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ESSENTIALLY REDUCTIVE HILBERT MODULES

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ABSTRACT. Consider a Hilbert space obtained as the completion of the polynomials $\mathbb{C}[\boldsymbol{z}]$ in m-variables for which the monomials are orthogonal. If the commuting weighted shifts defined by the coordinate functions are essentially normal, then the same is true for their restrictions to invariant subspaces spanned by monomials. This generalizes the result of Arveson [4] in which the Hilbert space is the m-shift Hardy space H^2_m. He establishes his result for the case of finite multiplicity and shows the self-commutators lie in the Schatten p-class for $p > m$. We establish our result at the same level of generality. We also discuss the K-homology invariant defined in these cases.

KEYWORDS: *Essentially reductive Hilbert modules, multivariate operator theory, essentially normal operators, commuting weighted shifts.*

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1. INTRODUCTION

The study of modules that are Hilbert spaces can be viewed as one approach to multivariate operator theory. While the underlying algebra could be almost anything, it is perhaps most natural to consider the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ or an algebra of holomorphic functions. In the case of a function algebra, such modules are called Hilbert modules and their study has been undertaken over the last two decades (cf. [13], [8]). In this paper, we will use the terminology Hilbert module to refer to any module that is a Hilbert space but we will keep track of the hypotheses being assumed about the algebra.

A Hilbert module $\mathcal{M}$ is said to be essentially reductive (cf. [13]) if the operators $\{M_\varphi\}$ in $\mathcal{L}(\mathcal{M})$ defined by module multiplication by elements φ in the algebra are all essentially normal, that is, the self-commutators $[M_\varphi^*, M_\varphi] = M_\varphi^* M_\varphi - M_\varphi M_\varphi^*$ are in $\mathcal{K}(\mathcal{M})$, the ideal of compact operators in $\mathcal{M}$, for all φ in the algebra A. (One could also refer to such Hilbert modules as "essentially normal.") In this case, there is a close relationship between the algebra A and the C^*-algebra, $\mathcal{J}(\mathcal{M})$, generated by the collection $\{M_\varphi : \varphi \in A\}$, particularly the quotient

$C(X_{\mathcal{M}}) = \mathcal{J}(\mathcal{M})/\mathcal{K}(\mathcal{M})$. In fact, the spectrum $X_{\mathcal{M}}$ can be identified as a subset of the maximal ideal space M_A of the algebra A, if A is a Banach algebra (cf. Theorem 1.6 of [10]). Moreover, the C^*-extension defined by $\mathcal{J}(\mathcal{M})$ yields an element in the odd K-homology group $K_1(X_{\mathcal{M}})$ of $X_{\mathcal{M}}$ (cf. [7]) which is an invariant for the Hilbert module $\mathcal{M}$.

In the classical case of the Hardy and Bergman modules over the disk algebra $A(\mathbb{D})$, both modules are essentially reductive as are the corresponding Hilbert modules for the Hardy and Bergman modules for the odd-dimensional spheres $\partial\mathbb{B}^n$ and balls $\mathbb{B}^n$. Moreover, in both cases the spectrum of the quotient C^*-algebras is the sphere, the boundary of $\mathbb{B}^n$, and the K-homology element is a generator for the group $K_1(\partial\mathbb{B}^n) \cong \mathbb{Z}$. However, for the polydisk $\mathbb{D}^n$, $n > 1$, neither the Hardy nor the Bergman module is essentially reductive. More generally, one obtains an essentially reductive Hilbert module for strongly pseudo-convex domains in $\mathbb{C}^n$, [6]. In a somewhat different direction, the m-shift Hardy space H^2_m, which is a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$, is essentially reductive [1].

Beyond the question of which Hilbert modules are essentially reductive, one can also ask which submodules and quotient modules are essentially reductive. In [11], Misra and I established by direct calculation that some quotient modules of the Hardy module for the bidisk algebra are essentially reductive and some are not. In this case, one can show that no nonzero submodule is essentially reductive using the fact that the coordinate functions define a pair of commuting isometries, both of infinite multiplicity. The question of essential reductivity for submodules and quotient modules of a given Hilbert module $\mathcal{M}$ is more likely to have an interesting answer, when $\mathcal{M}$ itself is essentially reductive. In this note, we show that for $\mathcal{M}$ essentially reductive, either both a submodule $\mathcal{N}$ and the corresponding quotient module $\mathcal{M}/\mathcal{N}$ are essentially reductive or neither is. Moreover, we extend this result, which concerns short exact sequences of Hilbert modules, to longer resolutions of Hilbert modules.

In [4] Arveson showed that submodules of $H^2_m \otimes \mathbb{C}^k$, $1 \leqslant k < \infty$, in a certain class are essentially reductive and raised a more general question. His question concerned all submodules generated by homogeneous polynomials in $\mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^k$, $1 \leqslant k < \infty$, and he established essential reductivity in case the submodule is generated by monomials. Further, Arveson has informed me that, based on a recent result of Guo [18], the question is answered in the affirmative for the general case when $m = 2$.

The action of the coordinate functions on $H^2_m \otimes \mathbb{C}^k$ can be seen to define commuting, contractive weighted shifts of multiplicity k. Our principal result is to extend Arveson's theorem to the case of general commuting weighted shifts so long as they define an essentially reductive Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Further, we will show that our results extend to the p-summable context which is what Arveson actually proves. Finally, we discuss the $\mathcal{K}$-homology class defined by this Hilbert module.

Before we begin, we want to thank the referee for pointing out a gap in our original proof in the p-summable case.

2. RESOLUTIONS AND ESSENTIAL REDUCTIVITY

We begin with the result relating the behavior of submodules and their respective quotient modules for an essentially reductive module.

THEOREM 2.1. *Let $\mathcal{M}$ be an essentially reductive Hilbert module over the algebra A, $\mathcal{N}$ be a submodule of $\mathcal{M}$ and $\mathcal{Q} = \mathcal{M}/\mathcal{N}$, the corresponding quotient module. Then $\mathcal{N}$ is essentially reductive if and only if $\mathcal{Q}$ is.*

Proof. The result depends on a simple matrix calculation. For φ in A we consider the matrix for M_φ relative to the decomposition $\mathcal{N} \oplus \mathcal{N}^\perp$ to obtain $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Since $\mathcal{N}$ is invariant for M_φ we have $C = 0$. Moreover, the action of φ on $\mathcal{N}$ defines the operator A, while the action of φ on $\mathcal{Q}$ defines an operator unitarily equivalent to D.

Then a simple calculation shows that the matrix for $[M_\varphi^*, M_\varphi]$ relative to $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}^\perp$ is $\begin{pmatrix} [A^*,A]-BB^* & A^*B-BD^* \\ B^*A-DB^* & [D^*,D]+B^*B \end{pmatrix}$. Since $\mathcal{M}$ is essentially reductive, it follows that $[M_\varphi^*, M_\varphi]$ is compact and hence so are the operators $[A^*, A] - BB^*$ and $[D^*, D] + B^*B$. If $\mathcal{N}$ is essentially reductive, then $[A^*, A]$ is compact and hence BB^* is compact. This implies B^*B is compact and that $[D^*, D]$ is compact. Since this is true for every φ in A, we see that $\mathcal{Q}$ is essentially reductive. The argument that $\mathcal{Q}$ essentially reductive implies that $\mathcal{N}$ is, proceeds similarly. ∎

The same argument shows the theorem holds if one uses p-reductive instead of essentially reductive. (See Section 5 for the definition of p-reductive.)

Now recall that by a resolution of length one of the Hilbert module $\mathcal{M}_0$ over A, we mean that there are Hilbert modules $\mathcal{M}_1$ and $\mathcal{M}_2$ and module maps $\mathcal{M}_1 \xrightarrow{X_0} \mathcal{M}_0$ and $M_2 \xrightarrow{X_1} \mathcal{M}_1$ such that range $(X_0) = \mathcal{M}_0$, kernel $(X_1) = (0)$, and range $(X_1) = \text{kernel}(X_0)$ (cf. [12]). If X_1 and X_0^* are isometries, then such a resolution can be seen to be equivalent to $\mathcal{M}_2$ being unitarily equivalent to a submodule of $\mathcal{M}_1$ with quotient module unitarily equivalent to $\mathcal{M}_0$.

