49}$$

Making use of the Monotone Convergence Theorem in the same manner as in Example 2.7 and taking into account condition (g), we get:

$$\lim_{t \downarrow 0} \int_0^{1/4} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t)}{s}\,\mathrm{d}s = \int_0^{1/4} \frac{\widetilde{\rho}(s) - \widetilde{\rho}(0)}{s}\,\mathrm{d}s = \infty.$$

Hence, by virtue of estimate (4.49), we get the desired limiting relation (4.48).

5. GLOBAL NECESSARY CONDITIONS

In this section we continue to study the case of a rank-one perturbation. Recall that the measure ρ, the space $\widetilde{\mathcal{H}}$ and the operators $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{A}_\gamma$ have been defined in Subsection 2.1 of Section 2. We shall prove a theorem, which links the necessary conditions for directional operator differentiability given by Theorem 4.5 with ones given by Theorem 3.8. To this end we need the following

LEMMA 5.1. *Let $\widetilde{\rho}$ be a non-negative continuous function defined on $\mathbb{R}$ and having a compact support. Assume that $\widetilde{\rho}(t) = 0$ in an interval (a, b) $(-\infty \leqslant a < b \leqslant \infty)$ and one of the following conditions is satisfied: either*

(A) *$a > -\infty$ and the function $\widetilde{\rho}$ is non-increasing and concave in a semi-interval $(a - \delta^-, a]$ $(\delta^- > 0)$, and, furthermore, it satisfies the condition*

$$\int_{-\infty}^{a} \frac{\widetilde{\rho}(u)}{a-u}\,\mathrm{d}u < \infty,$$

or

(B) *$b < \infty$ and the function $\widetilde{\rho}$ is non-decreasing and concave in a semi-interval $[b, b + \delta^+)$ $(\delta^+ > 0)$, and, furthermore, it satisfies the condition*

$$\int_b^{\infty} \frac{\widetilde{\rho}(u)}{u-b}\,\mathrm{d}u < \infty. \tag{5.1}$$

Then in the case (A) *there exists a semi-interval $I_-(a) = (a - \sigma^-, a]$ $(\sigma^- > 0)$, such that the function*

$$S(t) = \int_0^{\infty} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s}\,\mathrm{d}s$$

is bounded in $I_-(a)$, and in the case (B) *there exists a semi-interval $I_+(b) = [b, b + \sigma^+)$ $(\sigma^+ > 0)$, such that this function is bounded in $I_+(b)$.*

Proof. We shall consider only the case (B), because the case (A) is treated analogously. We have to prove that

$$\exists\, \sigma^+ > 0 : \sup_{t\in[b,b+\sigma^+)} S(t) < \infty. \tag{5.2}$$

Assume that $t \geqslant b$ and represent:

$$S(t) = I_1(t) + I_2(t), \tag{5.3}$$

where

$$I_1(t) = \int_0^{t-b} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s}\,\mathrm{d}s$$

and

$$I_2(t) = \int_{t-b}^{\infty} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s}\,\mathrm{d}s. \tag{5.4}$$

Let us estimate the integral $I_1(t)$. Observe that, since the function $\widetilde{\rho}$ is concave in the semi-interval $[b, b+\delta^+)$, it is absolutely continuous there and its derivative $\widetilde{\rho}'$ is non-increasing ([14], Chapter I, Section 1, Lemma 1.3 and Theorem 1.1 (p. 5)). Furthermore, observe that for $t \in [b, b+\frac{\delta^+}{2})$ and $s \in [0, t-b]$ the points $t+s$ and $t-s$ belong to the semi-interval $[b, b+\delta^+)$. These circumstances imply that for any $t \in [b, b+\frac{\delta^+}{2})$

$$|I_1(t)| \leqslant 2 \int_0^{t-b} \widetilde{\rho}'(t-s)\,\mathrm{d}s = 2(\widetilde{\rho}(t-b) - \widetilde{\rho}(0)).$$