THEOREM 2.2. *Consider a resolution of length one of the Hilbert module $\mathcal{M}_0$ over the algebra A:*

$$0 \longleftarrow \mathcal{M}_0 \xleftarrow{X_0} \mathcal{M}_1 \xleftarrow{X_1} \mathcal{M}_2 \longleftarrow 0.$$

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are essentially reductive and X_0^* is an isometry, then $\mathcal{M}_0$ is essentially reductive.

Proof. We work at the level of C^*-algebras modulo the compacts. Fix an element φ in A and let M_i be the operator on $\mathcal{L}(\mathcal{M}_i)$ defined by module multiplication by φ for $i = 0, 1, 2$. Moreover, let π denote the quotient maps $\mathcal{L}(\mathcal{M}_i) \longrightarrow \mathcal{Q}(\mathcal{M}_i) = \mathcal{L}(\mathcal{M}_i)/\mathcal{K}(\mathcal{M}_i)$ and $\mathcal{L}(\mathcal{M}_i, \mathcal{M}_j) \to \mathcal{Q}(\mathcal{M}_i, \mathcal{M}_j)$, for $0 \leqslant i, j \leqslant 2$.

Assuming $\mathcal{M}_1$ is essentially reductive, we have that $\pi(M_1)$ is a normal element of $\mathcal{Q}(\mathcal{M}_1)$. If $\mathcal{M}_2$ is essentially reductive, then $\pi(M_2)$ is a normal element of $\mathcal{Q}(\mathcal{M}_2)$. Moreover, $\pi(X_1)$ intertwines $\pi(M_2)$ and $\pi(M_1)$, that is, $\pi(M_2)\pi(X_1) = \pi(X_1)\pi(M_1)$. Since X_1 is one-to-one and has closed range, we can write $X_1 = V_1 P_1$, where V_1 is an isometry from $\mathcal{M}_1$ to $\mathcal{M}_2$ and P_1 is a positive invertible operator on $\mathcal{M}_1$. In view of the Fuglede Theorem, the intertwining relation for $\pi(X_1)$ yields $\pi(M_2)\pi(V_1) = \pi(V_1)\pi(M_1)$. If we decompose $\mathcal{Q}(\mathcal{M}_1)$ using the projections $p_1 = \pi(V_1)\pi(V_1)^*$ and $1 - p_1$, we obtain a matrix for $\pi(M_1)$ of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ as in the previous proof. Since $\pi(M_1)$ is normal, we see that d is normal. But $\pi(X_1)$ sets up a unitary equivalence between d and $\pi(M_0)$ and hence $\pi(M_0)$ is normal. Since this is true for all φ in A, we see that $\mathcal{M}_0$ is essentially reductive. ∎

If we weaken the hypotheses by not requiring X_0^* to be an isometry, then the previous proof fails. The Fuglede Theorem requires both of the operators intertwined to be normal and hence we can't replace the intertwining operator by its partially isometric part. Of course, it would be enough to assume that $\pi(X_0)^*$ is an isometry.

We can extend these results to longer resolutions if we assume that we have a strong resolution, that is, if the module maps are all partial isometries (cf. [12]). (Actually the last module map need not be an isometry but the others do or at least partial isometries modulo the compacts.)

THEOREM 2.3. *Consider a strong resolution of finite length of the Hilbert module $\mathcal{M}_0$*

$$0 \longleftarrow \mathcal{M}_0 \xleftarrow{X_0} \mathcal{M}_1 \longleftarrow \cdots \xleftarrow{X_n} \mathcal{M}_{n+1} \longleftarrow 0.$$

If each $\mathcal{M}_i$, $1 \leqslant i \leqslant n+1$, is essentially reductive, then so is $\mathcal{M}_0$.

Proof. The proof is the same as above once one observes that at each stage one not only concludes that modulo the compacts the operators defined by module multiplication are diagonal but so also are the connecting module maps. ∎

Extending these theorems to the p-reductive case would involve the consideration of the Fuglede Theorem in that context.

3. COMMUTING WEIGHTED SHIFTS — SCALAR CASE

We now turn to the question of establishing the essential reductivity of submodules of modules defined by multivariate weighted shifts. For a fixed positive integer m, let $\mathbb{C}[\boldsymbol{z}]$ denote the complex polynomials in m-variables with

$z = (z_1, \ldots, z_m)$. Let α be the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ with each α_i a non-negative integer, A_m be the set of multi-indices, $|\alpha_1| = \alpha_1 + \cdots + \alpha_m$, e_i the multi-index with 1 in the ith position and zero for all other entries, and Z^{α} the monomial $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_m^{\alpha_m}$ in $\mathbb{C}[z]$. Let $\Lambda = \{\lambda_{\alpha}\}_{\alpha \in A_m}$ be a set of weights, $0 < \lambda_{\alpha} < \infty$, and $\mathcal{M}_{\Lambda}$ be the Hilbert space spanned by the orthogonal set $\{Z^{\alpha}\}_{\alpha \in A_m}$ with $\|Z^{\alpha}\|_{\mathcal{M}_{\Lambda}} = \lambda_{\alpha}$. (This is the standard setup to define commuting weighted shifts in which the monomials are orthogonal.)

First, we make two basic assumptions about the set of weights. First, we assume that:

$(*)$ $\quad \lambda_{\alpha} \geqslant \lambda_{\alpha+e_i}$ for $\alpha \in A$ and $1 \leqslant i \leqslant m$,

which ensures that each operator Z_i defined by module multiplication by z_i is a contraction, $1 \leqslant i \leqslant m$. Therefore, $\mathcal{M}_{\Lambda}$ is a Hilbert module over $\mathbb{C}[z]$. (Actually, in almost all of what follows, the assumption that the Z_i are bounded is sufficient.) Moreover, one can show that $\mathcal{M}_{\Lambda}$ is a module for the algebra of functions holomorphic on any fixed polydisk of radius greater than one, although it is, in general, not a module over the polydisk algebra $A(\mathbb{D}^m)$. Further, we can show that the quotient algebra $\mathcal{J}(\mathcal{M}_{\Lambda})/\mathcal{C}(\mathcal{M}_{\Lambda})$, where $\mathcal{C}(\mathcal{M}_{\Lambda})$ is the commutator ideal for $\mathcal{J}(\mathcal{M}_{\Lambda})$, is isometrically isomorphic to $C(X_{\mathcal{M}_{\Lambda}})$ for some compact subset $X_{\mathcal{M}_{\Lambda}}$ of the closed unit polydisk ${}^{\mathrm{cl}}\mathbb{D}^m$ (cf. Theorem 1.6 of [10]).

Again we say that $\mathcal{M}_{\Lambda}$ is essentially reductive if the operators in $\mathcal{J}(\mathcal{M}_{\Lambda})$ are essentially normal. Our second assumption about the weight set Λ is:

$(**)$ $\quad [Z_i, Z_j^*]$ is compact for all i, j with $1 \leqslant i, j \leqslant m$.

It is enough to assume only that $[Z_i, Z_i^*]$ is compact for $1 \leqslant i \leqslant m$, since Fuglede's Theorem shows that this assumption together with the fact that $[Z_i, Z_j] = 0$ for $1 \leqslant i, j \leqslant m$ implies that the cross-commutators are also compact. We choose this form for $(**)$ to maintain parallelism with the later assumptions regarding p-summability.

In this case, since $\mathcal{J}(\mathcal{M}_{\Lambda})$ is irreducible, the compact operators $\mathcal{K}(\mathcal{M}_{\Lambda})$ on $\mathcal{M}_{\Lambda}$ are contained in $\mathcal{J}(\mathcal{M}_{\Lambda})$, it follows that $\mathcal{C}(\mathcal{M}_{\Lambda}) = \mathcal{K}(\mathcal{M}_{\Lambda})$ and the quotient algebra $\mathcal{J}(\mathcal{M}_{\Lambda})/\mathcal{K}(\mathcal{M}_{\Lambda}) \cong C(X_{\Lambda})$ for some compact subset X_{Λ} contained in ${}^{\mathrm{cl}}\mathbb{D}^m$ (see Theorem 5.3).

For B a subset of A_m, let $\mathcal{M}_{\Lambda}(B)$ be the subspace of $\mathcal{M}_{\Lambda}$ spanned by $\{Z^{\alpha}\}_{\alpha \in B}$. A subset B of A_m determines a submodule $\mathcal{M}_{\Lambda}(B)$ if and only if B is shift invariant which means that α in B implies $\alpha + e_i$ is in B for $1 \leqslant i \leqslant m$. For each i, $1 \leqslant i \leqslant m$, and non-negative integer k, set $\Sigma_i^k = \{\alpha \in A_m : \alpha_i = k\}$. Then $\mathcal{M}_{\Lambda}(\Sigma_i^k)$ is a reducing subspace for Z_j, $1 \leqslant j \leqslant m, j \neq i$.

For α in A_m let $B(\alpha)$ be the subset of A_m consisting of all β satisfying $\beta_i \geqslant \alpha_i$, $1 \leqslant i \leqslant m$. Observe that $B(\alpha)$ is a shift invariant subset of A_m which is naturally isomorphic to A_m and $\{\lambda_{\beta} : \beta \in B(\alpha)\}$ can be identified as a weight set for A_m using this identification.

We note that if the weight set Λ on A_m satisfies $(*)$ and $(**)$, then so does the weight set obtained by restricting Λ to $B(\alpha) \subset A_m$ for α in A_m with $B(\alpha)$

identified with A_m. Further, fix i and k, $1 \leqslant i \leqslant m$ and $0 < k < \infty$, and identify the polynomials in $\mathbb{C}[\boldsymbol{z}]$ that omit z_i with the polynomials in the $(m-1)$-variables $\{z_j\}_{j \neq i}$. Then the module for A_{m-1} with the weight set obtained by restricting Λ to Σ_i^k also satisfies $(*)$ and $(**)$.

We note that the weight set $\lambda_{\boldsymbol{\alpha}} = \frac{\alpha_1! \alpha_2! \cdots \alpha_m!}{|\boldsymbol{\alpha}|!}$ defines the m-shift Hardy module H_m^2 and Arveson established $(**)$ in [1] while $(*)$ is straightforward.

In [4] Arveson showed in this case that all submodules generated by monomials or, equivalently, those that are determined by a shift invariant subset of A_m (by Proposition 3.1 below), are essentially reductive. Our goal is to extend this result to the case of Hilbert modules defined by weighted shifts satisfying $(*)$ and $(**)$. Actually, Arveson establishes his result for the finite direct sum of copies of H_m^2 and showed that the self-commutators are in an appropriate Schatten p-class. We will do the same.

The following result shows that the collection of submodules generated by shift invariant subsets of A_m, is the same as the collection of submodules generated by monomials.

PROPOSITION 3.1. *A submodule of $\mathcal{M}_\Lambda$ is generated by a set of monomials $\{Z^{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in C}$ for $C \subset A_m$ if and only if it is of the form $\mathcal{M}(B)$ for some shift invariant subset B of A_m. Moreover, the generating set of monomials can be taken to be finite.*

Proof. If $\mathcal{S}$ is generated by the set $\{Z^{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in C}$, then let B be the shift invariant subset of A_m generated by C. Then $\{Z^{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in B}$ is contained in $\mathcal{S}$. Hence, $\mathcal{M}(B) \subset \mathcal{S}$ and since $\mathcal{M}(B)$ contains $\{Z^{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in C}$, we have equality. The converse proceeds in the same manner. Note also the proof follows from the fact that $B = \{\boldsymbol{\alpha} \in A_m : Z^{\boldsymbol{\alpha}} \in \mathcal{S}\}$.

The argument that the set C can be taken to be finite proceeds either using the finite basis result for $\mathbb{C}[\boldsymbol{z}]$ or the geometry of B. ∎

Before proceeding we need to identify a property of the weighted shifts acting on $\mathcal{M}_\Lambda$ which follows from $(**)$. In a preliminary version of this paper, the conclusion of the following lemma was assumption $(***)$ but Ken Davidson pointed out to me that it actually follows from $(**)$. We give his proof.

LEMMA 3.2. *Let Λ be a weight set for A_m, $1 \leqslant m < \infty$, satisfying $(*)$ and $(**)$. If X_i is the edge operator from $\mathcal{M}_\Lambda(\Sigma_i^k)$ to $\mathcal{M}_\Lambda$ defined by the action of the operator Z_i, then $X_i^* X_i$ is a compact operator on $M_\Lambda(\Sigma_i^k)$ for $1 \leqslant i \leqslant m$ and $0 < k < \infty$.*

Proof. Fix i and consider $k = 0$. Let $X_{i,0}$ be the operator defined by Z_i from $\mathcal{M}(\Sigma_i^0)$ to $\mathcal{M}_\Lambda$. Then $X_{i,0}^* X_{i,0}$ is equal to the restriction of $[Z_i^*, Z_i]$ to $\mathcal{M}(\Sigma_i^0)$ since $Z_i^*|_{\mathcal{M}(\Sigma_i^0)} = 0$. Because $[Z_i^*, Z_i]$ is compact, then so is $X_{i,0}^* X_{i,0}$. Now if we consider the case of Σ_i^1, then the restriction of $[Z_i^*, Z_i]$ to $\mathcal{M}(\Sigma_i^1)$, which is compact, is the sum of a compact operator, since $X_{i,0}^* X_{i,0}$ is compact, and $X_{i,1}^* X_{i,1}$. Thus the latter operator is compact and we can proceed inductively to complete the proof. ∎

Proposition 3.1 shows that submodules generated by monomials have a geometric description, that is, are determined by shift invariant subsets of A_m. Our proofs are accomplished by decomposing the subset that determines the submodule into sets invariant for one or more of the shifts and then reducing the compactness of the self-commutators to that of the operator acting on the entire space together with the compactness of the edge operators. Similar arguments allow us to conclude that the cross-commutators are also compact.

THEOREM 3.3. *Let Λ be a weight set for A_m, $1 \leqslant m < \infty$, satisfying $(*)$ and $(**)$. If B is a shift invariant subset of A_m, then the submodule $\mathcal{M}_\Lambda(B)$ is essentially reductive.*

Proof. We consider first the case $m = 2$ where the argument is more transparent since we can identify the multi-indices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ with the integral lattice points in the quarter plane. We reduce the general case to that of a single monomial. Let $\overline{\alpha}_i = \inf\{\alpha_i : (\alpha_1, \alpha_2) \in B\}$ for $i = 1, 2$, $\overline{\boldsymbol{\alpha}} = (\overline{\alpha}_1, \overline{\alpha}_2)$, and let $\overline{\mathcal{M}} = \mathcal{M}_\Lambda(\overline{B})$, where $\overline{B} = B(\overline{\boldsymbol{\alpha}})$. Since $\overline{B}$ is shift invariant and contains all (α_1, α_2) in B, it follows that $\overline{\mathcal{M}}$ contains $\mathcal{M}_\Lambda(B)$. We claim that $\overline{\mathcal{M}}/\mathcal{M}_\Lambda(B)$ is finite dimensional and hence the essential reductivity of $\mathcal{M}_\Lambda(B)$ is equivalent to that of $\overline{\mathcal{M}}$.