This estimate means that

$$\sup_{t\in[b,b+\frac{\delta^+}{2})} |I_1(t)| < \infty. \tag{5.5}$$

Now let us estimate the integral $I_2(t)$ defined by (5.4). To this end represent it in the form:

$$I_2(t) = I_2^+(t) + I_2^-(t), \tag{5.6}$$

where

$$I_2^+(t) = \int_{t-b}^{\infty} \frac{\widetilde{\rho}(t+s)}{s}\,\mathrm{d}s \quad \text{and} \quad I_2^-(t) = \int_{t-b}^{\infty} \frac{\widetilde{\rho}(t-s)}{s}\,\mathrm{d}s.$$

Observe that, if $t \in [b, b+(b-a))$ and $s \in (t-b, b-a)$, then $t-s \in (a, b)$, hence $\widetilde{\rho}(t-s) = 0$. Therefore for $t \in [b, b+(b-a))$

$$I_2^-(t) = \int_{b-a}^{\infty} \frac{\widetilde{\rho}(t-s)}{s}\,\mathrm{d}s,$$

hence

$$\sup_{t\in[b,b+(b-a))} |I_2^-(t)| < \infty. \tag{5.7}$$

Let us estimate the integral $I_2^+(t)$ representing it in the form for $t \in [b, b+\frac{\delta^+}{2})$:

$$I_2^+(t) = J_1(t) + J_2(t),$$

where

$$J_1(t) = \int_{t-b}^{\delta^+/2} \frac{\widetilde{\rho}(t+s)}{s}\, \mathrm{d}s \quad \text{and} \quad J_2(t) = \int_{\delta^+/2}^{\infty} \frac{\widetilde{\rho}(t+s)}{s}\, \mathrm{d}s.$$

It is evident that

$$\sup_{t\in[b,b+\frac{\delta^+}{2})} |J_2(t)| < \infty.$$

It remains only to estimate the integral $J_1(t)$. Taking into account condition (5.1), the equality $\widetilde{\rho}(b) = 0$ and the fact that the function $\widetilde{\rho}$ is non-decreasing and concave in $[b, b+\delta^+)$, we have for $t \in [b, b+\frac{\delta^+}{2})$:

$$|J_1(t)| \leqslant 2\int_{t-b}^{\delta^+/2} \frac{\widetilde{\rho}(t+s)-\widetilde{\rho}(b)}{t+s-b}\, \mathrm{d}s \leqslant 2\int_{t-b}^{\delta^+/2} \frac{\widetilde{\rho}(s+b)}{s}\, \mathrm{d}s \leqslant 2\int_{b}^{\infty} \frac{\widetilde{\rho}(u)\, \mathrm{d}u}{u-b} < \infty.$$

Hence the function $J_1(t)$ is bounded in the semi-interval $[b, b+\frac{\delta^+}{2})$. Thus, also the function $I_2^+(t)$ is bounded in this semi-interval too. The latter fact, property (5.7), representation (5.6), property (5.5) and representation (5.3) imply the desired property (5.2). The lemma is proven. ∎

We now turn to the theorem promised in the beginning of this section.

THEOREM 5.2. *Assume that the spectral measure ρ of the operator $\widetilde{A}$ is absolutely continuous and its density $\widetilde{\rho}$ has a compact support contained in an interval $(-M, M)$, $(M > 0)$. Assume also that the density $\widetilde{\rho}$ satisfies the following conditions:*

(a) $\widetilde{\rho} \in C(\mathbb{R})$;

(b) *$\widetilde{\rho}$ has bounded variation;*

(c) *the set*

$$Z_{\widetilde{\rho}} = \{t \in \mathbb{R} : \widetilde{\rho}(t) = 0\}$$

consists of two closed semi-axes and at most a finite number of compact intervals, such that each of them is not an one-point set. In other words,

$$Z_{\widetilde{\rho}} = (-\infty, b_1] \cup \bigcup_{j=2}^{N-1} [a_j, b_j] \cup [a_N, \infty),$$

where $N \geqslant 2$ and $a_j < b_j$ for any $j \in \{2, 3, \dots, N-1\}$ (if $N = 2$, the compact intervals $[a_j, b_j]$ are absent in the right hand side of the last formula);

(d) *for any* $\nu \in \{1, 2, \dots, N-1\}$ *the function* $\widetilde{\rho}$ *is non-decreasing and concave in a semi-interval* $[b_\nu, b_\nu + \delta_\nu^+)$ $(\delta_\nu^+ > 0)$ *and for any* $j \in \{2, 3, \dots, N\}$ *the function* $\widetilde{\rho}$ *is non-increasing and concave in a semi-interval* $(a_j - \delta_j^-, a_j]$ $(\delta_j^- > 0)$.

Furthermore, assume that the limit

$$\lim_{\gamma \to 0} \frac{\phi(\widetilde{A}_\gamma) - \phi(\widetilde{A})}{\gamma}$$

exists for any function $\phi \in C[-M, M]$ *in the strong operator topology. Then the integral*

$$S(t) = \int_0^\infty \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s} \, \mathrm{d}s$$

exists for almost all $t \in \mathbb{R}$ *and the function* S *belongs to the class* $L_\infty(\mathbb{R})$.

Proof. The first assertion of the theorem follows from the first assertion of Theorem 4.5. Let us prove the second assertion. Observe that, in view of Theorem 3.8 and Remark 3.9,

$$S|_{Z_{\widetilde{\rho}}} \in L_\infty(Z_{\widetilde{\rho}}).$$

Hence, in order to prove the membership $S \in L_\infty(\mathbb{R})$, it is enough to show that

$$S|_{\mathcal{P}_{\widetilde{\rho}}} \in L_\infty(\mathcal{P}_{\widetilde{\rho}}), \tag{5.8}$$

where $\mathcal{P}_{\widetilde{\rho}} = \mathbb{R} \setminus Z_{\widetilde{\rho}}$.

Let $I_-(a_j)$ $(j \in \{2, 3, \dots, N\})$ and $I_+(b_\nu)$ $(\nu \in \{1, 2, \dots, N-1\})$ be semi-intervals of the form:

$$I_-(a_j) = (a_j - \sigma_j^-, a_j] \quad (\sigma_j^- > 0) \quad \text{and} \quad I_+(b_\nu) = [b_\nu, b_\nu + \sigma_\nu^+) \quad (\sigma_\nu^+ > 0).$$

Consider the following set:

$$\mathcal{O} = \Big(\bigcup_{j=2}^{N} I_-(a_j) \Big) \bigcup \Big(\bigcup_{\nu=1}^{N-1} I_+(b_\nu) \Big).$$

It is clear that, if σ_ν^+ and σ_j^- are small enough, the set $\mathcal{P}_{\widetilde{\rho}} \setminus \mathcal{O}$ consists of a finite number of compact intervals I_k $(k = 1, 2, \dots, K)$, that is

$$\mathcal{P}_{\widetilde{\rho}} \setminus \mathcal{O} = \bigcup_{k=1}^{K} I_k.$$

Since $\widetilde{\rho}(t) > 0$ for $t \in \mathcal{P}_{\widetilde{\rho}} \setminus \mathcal{O}$, then, by the second assertion of Theorem 4.5, we have:

$$S|_{\mathcal{P}_{\widetilde{\rho}} \setminus \mathcal{O}} \in L_\infty(\mathcal{P}_{\widetilde{\rho}} \setminus \mathcal{O}). \tag{5.9}$$

On the other hand, by Theorem 3.8 and the second assertion of Proposition 3.7, we have that

$$\int_{b_\nu}^{\infty} \frac{\widetilde{\rho}(u)}{u-b_\nu}\,\mathrm{d}u < \infty \quad \text{for any } \nu \in \{1,2,\dots,N-1\},$$

$$\int_{-\infty}^{a_j} \frac{\widetilde{\rho}(u)}{a_j-u}\,\mathrm{d}u < \infty \quad \text{for any } j \in \{2,3\dots,N\}.$$