In this situation, the finite dimensionality of $\overline{\mathcal{M}}/\mathcal{M}_\Lambda(B)$ is equivalent to the cardinality of $\overline{B} \backslash B$ being finite. To see that the latter holds, there must exist nonnegative integers β_1 and β_2 such that $(\overline{\alpha}_1, \beta_2)$ and $(\beta_1, \overline{\alpha}_2)$ are in B. But then $\overline{B} \backslash B$ is contained in the set $\{(\gamma_1, \gamma_2) \in A_2 : \overline{\alpha}_1 \leqslant \gamma_1 \leqslant \beta_1, \overline{\alpha}_2 \leqslant \gamma_2 \leqslant \beta_2\}$, which is finite.

Now we must show that the restrictions of Z_1 and Z_2 to $\overline{\mathcal{M}}$ are essentially normal. Consider Z_1. Now the self-commutator of Z_1 on $\mathcal{M}_\Lambda$ is the direct sum of operators on the one-dimensional subspaces spanned by the monomials. The same is true for the restriction Y_1 of Z_1 to $\mathcal{M}_\Lambda(\overline{B})$. If we set $\overline{B} = \overline{B}_1 \cup \overline{B}_2$, where $\overline{B}_1 = \{(\gamma_1, \gamma_2) \in A_2 : \overline{\alpha}_1 < \gamma_1, \overline{\alpha}_2 \leqslant \gamma_2\}$ and $\overline{B}_2 = \{(\gamma_1, \gamma_2) \in A_2 : \overline{\alpha}_1 = \gamma_1, \overline{\alpha}_2 \leqslant \gamma_2\}$, then the restrictions of the self-commutators of Y_1 and Z_1 to $\mathcal{M}_\Lambda(\overline{B}_1)$ agree and hence the former is compact by $(**)$.

On $\mathcal{M}_\Lambda(\overline{B}_2)$, the restrictions of the self-commutators of Y_1 and Z_1 agree on $\mathcal{M}_\Lambda$ if $\overline{\alpha}_1 = 0$ and hence again are compact by $(**)$. If $\overline{\alpha}_1 > 0$, then the restriction of the self-commutator of Y_1 to $\mathcal{M}_\Lambda(\overline{B}_2)$ equals $X_1^* X_1$, where X_1 is the edge operator defined from $\mathcal{M}_\Lambda(\overline{B}_2)$ to $\mathcal{M}_\Lambda$ by the action of Z_1, which is compact by Lemma 3.2.

Now we repeat the argument for Z_2 noting that the decomposition used for $\overline{B}$ in this case is not the same as that used above. This completes the proof that the self-commutators are compact for the case $m = 2$.

To conclude that the cross-commutator $[Y_1^*, Y_2]$ is compact, we can either appeal to Fuglede's Theorem or note that the preceding analysis can be applied. In particular, $[Y_1^*, Y_2]$ takes a monomial $Z^{(\alpha_1, \alpha_2)}$ in the submodule to a multiple of $Z^{(\alpha_1-1,\alpha_2+1)}$ if the latter monomial is also in the submodule or to 0 otherwise. Thus on $\mathcal{M}_\Lambda(\overline{B}_1)$, we obtain a restriction of the cross-commutator $[Z_1^*, Z_2]$ which is compact by $(**)$ or an edge operator on $\mathcal{M}_\Lambda(\overline{B}_2)$ which is compact by Lemma 3.2. Hence, if we know that the cross-commutators on $\mathcal{M}_\Lambda$ and the edge operators are compact, then the same is true for the restrictions Y_1 and Y_2 to the submodule.

For $m > 2$ we will use induction and hence we assume the result holds for all $1 \leqslant m' < m$. Let Y_m be the restriction of Z_m to the submodule $\mathcal{S}$ generated by $\{Z^{\boldsymbol{\alpha}^i}\}$. We show that we can reduce the question of the essential normality of Y_m to the case in which all the α_1^i are constant. Repeating the argument, now focusing on the second component, allows us to assume not only are the α_1^i constant but also the α_2^i. Finally, we reach the point in which all α_j^i are constant for $j = 1, 2, \ldots, m-1$. In this case we have to consider Y_m on a submodule $\mathcal{S}$ generated by $\{Z^{\boldsymbol{\alpha}^i}\}$ with $\boldsymbol{\alpha}^i = (\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_{m-1}, \alpha_m^i)$ which equals $\mathcal{M}(B(\boldsymbol{\alpha}))$, where $\boldsymbol{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_m)$ with $\overline{\alpha}_m = \inf\{\alpha_m^i\}$. Thus we have reduced the proof to the case of a submodule generated by a single monomial just as for the case $m = 2$. The argument for showing Y_m is essentially normal proceeds in the same way as for the $m = 2$ case above, by decomposing $B(\boldsymbol{\alpha})$ into the two disjoint sets $\overline{B}_1 = \{(\gamma_1, \gamma_2, \ldots, \gamma_m) \in A_m : \overline{\alpha}_j \leqslant \gamma_j, 1 \leqslant j < m, \overline{\alpha}_m < \gamma_m\}$ and $\overline{B}_2 = \{(\gamma_1, \gamma_2, \ldots, \gamma_m) \in A_m : \overline{\alpha}_j \leqslant \gamma_j, 1 \leqslant j < m, \overline{\alpha}_m = \gamma_m\}$. The arguments for the compactness of the two parts of the self-commutator of Y_m are also the same as those given above. Also, the proof of the compactness of the cross-commutator $[Y_m^*, Y_i]$ for $1 \leqslant i \leqslant m-1$ proceeds in the same manner.

Now we return to the matter of reducing the general case of showing the essential normality of the restriction Y_m of Z_m to a submodule of $\mathcal{S}$ generated by the set $\{Z^{\boldsymbol{\alpha}^i}\}$ to the case in which the α_1^i are all constant. If $\overline{\alpha}_1$ is the maximum of the set α_1^i, then $\mathcal{S}$ can be written as the orthogonal direct sum of the submodule $\mathcal{S}^1$ generated by $\{Z^{(\overline{\alpha}_1, \alpha_2^i, \ldots, \alpha_m^i)}\}$ and subspaces $\mathcal{S}_\gamma^1$, defined for $0 \leqslant \gamma < \overline{\alpha}_1$, each of which reduces Y_m. Thus the self-commutator of Y_m is the direct sum of the self-commutator of Z_m restricted to each of these summands. The self-commutator of the restriction of Z_m to the first summand is the case in which all the first components are constant. To complete the reduction, we need to define the $\mathcal{S}_\gamma^1$ and show that the self-commutators of the restriction of Y_m to each of them are all compact.

For each $i, 1 \leqslant i \leqslant n$, the submodule of $\mathcal{M}_\Lambda$ generated by $Z^{\boldsymbol{\alpha}^i}$ has the form $\mathcal{M}(B(\boldsymbol{\alpha}^i))$. The subspace $\mathcal{S}_\gamma^1$ is spanned by the collection of monomials $N_1^\gamma = \bigcup_{i=1}^n \{Z^{\boldsymbol{\beta}} : \boldsymbol{\beta} \in B(\boldsymbol{\alpha}^i), \beta_1 = \gamma\}$. The fact that $\mathcal{S}_\gamma^1$ reduces Y_m follows from the fact that a monomial $Z^{\boldsymbol{\beta} - e_m}$ is in N_1^γ, and hence in $\mathcal{S}_\gamma^1$, if and only if it is in $\mathcal{S}$. Now we can view N_1^γ as a subset of all monomials that omit z_1. After identifying this set with the polynomials in $(m-1)$-variables, we obtain the weight set by restricting Λ. Then noting that it satisfies $(*)$ and $(**)$, we can apply the induction hypothesis to conclude that the restriction of the self-commutator of Y_m to $\mathcal{S}^1_\gamma$ is compact. Thus the restriction of the self-commutator of Y_m to each $\mathcal{S}^1_\gamma$ is compact which completes the reduction and the proof that Y_m is essentially normal. Further, the argument for the cross-commutators proceeds as above for $[Y_1^*, Y_i], 1 < i \leqslant m$, except for $\mathcal{S}^1_0$. Here, the argument also involves the fact that the cross-commutators $[Z_1^*, Z_i]$ are compact as well as the compactness of the edge operator for Z_1 for the $\mathcal{S}^1_\gamma$, $\gamma > 0$, by Lemma 3.2. With the next step, the reduction to the case in which both the first and second components, α_1^i and α_2^i, are each all constant, we conclude that the cross-commutators $[Y_1^*, Y_i]$ and $[Y_2^*, Y_j]$ are compact for $1 < i \leqslant m$ and $2 < j \leqslant m$. Hence, when we have completed the reduction, we know that all cross-commutators $[Y_i^*, Y_j]$ are compact for $1 \leqslant i, j \leqslant m$.