Thus, making use of condition (d) and Lemma 5.1, we can choose the semi-intervals $I_-(a_j)$, $j \in \{2,3,\dots,N\}$, and $I_+(b_\nu)$, $\nu \in \{1,2,\dots,N-1\}$, such that

$$S|_{\mathcal{O}} \in L_\infty(\mathcal{O}).$$

The latter membership together with (5.9) yields the desired membership (5.8). The theorem is proven. ∎

APPENDIX 1: A FORMULA FOR THE OPERATOR FUNCTION $\phi(A)$

In this section we shall obtain a formula for the operator function, which has been used in Sections 2 and 4. It is given by the following

PROPOSITION A1.1. *Let A be a bounded self-adjoint operator acting in a Hilbert space $\mathcal{H}$ and $M > 0$ be such that $\sigma(A) \subset (-M, M)$. Then for any function $\phi \in C[-M, M]$*

$$\phi(A) = -\frac{1}{2\pi\mathrm{i}} \lim_{\varepsilon\downarrow 0} \int_{C_\varepsilon} \phi_0(\Re\lambda) R_\lambda(A)\,\mathrm{d}\lambda \tag{A1.1}$$

where the limit is in the uniform operator topology, the function $\phi_0(t)$ is defined by (2.14), *the contour C_ε has the form:*

$$C_\varepsilon = \{\lambda \in \mathbb{C} : |\Im\lambda| = \varepsilon\} \quad (\varepsilon > 0)$$

and the orientation on the lines of C_ε is positive with respect to the strip $\{\lambda \in \mathbb{C} : |\Im\lambda| < \varepsilon\}$.

Proof. Let us represent the integral in the right hand side of (A1.1) in the form:

$$-\frac{1}{2\pi\mathrm{i}} \int_{C_\varepsilon} \phi_0(\Re\lambda) R_\lambda(A)\,\mathrm{d}\lambda = \frac{1}{2\pi\mathrm{i}} \int_{-M}^{M} \phi(t)(R_{t+\mathrm{i}\varepsilon}(A) - R_{t-\mathrm{i}\varepsilon}(A))\,\mathrm{d}t.$$

Consider the corresponding scalar integral replacing the operator A by a real number μ:

$$
\text{(A1.2)} \qquad f(\mu,\varepsilon) = \frac{1}{2\pi \mathrm{i}} \int_{-M}^{M} \phi(t)\Big(\frac{1}{\mu - t - \mathrm{i}\varepsilon} - \frac{1}{\mu - t + \mathrm{i}\varepsilon}\Big)\,\mathrm{d}t
$$

$$
= \int_{-M}^{M} \mathcal{P}_\varepsilon(\mu - t)\phi(t)\,\mathrm{d}t,
$$

where $\mathcal{P}_\varepsilon(u)$ is the Poisson kernel: $\mathcal{P}_\varepsilon(u) = \frac{\varepsilon}{\pi(u^2+\varepsilon^2)}$. Since $\sigma(A) \subset (-M, M)$ and the function $\phi(t)$ is continuous in $[-M, M]$, a well known property of Poisson kernel guarantee that the limiting relation

$$
\lim_{\varepsilon \downarrow 0} f(\mu,\varepsilon) = \phi(\mu)
$$

is valid uniformly on $\sigma(A)$. Then, by a property of the functional calculus of normal operators, we have the following limiting relation with respect to the uniform operator topology:

$$
\lim_{\varepsilon \downarrow 0} f(A,\varepsilon) = \phi(A)
$$

(see Chapter IX, Section 3, Corollary 15 of [10]). In order to prove equality (A1.1), it remains only to show that

$$
\text{(A1.3)} \qquad f(A,\varepsilon) = \frac{1}{2\pi \mathrm{i}} \int_{-M}^{M} \phi(t)(R_{t+\mathrm{i}\varepsilon}(A) - R_{t-\mathrm{i}\varepsilon}(A))\,\mathrm{d}t.
$$