Finally, we can repeat the argument for the restriction of each of the coordinate operators $\{Z_i\}_{1 \leqslant i \leqslant m-1}$. This also enables us to conclude that all cross-commutators $[Y_i^*, Y_j]$ are compact. This completes the proof. ∎

4. COMMUTING WEIGHTED SHIFTS — FINITE MULTIPLICITY

We can extend the above result trivially to the case of higher multiplicity in one elementary situation.

COROLLARY 4.1. *Let Λ be a weight set for A_m, $m \geqslant 1$, satisfying $(*)$ and $(**)$ and $1 \leqslant k < \infty$. If B is a shift invariant subset of A_m, then the submodule $\mathcal{M}_\Lambda(B) \otimes \mathbb{C}^k$ of $\mathcal{M}_\Lambda \otimes \mathbb{C}^k$ is essentially reductive. Equivalently, every submodule generated by $\{Z^{\boldsymbol{\alpha}} \otimes \mathbb{C}^k : \boldsymbol{\alpha} \in C\}$ for some subset C of A_m is essentially reductive.*

Proof. The result follows from the theorem since $\mathcal{M}_\Lambda(B) \otimes \mathbb{C}^k$ is the direct sum of finitely many copies of $\mathcal{M}_\Lambda(B)$ each of which reduces all of the Z_i. ∎

We now extend this result to general submodules of $\mathcal{M}_\Lambda \otimes \mathbb{C}^k$, $1 \leqslant k < \infty$, generated by monomials. We begin with the case $m = 2$.

THEOREM 4.2. *Let Λ be a weight set of A_2 satisfying $(*)$ and $(**)$ and let $1 \leqslant k < \infty$. Then the submodule $\mathcal{S}$ generated by the set of monomials $\{Z^{\boldsymbol{\alpha}^i} \otimes \boldsymbol{x}_i\}_{i=1}^n$ for $\{\boldsymbol{\alpha}^i\} \subset A_2$ and $\{\boldsymbol{x}_i\} \subset \mathbb{C}^k$ is essentially reductive.*

Proof. We show that the restriction Y_2 of Z_2 to $\mathcal{S}$ is essentially normal, by reducing to the case in which the α_1^i are all equal, where $\boldsymbol{\alpha}^i = (\alpha_1^i, \alpha_2^i)$.

This is the same argument used in the proof of Theorem 3.3 with one additional complication which arises from the multiplicity. If $\overline{\alpha}_1$ is the maximum of the set $\{\alpha_1^i\}$, then $\mathcal{S}$ can be written as the orthogonal direct sum of the submodule $\mathcal{S}^1$ generated by $\{Z^{(\overline{\alpha}_1, \alpha_2^i)} \otimes \boldsymbol{x}_i\}$ and subspaces $\mathcal{S}^1_\gamma$, defined for $0 \leqslant \gamma < \overline{\alpha}_1$, each of which reduces Y_2. Thus the self-commutator of Y_2 is the direct sum of the self-commutators of Z_2 restricted to each of these summands. The self-commutator of the restriction of Z_2 to the first summand is the reduction to the case in which all indices (α_1^i, α_2^i) have constant first entries. To complete this reduction, we need to define the $\mathcal{S}_\gamma^1$ and show that the self-commutators of the restriction of Z_2 to each of them are all compact.

For each $i, 1 \leqslant i \leqslant n$, the submodule of $\mathcal{M}_\Lambda \otimes \mathbb{C}^k$ generated by $Z^{\boldsymbol{\alpha}^i} \otimes \boldsymbol{x}_i$ has the form $\mathcal{M}(B(\boldsymbol{\alpha}^i)) \otimes (\boldsymbol{x}_i)$, where $(\boldsymbol{x}_i)$ denotes the subspace of $\mathbb{C}^k$ generated by the vector $\boldsymbol{x}_i$. The subspace $\mathcal{S}_\gamma^1$ is spanned by the collection of monomials $N_1^\gamma = \bigcup_{i=1}^{n} \{Z^{\boldsymbol{\beta}} \otimes \boldsymbol{x}_i : \boldsymbol{\beta} \in B(\boldsymbol{\alpha}_i), \beta_1 = \gamma\}$. The fact that $\mathcal{S}_\gamma^1$ reduces Y_2 follows from the fact that a monomial $Z^{\boldsymbol{\beta}-e_2} \otimes \boldsymbol{x}_i$ is in N_1^γ and hence in $\mathcal{S}_\gamma^1$ if and only if it is in $\mathcal{S}$. For each $0 \leqslant j$, let $\mathcal{H}_j^\gamma$ be the subspace of $\mathbb{C}^k$ spanned by the $\boldsymbol{x}_i$ for which $Z^{(\gamma,j)} \otimes \boldsymbol{x}_i$ is in N_1^γ. Then $\{\mathcal{H}_j^\gamma\}$ is a strictly increasing sequence of subspaces of $\mathbb{C}^k$ and there exists an increasing sequence $0 \leqslant n_1 < \cdots < n_\ell < \infty$ such that every $\mathcal{H}_j^\gamma$ is equal to one of the $\mathcal{H}_{n_i}^\gamma$ and the n_i are each chosen as small as possible. Then we can express $\mathcal{S}_\gamma^1$ as the direct sum of subspaces $\mathcal{S}_\gamma^1(i)$, $1 \leqslant i \leqslant \ell$, where $\mathcal{S}_\gamma^1(i)$ is the tensor product of the span of the monomials $\{Z^{\boldsymbol{\beta}} : \beta_1 = \gamma, \beta_2 \geqslant n_i\}$ with the subspace $\mathcal{H}_{n_i}^\gamma \cap (\mathcal{H}_{n_{i-1}}^\gamma)^\perp$ of $\mathbb{C}^k$, where we set $\mathcal{H}_{n_0}^\gamma = (0)$. For $n_1 = 0$, the self-commutator of Y_2 restricted to $\mathcal{S}_\gamma^1(1)$ is a direct summand of the self-commutator of $Z_2 \otimes I_{\mathcal{H}_{n_1}^\gamma}$ and hence is compact by $(**)$. For all $n_i > 0$, the restriction of the operator $Z_2 \otimes I_{\mathcal{H}_{n_i}^\gamma \cap (\mathcal{H}_{n_{i-1}}^\gamma)^\perp}$ to $\mathcal{S}_\gamma^1(i)$ is compact by Lemma 3.2 and hence the self-commutator is also compact.

Now let Y_1 denote the restriction of Z_1 to $\mathcal{S}$ and consider the cross-commutator $[Y_1^*, Y_2]$ relative to the foregoing decomposition of $\mathcal{S}$. For the first term $[Y_1^*, Y_2]$ agrees with $[Z_1^*, Z_2]$ except for an edge operator. For all other summands, the restriction of $[Y_1^*, Y_2]$ is compact because both terms involve a compact edge operator. This completes the reduction.