Consider a sequence of Riemann sums $S_n(\mu)$ for the second integral of (A1.2). Since the function $\Phi(\mu, t) = \mathcal{P}_\varepsilon(\mu - t)\phi(t)$ is continuous in the compact set $\sigma(A) \times [-M, M]$, these sums converge to the second integral of (A1.2) uniformly with respect to $\mu \in \sigma(A)$. On the other hand, each $S_n(\mu)$ is a rational function, hence replacing the variable μ by the operator A, we obtain $S_n(A)$. Using again the property of the functional calculus mentioned above, we obtain (A1.3). ∎

COROLLARY A1.2. *The formula* (A1.1) *can be written in terms of the Poisson kernel*

$$
P_\varepsilon(x) = \frac{\varepsilon}{\pi(x^2+\varepsilon^2)} \quad \varepsilon > 0
$$

in the form

$$
\phi(A) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \phi_0(t)\, P_\varepsilon(A - tI)\,\mathrm{d}t.
$$

APPENDIX 2: BOUNDARY VALUES OF BOREL TRANSFORM

We used a property of Borel transform

$$\Theta(\lambda) = \int_{-\infty}^{\infty} \frac{\widetilde{\rho}(\mu)\,\mathrm{d}\mu}{\mu - \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$$

of a function $\widetilde{\rho}$, defined on $\mathbb{R}$, which may be considered as a non-classical generalization of the well known Sokhotskii boundary property of the integral of Cauchy type ([12], Chapter 1, Section 4, $\mathrm{n}^o 4.2$).

PROPOSITION A2.1. *Assume that a function $\widetilde{\rho} \in L_2(\mathbb{R})$ has a compact support, contained in an interval $(-M, M)$ $(M > 0)$, and the following condition is satisfied :*

$$\text{(A2.1)} \qquad \int_0^{\infty} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s}\,\mathrm{d}s \;<\; \infty \quad \text{for almost all } t \in (-M, M).$$

Then the Borel transform $\Theta(\lambda)$ of the function $\widetilde{\rho}$ has the following properties:

(i) *The following limits*

$$\text{(A2.2)} \qquad \Theta(t + \mathrm{i}0) = \lim_{\varepsilon \downarrow 0} \Theta(t + \mathrm{i}\varepsilon),$$

$$\text{(A2.3)} \qquad \Theta(t - \mathrm{i}0) = \lim_{\varepsilon \downarrow 0} \Theta(t - \mathrm{i}\varepsilon),$$

exist for almost all $t \in \mathbb{R}$, the integral

$$\int_0^{\infty} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s}\,\mathrm{d}s$$

exists for almost all $t \in \mathbb{R}$, and the following formulas are valid :

$$\Theta(t + \mathrm{i}0) = \int_0^{\infty} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s}\,\mathrm{d}s + \mathrm{i}\pi\widetilde{\rho}(t),$$

$$\Theta(t - \mathrm{i}0) = \int_0^{\infty} \frac{\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)}{s}\,\mathrm{d}s - \mathrm{i}\pi\widetilde{\rho}(t).$$

Furthermore, the functions $\Theta(t + \mathrm{i}0)$ and $\Theta(t - \mathrm{i}0)$ belong to the class $L_2(\mathbb{R})$.

(ii) *Moreover, if $\widetilde{\rho} \in L_\infty(\mathbb{R})$ and the function*

$$\text{(A2.4)} \qquad \mathcal{R}(t) = \int_0^{\infty} \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s}\,\mathrm{d}s$$

belongs to the class $L_\infty(-M, M)$, then the families of functions

$$\text{(A2.5)} \qquad \{\Theta(t + \mathrm{i}\varepsilon)\}_{\varepsilon > 0} \quad \text{and} \quad \{\Theta(t - \mathrm{i}\varepsilon)\}_{\varepsilon > 0}$$

are uniformly bounded on $\mathbb{R}$.

Proof. We shall prove the assertion of the proposition only for the function $\Theta(t+\mathrm{i}\varepsilon)$, because the function $\Theta(t-\mathrm{i}\varepsilon)$ is treated analogously. Let us represent the function $\Theta(t+\mathrm{i}\varepsilon)$ $(t\in\mathbb{R})$ in the form:

$$\Theta(t+\mathrm{i}\varepsilon)=\widetilde{\Theta}(t,\varepsilon)+P(t,\varepsilon), \tag{A2.6}$$

where

$$\widetilde{\Theta}(t,\varepsilon)=\int_0^\infty \frac{\widetilde{\rho}(t+s)-\widetilde{\rho}(t-s)}{s-\mathrm{i}\varepsilon}\,\mathrm{d}s, \tag{A2.7}$$

$$P(t,\varepsilon)=\int_0^\infty \widetilde{\rho}(t-s)\Big(\frac{1}{s-\mathrm{i}\varepsilon}-\frac{1}{s+\mathrm{i}\varepsilon}\Big)\,\mathrm{d}s=\mathrm{i}\pi\int_{-\infty}^\infty \widetilde{\rho}(\mu)2\mathcal{P}_\varepsilon^-(\mu-t)\,\mathrm{d}\mu, \tag{A2.8}$$

and $\mathcal{P}_\varepsilon^-(s)$ is a "half Poisson" kernel:

$$\mathcal{P}_\varepsilon^-(s)=\begin{cases}\frac{\varepsilon}{\pi(s^2+\varepsilon^2)} & s\leqslant 0,\\ 0 & s>0.\end{cases}$$

Observe that

$$\int_{-\infty}^\infty 2\mathcal{P}_\varepsilon^-(s)\,\mathrm{d}s=1 \tag{A2.9}$$

and the function $2\mathcal{P}_\varepsilon^-(\mu-t)$ is a summability kernel (as $\varepsilon\downarrow 0$) of the integral operator $P(t,\varepsilon)$, defined by (A2.8). Furthermore, the function $2\mathcal{P}_\varepsilon^-(s)$ is non-negative, it is increasing for $s\leqslant 0$ and it is non-increasing for $s\geqslant 0$. These circumstances mean that the function $2\mathcal{P}_\varepsilon^-(\mu-t)$ is a convex majorant for itself, which satisfies the condition of Faddeyev Theorem 4.4. Thus, since the function $\widetilde{\rho}$ is compactly supported and belongs to $L_2(\mathbb{R})$, we have:

$$\lim_{\varepsilon\downarrow 0}P(t,\varepsilon)=\mathrm{i}\pi\widetilde{\rho}(t)\quad\text{a. e. in }\mathbb{R}. \tag{A2.10}$$

We now turn to the integral $\widetilde{\Theta}(t,\varepsilon)$ defined by (A2.7). Observe that, in view of the inclusion $\operatorname{supp}(\widetilde{\rho})\subset(-M,M)$ and condition (A2.1), the integral in (A2.1) exists for almost all $t\in\mathbb{R}$. Hence the integral $\widetilde{\Theta}(t,0)$ exists for almost all $t\in\mathbb{R}$ and, furthermore,

$$\lim_{\delta\downarrow 0}\int_0^\delta \frac{|\widetilde{\rho}(t+s)-\widetilde{\rho}(t-s)|}{s}\,\mathrm{d}s=0\quad\text{for almost all }t\in\mathbb{R}.$$