Thus we have a submodule $\mathcal{S}$ generated by a set of monomials $\{Z^{\boldsymbol{\alpha}^i} \otimes \boldsymbol{x}_i\}$, in which $\alpha_1^i = a_1$ for all i, $1 \leqslant i \leqslant n$. The remainder of the proof is similar to what was done in the preceding paragraph. Again, $\mathcal{S}$ is generated by the monomials $N_1 = \bigcup_{i=1}^{n} \{Z^{\boldsymbol{\beta}} \otimes \boldsymbol{x}_i : \boldsymbol{\beta} \in B(\boldsymbol{\alpha}^i)\}$ and for each $0 \leqslant j$, we let $\mathcal{H}_j$ be the subspace of $\mathbb{C}^k$ spanned by the $\boldsymbol{x}_i$ for which $Z^{(a_1,j)} \otimes \boldsymbol{x}_i$ is in N_1. Then $\{\mathcal{H}_j\}$ is an increasing sequence of subspaces of $\mathbb{C}^k$ and there exists an increasing sequence $0 \leqslant n_1 < n_2 < \cdots < n_\ell < \infty$ such that every $\mathcal{H}_j$ is equal to one of the $\mathcal{H}_{n_i}$ and each n_i is chosen as small as possible. Then we can express $\mathcal{S}$ as the direct sum of subspaces $\mathcal{S}(i)$, $1 \leqslant i \leqslant \ell$, where $\mathcal{S}(i)$ is the tensor product $\mathcal{M}(B(a_1, i)) \otimes (\mathcal{H}_{n_i} \cap (\mathcal{H}_{n_{i-1}})^\perp)$, again with $\mathcal{H}_{n_0} = (0)$. The self-commutator of the restriction of Z_2 to these subspaces is compact by Corollary 4.1 since $\mathcal{H}_{n_i} \cap (\mathcal{H}_{n_{i-1}})^\perp$ is finite dimensional. The argument for cross-commutators is similar. This completes the proof. ∎

We now extend this result to the case $m > 2$. While our argument is similar to that used above, it requires not only more elaborate decompositions of the subsets of A_m but also induction on m.

THEOREM 4.3. *Let Λ be a weight set for A_m, $1 \leqslant m < \infty$, satisfying $(*)$ and $(**)$ and let $1 \leqslant k < \infty$. Then the submodule $\mathcal{S}$ generated by the set of monomials $\{Z^{\boldsymbol{\alpha}^i} \otimes \boldsymbol{x}_i\}_{i=1}^n$ for $\{\boldsymbol{\alpha}^i\} \subset A_m$ and $\{\boldsymbol{x}_i\} \subset \mathbb{C}^k$ is essentially reductive.*

Proof. Fix m and assume the result holds for all $1 \leqslant m' < m$. The previous result fulfills the induction hypothesis.

We want to first reduce the result to the case in which the first components of the $\boldsymbol{\alpha}^i$ are all constant. Let $\overline{\alpha}_1$ be the largest integer in the given set $\{\alpha_1^i\}$. First we decompose $\mathcal{S}$ into the orthogonal direct sum of the submodule $\mathcal{S}_1$ spanned by the set $\{Z^{(\overline{\alpha}_1, \alpha_2^i, \dots, \alpha_m^i)} \otimes \boldsymbol{x}_i\}$ and $\mathcal{S}_1' = \mathcal{S} \cap (\mathcal{S}_1^{\perp})$, which reduces the restrictions $Y_2, \dots, Y_m$ of $Z_2, \dots, Z_m$, respectively, to $\mathcal{S}$. To see this consider the collection of monomials $N_1 = \bigcup_{i=1}^{n} \{Z^{\boldsymbol{\beta}} \otimes \boldsymbol{x}_i : \boldsymbol{\beta} \in B(\boldsymbol{\alpha}_i), \alpha_1^i \leqslant \beta_1 < \alpha_1\}$. Then $\mathcal{S}_1'$ is the span of N_1. Now we decompose $\mathcal{S}_1'$ into the orthogonal direct sum of $\mathcal{S}_1'(\gamma)$ for $0 \leqslant \gamma < \overline{\alpha}_1$, where $\mathcal{S}_1'(\gamma)$ is the span of the monomials $\{Z^{\boldsymbol{\beta}} \otimes \boldsymbol{x}_i \in N_1 : \beta_1 = \gamma\}$. Each subspace $\mathcal{S}_1'(\gamma)$ reduces the operators, $Y_2, \dots, Y_m$. Moreover, after identifying Σ_1^{γ} with A_{m-1}, we see that $\mathcal{S}_1'(\gamma)$ is a submodule of $\mathcal{M}_{\Lambda_1^{\gamma}} \otimes \mathbb{C}^k$ to which the induction hypothesis applies, where Λ_1^{γ} is the weight set for A_{m-1} obtained by restricting Λ to Σ_1^{γ}. Therefore, the self-commutators of the restrictions of $Z_2, \dots, Z_m$ to each $\mathcal{S}_1'(\gamma)$ are compact. Hence, we can assume the first components of all the multi-indices $\{\boldsymbol{\alpha}^i\}$ are the same. Again, this same decomposition can be used to show that the cross-commutators $[Y_1^*, Y_i]$ for $1 < i \leqslant m$ are compact.

Now starting with such a set $\{Z^{\boldsymbol{\alpha}^i} \otimes \boldsymbol{x}_i\}$, we can reduce the essential normality of $Z_3, \dots, Z_m$ and the compactness of the cross-commutators $[Y_1^*, Y_i]$, $2 \leqslant i \leqslant m$ and $[Y_2^*, Y_j]$, $3 \leqslant j \leqslant m$, to the case in which the first and second components of the $\{\boldsymbol{\alpha}^i\}$ are each constant. Continuing we eventually reduce the essential normality of the restriction of Z_m to $\mathcal{S}$ as well as the compactness of all cross-commutators to the case in which all the $\{\boldsymbol{\alpha}^i\}$ are constant and then the result follows from Corollary 4.1. Thus the restriction of Z_m to $\mathcal{S}$ is essentially normal and all cross-commutators are compact. By symmetry, we conclude that all the cross-commutators $[Y_i^*, Y_j]$, $1 \leqslant i, j \leqslant m$, are compact and hence $\mathcal{S}$ is essentially reductive, which concludes the proof. ∎

Using the geometry of A_m and the finite dimension of $\mathbb{C}^k$ one can show that every submodule $\mathcal{S}$ generated by a set of monomials $\{Z^{\boldsymbol{\alpha}} \otimes \boldsymbol{x}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in C}$, for $C \subset A_m$, is finitely generated. Hence, we can extend Theorems 4.3 and 5.1 (below) to submodules generated by arbitrary collections of monomials.

5. p-SUMMABLE SELF COMMUTATORS

We next consider these results in the p-summable context. Let $\mathcal{C}_p$ denote the Schatten p-class (cf. [17]). First, we modify condition $(**)$ as follows:

$(**)_p$ the cross-commutator $[Z_i, Z_j^*]$ is in $\mathcal{C}_p$ for $1 \leqslant i, j \leqslant m$.

In [15], where the Berger-Shaw Theorem was generalized from single operators to the context of Hilbert modules of Krull dimension one, Hilbert modules satisfying $(**)_p$ were called p-reductive. For $(**)$ we pointed out that the Fuglede Theorem allowed an apparent weakening in which only the self-commutators are assumed to be compact. But this argument involves the Calkin algebra. There is a generalization to the Schatten p-class of the Fuglede Theorem, called the Fuglede-Weiss Theorem [19]. However, that result does not seem adequate to reduce $(**)_p$ to assuming p-summability only for the self-commutators.

Note that H_m^2 satisfies $(**)_p$ when $p > m$. For a general weight set, $(**)$ often holds for smaller p, since it is possible for some of the Z_i to be compact or Schatten q-class. (However, compare the remark at the end of the paragraph following the proof of Corollary 6.2.)

If one examines the proofs of the previous four results, including the constructions in them, one sees that in the presence of the stronger hypothesis, one can draw stronger conclusions.