Let us take a point $t\in\mathbb{R}$, for which the latter limiting relation holds. Hence the integral $\widetilde{\Theta}(t,0)$ exists for this t. Consider the following representation for $\delta>0$:

$$\widetilde{\Theta}(t,\varepsilon)-\widetilde{\Theta}(t,0)=I_1(t,\varepsilon,\delta)+I_2(t,\varepsilon,\delta), \tag{A2.11}$$

where

$$I_1(t,\varepsilon,\delta) = \int_0^\delta (\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s))\Big(\frac{1}{s+\mathrm{i}\varepsilon} - \frac{1}{s}\Big)\,\mathrm{d}s$$

and

$$I_2(t,\varepsilon,\delta) = \int_\delta^\infty (\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s))\Big(\frac{1}{s+\mathrm{i}\varepsilon} - \frac{1}{s}\Big)\,\mathrm{d}s. \tag{A2.12}$$

We have:

$$|I_1(t,\varepsilon,\delta)| \leqslant 2\int_0^\delta \frac{|\widetilde{\rho}(t+s) - \widetilde{\rho}(t-s)|}{s}\,\mathrm{d}s,$$

therefore, in view of the choice of the point t,

$$\forall\, \nu > 0 \quad \exists\, \delta > 0, \quad \forall\, \varepsilon \geqslant 0: \quad |I_1(t,\varepsilon,\delta)| < \frac{\nu}{2}. \tag{A2.13}$$

Let us estimate the integral $I_2(t,\varepsilon,\delta)$ defined by (A2.12):

$$|I_2(t,\varepsilon,\delta)| \leqslant \frac{\varepsilon}{\delta^2}\int_\delta^\infty (|\widetilde{\rho}(t+s)| + |\widetilde{\rho}(t-s)|)\,\mathrm{d}s \leqslant \frac{2\varepsilon}{\delta^2}\int_{-M}^{M} |\widetilde{\rho}(s)|\,\mathrm{d}s.$$

Then

$$\exists\, \varepsilon_0 > 0, \;\; \forall\, \varepsilon \in (0,\varepsilon_0): \quad |I_2(t,\varepsilon,\delta)| < \frac{\nu}{2}.$$

Using the latter property, representation (A2.11) and property (A2.13), we get:

$$\lim_{\varepsilon\downarrow 0} \widetilde{\Theta}(t,\varepsilon) = \widetilde{\Theta}(t,0) \quad \text{a. e. in } \mathbb{R}.$$

The latter limiting relation, the relation (A2.10) and representation (A2.6) imply the first part of assertion (i). The membership $\Theta(\cdot + \mathrm{i}0) \in L_2(\mathbb{R})$ follows from the condition $\widetilde{\rho} \in L_2(\mathbb{R})$ and Proposition 2.1.

We now turn to the proof of assertion (ii). Making use of (A2.6), (A2.7), (A2.8) and (A2.9), we get for $\varepsilon > 0$:

$$\|\Theta(t+\mathrm{i}\varepsilon)\|_{L_\infty(-M,M)} \leqslant \|\mathcal{R}\|_{L_\infty(-M,M)} + \pi\|\widetilde{\rho}\|_{L_\infty(\mathbb{R})},$$

where the function $\mathcal{R}(t)$ is defined by (A2.4). Furthermore, in view of the inclusion $\mathrm{supp}(\widetilde{\rho}) \subset (-M,M)$, the family of functions $\{\Theta(t+\mathrm{i}\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded on the set $\mathbb{R}\setminus(-M,M)$. These circumstances and the fact that each function $\Theta(t+\mathrm{i}\varepsilon)$ $(\varepsilon > 0)$ is continuous imply assertion (ii). The proposition is proven. ∎

Proposition A2.1 implies the following

COROLLARY A2.2. *If condition* (A2.1) *of Proposition* A2.1 *is satisfied and the function $\mathcal{R}(t)$, defined by* (A2.4), *belongs to the class $L_\infty(-M,M)$, then the functions $\Theta(t+\mathrm{i}0)$ and $\Theta(t-\mathrm{i}0)$, defined by* (A2.2) *and* (A2.3), *belong to $L_\infty(\mathbb{R})$.*

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