THEOREM 5.1. *Let Λ be a weight set for A_m, $1 \leqslant m < \infty$, satisfying $(*)$ and $(**)_p$ and let $1 \leqslant k < \infty$. Then for the submodule $\mathcal{S}$ generated by the set of monomials $\{Z^{\boldsymbol{\alpha}} \otimes \boldsymbol{x}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in C}$ for $C \subset A_m$ and $\{\boldsymbol{x}_{\boldsymbol{\alpha}}\} \subset \mathbb{C}^k$, the cross-commutators of the co-ordinate functions and their adjoints lie in the Schatten p-class.*

Actually, one can show the same holds for operators defined by functions of the operators which are holomorphic on a polydisk of radius greater than one.

We omit the details of the proof of this theorem but they are precisely the same as before where the condition of being compact is replaced throughout by that of being in the Schatten p-class noting that Lemma 3.2 extends to this case. This is true for both the self-commutators and the cross-commutators.

Actually, one can often draw sharper conclusions for the cross-commutators of the operators defined by the action of the coordinate functions and their adjoints on the quotient module $\mathcal{M}_\Lambda/\mathcal{S}$ in the presence of a slightly stronger version of $(**)_p$ and a strengthened Lemma 3.2, which hold for example, for H_m^2. We will not attempt the maximum generality which would have to take into account degeneracies in the module action. We will use $(**)^p$ to denote the new assumption for a weight set $\Lambda_{\mathcal{M}}$ with the p tied to m. Before providing the statements, however, we need some additional notation.

Let $\boldsymbol{i} = \{i_1, i_2, \ldots, i_\ell\}$ be a subset of $\{1, 2, \ldots, m\}$ so that $i_1 < i_2 < \cdots < i_\ell$, $[\boldsymbol{i}] = \ell$, I^{c} the complement of the $\{i_j\}$ in $\{1, \ldots, m\}$, $\boldsymbol{k} = \{k_1, k_2, \ldots, k_\ell\}$ so that $k_j \geqslant 0$ for $1 \leqslant j \leqslant \ell$, and $\Sigma_{\boldsymbol{i}}^{\boldsymbol{k}} = \{\boldsymbol{\alpha} \in A_m : \alpha_{i_j} = k_j, 1 \leqslant j \leqslant \ell\}$. (Note for $\ell = 1$, we obtain simply the Σ_i^k introduced earlier.) Then $M_\Lambda(\Sigma_{\boldsymbol{i}}^{\boldsymbol{k}})$ is a reducing subspace for Z_p, p in I^c. Moreover, we can identify the polynomials in the $(m-\ell)$-variables, Z_p, for p in I^c with $\mathbb{C}[\boldsymbol{z}]$ and obtain a weight set $\Lambda_{\boldsymbol{i}}^{\boldsymbol{k}}$ by restricting $\Lambda_{\mathcal{M}}$. Our strengthened assumption is:

$(**)^p$ For every $\boldsymbol{i}$ and $\boldsymbol{k}$, the commutators of the restriction of Z_ℓ and the compression of Z_n^* to $\mathcal{M}(\Sigma_{\boldsymbol{i}}^{\boldsymbol{k}})$ for ℓ, n in I^c lies in the Schatten q-class for $q > m - [\boldsymbol{i}]$.

LEMMA 5.2. *Let Λ be a weight set for A_m, $1 \leqslant m < \infty$, satisfing $(*)$ and $(**)^p$. For every $\boldsymbol{i}$ and $\boldsymbol{k}$, if X_p is the operator defined by the action of Z_p from $\mathcal{M}_\Lambda(\Sigma_{\boldsymbol{i}}^{\boldsymbol{k}})$ to $\mathcal{M}_\Lambda$, then $X_p^* X_p$ lies in the Schatten q-class for $q > m - [\boldsymbol{i}]$.*

Now recall that one can define the Hilbert-Samuel polynomial $p_m(z)$ (a polynomial in one variable) for a Hilbert module $\mathcal{M}$ over $\mathbb{C}[\boldsymbol{z}]$ so long as it is finitely generated and the dimension of $\mathcal{M}/[\mathbb{C}_0[\boldsymbol{z}]\mathcal{M}]$ is finite, where $\mathbb{C}_0[\boldsymbol{z}]$ is the ideal of polynomials vanishing at $\mathbf{0}$ and $[\cdot]$ denotes the closure in $\mathcal{M}$ ([16], cf. Theorem 4.2 of [2]). The order of $p_m(z)$ is said to be the dimension of $\mathcal{M}$.

By analyzing the decompositions used in the proofs of the preceding theorems, one can show that the dimension of a quotient module $\mathcal{M} \otimes \mathbb{C}^k/\mathcal{S}$ for a submodule $\mathcal{S}$ generated by monomials is the same as the smallest $[\boldsymbol{i}]$, where $\Sigma_{\boldsymbol{i}}^{\boldsymbol{k}}$ ranges over the blocks used in the decompositions in the proofs. As a consequence one can obtain the following result.

THEOREM 5.3. *Let Λ be a weight set for A_m, $1 \leqslant m < \infty$ satisfying $(*)$ and $(**)^p$, and let $1 \leqslant k < \infty$. If $\mathcal{S}$ is a submodule of $\mathcal{M}_\Lambda \otimes \mathbb{C}^k$ generated by a set of monomials $\{Z^{\boldsymbol{a}} \otimes \boldsymbol{x}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in C}$ for $C \subset A_m$ and $\{x_{\boldsymbol{\alpha}}\} \subset \mathbb{C}^k$, then the commutators of the operators defined by the coordinate functions on the quotient module $\mathcal{M}_\Lambda \otimes \mathbb{C}^k/\mathcal{S}$ and their adjoints lie in the Schatten q-class for $q > d$, where d is the dimension of $\mathcal{M}_\Lambda \otimes \mathbb{C}_k/\mathcal{S}$.*

Since, as we pointed out above, the weight set for H_m^2 satisfies conditions $(*)$ and $(**)^p$ for $p > m$, it seems likely that the stronger conclusion of this theorem would hold for quotient modules $H_m^2/\mathcal{S}$ for an arbitrary submodule $\mathcal{S}$ generated by homogeneous polynomials, assuming that Arveson's conjecture is valid.

6. K-HOMOLOGY CLASSES

Let $\mathcal{M}_\Lambda$ be the Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ defined by a weight set Λ for A_m satisfying $(*)$ and $(**)$. As we pointed out earlier, since multiplication by the coordinate functions is contractive by $(*)$, $\mathcal{M}_\Lambda$ is a bounded Hilbert module over the algebra $A(\mathbb{D}_r^m)$ of functions holomorphic on a polydisk $\mathbb{D}_r^m$ in $\mathbb{C}^m$ of radius $r > 1$. Again $\mathcal{J}(M_\Lambda)$ denotes the C^*-algebra in $\mathcal{L}(\mathcal{M}_\Lambda)$ generated by $I_{\mathcal{M}_\Lambda}$, $\{Z_i\}$ and the compact operators $\mathcal{K}(\mathcal{M}_\Lambda)$, and $\mathcal{J}(\mathcal{M}_\Lambda)/\mathcal{K}(\mathcal{M}_\Lambda)$ denotes the quotient algebra. By $(**)$, the quotient algebra is commutative and there exists a compact metric space X_Λ such that $\mathcal{J}(\mathcal{M}_\Lambda)/\mathcal{K}(\mathcal{M}_\Lambda) \cong C(X_\Lambda)$. Using ideas from the proof of Theorem 1 in [10] one can show:

THEOREM 6.1. *For Λ a weight set for A_m satisfying $(*)$ and $(**)$, X_Λ can be identified as a subset of the closed polydisk ${}^{\mathrm{cl}}\mathbb{D}^m$ so that Z_i corresponds to the restriction of z_i to X_Λ.*

Proof. One obtains for every $r > 1$ a homomorphism from $A(\mathbb{D}^m_r)$ to $\mathcal{J}(\mathcal{M}_\Lambda)$ and then to $\mathcal{J}(\mathcal{M}_\Lambda)/\mathcal{K}(\mathcal{M}_\Lambda)$. Thus we have a bounded homomorphism from $A(\mathbb{D}^m_r)$ to $C(X_\Lambda)$. Thus X_Λ can be identified as a closed subset of ${}^{\mathrm{cl}}\mathbb{D}^m_r$ for $r > 1$. But the identifications for $r_1, r_2 > 1$ must be consistent which implies $X_\Lambda \subset {}^{\mathrm{cl}}\mathbb{D}^m$, which completes the proof. ∎

The set X_Λ is not an arbitrary one since it must be invariant under multiplication by $e^{i\boldsymbol{\theta}} \equiv (e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_m})$ because multiplying the m-tuple $(Z_1, \ldots, Z_m)$ by $e^{i\boldsymbol{\theta}}$ yields an m-tuple of operators which is unitarily equivalent to the original one. In the case of the Hardy or Bergman module over $\mathbb{B}^m$ or the m-shift Hardy module H^2_m, X_Λ is $\partial\mathbb{B}^m$. If the $(Z_1, Z_2, \ldots, Z_m)$ form a spherical contraction, (that is, $Z_1^*Z_1 + \cdots + Z_m^*Z_m \leqslant I$, then X_Λ is contained in ${}^{\mathrm{cl}}\mathbb{B}^m$. In general, it need not equal $\partial\mathbb{B}^m$.

COROLLARY 6.2. *A weight set Λ for A_m satisfying $(*)$ and $(**)$ determines a canonical element $[\Lambda]$ in $K_1(X_\Lambda)$.*

Proof. This follows from [7] since we have an extension $0 \to \mathcal{K}(\mathcal{M}_\Lambda) \to \mathcal{J}(\mathcal{M}_\Lambda) \to C(X_\Lambda) \to 0$ by the Theorem 6.1. ∎

This element is not interesting unless the ordinary homology of X_Λ is non-trivial. If X_Λ is contractible, for example, the closed ball ${}^{\mathrm{cl}}\mathbb{D}^m$ or a point, then $K_1(X_\Lambda) \cong (0)$ and there is no invariant. In fact, in this case, one can deform the m-tuple of operators $\{Z_i\}$ to a commuting m-tuple of normal operators [7]. On the other hand, if $X_\Lambda = \partial\mathbb{B}^m$, then $K_1(\partial\mathbb{B}^m) \cong \mathbb{Z}$, and there is a non-zero invariant. In particular, the invariant corresponds to -1, giving $\partial\mathbb{B}^m$ the standard orientation, for Λ the weight sets for the Bergman, Hardy or m-shift Hardy modules for $\mathbb{B}^m$. If we consider $\mathcal{M}_\Lambda \otimes \mathbb{C}^k$, then the K-homology element is multiplied by k. One knows for extensions over $\partial\mathbb{B}^m$, that the K-homology element is determined by the index of the Koszul complex defined by a commuting m-tuple of generators [5], [7] if such an m-tuple exists. In other cases, one must resort to different measures [9], [3]. Thus, in the case of H^2_m, the K-homology invariant coincides with the curvature invariant of Arveson [1]. Using the main result in [14], one can also show if $[\Lambda] \neq 0$ and Λ satisfies $(**)_p$, then $p > m$.

If $\mathcal{S}$ is a submodule of $\mathcal{M}_\Lambda \otimes \mathbb{C}^k$ which is essentially reductive, then repeating the construction in the Theorem 6.1 yields a closed subset $X_\mathcal{S}$ of X_Λ for which $\mathcal{J}(\mathcal{S})/K(\mathcal{S}) \cong C(X_\mathcal{S})$ and hence an element $[\mathcal{S}]$ in $K_1(X_\mathcal{S})$ is defined. Similarly the quotient module $\mathcal{S}^\perp = \mathcal{M}_\Lambda \otimes \mathbb{C}^k/\mathcal{S}$ yields an element $[\mathcal{S}^\perp]$ in $K_1(X_{\mathcal{S}^\perp})$. One can show that $X_\mathcal{S} \cup X_{\mathcal{S}^\perp} = X_\Lambda$ and that $i^1_*([\mathcal{S}]) + i^2_*([\mathcal{S}^\perp]) = [\Lambda]$, where i^1 and

i^2 are the inclusion maps of $X_{\mathcal{S}}$ and $X_{\mathcal{S}^\perp}$ into X_Λ, respectively. If $X_\Lambda = \partial\mathbb{B}^m$ and $[X_\Lambda] \neq 0$, then at least one of $[X_{\mathcal{S}}]$ and $[X_{\mathcal{S}^\perp}]$ is non-trivial and the corresponding m-tuple of operators defined by $\{Z_i\}$ can not be perturbed to a commuting m-tuple of normal operators. For $k = 1$, one might conjecture that $[\mathcal{S}^\perp] = 0$ in this case unless $\mathcal{S} = (0)$.

For $\mathcal{S}$ a submodule of $\mathcal{M}_\Lambda$ generated by monomials, that is, the case considered in this note, one can show that $X_{\mathcal{S}^\perp}$ is contained in the common zero set of the monomials generating it. If $X_\Lambda = \partial\mathbb{B}$, then $i^2_*[S^\perp]$. Hence, $[\mathcal{S}^\perp] = 0$. This argument should work also for $\mathcal{S}$ generated by homogeneous polynomials once one knows that $\mathcal{S}$ is essentially reductive.

One can use the decompositions introduced previously in Sections 3 and 4 to draw more conclusions about $X_{\mathcal{S}^\perp}$ and $[\mathcal{S}^\perp]$ for $\mathcal{S}$ generated by monomials. Let $\mathbb{C}^m_{\deg}$ denote all points in $\mathbb{C}^m$ with at least one coordinate zero and $Y_\Lambda = X_\Lambda \cap \mathbb{C}^m_{\deg}$. Then one can show

THEOREM 6.3. *If $\mathcal{S}$ is a submodule of $\mathcal{M}_\Lambda \otimes \mathbb{C}_k$ generated by the monomials $\{Z^{\boldsymbol{\alpha}} \otimes x_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in C}$ for $C \subset A_m$, where Λ is a weight set for A_m satisfying $(*)$ and $(**)$ with $X_\Lambda = \partial\mathbb{B}$, and such that the $\{\boldsymbol{x}_{\boldsymbol{\alpha}}\}$ span $\mathbb{C}^k$, then $X_{\mathcal{S}^\perp} \subseteq Y_\Lambda$. Hence, $i^2_*[\mathcal{S}^\perp] = 0$ and $i^1_*[\mathcal{S}] = [\Lambda]$.*

Proof. This argument is closely related to the one sketched for Theorem 5.3. One proceeds by obtaining decompositions for $\mathcal{S}^\perp$ analogous to those used in the preceding proofs for $\mathcal{S}$ and then noting that the pieces essentially commute and at least one of the operators $Z_1, \ldots, Z_m$ is compact for each piece. ∎

If the $\{\boldsymbol{x}_i\}$ do not span $\mathbb{C}^k$, then it is possible for $i^2_*[\mathcal{S}^\perp] \neq 0$. With a little more effort, one can say more. Again one would expect the same result to hold for quotient modules defined by submodules generated by homogeneous polynomials if Arveson's conjecture is valid.

THEOREM 6.4. *Under the same hypotheses as in Theorem 6.3, $i^2_*[\mathcal{S}^\perp] = (k - \ell)[\Lambda]$ and $i^1_*[\mathcal{S}] = \ell[\Lambda]$, where ℓ is the dimension of the subspace spanned by the $\{\boldsymbol{x}_{\boldsymbol{\alpha}}\}$ in $\mathbb{C}^k$.*

Note that without the assumption on X_Λ it is possible for $[\Lambda] = 0$ in $K_1(X_\Lambda)$. Hence these equations might be vacuous. However, for the weight set for H^2_m, we obtain another expression for the curvature invariant introduced by Arveson.

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